GRADED MODULES OVER SIMPLE LIE ALGEBRAS

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Abstract. The paper is devoted to the study of graded-simple modules and gradings on simple modules over finite-dimensional simple Lie algebras. In general, a connection between these two objects is given by the so-called loop construction. We review the main features of this construction as well as necessary and sufficient conditions under which finite-dimensional simple modules can be graded. Over the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$, we consider specific gradings on simple modules of arbitrary dimension.

1. Introduction

Let G be a non-empty set. A *G-grading* on a vector space V over a field $\mathbb{F}$ is a direct sum decomposition of the form

$$V = \bigoplus_{g\in G} V_g. \tag{1}$$

We will sometimes use Greek letters to refer to gradings, for example, we may write $\Gamma : V = \bigoplus_{g\in G} V_g$. If such a grading is fixed, V is called *G-graded*.

Note that the V_g are allowed to be zero subspaces. The subset $S \subset G$ consisting of those $g \in G$ for which $V_g \neq \{0\}$ is called the *support* of the grading Γ and denoted by $\operatorname{Supp}\Gamma$ or $\operatorname{Supp} V$. The subspaces V_g are called the *homogeneous components* of Γ, and the nonzero elements in V_g are called *homogeneous of degree g* (with respect to Γ). A *graded subspace* $U \subset V$ is an $\mathbb{F}$-subspace satisfying $U = \bigoplus_{g\in G} U \cap V_g$ (so U itself becomes G-graded).

Now let Γ and $\Gamma' : V = \bigoplus_{g'\in G'} V'_{g'}$ be two gradings on V with supports S and S', respectively. We say that Γ is a *refinement* of Γ' (or Γ' is a *coarsening* of Γ), if for any $s \in S$ there exists $s' \in S'$ such that $V_s \subset V'_{s'}$. The refinement is *proper* if this inclusion is strict for at least one $s \in S$.

An $\mathbb{F}$-algebra A (not necessarily associative) is said to be *graded by a set G*, or *G-graded* if A is a G-graded vector space and for any $g, h \in G$ such that $A_g A_h \neq \{0\}$ there is $k \in G$ (automatically unique) such that

$$A_g A_h \subset A_k. \tag{2}$$

In this paper, we will always assume that G is an *abelian group* and k in Equation (2) is determined by the operation of G. Thus, if G is written additively (as is commonly done in papers on Lie theory), then Equation (2) becomes $A_g A_h \subset A_{g+h}$.

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If G is written multiplicatively, then it becomes $A_g A_h \subset A_{gh}$. More generally, one can consider gradings by nonabelian groups (or semigroups). A grading on A is called *fine* if it does not have a proper refinement. Note that this concept depends on the class of gradings under consideration: by sets, groups, abelian groups, etc. It is well known that the latter two classes coincide for simple Lie algebras.

Given a grading $\Gamma : A = \bigoplus_{g\in G} A_g$ with support S, the *universal group* of Γ, denoted by G^u, is the group given in terms of generators and defining relations as follows: $G^u = \langle S \mid R\rangle$, where R consists of all relations of the form $gh = k$ with $\{0\} \neq A_g A_h \subset A_k$. If Γ is a group grading, then S is embedded in G^u and the identity map id_S extends to a homomorphism $G^u \to G$ so that Γ can be viewed as a G^u-grading Γ^u. In fact, any group grading $\Gamma' : A = \bigoplus_{g'\in G'} A'_{g'}$ that is a coarsening of Γ can be induced from Γ^u by a (unique) homomorphism $\nu : G^u \to G'$ in the sense that $A_{g'} = \bigoplus_{g\in\nu^{-1}(g')} A_g$ for all $g' \in G'$. In this situation, one may say that Γ' is a *quotient* of Γ^u. In the above considerations, we can replace "group" by "abelian group" and, in general, this leads to a different G^u. However, there is no difference for gradings on simple Lie algebras.

For example, choose the elements

$$x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

as a basis of $L = \mathfrak{sl}_2(\mathbb{C})$ and consider the following grading by $G = \mathbb{Z}_3$:

$$\Gamma:\ L_1 = \langle x\rangle,\ L_0 = \langle h\rangle,\ L_2 = \langle y\rangle.$$

The support of Γ is G itself, the universal group is $\mathbb{Z}$, and

$$\Gamma^u:\ L_{-1} = \langle x\rangle,\ L_0 = \langle h\rangle,\ L_1 = \langle y\rangle.$$

The following grading by $G' = \mathbb{Z}_2$ is a coarsening of Γ:

$$\Gamma':\ L_1 = \langle x, y\rangle,\ L_0 = \langle h\rangle.$$

Both Γ and Γ' are quotients of Γ^u, while Γ' is a coarsening but not a quotient of Γ.

A left module M over a G-graded associative algebra A is called *G-graded* if M is a G-graded vector space and

$$A_g M_h \subset M_{gh} \text{ for all } g, h \in G.$$

A G-graded left A-module M is called *graded-simple* if M has no graded submodules different from $\{0\}$ and M. Graded modules and graded-simple modules over a graded Lie algebra L are defined in the same way.

If a Lie algebra L is graded by an abelian group G, then its universal enveloping algebra $U(L)$ is also G-graded. Every graded L-module is a graded left $U(L)$-module and *vice versa*. The same is true for graded-simple modules.

A very general problem is the following: given a module V over a G-graded Lie algebra L, determine if V can be given a G-grading that is compatible with the G-grading on L, i.e., one that makes V a graded L-module. In this paper, we restrict ourselves to the case where L is a finite-dimensional simple Lie algebra over an algebraically closed field of characteristic zero and focus on simple L-modules.

For finite-dimensional V, the answer is given in [EK15a], where the authors classified finite-dimensional graded-simple modules up to isomorphism and, as a corollary, determined which finite-dimensional simple modules can be made graded and which finite-dimensional modules can be made graded-simple. The classification depends on the so-called graded Brauer invariants (see Subsections 4.3 and

4.4 for definitions), which were computed in [EK15a] for all classical simple Lie algebras except D_4 and for the remaining types in [EK15b, DEK17]. We note that it is difficult to obtain an explicit grading on V using this approach.

If we do not restrict ourselves to finite-dimensional modules, the first question that arises is that, in general, there is no classification of simple modules of arbitrary dimension for any simple Lie algebra, with the exception of $L = \mathfrak{sl}_2(\mathbb{C})$, for which a classification was suggested by R. Block [Blo81]. Despite this, in a number of more recent papers, the authors still try to give a more transparent description of simple $\mathfrak{sl}_2(\mathbb{C})$-modules. We refer the reader to the monograph [Maz09]; some other works in this area are [AP74, Bav92, EK15a, PT17, MZpr, Nil15].

We start this paper by reviewing the criteria of [EK15a, EK15b, DEK17] for the existence of a compatible grading on a finite-dimensional simple module V. Then we focus on the case $L = \mathfrak{sl}_2(\mathbb{C})$, where we give explicit gradings for those V that admit them.

After this we switch to infinite-dimensional simple $\mathfrak{sl}_2(\mathbb{C})$-modules. We review their construction and determine, for some of these modules, whether they can be made graded or not.

Finally, we turn our attention to reviewing the main results of [EK17]. Therein, it is described how the so-called loop construction could be used for the classification of graded-simple modules of arbitrary dimension. It should be noted that, even in the case $L = \mathfrak{sl}_2(\mathbb{C})$, this classification remains an interesting open problem.

2. Finite-dimensional simple modules over finite-dimensional simple Lie algebras

Let L be a finite-dimensional simple Lie algebra over an algebraically closed field $\mathbb{F}$ of characteristic 0 and suppose L is graded by an abelian group G. In this section, we will give necessary and sufficient conditions for the finite-dimensional simple L-module $V(\lambda)$ of highest weight λ to admit a structure of G-graded L-module.

All G-gradings on L are known (see e.g. the monograph [EK13, Ch. 3–6]): they have been classified up to isomorphism for all types except E_6, E_7 and E_8, and for these latter, the fine gradings have been classified ([DV16, Yu16, Eld16]), which gives a description of all G-gradings as follows. Every G-grading Γ on L is a coarsening of at least one fine grading Δ, so Γ is induced by a homomorphism $\nu : G^u \to G$, where G^u is the universal group of Δ. In other words, Γ is obtained by assigning the degree $\nu(s) \in G$ to all nonzero elements of L that are homogeneous of degree $s \in G^u$ with respect to Δ. The isomorphism problem for G-gradings on L of types E_6, E_7 and E_8 remains open.

Let $\widehat{G}$ be the group of characters of G, i.e., group homomorphisms $\chi : G \to \mathbb{F}^\times$. If W is a G-graded vector space then $\widehat{G}$ acts on W as follows:

$$\chi \cdot w = \chi(g)w \quad \forall \chi \in \widehat{G},\, g \in G,\, w \in W_g \tag{3}$$

(extended by linearity). For the given G-grading on the Lie algebra L, such action defines a homomorphism $\widehat{G} \to \mathrm{Aut}(L)$ sending $\chi \mapsto \alpha_\chi$ where $\alpha_\chi(x) := \chi \cdot x$ for all $x \in L$. The grading is called *inner* if all α_χ belong to the group of inner automorphisms $\mathrm{Int}(L)$, otherwise it is called *outer*. Let τ_χ be the image of α_χ in the outer automorphism group $\mathrm{Out}(L) := \mathrm{Aut}(L)/\mathrm{Int}(L)$.

Fixing a Cartan subalgebra and a system of simple roots $\alpha_1, \dots, \alpha_r$ for L, we may identify $\mathrm{Out}(L)$ with the group of automorphisms of the Dynkin diagram of

L, which permutes $\alpha_1, \dots, \alpha_r$ and hence acts on the lattice of integral weights. Let

$$K_\lambda = \{\chi \in \widehat{G} : \tau_\chi(\lambda) = \lambda\} \text{ and } H_\lambda = K_\lambda^\perp := \{h \in G : \chi(h) = 1 \ \forall \chi \in K_\lambda\}.$$

Observe that $|H_\lambda| = [\widehat{G} : K_\lambda]$ is the size of the $\widehat{G}$-orbit of λ. The nontriviality of H_λ is the first obstruction for $V(\lambda)$ becoming a G-graded L-module (see [EK15a, §3.1]).

Denote the fundamental weights of L by $\pi_1, \dots, \pi_r$ and write $\lambda = \sum_{i=1}^r m_i \pi_i$, $m_i \in \mathbb{Z}_{\geq 0}$. Our numbering of the simple roots is shown for each type of L on the diagrams below. In all cases, $V(\pi_1)$ has the lowest possible dimension among the nontrivial L-modules (which is the reason why we prefer C_2 over B_2). Let $H = H_{\pi_1}$. We have $|H| \leq 2$ for types A_r $(r \geq 2)$ and E_6, $|H| \leq 3$ for D_4, and $|H| = 1$ for all other types.

Consider the homomorphism $\varrho_\lambda : U(L) \to E := \mathrm{End}_{\mathbb{F}}(V(\lambda))$ associated to the L-action on $V(\lambda)$. It turns out that there is a unique G/H_λ-grading on the simple associative algebra E such that ϱ_λ becomes a homomorphism of graded algebras (see [EK15a, §3.2]). For this grading on E, there exist a graded-division algebra $\mathcal{D}$ and a graded right $\mathcal{D}$-module $\mathcal{V}$ such that E is isomorphic to $\mathrm{End}_{\mathcal{D}}(\mathcal{V})$ as a G-graded algebra (see e.g. [EK13, Theorem 2.6]), where $\mathcal{D}$ is unique up to graded isomorphism and $\mathcal{V}$ up to graded isomorphism and shift of grading (see e.g. [EK13, Theorem 2.10]). Here, a *graded-division algebra* is a graded unital associative algebra in which every nonzero homogeneous element is invertible, and the *shift of grading* by an element $g \in G$ replaces a G-graded vector space W with $W^{[g]}$, which equals W as a vector space, but the elements that had degree g' will now have degree $g'g$, for any $g' \in G$. The graded-division algebra $\mathcal{D}$ represents the *graded Brauer invariant* of $V(\lambda)$, and its nontriviality is the second obstruction for $V(\lambda)$ becoming a G-graded L-module (see [EK15a, §3.2]). A generalization of this analysis is outlined in Subsections 4.3 and 4.4 below, following [EK17].

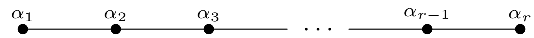
Group gradings on classical simple Lie algebras were classified by studying $\mathcal{D}$ and $\mathcal{V}$ associated to the 'natural module' $V(\pi_1)$. Since $\mathcal{D}$ is a graded-division algebra, we can find a $\mathcal{D}$-basis $\{v_1, \dots, v_k\}$ of $\mathcal{V}$ that consists of homogeneous elements. Let $g_1, \dots, g_k$ be the degrees of the basis elements. If H is nontrivial, we will write $\bar{g}_1, \dots, \bar{g}_k$ to remind ourselves that these degrees belong to G/H. Let T be the support of $\mathcal{D}$, which is a finite subgroup of G/H. Pick any nonzero elements X_t of $\mathcal{D}_t$, $t \in T$. Note that all homogeneous components of $\mathcal{D}$ are one-dimensional, because $\mathcal{D}_e = \mathbb{F}1$ (being a finite-dimensional division algebra over the algebraically closed field $\mathbb{F}$) and hence $\mathcal{D}_t = \mathcal{D}_e X_t = \mathbb{F} X_t$. Hence,

$$X_s X_t = \beta(s,t) X_t X_s \quad \forall s, t \in T, \tag{4}$$

where $\beta : T \times T \to \mathbb{F}^\times$ is an alternating bicharacter, i.e., $\beta(s_1 s_2, t) = \beta(s_1, t)\beta(s_2, t)$, $\beta(t, s_1 s_2) = \beta(t, s_1)\beta(t, s_2)$, and $\beta(t,t) = 1$ for all $s_1, s_2, t \in T$. Bicharacters are analogous to bilinear forms, so we are using the same terminology. In particular, the *radical* of β is the subgroup $\{s \in T : \beta(s,t) = 1 \ \forall t \in T\}$. Since the algebra $\mathrm{End}_{\mathbb{F}}(V(\pi_1))$ is central simple, so is $\mathcal{D}$, and hence the radical of β must be trivial. Alternating bicharacters with trivial radical are said to be *nondegenerate*. They admit a 'symplectic basis' (see e.g. [EK13, Ch. 2, §2]), which implies that there exist subgroups P and Q of T such that $T = P \times Q$, the restrictions of β to these subgroups are trivial, and the mapping $P \to \widehat{Q}$ sending $p \mapsto \beta(p, \cdot)$ is an isomorphism. Therefore, $|T| = \ell^2$ where $\ell = |P| = |Q|$. Note that in our case ℓ is the degree of the matrix algebra $\mathcal{D}$, hence $k\ell = n := \dim V(\pi_1)$.

The bicharacter β is clearly independent of the choice of the elements X_t. Even though the k-tuple $(g_1, \ldots, g_k)$ depends on the choice of the basis $\{v_1, \ldots, v_k\}$, the multiset $\Xi := \{g_1T, \ldots, g_kT\}$ in G/T is uniquely determined by $\mathcal{V}$. T, β and Ξ are among the parameters that define the grading on L up to isomorphism. Some other parameters will be introduced later as needed.

$A_r \quad (r \geq 1)$



For this type $n = r + 1$. Note that if $r \geq 2$ then there are two possibilities for π_1, which lead to L-modules that are dual to one another.

We have $|H| = 1$ if the grading on L is inner and $|H| = 2$ if it is outer. In the latter case, the grading determines a nondegenerate homogeneous φ_0-sesquilinear form $B : \mathcal{V} \times \mathcal{V} \to \mathcal{D}$, where φ_0 is an orthogonal involution on the G/H-graded matrix algebra $\mathcal{D}$ (see [EK13, Ch. 2, §4 and Ch. 3, §1]). The existence of φ_0 implies that T is an elementary 2-group, so ℓ is a power of 2. The degree $\overline{g}_0 \in G/H$ of B is another parameter of the grading on L. If n is even, set

$$g_{\Xi,\bar{g}_0} := \begin{cases} \overline{g}_0^{\,n/2}(\overline{g}_1 \cdots \overline{g}_k)^{\ell} & \text{if } \ell \neq 2, \\ (\bar{c}\overline{g}_0)^{n/2}(\overline{g}_1 \cdots \overline{g}_k)^{\ell} & \text{if } \ell = 2, \end{cases} \tag{5}$$

where, for $\ell = 2$, $\bar{c}$ is the unique element of T such that $\varphi_0(X_{\bar{c}}) = -X_{\bar{c}}$.

Theorem 2.1 ([EK15a, Corollaries 16 and 24])**.** Suppose a simple Lie algebra L of type A_r is given a G-grading with parameters T, β, Ξ and, if the grading is outer, also $\overline{g}_0 \in G/H$ as described above. Consider the finite-dimensional simple L-module $V(\lambda)$ of highest weight $\lambda = \sum_{i=1}^{r} m_i\pi_i$.

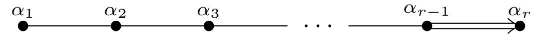
I If the grading on L is inner, then $V(\lambda)$ admits a G-grading making it a graded L-module if and only if the number $\sum_{i=1}^{r} im_i$ is divisible by the exponent of the group T.

