WEYL GROUPS OF FINE GRADINGS ON SIMPLE LIE ALGEBRAS OF TYPES A, B, C AND D

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Dedicated to Professor Yuri Bahturin on the occasion of his sixty fifth birthday.

Abstract. Given a grading $\Gamma : \mathcal{L} = \bigoplus_{g\in G} \mathcal{L}_g$ on a nonassociative algebra $\mathcal{L}$ by an abelian group G, we have two subgroups of $\mathrm{Aut}(\mathcal{L})$: the automorphisms that stabilize each component $\mathcal{L}_g$ (as a subspace) and the automorphisms that permute the components. By the Weyl group of Γ we mean the quotient of the latter subgroup by the former. In the case of a Cartan decomposition of a semisimple complex Lie algebra, this is the automorphism group of the root system, i.e., the so-called extended Weyl group. A grading is called fine if it cannot be refined. We compute the Weyl groups of all fine gradings on simple Lie algebras of types A, B, C and D (except D_4) over an algebraically closed field of characteristic different from 2.

1. Introduction

In [EKb], we computed the Weyl groups of all fine gradings on matrix algebras, the Cayley algebra $\mathcal{C}$ and the Albert algebra $\mathcal{A}$ over an algebraically closed field $\mathbb{F}$ ($\mathrm{char}\,\mathbb{F} \neq 2$ in the case of the Albert algebra). It is well known that $\mathrm{Der}(\mathcal{C})$ is a simple Lie algebra of type G_2 ($\mathrm{char}\,\mathbb{F} \neq 2, 3$) and $\mathrm{Der}(\mathcal{A})$ is a simple Lie algebra of type F_4 ($\mathrm{char}\,\mathbb{F} \neq 2$). Since the automorphism group schemes of $\mathcal{C}$ and $\mathrm{Der}(\mathcal{C})$, respectively $\mathcal{A}$ and $\mathrm{Der}(\mathcal{A})$, are isomorphic, the classification of fine gradings on $\mathrm{Der}(\mathcal{C})$, respectively $\mathrm{Der}(\mathcal{A})$, is the same as that on $\mathcal{C}$, respectively $\mathcal{A}$ [EKa] and, moreover, the Weyl groups of the corresponding fine gradings are isomorphic. The situation with fine gradings on the simple Lie algebras belonging to series A, B, C and D is more complicated, because the fine gradings on matrix algebras yield only a part of the fine gradings on the simple Lie algebras of series A (so-called Type I gradings). In order to obtain the fine gradings for series B, C and D and the remaining (Type II) fine gradings for series A, one has to consider fine φ-gradings on matrix algebras, which were introduced and classified in [Eld10].

The purpose of this paper is to compute the Weyl groups of all fine gradings on the simple Lie algebras of series A, B, C and D, with the sole exception of type D_4 (which differs from the other types due to the triality phenomenon), over an algebraically closed field $\mathbb{F}$ of characteristic different from 2. To achieve this, we first determine the automorphisms of each fine φ-grading on the matrix algebra $\mathcal{R} = M_n(\mathbb{F})$, $n \geq 3$, and then use the transfer technique of [BK10] to obtain

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the Weyl group of the corresponding fine grading on the simple Lie algebra $\mathcal{L} = [\mathcal{R},\mathcal{R}]/(Z(\mathcal{R})\cap[\mathcal{R},\mathcal{R}])$ or $\mathcal{K}(\mathcal{R},\varphi)$, where in the second case φ is an involution on $\mathcal{R}$ and $\mathcal{K}(\mathcal{R},\varphi)$ stands for the set of skew-symmetric elements with respect to φ.

We adopt the terminology and notation of [EKb], which is recalled in Section 2 for convenience of the reader. In Section 3, we restate the classification of fine φ-gradings on matrix algebras [Eld10] in more explicit terms and determine the relevant automorphism groups of each fine φ-grading (Theorem 3.12). In Section 4, we deal with the simple Lie algebras of series A (Theorems 4.6 and 4.7) and, in Section 5, with those of series B, C and D (Theorems 5.6 and 5.7).

2. Generalities on gradings

Let $\mathcal{A}$ be an algebra (not necessarily associative) over a field $\mathbb{F}$ and let G be a group (written multiplicatively).

Definition 2.1. A *G-grading* on $\mathcal{A}$ is a vector space decomposition

$$\Gamma:\ \mathcal{A}=\bigoplus_{g\in G}\mathcal{A}_g$$

such that

$$\mathcal{A}_g\mathcal{A}_h\subset\mathcal{A}_{gh}\quad\text{for all}\quad g,h\in G.$$

If such a decomposition is fixed, we will refer to $\mathcal{A}$ as a *G-graded algebra*. The nonzero elements $a\in\mathcal{A}_g$ are said to be *homogeneous of degree g*; we will write $\deg a = g$. The *support* of Γ is the set $\operatorname{Supp}\Gamma := \{g\in G \mid \mathcal{A}_g\neq 0\}$.

There are two natural ways to define equivalence relation on graded algebras. We will use the term "isomorphism" for the case when the grading group is a part of definition and "equivalence" for the case when the grading group plays a secondary role. Let

$$\Gamma:\ \mathcal{A}