II If the grading on L is outer (hence $r \geq 2$), then $V(\lambda)$ admits a G-grading making it a graded L-module if and only if the following two conditions are satisfied:

1) $m_i = m_{r+1-i}$ for all i;
2) either r is even or r is odd and at least one of the following holds:
   (i) $m_{(r+1)/2}$ is even, or
   (ii) $r \equiv 3 \pmod 4$ and $g_{\Xi,\bar{g}_0}$ is the trivial element of G/H, or
   (iii) $r \equiv 1 \pmod 4$, $|T| = 1$, and $g_{\Xi,\bar{g}_0}$ is the trivial element of G/H,

where $g_{\Xi,\bar{g}_0}$ is defined by Equation (5). □

$B_r \quad (r \geq 3)$



For this type $n = 2r+1$ is odd and $|H| = 1$. The existence of an involution on $\mathcal{D}$ implies that T is an elementary 2-group, so ℓ is a power of 2 dividing n, hence $\ell = 1$, $k = n$ and $\mathcal{D} = \mathbb{F}$. The grading on L determines a nondegenerate homogeneous symmetric bilinear form $B : \mathcal{V} \times \mathcal{V} \to \mathbb{F}$, which may be assumed to have degree e (at the expense of shifting the grading on $\mathcal{V}$, see [EK13, Ch. 3, §4]). This implies that the multiset $\Xi = \{g_1, \ldots, g_n\}$ is 'balanced' in the sense that, for any $g \in G$, the multiplicities of g and g^{-1} in Ξ are equal to one another. We order the n-tuple

$(g_1,\dots,g_n)$ so that $g_i^2=e$ for $1\le i\le q$ and $g_i^2\ne e$ for $i>q$, where $1\le q\le n$ and q is odd. For $i=1,\dots,q$, set

$$\tilde g_i := g_1\cdots g_{i-1}g_{i+1}\cdots g_q. \tag{6}$$

Then $\tilde g_i^2=e$ and $\tilde g_1\cdots\tilde g_q=e$. Consider the group homomorphism $f_\Xi:\widehat G\to\mathbb{Z}_2^q$ given by

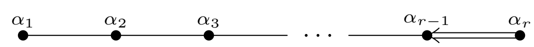
$$f_\Xi(\chi):=(x_1,\dots,x_q) \text{ where } \chi(\tilde g_i)=(-1)^{x_i}. \tag{7}$$

It determines the graded Brauer invariant of the *spin module* $V(\omega_r)$ (see [EK15a, §5]), but here we only state the following:

Theorem 2.2 ([EK15a, Corollary 29])**.** Suppose a simple Lie algebra L of type B_r is given a G-grading with parameter Ξ as described above. The finite-dimensional simple L-module $V(\lambda)$ of highest weight $\lambda=\sum_{i=1}^r m_i\pi_i$ admits a G-grading making it a graded L-module if and only if at least one of the following holds:

(i) m_r is even, or
(ii) the elements $\tilde g_1,\dots,\tilde g_q$ of G defined by Equation (6) and the homomorphism $f_\Xi:\widehat G\to\mathbb{Z}_2^q$ defined by Equation (7) have the following property: for any $x\in f_\Xi(\widehat G)$, $\tilde g_1^{x_1}\cdots\tilde g_q^{x_q}=e$. □

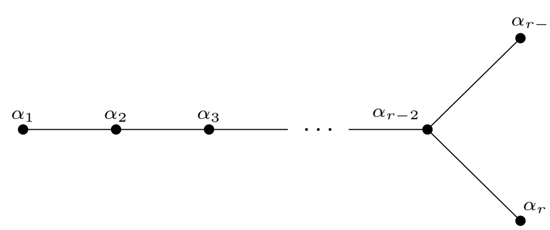
$C_r\quad (r\ge 2)$



For this type $n=2r$, $|H|=1$, and again the existence of an involution on $\mathcal{D}$ implies that T is an elementary 2-group.

Theorem 2.3 ([EK15a, Corollary 32])**.** Suppose a simple Lie algebra L of type C_r is given a G-grading with parameter T as described above. The finite-dimensional simple L-module $V(\lambda)$ of highest weight $\lambda=\sum_{i=1}^r m_i\pi_i$ admits a G-grading making it a graded L-module if and only if either $|T|=1$ or $\sum_{i=1}^{\lfloor (r+1)/2\rfloor} m_{2i-1}$ is even. □

$D_r\quad (r\ge 4)$



For this type $n=2r$ and, unless $r=4$, $|H|=1$. For type D_4, we have $|H|\le 3$ and we can avoid the case $|H|=2$: if $\widehat G$ interchanges two of the outer vertices of the Dynkin diagram, we label by 1 the fixed outer vertex.

Assume that the grading on L is inner. Then $|H|=1$ and the grading determines a nondegenerate homogeneous φ_0-hermitian form $B:\mathcal{V}\times\mathcal{V}\to\mathcal{D}$, where φ_0 is an orthogonal involution on the G-graded matrix algebra $\mathcal{D}$. Let $g_0\in G$ be the degree of B. The existence of φ_0 again implies that T is an elementary 2-group, so ℓ is a power of 2 dividing n.

We need to take a closer look at φ_0. Since it preserves degree and all components of $\mathcal{D}$ are one-dimensional, we have

$$\varphi(X_t)=\beta(t)X_t \quad \forall t\in T$$

where $\beta : T \to \{\pm 1\}$, and Equation (4) shows that $\beta(st) = \beta(s)\beta(t)\beta(s,t)$ for all $s, t \in T$, i.e., $\beta(\cdot)$ is a quadratic form with polar form $\beta(\cdot,\cdot)$ if we regard T as a vector space over the field $\mathbb{Z}_2$. Moreover, this quadratic form has Arf invariant 0 because φ_0 is orthogonal.

The multiset $\Xi = \{g_1T, \dots, g_kT\}$ is 'g_0-balanced' in the following sense: if g' and g'' in G satisfy $g_0g'g'' \in T$ then $g'T$ and $g''T$ have the same mutliplicity in Ξ. We order the k-tuple $(g_1, \dots, g_k)$ so that $g_0g_i^2 \in T$ for $1 \le i \le q$ and $g_0g_i^2 \notin T$ for $i > q$, where $0 \le q \le k$ and q has the same parity as k. The cases $\ell = 1$, $\ell = 2$, $\ell = 4$, and $\ell > 4$ require different computations to find the graded Brauer invariants of the *half-spin modules* $V(\pi_{r-1})$ and $V(\pi_r)$ (see [EK15a, §7.3]), so we consider these cases separately. If $q = 0$, the invariants are trivial, so we assume $q \ge 1$.

$\boxed{\ell = 1}$ This case is similar to type B_r: $k = n$, $\mathcal{D} = \mathbb{F}$, and we may assume $g_0 = e$ at the expense of shifting the grading on $\mathcal{V}$ (see [EK15a, Remark 42]). For $i = 1, \dots, q$, we have $g_i^2 = e$, and it can be shown that $g_1 \cdots g_q = e$. Consider the group homomorphism $f_\Xi : \widehat{G} \to \mathbb{Z}_2^q$ given by

$$f_\Xi(\chi) := (x_1, \dots, x_q) \text{ where } \chi(g_i) = (-1)^{x_i}. \tag{8}$$

It determines the graded Brauer invariants of the half-spin modules, which in this case are equal to one another.

In all remaining cases, these invariants are distinct (although related), and the grading on L can be used to define a specific nonscalar central element of the spin group (see [EK15a, §7.3]), whose action determines the designation of one of the half-spin modules as S^+ and the other as S^-. For $i = 1, \dots, q$, set

$$t_i := g_0g_i^2.$$

These elements of T determine the canonical form of $B : \mathcal{V} \times \mathcal{V} \to \mathcal{D}$ and satisfy $\beta(t_i) = 1$ for all i.

$\boxed{\ell = 2}$ Write $T = \{e, a, b, c\} \simeq \mathbb{Z}_2^2$ where $\beta(a) = \beta(b) = 1$ and $\beta(c) = -1$, so $t_i \in \{e, a, b\}$. For any $t \in T$, define

$$I_t = \{1 \le i \le q : t_i = t\}.$$

Then $I_c = \emptyset$ and the sets I_e, I_a and I_b form a partition of $\{1, \dots, q\}$. It can be seen that $|I_e|$, $|I_a|$ and $|I_b|$ have the same parity as r. Set

$$g_a = g_0^{(|I_e|+|I_a|)/2} \prod_{i \in I_e \cup I_a} g_i \quad \text{and} \quad g_b = g_0^{(|I_e|+|I_b|)/2} \prod_{i \in I_e \cup I_b} g_i. \tag{9}$$

These elements determine the graded Brauer invariant of S^+ and hence of S^-.

$\boxed{\ell = 4}$ Recall that T has a 'symplectic basis': $T = \langle a_1, a_2, b_1, b_2 \rangle \simeq \mathbb{Z}_2^4$ where $\beta(a_i, b_j) = (-1)^{\delta_{ij}}$ and the values of $\beta(\cdot,\cdot)$ on the remaining pairs of basis elements are equal to 1. We choose the basis in such a way that $\beta(a_j) = \beta(b_j) = 1$ for $j = 1, 2$ (in other words, with respect to the quadratic form $\beta(\cdot)$, the subgroups $\langle a_1, a_2 \rangle$ and $\langle b_1, b_2 \rangle$ are totally isotropic). Then the following 4×4 matrix with entries in $\mathbb{Z}_2$ determines the graded Brauer invariant of S^+:

$$M^+_{\Xi, g_0} = \sum_{i=1}^q M^+(t_i), \tag{10}$$

where, for any $t = a_1^{x_1} a_2^{x_2} b_1^{y_1} b_2^{y_2}$, the symmetric matrix $M^+(t)$ is given by

$$M^+(t) = \begin{bmatrix} 0 & (x_1+1)(x_2+1) & 0 & (x_1+1)(y_2+1) \\ & 0 & (x_2+1)(y_1+1) & 1 \\ & & 0 & (y_1+1)(y_2+1) \\ \text{sym} & & & 0 \end{bmatrix}.$$

$\boxed{\ell > 4}$ In this case, the graded Brauer invariant of S^+ is trivial.

Theorem 2.4 ([EK15a, Corollaries 47 and 49] and [EK15b, Corollary 24])**.** Suppose a simple Lie algebra L of type D_r is given a G-grading and consider the finite-dimensional simple L-module $V(\lambda)$ of highest weight $\lambda = \sum_{i=1}^r m_i \pi_i$.

I If the grading on L is inner, with parameters T, β, Ξ and g_0 as described above, then $V(\lambda)$ admits a G-grading making it a graded L-module if and only if one of the following conditions is satisfied:

1) $|T| = 1$ and at least one of the following holds:
   (i) $m_{r-1} \equiv m_r \pmod 2$, or
   (ii) the elements $g_1, \ldots, g_q$ and the homomorphism $f_\Xi : \widehat{G} \to \mathbb{Z}_2^q$ defined by Equation (8) have the following property: for any $x \in f_\Xi(\widehat{G})$, $g_1^{x_1} \cdots g_q^{x_q} = e$;
2) $|T| > 1$, $m_{r-1} \equiv m_r \pmod 2$ and one of the following holds:
   (i) r is even and $\sum_{i=1}^{r/2} m_{2i-1}$ is even, or
   (ii) r is odd and $\sum_{i=1}^{(r-1)/2} m_{2i-1} - (m_{r-1} - m_r)/2$ is even;
3) $m_{r-1} \not\equiv m_r \pmod 2$, r is even, $\sum_{i=1}^{r/2} m_{2i-1}$ is even, and one of the following holds:
   (i) $|T| = 4$ and the elements g_a and g_b defined by Equation (9) belong to T, or
   (ii) $|T| = 16$ and the matrix M^+_{Ξ, g_0} defined by Equation (10) is 0, or
   (iii) $|T| > 16$,
   where in 3) we assume that the numbering of the simple roots is chosen so that $V(\pi_r) = S^+$.

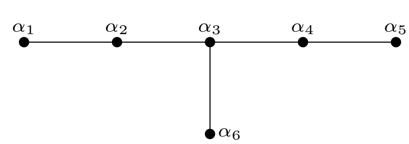
II If the grading on L is outer and, in the case $r = 4$, the $\widehat{G}$-action is not transitive on the outer vertices of the Dynkin diagram, then $V(\lambda)$ admits a G-grading making it a graded L-module if and only if the following two conditions are satisfied:

1) $m_{r-1} = m_r$;
2) $|T| = 1$ or $\sum_{i=1}^{\lfloor r/2 \rfloor} m_{2i-1}$ is even;

where in the case $r = 4$ we assume that the numbering of the simple roots is chosen so that π_1 is fixed by $\widehat{G}$.

III If $r = 4$ and the $\widehat{G}$-action is transitive on the outer vertices of the Dynkin diagram, then $V(\lambda)$ admits a G-grading making it a graded L-module if and only if $m_1 = m_3 = m_4$. $\square$

E_6



For this type the dimension of $V(\pi_1)$ is 27 (there are two possibilities for π_1, which lead to dual modules), and we have $|H| = 1$ if the grading on L is inner and $|H| = 2$ if it is outer.

Out of the 14 fine gradings on L (up to equivalence), 5 are inner, with universal groups $\mathbb{Z}^6$, $\mathbb{Z}^2 \times \mathbb{Z}_3^2$, $\mathbb{Z}^2 \times \mathbb{Z}_2^3$, $\mathbb{Z}_3^4$ and $\mathbb{Z}_2^3 \times \mathbb{Z}_3^2$, and 9 are outer, with universal groups $\mathbb{Z}^4 \times \mathbb{Z}_2$, $\mathbb{Z}^2 \times \mathbb{Z}_2^3$, $\mathbb{Z} \times \mathbb{Z}_2^5$, $\mathbb{Z} \times \mathbb{Z}_2^4$, $\mathbb{Z}_2 \times \mathbb{Z}_3^3$, $\mathbb{Z}_2^7$, $\mathbb{Z}_2^6$, $\mathbb{Z}_4^3$ and $\mathbb{Z}_4 \times \mathbb{Z}_2^4$.

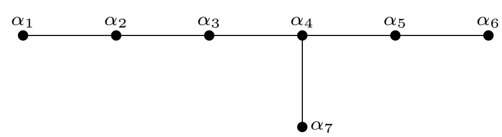
For each of the inner fine gradings on L with $G^u = \mathbb{Z}^2 \times \mathbb{Z}_3^2$, $\mathbb{Z}_2^3 \times \mathbb{Z}_3^2$ and $\mathbb{Z}_3^4$, there is a distinguished subgroup $T \simeq \mathbb{Z}_3^2$ of G^u, which is associated to the graded Brauer invariant of $V(\pi_1)$. For all other fine gradings, this invariant is trivial (see [DEK17, §4]).

Theorem 2.5 ([DEK17, Corollaries 4.2 and 4.5])**.** Suppose a simple Lie algebra L of type E_6 is given a G-grading induced by a homomorphism $\nu : G^u \to G$ from one of the fine gradings. Consider the finite-dimensional simple L-module $V(\lambda)$ of highest weight $\lambda = \sum_{i=1}^{6} m_i\pi_i$.

I If the grading on L is inner, then $V(\lambda)$ admits a G-grading making it a graded L-module if and only if one of the following conditions is satisfied:
 1) G^u is not one of the groups $\mathbb{Z}^2 \times \mathbb{Z}_3^2$, $\mathbb{Z}_2^3 \times \mathbb{Z}_3^2$ and $\mathbb{Z}_3^4$;
 2) G^u is $\mathbb{Z}^2 \times \mathbb{Z}_3^2$, $\mathbb{Z}_2^3 \times \mathbb{Z}_3^2$ or $\mathbb{Z}_3^4$ and at least one of the following holds:
  (i) $m_1 - m_2 + m_4 - m_5 \equiv 0 \pmod 3$, or
  (ii) ν is not injective on the distinguished subgroup $T \subset G^u$.

II If the grading on L is outer, then $V(\lambda)$ admits a G-grading making it a graded L-module if and only if $m_1 = m_5$ and $m_2 = m_4$. □

E_7



For this type the dimension of $V(\pi_1)$ is 56 and we have $|H| = 1$. There are 14 fine gradings on L (up to equivalence), with universal groups $\mathbb{Z}^7$, $\mathbb{Z}^3 \times \mathbb{Z}_2^3$, $\mathbb{Z} \times \mathbb{Z}_3^3$, $\mathbb{Z}_2^2 \times \mathbb{Z}_3^3$, $\mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_4^2$, $\mathbb{Z}_2^3 \times \mathbb{Z}_4^2$, $\mathbb{Z}_2 \times \mathbb{Z}_4^3$, $\mathbb{Z}^4 \times \mathbb{Z}_2^2$, $\mathbb{Z}^2 \times \mathbb{Z}_2^4$, $\mathbb{Z} \times \mathbb{Z}_2^5$, $\mathbb{Z} \times \mathbb{Z}_2^6$, $\mathbb{Z}_2^7$, $\mathbb{Z}_2^5 \times \mathbb{Z}_4$ and $\mathbb{Z}_2^8$.

For the fine gradings on L with $G^u = \mathbb{Z}^7$, $\mathbb{Z}^3 \times \mathbb{Z}_2^3$ and $\mathbb{Z} \times \mathbb{Z}_3^3$, the graded Brauer invariant of $V(\pi_1)$ is trivial. For each of the remaining fine gradings, this invariant gives a distinguished subgroup $T \simeq \mathbb{Z}_2^2$ of G^u (see [DEK17, §5]).

Theorem 2.6 ([DEK17, Corollary 5.7])**.** Suppose a simple Lie algebra L of type E_7 is given a G-grading induced by a homomorphism $\nu : G^u \to G$ from one of the fine gradings. The finite-dimensional simple L-module $V(\lambda)$ of highest weight $\lambda = \sum_{i=1}^{7} m_i\pi_i$ admits a G-grading making it a graded L-module if and only if one of the following conditions is satisfied:

1) G^u is $\mathbb{Z}^7$, $\mathbb{Z}^3 \times \mathbb{Z}_2^3$ or $\mathbb{Z} \times \mathbb{Z}_3^3$;
2) G^u is not one of the groups $\mathbb{Z}^7$, $\mathbb{Z}^3 \times \mathbb{Z}_2^3$ and $\mathbb{Z} \times \mathbb{Z}_3^3$ and at least one of the following holds:
 (i) $m_1 + m_3 + m_7 \equiv 0 \pmod 2$, or
 (ii) ν is not injective on the distinguished subgroup $T \subset G^u$. □

For the remaining types, the algebraic group $\mathrm{Aut}(L)$ is connected and simply connected, which implies that every dominant integral weight λ is fixed by $\widehat{G}$ and the graded Brauer invariant of $V(\lambda)$ is trivial (see [EK15b, Appendix A]).

Theorem 2.7 ([EK15b, Corollary 22])**.** Suppose a simple Lie algebra L of type E_8, F_4 or G_2 is given a G-grading. Then any finite-dimensional L-module admits a G-grading making it a graded L-module. □

3. Group gradings of $\mathfrak{sl}_2(\mathbb{C})$-modules

In this section we restrict our attention to modules over the Lie algebra of type A_1, which can be realized as $\mathfrak{sl}_2(\mathbb{C})$.

3.1. **Group gradings of $\mathfrak{sl}_2(\mathbb{C})$.** All group gradings on $\mathfrak{sl}_2(\mathbb{C})$ are well-known, see e.g [EK13]. We will use the following bases:

$$x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}. \tag{11}$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = h, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{12}$$

Up to equivalence, there are precisely two fine gradings on $\mathfrak{sl}_2(\mathbb{C})$ (see [EK13, Theorem 3.55]):

(1) *the Cartan grading* with the universal group $\mathbb{Z}$,

$$\Gamma^1_{\mathfrak{sl}_2} : \mathfrak{sl}_2(\mathbb{C}) = L_{-1} \oplus L_0 \oplus L_1 \text{ where } L_0 = \langle h\rangle, L_1 = \langle x\rangle, L_{-1} = \langle y\rangle;$$

(2) *the Pauli grading* with the universal group $\mathbb{Z}_2^2$,

$$\Gamma^2_{\mathfrak{sl}_2} : \mathfrak{sl}_2(\mathbb{C}) = L_{(1,0)} \oplus L_{(0,1)} \oplus L_{(1,1)} \text{ where } L_{(1,0)} = \langle A\rangle, L_{(0,1)} = \langle B\rangle, L_{(1,1)} = \langle C\rangle.$$

Hence, up to isomorphism, any G-grading on $\mathfrak{sl}_2(\mathbb{C})$ is a coarsening of one of the two gradings: Cartan or Pauli.

Note that any grading Γ of a Lie algebra L uniquely extends to a grading $U(\Gamma)$ of its universal enveloping algebra $U(L)$. The grading $U(\Gamma)$ is a grading in the sense of associative algebras but also as L-modules where $U(L)$ is either a (left) regular L-module or an adjoint L-module. In our study of gradings on $\mathfrak{sl}_2(\mathbb{C})$-modules we will often consider a $\mathbb{Z}_2$-coarsening of $U(\Gamma^2_{\mathfrak{sl}_2})$, in which the component of the coarsening labeled by 0 is the sum of components of the original grading labeled by $(0,0)$ and $(1,0)$ while the component labeled by 1 is the sum of components labeled by $(0,1)$ and $(1,1)$.

3.2. **Algebras $U(I_\lambda)$.** Let $c \in U(\mathfrak{sl}_2(\mathbb{C}))$ be the Casimir element for $\mathfrak{sl}_2(\mathbb{C})$. With respect to the basis $\{h, x, y\}$ of $\mathfrak{sl}_2(\mathbb{C})$, this element can be written as

$$c = (h+1)^2 + 4yx = h^2 + 1 + 2xy + 2yx. \tag{13}$$

It is well-known that the center of $U(\mathfrak{sl}_2(\mathbb{C}))$ is the polynomial ring $\mathbb{C}[c]$. Note that c is a homogeneous element of degree zero, with respect to the Cartan grading of $U(\mathfrak{sl}_2(\mathbb{C}))$. One can write the Casimir element with respect to the basis $\{h, B, C\}$ of $\mathfrak{sl}_2(\mathbb{C})$.

Namely,

$$\begin{aligned} c &= 2xy+2yx+h^2+1 \\ &= 2\left(\frac{B+C}{2}\right)\left(\frac{B-C}{2}\right)+2\left(\frac{B-C}{2}\right)\left(\frac{B+C}{2}\right)+h^2+1 \\ &= \frac{1}{2}(B^2+CB-BC-C^2)+\frac{1}{2}(B^2+BC-CB-C^2)+h^2+1, \end{aligned}$$

and so

$$c = B^2-C^2+h^2+I = A^2+B^2-C^2+1.$$

It follows that c is also homogeneous, of degree (0,0), with respect to the Pauli grading of $U(\mathfrak{sl}_2(\mathbb{C}))$.

Let R be an associative algebra (or just an associative ring), and V be a left R-module. The annihilator of V, denoted by $\mathrm{Ann}_R(V)$, is the set of all elements r in R such that, for all v in V, $r.v = 0$:

$$\mathrm{Ann}_R(V) = \{r \in R \mid r.v = 0 \text{ for all } v \in V\}.$$

Given $\lambda \in \mathbb{C}$, let I_λ be the two-side ideal of $U(\mathfrak{sl}_2(\mathbb{C}))$, generated by the central element $c-(\lambda+1)^2$.

Theorem 3.1 ([Maz09, Theorem 4.7])**.** *For any simple $U(\mathfrak{sl}_2(\mathbb{C}))$-module M, there exists $\lambda \in \mathbb{C}$ such that $I_\lambda \subset \mathrm{Ann}_{U(\mathfrak{sl}_2(\mathbb{C}))}(M)$.*

Proposition 3.2. Let R be a graded algebra and M be a graded R-module, then $\mathrm{Ann}_R(M)$ is graded.

Proof. Let $I = \mathrm{Ann}_R(M) = \{x \in R \mid x.M = 0\}$, and $0 \neq x \in I \subseteq R$, then $x = x_1 + x_2 + \cdots x_k$, where x_i are homogeneous elements in R (belonging to different homogeneous components). Let $v \in M$ be an arbitrary homogeneous element, then $0 = x.v = x_1.v + x_2.v + \cdots x_k.v$. Since the components $x_i.v$ belong to different homogeneous subspaces, it follows that $x_i.v = 0$ for all i. and since v is an arbitrary homogeneous element, then $x_i \in I$ for all i. □

Proposition 3.3. The ideal I_λ is both $\mathbb{Z}$- and $\mathbb{Z}_2^2$-graded ideal.

Proof. Since $c-(\lambda+1)^2$ is homogeneous of degree 0 (resp., (0,0)) with respect to the $\mathbb{Z}$-grading (resp., $\mathbb{Z}_2^2$- grading), then I_λ is graded. □

Now for any $\lambda \in \mathbb{C}$, we write $U(I_\lambda) := U(\mathfrak{sl}_2(\mathbb{C}))/I_\lambda$. Using Proposition 3.3, $U(I_\lambda)$ is a $\mathbb{Z}$-graded algebra and $\mathbb{Z}_2^2$-graded algebra. It is well-known (see e.g. [Maz09]) that the algebra $U(I_\lambda)$ is a free $\mathbb{C}[h]$-module with basis

$$\mathcal{B}_0 = \left\{1, x, y, x^2, y^2, \ldots\right\},$$

and so it is free over $\mathbb{C}$ with basis $\mathcal{B} = \left\{1, h, h^2, \ldots\right\}.\mathcal{B}_0$. Note that the basis $\mathcal{B}$ is a basis of $U(I_\lambda)$ consisting of homogeneous elements with respect to the Cartan grading by $\mathbb{Z}$. A basis of $U(I_\lambda)$ over $\mathbb{C}$ consisting of homogeneous elements with respect to the Pauli grading by $\mathbb{Z}_2^2$ can be computed as follows. Set

$$\widehat{B}_0 = \{1, B, C, BC, B^2, B^2C, B^3, B^3C, \ldots\}.$$

Then easy calculations, using induction by the natural filtration in B and the relation $C^2 = h^2 + B^2 - \lambda^2 - 2\lambda$ show that the set $\widehat{B} = \{1, h, h^2, \ldots\} \cdot \widehat{B}_0$ is a $\mathbb{Z}_2^2$-homogeneous basis of $U(I_\lambda)$.

Let $p(t) = \frac{1}{4}((\lambda^2+2\lambda) - 2t - t^2) \in \mathbb{C}[t]$. Then, inside $U(I_\lambda)$, for any $q(t) \in \mathbb{C}[t]$, we have the following relations:

$$x^k q(h) = q(h-2k)x^k$$
$$y^j q(h) = q(h+2j)y^j.$$

If $k \geq j$ then

$$x^k y^j = p(h-2k)\cdots p(h-2(k-j+1))x^{k-j}$$
$$y^j x^k = p(h+2(j-1))\cdots p(h)x^{k-j}.$$

If $j \geq k$ then

$$x^k y^j = p(h-2k)\cdots p(h-2)y^{j-k}$$
$$y^j x^k = p(h+2(j-1))\cdots p(h+2(j-k))y^{j-k}.$$

Moreover, $U(I_\lambda)$ is a generalized Weyl algebra (see e.g [Bav92]) and has the following properties.

Theorem 3.4 ([Maz09, Theorem 4.15])**.**

(1) *$U(I_\lambda)$ is both left and right Noetherian.*
(2) *$U(I_\lambda)$ is a domain.*
(3) *The algebra $U(I_\lambda)$ is simple for all $\lambda \in \mathbb{C} \setminus \mathbb{Z}$.*
(4) *For every $n \in \mathbb{N}_0$, the algebra $U(I_n)$ has a unique proper ideal.*

One more property that is important for us is the following.

Theorem 3.5 ([Maz09, Theorem 4.26])**.** *For any non-zero left ideal $I \subset U(I_\lambda)$, the $U(I_\lambda)$-module $U(I_\lambda)/I$ has finite length.*

3.3. **Weight modules over $\mathfrak{sl}_2(\mathbb{C})$.** Let V be an $\mathfrak{sl}_2(\mathbb{C})$-module, $\mathfrak{h} = \langle h \rangle$ be the Cartan subalgebra of $\mathfrak{sl}_2(\mathbb{C})$. Since $\dim(\mathfrak{h}) = 1$, we can think of $\mathfrak{h}^*$ as $\mathbb{C}$. We call

$$V_\mu = \{v \in V \mid h.v = \mu v\}, \text{ for } \mu \in \mathbb{C},$$

the *weight spaces* for V, and if V_μ is nontrivial we call $\mu \in \mathbb{C}$ the *weight* of V. If V is the direct sum of these weight spaces, we say that V is *a weight module*. The set of all weights is called the *support* of V, denoted $\mathrm{Supp}(V)$. In the case of a weight module, if $\lambda \in \mathrm{Supp}(V)$ and $\lambda + 2 \notin \mathrm{Supp}(V)$, λ is called the *highest weight* of V, and the elements of the space V_λ are called *highest weight vectors*. Similarly, if $\lambda \in \mathrm{Supp}(V)$ and $\lambda - 2 \notin \mathrm{Supp}(V)$, then λ is called the *lowest weight* and the elements of the space V_λ are called *lowest weight vectors*. If the weight module is generated by v_λ, where v_λ is a highest (resp., lowest) weight vector, then V is called *highest (resp., lowest) weight module* of weight λ.

Lemma 3.6. Any h-invariant subspace of a weight $\mathfrak{sl}_2(\mathbb{C})$-module is spanned by weight vectors.

Proof. Let V be a weight $\mathfrak{sl}_2(\mathbb{C})$-module and W an h-invariant subspace of V. Let $w \in W \subset V$, so $w = v_1 + v_2 + \cdots + v_k$, where v_i is a nonzero weight vector of weight $\mu_i \in \mathbb{C}$, for all $i = 1, 2, \ldots, k$, where we may assume that $\mu_1, \mu_2, \ldots, \mu_k$ are distinct. Define the elements $h_i \in U(\mathfrak{h})$, $i = 1, \ldots, k$, by

$$h_i = \textstyle\prod_{l \neq i}(h - \mu_l).$$

Then

$$h_i.v_j = \begin{cases} 0 & \text{if } i \neq j; \\ \prod_{l \neq i}(\mu_i - \mu_l)v_i & \text{if } i = j. \end{cases}$$

Hence,

$$W \ni h_i.w = \sum_{j=1}^{k} h_i.v_j = h_i.v_i = \prod_{l\neq i}(\mu_i - \mu_l)v_i,$$

which means that $v_i \in W$.

□

3.3.1. *Simple finite-dimensional* $\mathfrak{sl}_2(\mathbb{C})$*-modules.* Let $V = V(n)$ be a finite - dimensional simple $\mathfrak{sl}_2(\mathbb{C})$-module of dimension $n+1$, with a highest weight vector $v_0 \in V_n$ and highest weight n. Define $v_i = \frac{1}{i!}y^i.v_0$ for $i = 0, 1, \dots, n$. This is a basis of V. It is convenient to set $v_{-1} = 0$. The module action is given by

$$\begin{aligned} h.v_i &= (n-2i)v_i, \\ x.v_i &= (n-(i-1))v_{i-1}, \\ y.v_i &= (i+1)v_{i+1}, \end{aligned} \tag{14}$$

hence

$$V(n) = V_n \oplus V_{n-2} \oplus \dots\dots \oplus V_{-(n-2)} \oplus V_{-n}.$$

Note that any finite-dimensional simple $\mathfrak{sl}_2(\mathbb{C})$-module is a highest weight module of weight $n = \dim(V) - 1$, see e.g [Hum78, Maz09].

3.3.2. *Verma modules of* $\mathfrak{sl}_2(\mathbb{C})$. The general construction for the Verma modules over a semisimple Lie algebra L is given by the following: consider $B(\Delta) = \mathfrak{h} \oplus N$ be the standard Borel subalgebra of the semisimple Lie algebra L, where $\mathfrak{h}$ is the Cartan subalgebra of L, Δ is the basis of the root system of L with respect to $\mathfrak{h}$, and N the sum of the positive root spaces. For any $\lambda \in \mathfrak{h}^*$, start with a 1-dimensional $B(\Delta)$-module, say D_λ, with trivial N-action and $\mathfrak{h}$ acting through λ, and set $Z(\lambda) = U(L) \otimes_{U(B(\Delta))} D_\lambda$. Then $Z(\lambda)$ is a $U(L)$-module called *the Verma module of weight* λ. In the case of $L = \mathfrak{sl}_2(\mathbb{C})$, we have $B(\Delta) = \langle h, x\rangle$ and $N = \langle x\rangle$. In view of the general Definition of the Verma module, Verma $\mathfrak{sl}_2(\mathbb{C})$-module of highest weight $\lambda \in \mathbb{C}$, is

$$Z(\lambda) = U(\mathfrak{sl}_2(\mathbb{C})) \otimes_{U(B(\Delta))} D_\lambda.$$

In [Maz09], Mazorchuk introduces the Verma $\mathfrak{sl}_2(\mathbb{C})$-module explicitly, he just uses the mathematical induction to generalize from the case of simple finite-dimensional $\mathfrak{sl}_2(\mathbb{C})$-modules to the Verma $\mathfrak{sl}_2(\mathbb{C})$-modules, and takes $v_i = \frac{1}{i!}y^i.v_0$, for $i \in \mathbb{N}_0$. Then

$$Z(\lambda) = \langle v_0, v_1, v_2, \dots\rangle$$

and the action is given by the formulas (14). Thus,

$$Z(\lambda) = \bigoplus_{i\in\mathbb{N}_0} V_{\lambda-2i},$$

where $V_{\lambda-2i} = \mathbb{C}v_i$.

The module $Z(\lambda)$ is a simple $\mathfrak{sl}_2(\mathbb{C})$-module if and only if $\lambda \notin \mathbb{N}_0$. If n is a non-negative integer, then $Z(n)$ is indecomposable and has a unique nontrivial submodule $Z(-n-2)$, with $V(n) \cong Z(n)/Z(-n-2)$. It is well-known see e.g. [Maz09] that I_λ is the annihilator of the Verma module $Z(\lambda)$.

3.3.3. *Anti-Verma modules of* $\mathfrak{sl}_2(\mathbb{C})$. Let V be the formal vector space with the basis $\{v_i \mid i \in \mathbb{N}_0\}$. Now set $v_{-1} = 0$ and define the action on V for $\lambda \in \mathbb{C}$ as:

$$\begin{aligned} h.v_i &= (\lambda+2i)v_i, \\ x.v_i &= v_{i+1}, \\ y.v_i &= -i(\lambda+i-1)v_{i-1}, \end{aligned} \tag{15}$$

then V is a lowest weight $\mathfrak{sl}_2(\mathbb{C})$-module with lowest weight λ, denoted by $\overline{Z}(\lambda)$ and called *anti-Verma module*.

The support of the anti-Verma module is

$$\mathrm{Supp}(\overline{Z}(\lambda)) = \{\lambda + 2i \mid i \in \mathbb{N}_0\}$$

and the Casimir element acts on it as the scalar $(\lambda - 1)^2$. The module $\overline{Z}(\lambda)$ is a simple $\mathfrak{sl}_2(\mathbb{C})$-module if and only if $-\lambda \notin \mathbb{N}_0$. If n is a negative integer, then $\overline{Z}(n)$ has a unique maximal submodule $\overline{Z}(-n+2)$, with $V(n) \cong \overline{Z}(n)/Z(-n+2)$.

3.3.4. *Dense modules of* $\mathfrak{sl}_2(\mathbb{C})$*.* A weight $\mathfrak{sl}_2(\mathbb{C})$-module is called a *dense module* if it has no highest nor lowest weights. In other words, the weight module V is dense if $\mathrm{Supp}(V) = \lambda + 2\mathbb{Z}$ for some $\lambda \in \mathbb{C}$. Now we will study a big class of the dense modules.

For $\xi \in \mathbb{C}/2\mathbb{Z}$ and $\tau \in \mathbb{C}$, consider V to be the formal vector space with the basis $\{v_\mu \mid \mu \in \xi\}$. Define the action on V as:

$$\begin{aligned} h.v_\mu &= \mu v_\mu, \\ x.v_\mu &= \tfrac{1}{4}(\tau - (\mu+1)^2)v_{\mu+2}, \\ y.v_\mu &= v_{\mu-2}, \end{aligned} \tag{16}$$

then V is a dense weight $\mathfrak{sl}_2(\mathbb{C})$-module, denoted by $V(\xi, \tau)$. In this case the module $V(\xi, \tau)$ is simple if and only if $\tau \neq (\lambda+1)^2$ for all $\lambda \in \xi$, but if the module $V(\xi, \tau)$ is not simple, then it contains a unique maximal submodule isomorphic to a Verma module for some highest weight.

Theorem 3.7 ([Maz09, Theorem 3.32])**.** *Up to isomorphism, any simple weight* $\mathfrak{sl}_2(\mathbb{C})$*-module is one of the following modules*

(1) $V(n)$ *for some* $n \in \mathbb{N}$*;*
(2) $Z(\lambda)$ *for some* $\lambda \in \mathbb{C}\backslash\mathbb{N}_0$*;*
(3) $\overline{Z}(-\lambda)$ *for some* $\lambda \in \mathbb{C}\backslash\mathbb{N}_0$*;*
(4) $V(\xi, \tau)$ *for some* $\xi \in \mathbb{C}/2\mathbb{Z}$ *and* $\tau \in \mathbb{C}$*, with* $\tau \neq (\lambda+1)^2$ *for all* $\lambda \in \xi$*.* □

Proposition 3.8 ([Maz09])**.** Let $J_n := \mathrm{Ann}_{U(\mathfrak{sl}_2(\mathbb{C}))}(V(n))$, where $V(n)$ is a finite-dimensional simple $\mathfrak{sl}_2(\mathbb{C})$-module. Then

(1) $I_n \subset J_n$.
(2) $\mathrm{Ann}_{U(\mathfrak{sl}_2(\mathbb{C}))}(\overline{Z}(\lambda)) = I_{\lambda-2}$.
(3) Let $\xi \in \mathbb{C}/2\mathbb{Z}$ and $\tau = (\lambda+1)^2 \in \mathbb{C}$, then $\mathrm{Ann}_{U(\mathfrak{sl}_2(\mathbb{C}))}(V(\xi, \tau)) = I_\lambda$. □

3.4. **Torsion-free modules over** $\mathfrak{sl}_2(\mathbb{C})$**.**

Definition 3.9. Let M be an $\mathfrak{sl}_2(\mathbb{C})$-module, then the module M is called *torsion* if for any $m \in M$ there exists non-zero $p(t) \in \mathbb{C}[t]$ such that $p(h).m = 0$. The module M is *torsion-free* if $M \neq 0$ and $p(h).m \neq 0$ for all $0 \neq m \in M$ and all non-zero $p(t) \in \mathbb{C}[t]$. If M a torsion-free $\mathbb{C}[h]$-module of rank n, we say that M is of *rank* n.

Theorem 3.10 ([Maz09, Theorem 6.3])**.** *A simple* $\mathfrak{sl}_2(\mathbb{C})$*-module is either a weight or a torsion-free module.* □

Theorem 3.10 means that if h has at least one eigenvector on M, then M is a weight module.

As a consequence of Theorem 3.1, it is sufficient to describe simple torsion-free $U(I_\lambda)$-modules instead of simple $U(\mathfrak{sl}_2(\mathbb{C}))$-modules (see e.g [Maz09]).

A further reduction can be achieved as follows. We consider the field of rational functions in h, $\mathbb{K} = \mathbb{C}(h)$, and set $\mathbb{A}$ to be the *algebra of skew Laurent polynomials* over $\mathbb{K}$, that is

$$\mathbb{A} = \mathbb{K}[X, X^{-1}, \sigma] = \left\{ \sum_{i\in\mathbb{Z}} q_i(h)X^i \mid q_i(h) \in \mathbb{K},\ \text{almost all } q_i(h) = 0 \right\},$$

with the usual addition and scalar multiplication, and the product

$$(\sum_{i\in\mathbb{Z}} p_i(h)X^i)(\sum_{j\in\mathbb{Z}} q_j(h)X^j) = \sum_{i,j\in\mathbb{Z}} p_i(h)\sigma^i(q_j(h))X^{i+j},$$

where $\sigma(h) = h - 2$. Note that $\mathbb{A}$ is an Euclidean domain and it is isomorphic to $S^{-1}U(I_\lambda)$, the localization of the generalized Weyl algebra $U(I_\lambda)$, where $S = \mathbb{C}[h]\setminus\{0\}$. An embedding of $\Phi_\lambda : U(I_\lambda) \to \mathbb{A}$ is the unique extension of the following map:

$$\Phi_\lambda(h) = h,\ \Phi_\lambda(x) = X,\ \Phi_\lambda(y) = \frac{(\lambda+1)^2 - (h+1)^2}{4} X^{-1}.$$

Thanks to this embedding, $\mathbb{A}$ becomes a $\mathbb{A} - U(I_\lambda)$-bimodule and given an $U(I_\lambda)$-module M one can define an $\mathbb{A}$-module $\mathcal{F}(M)$ by

$$\mathcal{F}(M) = \mathbb{A} \underset{U(I_\lambda)}{\otimes} M.$$

Theorem 3.11 ([Maz09, Theorem 6.24])**.**

(i) *The functor $\mathcal{F}$ induces a bijection $\widehat{\mathcal{F}}$ between the isomorphism classes of simple torsion-free $U(I_\lambda)$-modules to the set of isomorphism classes of simple $\mathbb{A}$-modules;*

(ii) *The inverse of the bijection from (i) is the map that sends a simple $\mathbb{A}$-module N to its $U(I_\lambda)$-socle $\operatorname{soc}_{U(I_\lambda)}(N)$.*

Theorem 3.12 ([Bav90, Proposition 3])**.** *Let M be a simple torsion-free $U(I_\lambda)$-module, them $M \cong U(I_\lambda)/(U(I_\lambda) \cap \mathbb{A}\alpha)$, for some $\alpha \in U(I_\lambda)$ which is irreducible as an element of $\mathbb{A}$.*

Many examples of torsion-free $\mathfrak{sl}_2(\mathbb{C})$-modules have been introduced, see e.g [Maz09, PT17, Nil15]. We will highlight those of them for which we can decide if those modules are graded or not.

Let us define a family of $U(I_\lambda)$-modules modules, as follows. Given two polynomials $p(t), g(t) \in \mathbb{C}[t]$, we set

$$M(p(t), g(t), \lambda) := U(I_\lambda)/U(I_\lambda)(g(h)x + p(h)$$

and

$$M'(p(t), g(t), \lambda) := U(I_\lambda)/U(I_\lambda)(g(h)y + p(h).$$

Theorem 3.13 ([Maz09, Theorem 6.50])**.** *Let $\lambda \in \mathbb{C}$, and $g(t), p(t)$ be non-zero polynomials in $\mathbb{C}[t]$ such that if $r \in \mathbb{C}$ is a root of $p(t)$ then*

(1) *$r + n$ is not a root for $g(t)$ for all $n \in \mathbb{Z}$,*
(2) *$(\lambda+1)^2 \neq (r+n+1)^2$ for all $n \in \mathbb{Z}$.*

Then the $U(I_\lambda)$-modules $M(p(t), g(t), \lambda)$ and $M'(p(t), g(t), \lambda)$ are simple. □

The so called Whittaker modules are a special case of Theorem 3.13. They are defined as follows:

Definition 3.14. Let $\alpha \in \mathbb{C}\setminus\{0\}$ and $\lambda \in \mathbb{C}$, then the *Whittaker modules* are the modules $M_\alpha = U(I_\lambda)/U(I_\lambda)(1-\alpha x) = U(I_\lambda)/U(I_\lambda)(1-\frac{\alpha}{2}B-\frac{\alpha}{2}C)$.

A full description of torsion-free $\mathfrak{sl}_2(\mathbb{C})$-modules of rank 1 (over $\mathbb{C}[h]$) was given in [Nil15].

Definition 3.15. Let $\alpha \in \mathbb{C}\setminus\{0\}$ and $\beta \in \mathbb{C}$. Let us define an $\mathfrak{sl}_2(\mathbb{C})$-module $N(\alpha,\beta)$ as a vector space $\mathbb{C}[h]$ equipped with the following action: for $f(h) \in \mathbb{C}[h]$

$$\begin{gathered} h.f(h) = hf(h), \\ x.f(h) = \alpha(\tfrac{h}{2}+\beta)f(h-2), \\ y.f(h) = -\tfrac{1}{\alpha}(\tfrac{h}{2}-\beta)f(h+2). \end{gathered} \tag{17}$$

Note that $N(\alpha,\beta)$ is simple if and only if $2\beta \notin \mathbb{N}_0$, see [Nil15].

Definition 3.16. Let $\alpha \in \mathbb{C}\setminus\{0\}$ and $\beta \in \mathbb{C}$ with $\mathrm{Re}(\beta) \geq -\frac{1}{2}$. Let us define an $\mathfrak{sl}_2(\mathbb{C})$-module $N'(\alpha,\beta)$ as a vector space $\mathbb{C}[h]$ equipped with the following action: for $f(h) \in \mathbb{C}[h]$

$$\begin{gathered} h.f(h) = hf(h), \\ x.f(h) = \alpha f(h-2), \\ y.f(h) = -\tfrac{1}{\alpha}(\tfrac{h}{2}+\beta+1)(\tfrac{h}{2}-\beta)f(h+2). \end{gathered} \tag{18}$$

Definition 3.17. Let $\alpha \in \mathbb{C}\setminus\{0\}$ and $\beta \in \mathbb{C}$, with $\mathrm{Re}(\beta) \geq -\frac{1}{2}$. Let us define an $\mathfrak{sl}_2(\mathbb{C})$-module $\bar{N}(\alpha,\beta)$ as a vector space $\mathbb{C}[h]$ equipped with the following action: for $f(h) \in \mathbb{C}[h]$

$$\begin{gathered} h.f(h) = -hf(h), \\ x.f(h) = \tfrac{1}{\alpha}(\tfrac{h}{2}+\beta+1)(\tfrac{h}{2}-\beta)f(h+2), \\ y.f(h) = -\alpha f(h-2). \end{gathered} \tag{19}$$

Note that the Whittaker modules are torsion-free $\mathfrak{sl}_2(\mathbb{C})$-modules of rank 1 with type $N'(\frac{1}{\alpha},\frac{\lambda}{2})$.

Theorem 3.18 ([Nil15, Theorem 9, Lemma 12])**.** *Each simple torsion-free* $\mathfrak{sl}_2(\mathbb{C})$*-module of rank 1 is isomorphic to one of the following (pairwise non-isomorphic) modules:*

(1) $N(\alpha,\beta)$ *for some* $\alpha \in \mathbb{C}\setminus\{0\}$ *and* $\beta \in \mathbb{C}$ *with* $2\beta \notin \mathbb{N}_0$.
(2) $N'(\alpha,\beta)$ *for some* $\alpha \in \mathbb{C}\setminus\{0\}$ *and* $\beta \in \mathbb{C}$ *with* $\mathrm{Re}(\beta) \geq -\frac{1}{2}$.
(3) $\bar{N}(\alpha,\beta)$ *for some* $\alpha \in \mathbb{C}\setminus\{0\}$ *and* $\beta \in \mathbb{C}$ *with* $\mathrm{Re}(\beta) \geq -\frac{1}{2}$.

3.5. **Gradings on the weight modules.**

3.5.1. *Gradings on simple finite-dimensional* $\mathfrak{sl}_2(\mathbb{C})$*-modules.* It is obvious that every simple finite-dimensional module of $\mathfrak{sl}_2(\mathbb{C})$ is a weight module, i.e., it decomposes as the direct sum of weight spaces and this decomposition is a grading compatible with the Cartan grading on $\mathfrak{sl}_2(\mathbb{C})$. In [EK15a], the authors show that the finite-dimensional simple modules with even highest weight have a grading compatible with the Pauli grading on $\mathfrak{sl}_2(\mathbb{C})$, while those ones with the odd highest weight do not. Here we will give an explicit construction of the grading in the even case.

Let $V = V(n)$ be a simple $\mathfrak{sl}_2(\mathbb{C})$-module with an even highest weight $n = 2m$ and basis $\{v_0, v_1, ..., v_n\}$. To construct a $\mathbb{Z}_2^2$-grading on V, we first define a new basis of V as follows. Set

$$e_i = v_i + v_{n-i} \text{ for all } i = 0, 1, \ldots, m,$$

and

$$d_i = v_i - v_{n-i} \text{ for all } i = 0, 1, \ldots, m-1.$$

Then $\{e_0, e_1, \ldots, e_m, d_0, d_1, \ldots, d_{m-1}\}$ is a basis of V and the module action is given as follows.

$$\begin{aligned}
h.e_i &= (n-2i)d_i \text{ for all } i = 0,1,\ldots,m;\\
B.e_i &= \begin{cases}(n-i+1)e_{i-1} + (i+1)e_{i+1}, & \text{if } i = 0,1,\ldots,m-1;\\ 2(m+1)e_m, & \text{if } i = m;\end{cases}\\
C.e_i &= \begin{cases}(n-i+1)d_{i-1} - (i+1)d_{i+1}, & \text{if } i = 0,1,\ldots,m-1;\\ 2(m+1)d_{m-1}, & \text{if } i = m;\end{cases}\\
h.d_i &= (n-2i)e_i \text{ for all } i = 0,1,\ldots,m-1;\\
B.d_i &= (n-i+1)d_{i-1} + (i+1)d_{i+1} \text{ if } i = 0,1,\ldots,m-1;\\
C.d_i &= (n-i+1)e_{i-1} - (i+1)e_{i+1} \text{ if } i = 0,1,\ldots,m-1.
\end{aligned}$$

Let $V_{(0,0)} = \langle e_i \mid i \text{ even}\rangle$, $V_{(0,1)} = \langle e_i \mid i \text{ odd}\rangle$, $V_{(1,0)} = \langle d_i \mid i \text{ even }\rangle$, and $V_{(1,1)} = \langle d_i \mid i \text{ odd}\rangle$. One now easily checks the following.

Proposition 3.19. The above formulas provide a $\mathbb{Z}_2^2$-grading

$$\Gamma : V = V_{(0,0)} \oplus V_{(1,0)} \oplus V_{(0,1)} \oplus V_{(1,1)}$$

on the highest weight module $V = V(n)$, n even, which is compatible with the Pauli grading on $\mathfrak{sl}_2(\mathbb{C})$. □

3.5.2. *Gradings on Verma $\mathfrak{sl}_2(\mathbb{C})$-modules.* As we mentioned above, any weight $\mathfrak{sl}_2(\mathbb{C})$-module has a grading compatible with the Cartan grading on $\mathfrak{sl}_2(\mathbb{C})$ via the weight decomposition. As a special case, we will explicitly describe the Cartan gradings on the Verma modules.

Let $\{v_0, v_1, \ldots, v_k, \ldots\}$ be a basis of $V(\lambda)$, as described in Subsection 3.3.2. Consider the canonical basis $\{x, y, h\}$ of $\mathfrak{sl}_2(\mathbb{C})$ with the Cartan grading by $\mathbb{Z}$, that is, $\deg(x) = 1$, $\deg(y) = -1$, $\deg(h) = 0$. The action of $\mathfrak{sl}_2(\mathbb{C})$ on V is the following:

.	v_0	v_1	v_2	$\ldots$	v_k	$\ldots$
h	λv_0	$(\lambda-2)v_1$	$(\lambda-4)v_2$	$\ldots$	$(\lambda-2k)v_k$	$\ldots$
x	0	λv_0	$(\lambda-1)v_1$	$\ldots$	$(\lambda-k+1)v_{k-1}$	$\ldots$
y	v_1	$2v_2$	$3v_3$	$\ldots$	$(k+1)v_{k+1}$	$\ldots$

Let $V_{-k} = \langle v_k \rangle$ for $k = 0, 1, 2, ...$, and $V_k = \{0\}$ for $k = 1, 2, \ldots$, then the grading $V = \bigoplus_{k=0}^{\infty} V_{-k}$ makes V a graded $\mathfrak{sl}_2(\mathbb{C})$-module.

Theorem 3.20. *Let V be a Verma $\mathfrak{sl}_2(\mathbb{C})$-module with highest weight $\lambda \in \mathbb{C} \setminus 2\mathbb{N}_0$. Then V is not a $\mathbb{Z}_2^2$-graded $\mathfrak{sl}_2(\mathbb{C})$-module.*

Proof. Let $V = \bigoplus\limits_{\mu\in\mathbb{C}} V_\mu$, with a maximal vector $v_0 \in V_\lambda$. Then V has a basis $\{v_0, v_1, v_2, \ldots\}$ given in Subsection 3.3.2. Assume that V has a grading compatible with the Pauli grading on $\mathfrak{sl}_2(\mathbb{C})$, so it can written as $V = V_{(0,0)} \oplus V_{(1,0)} \oplus V_{(0,1)} \oplus V_{(1,1)}$. Now let $V^0 = V_{(0,0)} \oplus V_{(1,0)}$, and $V^1 = V_{(0,1)} \oplus V_{(1,1)}$. The modules V^0 and V^1 are thus h-invariant, with the action of B sending V^0 to V^1 and vice versa. By Lemma 3.6, V^0 and V^1 are spanned by weight vectors. Since $V_\lambda = \mathbb{C}v_0$, we must have either $v_0 \in V^0$ or $v_0 \in V^1$.

Without loss of generality, suppose $v_0 \in V^0$ (otherwise apply the shift of grading), then $V^1 \ni B.v_0 = v_1$, so $v_1 \in V^1$. Hence $V^0 \ni B.v_1 = \lambda v_0 + 2v_2$. Since $v_0 \in V^0$ we get $v_2 \in V^0$. Again $V^1 \ni B.v_2 = (\lambda - 1)v_1 + 3v_3$, which implies $v_3 \in V^1$, and so on. We have shown that V^0 is spanned by the set $\{v_0, v_2, v_4, \ldots\}$ and V^1 by $\{v_1, v_3, v_5, \ldots\}$.
Now let $0 \neq v \in V_{(0,0)} \subseteq V^0$. Then v can be written as $v = \alpha_0 v_0 + \alpha_2 v_2 + \cdots + \alpha_{2k} v_{2k}$, for some non-negative integer k, and some $\alpha_i \in \mathbb{C}$. Since $V_{(0,0)}$ is h^2-invariant, the elements

$$\begin{aligned} h^2.v &= \alpha_0\lambda^2 v_0 + \alpha_2(\lambda-4)^2 v_2 + \cdots + \alpha_{2k}(\lambda-4k)^2 v_{2k}, \\ h^4.v &= \alpha_0\lambda^4 v_0 + \alpha_2(\lambda-4)^4 v_2 + \cdots + \alpha_{2k}(\lambda-4k)^4 v_{2k}, \\ &\cdots \\ h^{2k}.v &= \alpha_0\lambda^{2k} v_0 + \alpha_2(\lambda-4)^{2k} v_2 + \ldots.. + \alpha_{2k}(\lambda-4k)^{2k} v_{2k} \end{aligned}$$

all belong to $V_{(0,0)}$. In order to use the Vandermonde's argument, we have to show that $\lambda^2, (\lambda-4)^2, \ldots, (\lambda-4k)^2$ are all distinct. Assume that we have two different weights, $(\lambda - 4n)$ and $(\lambda - 4m)$ such that $(\lambda-4n)^2 = (\lambda-4m)^2$. Then $|\lambda - 4n| = |\lambda - 4m|$. Hence either $\lambda - 4n = \lambda - 4m$ or $\lambda - 4n = 4m - \lambda$, the first case being impossible. This means that $\lambda = 2(n+m) \in 2\mathbb{N}_0$, which is a contradiction. Hence,

$$\begin{vmatrix} 1 & 1 & \ldots & 1 \\ \lambda^2 & (\lambda-4)^2 & \ldots & (\lambda-4k)^2 \\ \vdots & \vdots & \ldots & \vdots \\ \lambda^{2k} & (\lambda-4)^{2k} & \ldots & (\lambda-4k)^{2k} \end{vmatrix} \neq 0.$$

It follows that $V_{(0,0)}$ is spanned by the weight vectors, which means that there is $v_s \in V_{(0,0)}$ for some s. Then $h.v_s = (\lambda - 2s)v_s \in V_{(1,0)}$, a contradiction. □

Corollary 3.21. Let V be a Verma $\mathfrak{sl}_2(\mathbb{C})$-module with a non-negative even integer highest weight n. Then V cannot be a $\mathbb{Z}_2^2$-graded module.

Proof. Assume that V is $\mathbb{Z}_2^2$-graded module. Since the highest weight is an integer number then V is not simple and has a unique maximal submodule $Z(-n-2)$, which therefore must be a graded submodule. But $(-n-2)$ is a negative number, so we get a contradiction with Theorem 3.20. □

3.5.3. *Gradings on Anti-Verma $\mathfrak{sl}_2(\mathbb{C})$-modules.* From what we said above about $\mathbb{Z}$-gradings on the weight modules, it follows that $\overline{Z}(\lambda)$ is a $\mathbb{Z}$-graded $\mathfrak{sl}_2(\mathbb{C})$-module. Let $V = \overline{Z}(\lambda)$ with the basis $\{v_0, v_1, \ldots, v_k, \ldots\}$. Consider the basis $\{x, y, h\}$ of $\mathfrak{sl}_2(\mathbb{C})$ with the Cartan grading by $\mathbb{Z}$, that is, $\deg(x) = 1$, $\deg(y) = -1$, $\deg(h) = 0$. The action of $\mathfrak{sl}_2(\mathbb{C})$ on V is the following:

.	v_0	v_1	v_2	$\ldots$	v_k	$\ldots$
h	λv_0	$(\lambda+2)v_1$	$(\lambda+4)v_2$	$\ldots$	$(\lambda+2k)v_k$	$\ldots$
x	v_1	v_2	v_3	$\ldots$	v_{k+1}	$\ldots$
y	0	$-\lambda v_0$	$-2(\lambda+1)v_1$	$\ldots$	$-k(\lambda+k-1)v_{k-1}$	$\ldots$

Let $V_k = \mathbb{C}v_k$ for $k = 0, 1, 2, ...$, and $V_k = 0$ for $k = -1, -2, \ldots$, then the grading $V = \bigoplus_{k=0}^{\infty} V_k$ makes V a $\mathbb{Z}$-graded $\mathfrak{sl}_2(\mathbb{C})$-module.

Theorem 3.22. *Let V be an anti-Verma $\mathfrak{sl}_2(\mathbb{C})$-module with lowest weight $\lambda \in \mathbb{C}$. Then V cannot be a $\mathbb{Z}_2^2$-graded $\mathfrak{sl}_2(\mathbb{C})$-module.*

Proof. Let $V = \bigoplus_{k=0}^{\infty} V_k$ where $V_k = \mathbb{C}v_k$ for $k = 0, 1, 2, ...,$ and $\{v_0, v_1, v_2, \ldots\}$ be the basis of V. Assume that V has a grading compatible with the Pauli grading on $\mathfrak{sl}_2(\mathbb{C})$, so it can written as $V = V_{(0,0)} \oplus V_{(1,0)} \oplus V_{(0,1)} \oplus V_{(1,1)}$. Now let $V^0 = V_{(0,0)} \oplus V_{(1,0)}$, and $V^1 = V_{(0,1)} \oplus V_{(1,1)}$. We have that V^0 and V^1 are thus h-invariant, with the action of B and C sending V^0 to V^1 and vice versa. By Lemma 3.6, V^0 and V^1 are spanned by the weight vectors. Since $V_0 = \mathbb{C}v_0$, we must have $v_0 \in V^0$ or $v_0 \in V^1$.

Without loss of generality, suppose $v_0 \in V^0$ (otherwise apply the shift of grading), then $V^1 \ni B.v_0 = v_1$, so $v_1 \in V^1$. Hence $V^0 \ni B.v_1 = v_2 - \lambda v_0$. Since $v_0 \in V^0$ we get $v_2 \in V^0$. Again $V^1 \ni B.v_2 = v_3 - 2(\lambda+1)v_1$, which implies $v_3 \in V^1$, and so on. We have shown that V^0 is spanned by the set $\{v_0, v_2, v_4, \ldots\}$ and V^1 by $\{v_1, v_3, v_5, \ldots\}$.

Now let $0 \neq v \in V_{(0,0)} \subseteq V^0$. Then v can be written as $v = \alpha_0 v_0 + \alpha_2 v_2 + \cdots + \alpha_{2k} v_{2k}$ for some non-negative integer k and some $\alpha_i \in \mathbb{C}$. But since $V_{(0,0)}$ is h^2-invariant, the elements

$$\begin{aligned}
h^2.v &= \alpha_0 \lambda^2 v_0 + \alpha_2 (\lambda+4)^2 v_2 + \cdots + \alpha_{2k} (\lambda+4k)^2 v_{2k}, \\
h^4.v &= \alpha_0 \lambda^4 v_0 + \alpha_2 (\lambda+4)^4 v_2 + \cdots + \alpha_{2k} (\lambda+4k)^4 v_{2k}, \\
&\cdots \\
h^{2k}.v &= \alpha_0 \lambda^{2k} v_0 + \alpha_2 (\lambda+4)^{2k} v_2 + \cdots + \alpha_{2k} (\lambda+4k)^{2k} v_{2k},
\end{aligned}$$

all belong to $V_{(0,0)}$.

Now we have two cases:

Case 1 Assume that $-\lambda \notin 2\mathbb{N}_0$. In order to use the Vandermonde's argument, we need to show that $\lambda^2, (\lambda+4)^2, \ldots, (\lambda+4k)^2$ are all distinct. Assume that we have two different weights, $(\lambda+4n)$ and $(\lambda+4m)$ such that $(\lambda+4n)^2 = (\lambda+4m)^2$. Then $|\lambda+4n| = |\lambda+4m|$. Hence either $\lambda+4n = \lambda+4m$ or $\lambda+4n = -4m-\lambda$, but the first case is impossible. Therefore $-\lambda = 2(n+m) \in 2\mathbb{N}_0$, a contradiction. Hence,

$$\begin{vmatrix}
1 & 1 & \ldots & 1 \\
\lambda^2 & (\lambda+4)^2 & \ldots & (\lambda+4k)^2 \\
\vdots & \vdots & \ldots & \vdots \\
\lambda^{2k} & (\lambda+4)^{2k} & \ldots & (\lambda+4k)^{2k}
\end{vmatrix} \neq 0.$$

It follows that $V_{(0,0)}$ is spanned by the weight vectors, which means that there is $v_s \in V_{(0,0)}$ for some s. Note that $h.v_s = (\lambda+2s)v_s \in V_{(1,0)}$, which is a contradiction.

Case 2 Assume that $-\lambda \in 2\mathbb{N}_0$. Then V is not simple and has a unique maximal submodule $\overline{Z}(-\lambda+2)$. If V is graded by $\mathbb{Z}_2^2$, then the unique maximal submodule of V must be graded. However, this contradicts Case 1 since $(-(-\lambda+2)) \notin 2\mathbb{N}_0$. □

3.5.4. *Gradings on dense $\mathfrak{sl}_2(\mathbb{C})$-modules.* As usual, the weight modules are graded by $\mathbb{Z}$. Let $\xi \in \mathbb{C}/2\mathbb{Z}$ and $\tau \in \mathbb{C}$, and let $V = V(\xi, \tau)$ with basis $\{v_\mu \mid \mu \in \xi\}$ as in Definition 3.3.4, and consider the basis $\{x, y, h\}$ of $\mathfrak{sl}_2(\mathbb{C})$ with a Cartan grading by $\mathbb{Z}$, that is, $\deg(x) = 1$, $\deg(y) = -1$, $\deg(h) = 0$. Now, since $\xi \in \mathbb{C}/2\mathbb{Z}$ then $\xi = \lambda + 2\mathbb{Z}$ for some $\lambda \in \mathbb{C}$ and hence, for any $\mu \in \xi$, $\mu = \lambda + 2i$ for some $i \in \mathbb{Z}$. Let $V_i = \mathbb{C}v_{\lambda+2i}$, $i \in \mathbb{Z}$, then the grading $V = \bigoplus_{i\in\mathbb{Z}} V_i$ makes V a $\mathbb{Z}$-graded $\mathfrak{sl}_2(\mathbb{C})$-module with $\deg(V_i) = i$.

As for the grading by $\mathbb{Z}_2^2$, some of the dense modules can be graded while some others can not.

Let us study the case where $\xi = \bar{0}$.

Proposition 3.23. Let $\tau \in \mathbb{C}$ be such that the module $V = V(\bar{0}, \tau)$ is simple, then V can be made a $\mathbb{Z}_2^2$-graded $\mathfrak{sl}_2(\mathbb{C})$-module.

Proof. Since $\xi = \bar{0}$, we can choose $\lambda = 0 \in \xi$. Then $V = \bigoplus_{i\in\mathbb{Z}} V_i$, where $V_i = \mathbb{C}v_{2i}$, being $\{v_{2i} \mid i \in \mathbb{Z} \ldots\}$ the basis of V. We set $e_0 = v_0, e_{-1} = 0$ and $e_k = \frac{1}{4^k}(\prod_{j=0}^{k}(\tau - (2j-1)^2))v_{2k} + v_{-2k}$, and also $d_0 = 0$ and $d_k = \frac{1}{4^k}(\prod_{j=0}^{k}(\tau - (2j-1)^2))v_{2k} - v_{-2k}$, for $k \in \mathbb{N}$. Since V is simple, the set $\{e_0, e_1, \ldots, d_1, d_2, \ldots\}$ is a basis for V with a module action given by:

$$\begin{aligned} h.e_k &= 2kd_k, \\ h.d_k &= 2ke_k, \\ B.e_k &= e_{k+1} + \tfrac{1}{4}(\tau - (2k-1)^2)e_{k-1}, \\ B.d_k &= d_{k+1} + \tfrac{1}{4}(\tau - (2k-1)^2)d_{k-1}, \\ C.e_k &= d_{k+1} - \tfrac{1}{4}(\tau - (2k-1)^2)d_{k-1}, \\ C.d_k &= e_{k+1} - \tfrac{1}{4}(\tau - (2k-1)^2)e_{k-1}, \end{aligned} \tag{20}$$

Let $V_{(0,0)} = \langle e_i \mid i \text{ is even}\rangle$, $V_{(0,1)} = \langle e_i \mid i \text{ is odd}\rangle$, $V_{(1,0)} = \langle d_i \mid i \text{ is even }\rangle$, and $V_{(1,1)} = \langle d_i \mid i \text{ is odd}\rangle$. Then $\Gamma : V = \bigoplus_{g\in\mathbb{Z}_2^2} V_g$ is a $\mathbb{Z}_2^2$-grading of V making V a graded $\mathfrak{sl}_2(\mathbb{C})$-module. □

Theorem 3.24. *Let $\bar{0} \neq \xi \in \mathbb{C}/2\mathbb{Z}$ and $\tau \in \mathbb{C}$ be such that the module $V = V(\xi, \tau)$ is simple. Then V is not a $\mathbb{Z}_2^2$-graded $\mathfrak{sl}_2(\mathbb{C})$-module.*

Proof. If $\lambda \in \xi$ then $V = \bigoplus_{k\in\mathbb{Z}} V_k$, where $V_k = \mathbb{C}v_{\lambda+2k}$ being $\{v_{\lambda+2i} \mid i \in \mathbb{Z} \ldots\}$ the basis of V given in Definition 3.3.4. Assume that V has a grading compatible with the Pauli grading on $\mathfrak{sl}_2(\mathbb{C})$, so it can written as $V = V_{(0,0)} \oplus V_{(1,0)} \oplus V_{(0,1)} \oplus V_{(1,1)}$. Now let $V^0 = V_{(0,0)} \oplus V_{(1,0)}$, and $V^1 = V_{(0,1)} \oplus V_{(1,1)}$. Then V^0 and V^1 are thus h-invariant, with the action of B and C sending V^0 to V^1 and vice versa. By Lemma 3.6, V^0 and V^1 are spanned by the weight vectors. Since $V_\lambda = \mathbb{C}v_\lambda$, we must have $v_\lambda \in V^0$ or $v_\lambda \in V^1$.

Without loss of generality, suppose $v_\lambda \in V^0$ (otherwise apply the shift of grading), then $V^1 \ni B.v_\lambda = \frac{1}{4}(\tau-(\lambda+1)^2)v_{\lambda+2}+v_{\lambda-2}$ and $V^1 \ni C.v_\lambda = \frac{1}{4}(\tau-(\lambda+1)^2)v_{\lambda+2} - v_{\lambda-2}$, and since V is simple then $(\tau-(\lambda+1)^2 \neq 0$ and hence $v_{\lambda+2}, v_{\lambda-2} \in V^1$. Now $B.v_{\lambda+2} = \frac{1}{4}(\tau-(\lambda+3)^2)v_{\lambda+4}+v_\lambda$ and $B.v_{\lambda-2} = \frac{1}{4}(\tau-(\lambda-1)^2)v_\lambda+v_{\lambda-4}$ are both in V^0. Since V is simple and $v_{\lambda\in V^0}$ then $v_{\lambda+4}, v_{\lambda-4} \in V^0$. Apply B again to $v_{\lambda+4}, v_{\lambda-4}$ to get that $v_{\lambda+6}, v_{\lambda-6} \in V^1$, and so on. We have shown that V^0 is spanned by the set $\{\ldots, v_{\lambda-8}, v_{\lambda-4}, v_\lambda, v_{\lambda+4}, v_{\lambda+8}, \ldots\}$ and V^1 by $\{\ldots, v_{\lambda-6}, v_{\lambda-2}, v_{\lambda+2}, v_{\lambda+6}, \ldots\}$. Now let $0 \neq v \in V_{(0,0)} \subseteq V^0$. Then v can be written as $v = \alpha_{-m}v_{\lambda-4m} + \cdots + \alpha_{-1}v_{\lambda-4} + \alpha_0 v_\lambda + \cdots + \alpha_n v_{\lambda+4n}$ for some non-negative integers m, n and some $\alpha_i \in \mathbb{C}$. But since $V_{(0,0)}$ is h^2-invariant, the elements

$$\begin{aligned} h^2.v &= \alpha_{-m}(\lambda-4m)^2 v_{\lambda-4m} + \cdots + \alpha_0\lambda^2 v_\lambda + \cdots + \alpha_n(\lambda+4n)^2 v_{\lambda+4n}, \\ h^4.v &= \alpha_{-m}(\lambda-4m)^4 v_{\lambda-4m} + \cdots + \alpha_0\lambda^4 v_\lambda + \cdots + \alpha_n(\lambda+4n)^4 v_{\lambda+4n}, \\ &\cdots \\ h^{2(m+n)}.v &= \alpha_{-m}(\lambda-4m)^{2(m+n)} v_{\lambda-4m} + \cdots + \alpha_0\lambda^{2(m+n)} v_\lambda + \cdots \\ &\quad +\alpha_n(\lambda+4n)^{2(m+n)} v_{\lambda+4n}. \end{aligned}$$

are in $V_{(0,0)}$. Now, to use the Vandermonde's determinant we have to show that $(\lambda-4m)^2,\dots,\lambda^2,(\lambda+4)^2,\dots,(\lambda+4n)^2$ are all distinct. Assume that we have two different weights, $(\lambda+4k_1)$ and $(\lambda+4k_2)$, such that $(\lambda+4k_1)^2=(\lambda+4k_2)^2$, then $|\lambda+4k_1|=|\lambda+4k_2|$. Hence either $\lambda+4k_1=\lambda+4k_2$ or $\lambda+4k_1=-4k_2-\lambda$, but the first one is impossible. This means that $\lambda=-2(k_1+k_2)\in 2\mathbb{Z}$, which is not the case since $\xi\neq\bar{0}$. Hence,

$$\begin{vmatrix} 1 & \dots & 1 & \dots & 1 \\ (\lambda-4m)^2 & \dots & \lambda^2 & \dots & (\lambda+4n)^2 \\ \vdots & \vdots & \dots & \vdots & \\ (\lambda-4m)^{2(m+n)} & \dots & \lambda^{2(m+n)} & \dots & (\lambda+4n)^{2(m+n)} \end{vmatrix} \neq 0.$$

It follows that $V_{(0,0)}$ is spanned by weight vectors, which means that there is $v_{\lambda+4s}\in V_{(0,0)}$ for some $s\in\mathbb{Z}$, but $h.v_s=(\lambda+4s)v_s\in V_{(1,0)}$, which is a contradiction. □

Corollary 3.25. Let $\bar{0}\neq\xi\in\mathbb{C}/2\mathbb{Z}$ and $\tau\in\mathbb{C}$. Then the module $V=V(\xi,\tau)$ cannot be a $\mathbb{Z}_2^2$-graded $\mathfrak{sl}_2(\mathbb{C})$-module.

Proof. Theorem 3.5.4 covers the case where V is simple, so it is enough to prove this fact when V is non-simple. Suppose that V is a non-simple $\mathbb{Z}_2^2$-graded $\mathfrak{sl}_2(\mathbb{C})$-module; then V has a unique maximal Verma submodule (see e.g. [Maz09, Theorem 3.29]), which has to be graded; this is a contradiction since Verma modules cannot be a $\mathbb{Z}_2^2$-graded $\mathfrak{sl}_2(\mathbb{C})$-modules. □

3.6. **Gradings on torsion-free modules.** Let V be a G-graded vector space, U a G-graded subspace of V, then V/U is canonically G-graded with $V/U=\bigoplus_{g\in G}(V/U)_g$, where $(V/U)_g=(V_g+U)/U$.

Lemma 3.26. Let V be a G-graded vector space, U a subspace of V such that V/U is canonically G-graded. Then U is graded.

Proof. Let $V=\bigoplus_{g\in G}V_g$, and $V/U=\bigoplus_{g\in G}(V/U)_g$, where $(V/U)_g=V_g+U/U$. Now let $u\in U$, then u can be written as $u=v_1+v_2+\cdots+v_m$, where $v_i\in V_{g_i}$ for some $g_i\in G$. Now

$$\begin{aligned} U=u+U&=(v_1+v_2+\cdots v_m)+U\\ &=(v_1+U)+(v_2+U)+\cdots+(v_m+U), \end{aligned}$$

but $\bar{v_i}=(v_i+U)\in(V/U)_{g_i}$, so in the algebra (V/U)

$$\overline{v_1}+\bar{v_2}+\cdots+\bar{v_m}=\bar{0},$$

and since the sum is direct, then $\overline{v_i}=\bar{0}$ for all $1\le i\le m$, and hence $v_i\in U$ for all $1\le i\le m$, showing that U is graded.

□

We will study now the canonical gradings of the modules described in Theorem 3.13. These gradings depend on the degree of the polynomial $p(t)$. Since the gradings of $M(p(t),g(t),\lambda)$ and $M'(p(t),g(t),\lambda)$ are similar, we will study only one of them.

Theorem 3.27. *Let $M(p(t),g(t),\lambda)$ be as in Theorem 3.13, with $p(t)=\mu$ a non-zero constant. Then $M(p(t),g(t),\lambda)$ is not a canonically $\mathbb{Z}_2^2$-graded $\mathfrak{sl}_2(\mathbb{C})$-module.*

Proof. Suppose that the left ideal $I = U(I_\lambda)(g(h)x + \mu)$ is graded by $\mathbb{Z}_2^2$, then the element $g(h)x + \mu = g(h)(\frac{B+C}{2}) + \mu = g(h)\frac{B}{2} + g(h)\frac{C}{2} + \mu$ belongs to I. The polynomial $g(h)$ has a linear combination of elements of degrees $(0,0)$ or $(1,0)$, so the term $g(h)\frac{B}{2}$ is a linear combination of elements of degrees $(0,1)$ or $(1,1)$; similarly $g(h)\frac{C}{2}$ is a linear combination of elements of degrees $(0,1)$ or $(1,1)$. Since only μ has degree $(0,0)$ it follows that $\mu \in I$ or, in other words, $1 \in I$, which means that $I = U(I_\lambda)$, so $M(p(t), g(t), \lambda)$ is trivial, a contradiction. As a result, I is not graded. Using Lemma 3.26, we conclude that $M(p(t), g(t), \lambda)$ is not canonically graded. □

Theorem 3.28. *Let $M(p(t), g(t), \lambda)$ be as in Theorem 3.13, with $p(t) = \mu$, a nonzero constant. Then $M(p(t), g(t), \lambda)$ is not a canonically $\mathbb{Z}$-graded $\mathfrak{sl}_2(\mathbb{C})$-module.*

Proof. Suppose that the left ideal $I = U(I_\lambda)(g(h)x + \mu)$ is graded by $\mathbb{Z}$, so that the element $g(h)x + \mu \in I$. Then the polynomial $g(h)$ has degree 0, so the term $g(h)x$ has degree 1. As before, μ is the only element of degree 0, which implies that $\mu \in I$, a contradiction. Using Lemma 3.26 again, we can see that $M(p(t), g(t), \lambda)$ is not canonically graded by $\mathbb{Z}$. □

Theorem 3.29. *Let $M(p(t), g(t), \lambda)$ be as in Theorem 3.13, with* $\deg p(t) \geq 1$. *Then $M(p(t), g(t), \lambda)$ is not a canonically $\mathbb{Z}$-graded $\mathfrak{sl}_2(\mathbb{C})$-module.*

Proof. Suppose that the left ideal $I = U(I_\lambda)(g(h)x + \mu)$ is graded by $\mathbb{Z}$. Since $g(h)x$ has degree 1, $p(h)$ is the only term of degree 0, which implies $p(h) \in I$, so for any nonzero generator v of $M(p(t), g(t), \lambda)$, $p(h).v = 0$. Now let we have that $p(h) = (h - \beta_1)(h - \beta_2)\cdots(h - \beta_k)$, and let $(h - \beta_j)$ be the last term with $(h - \beta_{j+1})(h - \beta_{j+2})\cdots(h - \beta_k).v \neq 0$, then $(h - \beta_j)$ annihilates a nonzero vector and hence h has an eigenvector, which implies that $M(p(t), g(t), \lambda)$ has a weight vector. So $M(p(t), g(t), \lambda)$ now is a simple $\mathbb{Z}$-graded *weight* module, which is a contradiction. □

Theorem 3.30. *Let $M(p(t), g(t), \lambda)$ be as in Theorem 3.13, with* $\deg p(t) \geq 1$. *Then $M(p(t), g(t), \lambda)$ is not a canonically $\mathbb{Z}_2^2$-graded $\mathfrak{sl}_2(\mathbb{C})$-module.*

Proof. Since $g(h)x$ is a linear combination of elements of degrees $(0,0)$ and $(1,1)$, and $p(h)$ is a linear combination of elements of degrees $(0,0)$ and $(1,0)$, it follows that $p(h) \in I = U(I_\lambda)(g(h)x + p(h))$. We can factor the polynomial $p(t) = (t - \beta_1)(t - \beta_2)\cdots(t - \beta_k)$, for a generator $u \in M(p(t), g(t), \lambda)$, $p(h).u = (h - \beta_1)(h - \beta_2)\cdots(h - \beta_k).u = 0$. Now let $(h - \beta_j)$ be the last term with $(h - \beta_{j+1})(h - \beta_{j+2})\cdots(h - \beta_k).u \neq 0$, which implies that $h - \beta_j$ annihilates a nonzero vector and hence h has an eigenvector, which means that $M(p(t), g(t), \lambda)$ is a weight module of $\mathfrak{sl}_2(\mathbb{C})$, a contradiction. □

Now we will study the gradings of the torsion-free modules of rank 1.

Lemma 3.31. Let M be a G-graded torsion-free $\mathfrak{sl}_2(\mathbb{C})$-module, $p(h) \in \mathbb{C}[h]$ a homogeneous element in $U(\mathfrak{sl}_2(\mathbb{C}))$, and $v \in M$ a non-homogeneous element. Then the element $p(h).v \in M$ is non-homogeneous.

Proof. Since $p(h)$ is homogeneous, then $p(h) \in (U(\mathfrak{sl}_2(\mathbb{C})))_g$ for some $g \in G$. Since v is non-homogeneous, then $v = v_{g_1} + v_{g_2} + \cdots + v_{g_k}$ for some $k > 1$ and $g_1, g_2, \ldots, g_k$ are distinct in G, where $v_{g_i} \in M_{g_i}$, with at least two of them nonzero (say v_{g_1}, v_{g_2} are non-zero). Now $p(h).v = p(h).v_{g_1} + p(h).v_{g_2} + \cdots + p(h).v_{g_k}$, where $p(h).v_{g_i} \in M_{g_i+g}$. But $g_1+g, g_2+g, \ldots, g_k+g$ are distinct in G. Since M is torsion-free, $p(h).v_{g_1}, p(h).v_{g_2}$ are non-zero, which means that $p(h).v$ is non-homogeneous. □

Theorem 3.32. *Torsion free $\mathfrak{sl}_2(\mathbb{C})$-modules of rank 1 cannot be $\mathbb{Z}$ or $\mathbb{Z}_2^2$-graded.*

We will prove this theorem for every kind of torsion-free module of rank 1 separately. A useful property is the following.

Lemma 3.33. Let M be a torsion-free $\mathfrak{sl}_2(\mathbb{C})$-module, and $0 \neq v \in M$. Then one of $x.v$ or $y.v$ is non-zero.

Proof. Assume that $x.v = 0$ and $y.v = 0$, then $0 = (xy - yx).v = h.v$, which means that $h.v = 0$, a contradiction. □

Proposition 3.34. The module $N(\alpha, \beta)$, as in Definition 3.15, is not a $\mathbb{Z}_2^2$-graded $\mathfrak{sl}_2(\mathbb{C})$-module.

Proof. Assume that $N = N(\alpha, \beta)$ is a $\mathbb{Z}_2^2$-graded $\mathfrak{sl}_2(\mathbb{C})$-module, so that $N = N_{(0,0)} + N_{(1,0)} + N_{(0,1)} + N_{(1,1)}$. Given a non-zero homogeneous element $f(h) \in N$, we define $\overline{f}(h)$ to be the same as $f(h)$ but computed in the algebra $U(\mathfrak{sl}_2(\mathbb{C}))$. Now $\overline{f}(h)$ can be written as the sum of a linear combination of monomials in h^{2k+1}, for $k = 0, 1, 2, \ldots$, and a linear combination of the monomials h^{2k}, for $k = 0, 1, 2, \ldots$, of degrees $(1, 0)$ and $(0, 0)$, respectively. As a result, $\overline{f}(h)$ is a homogeneous element in $U(\mathfrak{sl}_2)$ of degree 0 with respect to the $\mathbb{Z}_2$-grading on $U(\mathfrak{sl}_2)$ given by

$$U(\mathfrak{sl}_2) = (U(\mathfrak{sl}_2))^0 \oplus (U(\mathfrak{sl}_2))^1,$$

where

$$(U(\mathfrak{sl}_2))^0 = (U(\mathfrak{sl}_2))_{(0,0)} \oplus (U(\mathfrak{sl}_2))_{(1,0)}$$

and

$$(U(\mathfrak{sl}_2))^1 = (U(\mathfrak{sl}_2))_{(0,1)} \oplus (U(\mathfrak{sl}_2))_{(1,1)}.$$

Since $f(h)$ is homogeneous with respect to the $\mathbb{Z}_2^2$-grading, it will be homogeneous in the coarsening grading over $\mathbb{Z}_2$, where $N = N^0 \oplus N^1$, being $N^0 = N_{(0,0)} + N_{(1,0)}$ and $N^1 = N_{(0,1)} + N_{(1,1)}$. Thus either $f(h) \in N^0$ or $f(h) \in N^1$. But $\overline{f}(h).1 = f(h)$. Since $\overline{f}(h)$ is homogeneous in $U(\mathfrak{sl}_2(\mathbb{C}))$, with respect to the $\mathbb{Z}_2$-grading, and $f(h)$ is homogeneous in N with respect to the $\mathbb{Z}_2$-grading, using Lemma 3.31 we conclude that 1 is homogeneous in N with respect to the $\mathbb{Z}_2$-grading. Now either $1 \in N^0$ or $1 \in N^1$. Without loss of generality assume that $1 \in N^0$, which means that $N = N^0$ and N^1 is trivial. But using Lemma 3.33, we have either $B.1 \neq 0$ or $C.1 \neq 0$. These elements belong to N^1, which provides the desired contradiction. □

Proposition 3.35. The module $N(\alpha, \beta)$ as in Definition 3.15 is not a $\mathbb{Z}$-graded $\mathfrak{sl}_2(\mathbb{C})$-module.

Proof. Assume that $N = N(\alpha, \beta)$ is a $\mathbb{Z}$-graded $\mathfrak{sl}_2(\mathbb{C})$-module, hence $N = \bigoplus_{i \in \mathbb{Z}} N_i$. Let $f(h) \in N$ be a non-zero homogeneous element, define $\overline{f}(h)$ to be the same as $f(h)$ but computed in the algebra $U(\mathfrak{sl}_2(\mathbb{C}))$. Now $\overline{f}(h)$ is a homogeneous element in $U(\mathfrak{sl}_2(\mathbb{C}))$ of degree 0 with respect to the $\mathbb{Z}$-grading on $U(\mathfrak{sl}_2(\mathbb{C}))$. Now $\overline{f}(h).1 = f(h)$. Since $\overline{f}(h)$ is homogeneous in $U(\mathfrak{sl}_2)$ and $f(h)$ is homogeneous in N, it follows that 1 is homogeneous in N . Hence $1 \in N_k$ for some $k \in \mathbb{Z}$, which means that $N = N_k$ and N^i is trivial for all $i \neq k$. But using Lemma 3.33, we have either $0 \neq x.1 \in N_{k-1}$ or $0 \neq y.1 \in N_{k+1}$, a contradiction in any case. □

Proposition 3.36. The module $N'(\alpha, \beta)$ as in Definition 3.16 is not a $\mathbb{Z}_2^2$-graded $\mathfrak{sl}_2(\mathbb{C})$-module.

Proof. Assume that $N = N'(\alpha, \beta)$ is a $\mathbb{Z}_2^2$-graded $\mathfrak{sl}_2(\mathbb{C})$-module. Let $f(h) \in N$ be a non-zero homogeneous element, and define $\overline{f}(h)$ to be the same as $f(h)$ but computed in the algebra $U(\mathfrak{sl}_2)$. It follows that $\overline{f}(h)$ is a homogeneous element in $U(\mathfrak{sl}_2(\mathbb{C}))$ of degree 0 with respect to the $\mathbb{Z}_2$-grading on $U(\mathfrak{sl}_2(\mathbb{C}))$. Then $f(h)\overline{f}(h).1 = f(h)$ is homogeneous with respect to the coarsening grading by $\mathbb{Z}_2$. Either $f(h) \in N^0$ or $f(h) \in N^1$. But $\overline{f}(h)$ is homogeneous in $U(\mathfrak{sl}_2(\mathbb{C}))$ with respect to the $\mathbb{Z}_2$-grading, and $f(h)$ is homogeneous in N with respect to the $\mathbb{Z}_2$-grading. Using Lemma 3.31, it follows that 1 is homogeneous in N with respect to the $\mathbb{Z}_2$-grading. Hence either $1 \in N^0$ or $1 \in N^1$. Without loss of generality assume that $1 \in N^0$, which means that $N = N^0$ and N^1 is trivial. But using Lemma 3.33, we have either $0 \neq B.1 \in N^1$ or $0 \neq C.1 \in N^1$, a contradiction in both cases. □

Proposition 3.37. The module $N'(\alpha, \beta)$, as in Definition 3.16, is not a $\mathbb{Z}$-graded $\mathfrak{sl}_2(\mathbb{C})$-module.

Proof. Assume that $N = N'(\alpha, \beta)$ is a $\mathbb{Z}$-graded $\mathfrak{sl}_2(\mathbb{C})$-module. Let $f(h) \in N$ be a non-zero homogeneous element, and let $\overline{f}(h)$ be the same as $f(h)$ but computed in the algebra $U(\mathfrak{sl}_2(\mathbb{C}))$. Now $\overline{f}(h)$ is a homogeneous element in $U(\mathfrak{sl}_2(\mathbb{C}))$ of degree 0 with respect to the $\mathbb{Z}$-grading on $U(\mathfrak{sl}_2(\mathbb{C}))$. Now $\overline{f}(h).1 = f(h)$, and since $\overline{f}(h)$ is homogeneous in $U(\mathfrak{sl}_2(\mathbb{C}))$ and $f(h)$ is homogeneous in N then 1 is homogeneous in N. Hence $1 \in N_k$ for some $k \in \mathbb{Z}$, which means that $N = N_k$ and N^i is trivial for all $i \neq k$. But using Lemma 3.33, we have either $0 \neq x.1 \in N_{k-1}$ or $0 \neq y.1 \in N_{k+1}$, which is a contradiction in any case. □

Proposition 3.38. The module $\bar{N}(\alpha, \beta)$ is not a $\mathbb{Z}_2^2$-graded $\mathfrak{sl}_2(\mathbb{C})$-module.

Proof. Use the argument from the proof of Propositions 3.34 and 3.36. □

Proposition 3.39. The module $\bar{N}(\alpha, \beta)$ is not a $\mathbb{Z}$-graded $\mathfrak{sl}_2(\mathbb{C})$-module.

Proof. Use the argument from the proof of Propositions 3.35 and 3.37. □

In view of Theorem 3.18, the above propositions complete the proof of Theorem 3.32.

Corollary 3.40. The Whittaker modules cannot be $\mathbb{Z}$-graded $\mathfrak{sl}_2(\mathbb{C})$-modules.

Corollary 3.41. The Whittaker modules cannot be $\mathbb{Z}_2^2$-graded $\mathfrak{sl}_2(\mathbb{C})$-modules.

3.7. **Transition to graded-simple modules.** In conclusion, we remark that it is easy to construct a graded $U(I_\lambda)$-module. For example one might consider the module $M = U(I_\lambda)/U(I_\lambda)\alpha$ for some homogeneous element $0 \neq \alpha \in U(I_\lambda)$. For instance, one could take $\alpha = C$, in which case also $U(I_\lambda)\alpha \neq U(I_\lambda)$. Of course, such modules need not be simple. At the same time, using Theorem 3.5, one can construct the series

$$\{0\} \neq U(I_\lambda)\alpha = J_0 \subset J_1 \subset \cdots \subset J_n = U(I_\lambda)$$

of graded left ideals, where the quotients J_{i+1}/J_i are graded-simple $\mathfrak{sl}_2$-modules. The technique developed for the study of graded-simple modules (the *loop construction*) is provided in the next section of this paper. It describes the connection between graded-simple and simple graded modules.

4. Graded-simple modules via the loop construction

Let G be an abelian group and let R be a G-graded unital associative algebra, for example, $R = U(L)$, where L is a G-graded Lie algebra. In this section, we review the relation between simple R-modules and graded-simple R-modules given by the so-called loop construction. Under some restrictions, this construction reduces the classification of graded-simple R-modules to that of gradings by certain quotient groups of G on simple R-modules.

4.1. **Loop algebras and loop modules.** Let $\pi : G \to \overline{G}$ be an epimorphism of abelian groups and let H be the kernel of π. Any G-graded vector space W (in particular, a G-graded algebra or module) over a field $\mathbb{F}$ can be regarded as $\overline{G}$-graded using the grading induced by π, i.e., $W_{\overline{g}} = \bigoplus_{g\in\pi^{-1}(\overline{g})} W_g$ for any $\overline{g} \in \overline{G}$, and this gives us a 'forgetful' functor from the category of G-graded vector spaces (respectively, algebras or modules) to the category of $\overline{G}$-graded vector spaces (respectively, algebras or modules). The loop construction, defined as follows, is the right adjoint of this functor (see [EK17, Remark 3.3]). For a given $\overline{G}$-graded vector space V, consider the tensor product $V \otimes \mathbb{F}G$, where $\mathbb{F}G$ denotes the group algebra of G with coefficients in $\mathbb{F}$. Define $L_\pi(V)$ as the following subspace of $V \otimes \mathbb{F}G$:

$$L_\pi(V) := \bigoplus_{g\in G} V_{\pi(g)} \otimes g,$$

which is naturally G-graded: $L_\pi(V)_g = V_{\pi(g)} \otimes g$.

If A is a $\overline{G}$-graded algebra (not necessarily associative) then $L_\pi(A)$ is a G-graded algebra with respect to the usual product on $A \otimes \mathbb{F}G$, defined by $(a_1 \otimes g_1)(a_2 \otimes g_2) := a_1a_2 \otimes g_1g_2$. If $\mathbb{F}$ is infinite, then $L_\pi(A)$ belongs to a given variety of algebras (for example, associative or Lie) if and only if so does A. A classical example is the so-called *twisted loop algebra* $L(\mathfrak{g}, \Gamma)$ in Lie theory: given a semisimple complex Lie algebra $\mathfrak{g}$ and a $\mathbb{Z}/m\mathbb{Z}$-grading $\Gamma : \mathfrak{g} = \bigoplus_{\bar{k}\in\mathbb{Z}/m\mathbb{Z}} \mathfrak{g}_{\bar{k}}$, one defines $L(\mathfrak{g}, \Gamma) := \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_{\bar{k}} \otimes t^k$, which is a subalgebra of $\mathfrak{g}[t, t^{-1}] := \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$, so in our notation $L(\mathfrak{g}, \Gamma) = L_\pi(\mathfrak{g})$, where $\pi : \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ is the natural homomorphism.

Similarly, if R is a G-graded associative algebra and V is a $\overline{G}$-graded left R-module (where we regard R as a $\overline{G}$-graded algebra) then $L_\pi(V)$ is a G-graded left R-module through $r(v \otimes g) := rv \otimes g'g$ for all $g, g' \in G$, $v \in V_{\pi(g)}$, $r \in R_{g'}$.

Moreover, if $\psi : V \to V'$ is a homomorphism of $\overline{G}$-graded vector spaces (respectively, algebras or modules) then the linear map $L_\pi(\psi) : L_\pi(V) \to L_\pi(V)$ that sends $v \otimes g \mapsto \psi(v) \otimes g$, for all $g \in G$ and $v \in V_{\pi(g)}$, is a homomorphism of G-graded vector spaces (respectively, algebras or modules).

If H is finite and $\mathbb{F}$ is sufficiently good then there is an alternative definition of the loop functor as follows. Recall that the group of characters $\widehat{G}$ acts on any G-graded vector space (see Equation (3)). Similarly, a $\overline{G}$-graded vector space V becomes a module over the group algebra $\mathbb{F}(H^\perp)$, where the subgroup

$$H^\perp := \{\chi \in \widehat{G} : \chi(h) = 1 \ \forall h \in H\}$$

is naturally isomorphic to the group of characters of $\overline{G}$. Assume for now that $|H| = n < \infty$ and that $\mathbb{F}$ is algebraically closed and its characteristic does not divide n. Then we have $|\widehat{H}| = n$ and, moreover, any character of H extends to a character of G. Fix such extensions, $\chi_1, \dots, \chi_n$, for all characters of H, so $\widehat{H} = \{\chi_1|_H, \dots, \chi_n|_H\}$. Then $\{\chi_1, \dots, \chi_n\}$ is a transversal of $H^\perp$ in $\widehat{G}$ (i.e., a set of coset representatives of $H^\perp$ in $\widehat{G}$), hence $\mathbb{F}\widehat{G} = \chi_1\mathbb{F}(H^\perp) \oplus \cdots \oplus \chi_n\mathbb{F}(H^\perp)$.

If V is a $\overline{G}$-graded vector space then we can consider the induced $\mathbb{F}\widehat{G}$-module,

$$I_\pi(V) := \mathrm{Ind}_{H^\perp}^{\widehat{G}}(V) = \mathbb{F}\widehat{G} \otimes_{\mathbb{F}(H^\perp)} V = \chi_1 \otimes V \oplus \cdots \oplus \chi_n \otimes V,$$

which is clearly $\overline{G}$-graded, with the homogeneous component of degree $\bar{g}$ being $\chi_1 \otimes V_{\bar{g}} \oplus \cdots \oplus \chi_n \otimes V_{\bar{g}}$. In fact, this $\overline{G}$-grading on $I_\pi(V)$ can be refined to a G-grading:

$$I_\pi(V)_g := \{x \in \chi_1 \otimes V_{\pi(g)} \oplus \cdots \oplus \chi_n \otimes V_{\pi(g)} : \chi \cdot x = \chi(g)x \ \forall \chi \in \widehat{G}\}.$$

Now, if A is a $\overline{G}$-graded algebra then $I_\pi(A)$ is a G-graded algebra with multiplication defined by $(\chi_i \otimes a')(\chi_j \otimes a'') := \delta_{ij}\chi_i \otimes a'a''$ for $1 \le i, j \le n$ and $a', a'' \in A$, so each of the direct summands $\chi_j \otimes A$ is a $\overline{G}$-graded ideal isomorphic to A as a $\overline{G}$-graded algebra. If R is a G-graded associative algebra and V is a $\overline{G}$-graded left R-module then $I_\pi(V)$ is a G-graded left R-module by means of

$$r(\chi_j \otimes v) = \chi_j(g')^{-1}\chi_j \otimes rv \quad \forall r \in R_{g'},\, v \in V_{\pi(g)},\, g, g' \in G,\, 1 \le j \le n. \tag{21}$$

Note that the direct summands $\chi_j \otimes V$ are $\overline{G}$-graded R-submodules, but they are not necessarily isomorphic. In fact, Equation (21) tells us that, as a left R-module, $\chi_j \otimes V$ is isomorphic to V twisted by $\alpha_{\chi_j}^{-1}$, where α_χ, for any $\chi \in \widehat{G}$, are the automorphisms of R given by the action of $\widehat{G}$, and the twists of a module are defined as follows:

Definition 4.1. Given an automorphism α of R and a left R-module V, we define a new left R-module $V^\alpha = (V, *)$ which equals V as a vector space, but with the new action given by $r * v = \alpha(r)v$. This module V^α is referred to as V *twisted by* α.

It turns out that, under the above assumptions on H and $\mathbb{F}$, $I_\pi(V)$ is isomorphic to $L_\pi(V)$ as a G-graded vector space (respectively, algebra or module). An isomorphism $L_\pi(V) \to I_\pi(V)$ is given by

$$v \otimes g \mapsto \sum_{j=1}^{n} \chi_j(g)^{-1}\chi_j \otimes v \text{ for all } v \in V_{\pi(g)},\, g \in G,$$

it does not depend on the choice of the transversal $\{\chi_1, \dots, \chi_n\}$, and its inverse $I_\pi(V) \to L_\pi(V)$ is given by

$$\chi_j \otimes v \mapsto \frac{1}{n}\sum_{h \in H} \chi_j(gh)v \otimes gh \text{ for all } v \in V_{\pi(g)},\, g \in G,\, 1 \le j \le n$$

(see [EK17, Proposition 3.8] for the case of R-modules).

4.2. **Correspondence Theorem.** The loop functor L_π associated to an epimorphism $\pi : G \to \overline{G}$, as described in the previous subsection, can be used to establish a correspondence between, on the one hand, the class $\mathfrak{A}(\pi)$ of $\overline{G}$-graded algebras that are simple and central (disregarding the grading) and, on the other hand, the class $\mathfrak{B}(\pi)$ of G-graded algebras that are graded-simple and whose centroid is isomorphic to $\mathbb{F}H$ as a graded algebra, where H is the kernel of π. This correspondence was established in [ABFP08, Theorem 7.1.1] over an arbitrary field $\mathbb{F}$, but the result is easier to state if $\mathbb{F}$ is algebraically closed (thanks to [ABFP08, Lemmas 4.3.8 and 6.3.4(v)]). Then the above condition on the centroid is equivalent to its identity component being $\mathbb{F}$ (i.e., the algebra being graded-central) and its support being H, while the Correspondence Theorem says that L_π is a functor $\mathfrak{A}(\pi) \to \mathfrak{B}(\pi)$ that gives a bijection between the isomorphism classes in these categories. (Under some restrictions, the surjectivity was already established in [BSZ01, Theorem 7].) Thus, the classification of G-graded-central-simple algebras reduces to the classification of gradings on central simple algebras by the quotient groups of G.

A similar approach works for graded modules, although with some additional difficulties arising from the fact that the centralizer of a graded-simple module, unlike the centroid of a graded-simple algebra, need not be commutative. The use of the loop construction in this context was started in [MZpr] and the Correspondence Theorem was obtained in [EK17]. Before we state the result, we need to introduce some terminology and notation.

Let R be a G-graded unital associateive algebra. We denote the centralizer of a left R-module V by $C(V) := \mathrm{End}_R(V)$ and apply the elements of $C(V)$ to the elements of V on the right. Recall that a linear map $W \to W'$ of G-graded vector spaces is said to be *homogeneous of degree* g if it sends W_k to W'_{gk} for all $k \in G$. In particular, for a G-graded left R-module W, let $C(W)_g$ be the set of all elements of $C(W)$ that are homogeneous of degree g. It is clear from the definition that $C^{\mathrm{gr}}(W) := \bigoplus_{g\in G} C(W)_g$ is a G-graded algebra and W is a G-graded right $C^{\mathrm{gr}}(W)$-module. Moreover, if W is graded-simple then $C^{\mathrm{gr}}(W) = C(W)$ (see [EK17, Proposition 2.1]).

Note that if V is a $\overline{G}$-graded left R-module then $C^{\mathrm{gr}}(V)$ is a $\overline{G}$-graded algebra, so $L_\pi(C^{\mathrm{gr}}(V))$ is a G-graded algebra, which acts naturally on the G-graded left R-module $L_\pi(V)$:

$$(v \otimes g)(\delta \otimes g') := v\delta \otimes gg' \quad \forall v \in V_{\pi(g)},\, \delta \in C(V)_{\pi(g')},\, g, g' \in G,$$

and this action centralizes that of R. Thus, we can identify $L_\pi(C^{\mathrm{gr}}(V))$ with a G-graded subalgebra of $C^{\mathrm{gr}}(L_\pi(V))$.

The classical Schur's Lemma, which says that the centralizer of a simple module is a division algebra, has a graded analog: the centralizer of a graded-simple module is a graded-division algebra (see, for instance, [EK13, Lemma 2.4]), and hence the module is free over its centralizer. Commutative graded-division algebras are called *graded-fields* (not to be confused with fields that are graded!).

A module V is called *central* (or *Schurian*) if $C(V) = \mathbb{F}1$, i.e., $C(V)$ consists of the scalar multiples of the identity map. Similarly, a graded module W is called *graded-central* if $C(W)_e = \mathbb{F}1$.

We need one more concept, which is a generalization of G-grading and is called *G-pregrading* or *G-covering* (see [Smi97] and [BL07]).

Definition 4.2. Let V be a left R-module.

(1) A family of subspaces $\Sigma = \{V_g : g \in G\}$ is called a *G-pregrading* on V if $V = \sum_{g \in G} V_g$ and $R_g V_k \subset V_{gk}$ for all $g, k \in G$.
(2) Given two pregradings $\Sigma^i = \{V_g^i : g \in G\}$, $i = 1, 2$, Σ^1 is said to be a *refinement* of Σ^2 (or Σ^2 a *coarsening* of Σ^1) if $V_g^1 \subset V_g^2$ for all $g \in G$. If at least one of these inclusions is strict, the refinement is said to be *proper*.
(3) A G-pregrading Σ is called *thin* if it admits no proper refinement.

Example 4.3. Let S be a subgroup of G and suppose $V = \bigoplus_{\bar{g} \in G/S} V_{\bar{g}}$ is a G/S-graded left R-module. Then the family $\Sigma := \{V'_g : g \in G\}$, where $V'_g = V_{gS}$ for all $g \in G$, is a G-pregrading on V, which will be referred to as the *G-pregrading associated to the given G/S-grading on V*.

The importance of thin coverings in our context stems from the next result:

Proposition 4.4 ([MZpr, Lemma 27])**.** Let $\pi : G \to G/S$ be the natural homomorphism and let V be a G/S-graded left R-module. The following are equivalent:
(i) $L_\pi(V)$ is G-graded-simple;
(ii) V is G/S-graded-simple and the G-pregrading on V associated to its G/S-grading is thin. □

The Correspondence Theorem we are about to state relates the following two categories.

Definition 4.5. Fix a subgroup S of G and let $\pi : G \to \overline{G} = G/S$ be the natural homomorphism.
(1) $\mathfrak{M}(\pi)$ is the category whose objects are the simple, central, $\overline{G}$-graded left R-modules such that the G-pregrading associated to the $\overline{G}$-grading is thin, and whose morphisms are the isomorphisms of $\overline{G}$-graded modules.
(2) $\mathfrak{N}(\pi)$ is the category whose objects are the pairs $(W, \mathcal{F})$, where W is a G-graded-simple left R-module and $\mathcal{F}$ is a maximal graded-subfield of $C(W)$, which is isomorphic to the group algebra $\mathbb{F}S$ as a G-graded algebra, and the morphisms $(W, \mathcal{F}) \to (W', \mathcal{F}')$ are the isomorphism of G-graded modules $\phi : \mathcal{W} \to \mathcal{W}'$ such that $\phi \mathcal{F} \phi^{-1} = \mathcal{F}'$.

Theorem 4.6 ([EK17, Proposition 4.5 and Theorem 4.14])**.** If V is an object of $\mathfrak{M}(\pi)$ then $\big(L_\pi(V), L_\pi(\mathbb{F}1)\big)$ is an object of $\mathfrak{N}(\pi)$, and if $\varphi : V \to V'$ is a morphism in $\mathfrak{M}(\pi)$, then $L_\pi(\varphi)$ is a morphism in $\mathfrak{N}(\pi)$, so we have the loop functor $L_\pi : \mathfrak{M}(\pi) \to \mathfrak{N}(\pi)$. This functor has the following properties:
(i) L_π is faithful, i.e., injective on the set of morphisms $V \to V'$, for any objects V and V' in $\mathfrak{M}(\pi)$.
(ii) L_π is essentially surjective, i.e., any object $(W, \mathcal{F})$ in $\mathfrak{N}(\pi)$ is isomorphic to $\big(L_\pi(V), L_\pi(\mathbb{F}1)\big)$ for some object V in $\mathfrak{M}(\pi)$.
(iii) If V and V' are objects in $\mathfrak{M}(\pi)$ such that their images under L_π are isomorphic in $\mathfrak{N}(\pi)$, then there is a character $\chi \in \widehat{S}$ such that V' is isomorphic to V^χ in $\mathfrak{M}(\pi)$. □

The definition of the twisted module V^χ, $\chi \in \widehat{S}$, is technical (see [EK17, Definition 4.10], which is analogous to [ABFP08, Definition 6.3.1]), but if χ can be extended to a character of G (which is guaranteed if $\mathbb{F}$ is algebraically closed) then V^χ is isomorphic to V^{α_χ}, where α_χ is the automorphism of R given by the action of the extended χ (see [EK17, Proposition 4.11]).

If $\mathbb{F}$ is algebraically closed, this Correspondence Theorem gives a classification of G-graded-central-simple R-modules up to isomorphism as follows. The centralizer of any such module contains a maximal graded-subfield $\mathcal{F}$ isomorphic to $\mathbb{F}S$ for some subgroup S of G (see [EK17, Proposition 3.5]). We partition all G-graded-central-simple modules according to the graded isomorphism class of their centralizer and, for each class, make a choice of $\mathcal{F}$ (equivalently, of S) and let $\pi : G \to \overline{G} = G/S$ be the natural homomorphism. Then for every G-graded-central-simple W with a fixed centralizer, there exists a simple, central, $\overline{G}$-graded module V such that $W \simeq L_\pi(V)$, and this V is unique up to isomorphism of $\overline{G}$-graded modules and twisting by the action of $\widehat{G}$ on R. Thus, we can obtain the classification of G-graded-central-simple modules if we know the classification of gradings on central-simple modules by the quotient groups of G. Finally, we observe that, assuming $\mathbb{F}$ is algebraically closed, the condition of graded-centrality is automatic for graded-simple modules whose dimension (as a vector space) is less than the cardinality of $\mathbb{F}$ (see [MZpr, Theorem 14]).

4.3. **Graded Brauer invariants of graded-simple modules with a semisimple finite-dimensional centralizer.** The Brauer invariants that we are going to define belong to the graded version of Brauer group introduced in [PP70]. Given a field $\mathbb{F}$ and an abelian group G, the group $B_G(\mathbb{F})$ consists of the equivalence classes of finite-dimensional associative $\mathbb{F}$-algebras that are central, simple, and G-graded, where $A_1 \sim A_2$ if and only if there exist finite-dimensional G-graded $\mathbb{F}$-vector spaces V_1 and V_2 such that $A_1 \otimes \mathrm{End}_{\mathbb{F}}(V_1) \simeq A_2 \otimes \mathrm{End}_{\mathbb{F}}(V_2)$ as G-graded algebras. Here, unlike for some more general versions of the graded Brauer group, $A \otimes B$ denotes the usual (untwisted) tensor product of $\mathbb{F}$-algebras, equipped with the natural G-grading: $(a_1 \otimes b_1)(a_2 \otimes b_2) := a_1a_2 \otimes b_1b_2$ and $\deg(a \otimes b) := \deg(a)\deg(b)$ for nonzero homogeneous $a \in A$ and $b \in B$. This tensor product induces a group structure on the set of equivalence classes: $[A][B] := [A \otimes B]$.

Every class $[A]$ contains a unique graded-division algebra (up to isomorphism). Indeed, recall that there exist a graded-division algebra $\mathcal{D}$ and a graded right $\mathcal{D}$-module $\mathcal{V}$ such that A is isomorphic to $\mathrm{End}_{\mathcal{D}}(\mathcal{V})$ as a G-graded algebra, where $\mathcal{D}$ is unique up to graded isomorphism and $\mathcal{V}$ up to graded isomorphism and shift of grading. Pick a $\mathcal{D}$-basis $\{v_1, \ldots, v_k\}$ of $\mathcal{V}$ that consists of homogeneous elements. Let $\widetilde{\mathcal{V}} = \mathbb{F}v_1 \oplus \cdots \oplus \mathbb{F}v_k$. Then $\widetilde{\mathcal{V}}$ is a G-graded vector space, and the map

$$\widetilde{\mathcal{V}} \otimes \mathcal{D} \to \mathcal{V},\ v \otimes d \mapsto vd,$$

is a graded isomorphism. Thus we can assume $\mathcal{V} = \widetilde{\mathcal{V}} \otimes \mathcal{D}$ and hence identify

$$\mathrm{End}_{\mathcal{D}}(\mathcal{V}) \simeq \mathrm{End}_{\mathbb{F}}(\widetilde{\mathcal{V}}) \otimes \mathcal{D}.$$

Now the isomorphism $A \simeq \mathrm{End}_{\mathbb{F}}(\widetilde{\mathcal{V}}) \otimes \mathcal{D}$ implies that $\mathcal{D}$ is central simple and that $[A] = [\mathcal{D}]$, while the uniqueness of $\mathcal{D}$ mentioned above implies that $[\mathcal{D}_1] = [\mathcal{D}_2]$ if and only if $\mathcal{D}_1 \simeq \mathcal{D}_2$ as graded algebras.

In general, the graded Brauer group $B_G(\mathbb{F})$ can be complicated because it contains the classical Brauer group $B(\mathbb{F})$ as the classes of central division algebras with trivial G-grading. But if $\mathbb{F}$ is algebraically closed then, for any abelian group G, $B_G(\mathbb{F})$ is isomorphic to the group of alternating continuous bicharacters of the pro-finite group $\widehat{G_0}$, where G_0 is the torsion subgroup of G if $\mathrm{char}\,\mathbb{F} = 0$ and the p'-torsion subgroup of G if $\mathrm{char}\,\mathbb{F} = p > 0$ (i.e., the set of all elements whose order is finite and coprime with p) —see [EK15a, §2]. By means of duality, each such bicharacter corresponds to a pair (T, β) where T is a finite subgroup of G and $\beta : T \times T \to \mathbb{F}^\times$ is a nondegenerate alternating bicharacter. This pair is connected with the corresponding unique graded-division algebra $\mathcal{D}$ as follows: T is the support of $\mathcal{D}$ and β is defined by Equation (4).

From now on, we assume that $\mathbb{F}$ is algebraically closed and restrict our attention to G-graded-simple left R-modules W such that $\dim C(W)$ is finite and not divisible by $\operatorname{char} \mathbb{F}$. This is necessary and sufficient to guarantee that $\mathcal{D} := C(W)$ contains a maximal graded-subfield $\mathcal{F}$ isomorphic to $\mathbb{F}S$ where $|S|$ is finite and not divisible by $\operatorname{char} \mathbb{F}$; it also implies that W is semisimple as an ungraded module (see [EK17, Corollary 5.4]). Let T be the support of $\mathcal{D}$ and let $\beta : T \times T \to \mathbb{F}^\times$ be the alternating bicharacter defined by Equation (4). It is not necessarily nondegenerate: its radical is precisely the support of the center of $\mathcal{D}$, which we denote by H. The subgroup S is a maximal isotropic subgroup of T (i.e., a maximal subgroup with the property $\beta|_{S\times S} = 1$), and it contains H (see [EK17, Proposition 5.3], where our H is denoted by Z and our S by H; here we follow the notation of [EK15a]).

Definition 4.7. Let W be a G-graded-simple left R-module such that $\dim C(W)$ is finite and not divisible by $\operatorname{char} \mathbb{F}$.

(1) The *inertia group* of W is $K_W := H^\perp \subset \widehat{G}$, where H is the support of the center of $\mathcal{D} := C(W)$.
(2) The *(graded) Brauer invariant* of W is the class of the G/H-graded-division algebra $\mathcal{D}\varepsilon$ in $B_{G/H}(\mathbb{F})$, where ε is any primitive central idempotent of $\mathcal{D}$.
(3) The *(graded) Schur index* of W is the degree of the matrix algebra $\mathcal{D}\varepsilon$.

We note that $\mathcal{D}\varepsilon$ is a G/H-graded-division algebra that is central simple (disregarding the grading), so $[\mathcal{D}\varepsilon]$ is indeed an element of $B_{G/H}(\mathbb{F})$, and this element does not depend on the choice of ε (see [EK17, Theorem 5.7]). It corresponds to the pair (T', β'), where $T' = T/H$ and β' is the nondegenerate bicharacter $T' \times T' \to \mathbb{F}^\times$ induced by β (i.e., $\beta'(sH, tH) := \beta(s, t)$ for all $s, t \in T$). The Schur index equals $|S/H| = \sqrt{|T/H|}$ and has the meaning of the multiplicity of any simple constituent of W. The number of non-isomorphic simple constituents is $|H|$, they form an orbit under the action of $\widehat{G}$ on the isomorphism classes of R-modules by twisting, and the inertia group K_W is the stabilizer of each point in this orbit (see [EK17, Proposition 5.12]). By the Correspondence Theorem, $W \simeq L_\pi(V) \simeq I_\pi(V)$ for some object V of $\mathfrak{M}(\pi)$, where $\pi : G \to G/S$ is the natural homomorphism. Disregarding the G/S-grading, V is isomorphic to a simple constituent of W. In fact, any of these constituents can serve as V, since they are twists of each other.

4.4. **Finite-dimensional graded-simple modules.** We have already seen that the inertia group of a G-graded-simple left R-module W can be expressed in terms of any (ungraded) simple constituent V of W: $K_W = K_V$, where

$$K_V := \{\chi \in \widehat{G} : V^{\alpha_\chi} \text{ is isomorphic to } V\}.$$

If W is finite-dimensional then also its Brauer invariant can be expressed in terms of V. In fact, this is the way Brauer invariants were defined in [EK15a] (for the case $R = U(L)$, where L is a semisimple finite-dimensional Lie algebra equipped with a G-grading). We continue assuming that $\mathbb{F}$ is algebraically closed.

Theorem 4.8 ([EK17, Corollary 6.4])**.** Let W be a finite-dimensional G-graded-simple left R-module such that $\operatorname{char} \mathbb{F}$ does not divide the dimension of $C(W)$.

Let V be a simple (ungraded) submodule of W and let $\varrho_V : R \to \mathrm{End}_{\mathbb{F}}(V)$ be the associated representation. Let H be the support of the center of $C(W)$. Then there is a unique G/H-grading on $\mathrm{End}_{\mathbb{F}}(V)$ that makes ϱ_V a homomorphism of G/H-graded algebras. With respect to this grading, the class of $\mathrm{End}_{\mathbb{F}}(V)$ is precisely the Brauer invariant of W. □

The G-graded-simple module W can be reconstructed from V if we compute the pair (T', β') corresponding to the unique G/H-graded-division algebra $\mathcal{D}'$ in $[\mathrm{End}_{\mathbb{F}}(V)] \in B_{G/H}(\mathbb{F})$. As mentioned in the previous subsection, the support T and bicharacter $\beta : T \times T \to \mathbb{F}^\times$ of the G-graded-division algebra $\mathcal{D} := C(W)$ are given by $T = (\pi')^{-1}(T')$ and $\beta = \beta' \circ (\pi' \times \pi')$, where $\pi' : G \to G/H$ is the natural homomorphism. In fact, $\mathcal{D} \simeq L_{\pi'}(\mathcal{D}')$ by [EK17, Remark 5.10]. Now fix any maximal isotropic subgroup S' of T' (with respect to β'), then $S := (\pi')^{-1}(S')$ is a maximal isotropic subgroup of T (with respect to β), so $\mathcal{F} := \bigoplus_{s \in S} \mathcal{D}_s$ is a maximal graded-subfield of $\mathcal{D}$ isomorphic to $\mathbb{F}S$. Hence, it follows from the Correspondence Theorem that V admits a structure of G/S-graded R-module such that V becomes an object in $\mathfrak{M}(\pi)$ and $W \simeq L_\pi(V)$, where $\pi : G \to G/S$ is the natural homomorphism.

Remark 4.9. All G/S-gradings that make V a graded R-module are shifts of each other.

Proof. Suppose we have two such gradings, Γ and Γ'. Since R acts on V through ϱ_V and the simple, G/S-graded algebra $\mathrm{End}_{\mathbb{F}}(V)$ admits a unique G/S-simple-graded module up to isomorphism and shift, there exist $g \in G$ and an isomorphism of G/S-graded modules $f : (V, \Gamma)^{[g]} \to (V, \Gamma')$. Forgetting the gradings, f is an element of $\mathrm{End}_R(V)$, so f is a scalar multiple of the identity map and thus $\Gamma' = \Gamma^{[g]}$. □

Remark 4.10. W can be obtained from V by a two-step loop construction: first we get the G/H-graded module $W' := L_{\pi''}(V)$, where $\pi'' : G/H \to G/S$ is the natural homomorphism (so $\pi = \pi'' \circ \pi'$), and then $W \simeq L_{\pi'}(W')$ (see [EK17, p. 83]). The centralizer of W' is isomorphic to $\mathcal{D}'$ (the Brauer invariant) as a G/H-graded algebra, and V is the only simple constituent of W', with multiplicity equal to the Schur index. This two-step approach was taken in [EK15a].

There remains the question which simple R-modules appear as simple constituents of G-graded-simple modules. Assume $\operatorname{char} \mathbb{F} = 0$.

Theorem 4.11 ([EK17, Theorem 7.1])**.** A finite-dimensional simple left R-module V is isomorphic to a simple submodule of a finite-dimensional G-graded-simple left R-module if and only if the index $[\widehat{G} : K_V]$ is finite. □

Thus, the loop functor gives a bijection between, on the one hand, the classes of finite-dimensional G-graded-simple R-modules under isomorphism and shift and, on the other hand, the finite $\widehat{G}$-orbits of isomorphism classes of finite-dimensional simple R-modules. (Note that W and $W^{[g]}$ are isomorphic if and only if $g \in T$.)

Knowing the structure of G-graded-simple modules allows us to determine which semisimple modules admit a G-grading that makes them graded modules because, with such a grading, the module must be isomorphic to a direct sum of graded-simple modules. Hence, assuming $\mathbb{F}$ is algebraically closed and $\operatorname{char} \mathbb{F} = 0$, a finite-dimensional semisimple R-module M admits a G-grading if and only if, for each of its simple constituents V, the $\widehat{G}$-orbit is finite and all simple modules in the orbit occur in M with the same multiplicity that is divisible by the Schur index of V.

In the case $R = U(L)$, where L is a semisimple finite-dimensional Lie algebra, all orbits are finite because V^α is isomorphic to V for any inner automorphism α of L, and the outer automorphism group is finite. The Brauer invariants of finite-dimensional simple modules for all simple finite-dimensional Lie algebras, endowed with all possible G-gradings, were computed in [EK15a, EK15b, DEK17].

References

[ABFP08] Allison, B.; Berman, S.; Faulkner, J.; Pianzola, A. *Realization of graded-simple algebras as loop algebras*, Forum Math. **20** (2008), 395–432.

[AP74] Arnal, D.; Pinczon, G. *On algebraically irreducible representations of the Lie algebra sl(2)*, J. Math. Phys. **15** (1974), 350–359.

[BSZ01] Bahturin, Y.; Sehgal, S.; Zaicev, M. *Group gradings on associative algebras,* J. Algebra, **241**(2001), 667-698.

[Bav90] Bavula,V. *Classification of simple sl(2)-modules and the finite dimensionality of the module of extensions of simple sl(2)-modules.* (Russian), Ukrain. Mat. Zh. **42** (1990), 1174–1180; translation in Ukrainian Math. J. **42** (1990), 1044–1049 (1991).

[Bav92] Bavula,V. *Generalized Weyl algebras and their representations* (Russian), Algebra i Analiz **4** (1992), 75–97; translation in St. Petersburg Math. J. **4** (1993), 71–92.

[BL07] Billig, Y.; Lau, M. *Thin coverings of modules*, J. Algebra **316** (2007), 147–173.

[Blo81] Block R.E. *The irreducible representations of the Lie algebra* $\mathfrak{sl}_2$ *and of the Weyl algebra*, Advances Math. **39** (1981), 69–110.

[DEK17] Draper, C.; Elduque, A.; Kochetov, M. *Gradings on modules over Lie algebras of E types*, Algebr. Represent. Theory **20** (2017), 1085–1107.

[DV16] Draper, C.; Viruel, A. *Fine gradings on* $\mathfrak{e}_6$, Publ. Mat. **60** (2016), 113–170.

[Eld16] Elduque, A. *Gradings on algebras over algebraically closed fields*, Non-associative and non-commutative algebra and operator theory, 113–121, Springer Proc. Math. Stat., **160**, Springer, Cham, 2016.

[EK13] Elduque, A. and Kochetov, M. *Gradings on simple Lie algebras.* Mathematical Surveys and Monographs **189**, American Mathematical Society, Providence, RI; Atlantic Association for Research in the Mathematical Sciences (AARMS), Halifax, NS, 2013. xiv+336 pp., 2013.

[EK15a] Elduque, A.; Kochetov, M. *Graded modules over classical simple Lie algebras with a grading*, Israel J. Math. **207** (2015), no. 1, 229–280.

[EK15b] Elduque, A. and Kochetov, M. *Gradings on the Lie algebra* D_4 *revisited*, J. Algebra **441** (2015), 441–474.

[EK17] Elduque, A.; Kochetov, M. *Graded simple modules and loop modules*, in: Groups, rings, group rings, and Hopf algebras, Contemp. Math. **688** (2017), 53–85.

[Hum78] Humphreys, J. *Introduction to Lie algebras and representation theory.* Second printing, revised. Graduate Texts in Mathematics **9**, Springer-Verlag, New York–Berlin, 1978. xii+171 pp.

[Maz09] Mazorchuk, V. *Lectures on* $\mathfrak{sl}_2(\mathbb{C})$*-modules,* Imperial College Press, London, 2010, x+263 pp.

[MZpr] Mazorchuk, V.; Zhao, K. *Graded simple Lie algebras and graded simple representations.* Manuscripta Math. **156** (2018), 215–240.

[Nil15] Nilsson, J. *Simple* $\mathfrak{sl}_{n+1}$*-module structures on* $\mathcal{U}(\mathfrak{h})$, J.Algebra, **424** (2015), 294–329.

[PP70] Picco, D.; Platzeck, M. *Graded algebras and Galois extensions*, Collection of articles dedicated to Alberto González Domínguez on his sixty-fifth birthday, Rev. Un. Mat. Argentina **25** (1970/71), 401–415.

[PT17] Plaza Martín, F.J.; Tejero Prieto, C. *Construction of simple non-weight* $\mathfrak{sl}(2)$*-modules of arbitrary rank*, J. Algebra, **472** (2017), 172–194.

[Smi97] Smirnov, O. *Simple associative algebras with finite* $\mathbb{Z}$*-grading*, J. Algebra **196** (1997), 171–184.

[Yu16] Yu, J. *Maximal abelian subgroups of compact simple Lie groups of type E*, Geom. Dedicata **185** (2016), 205–269.

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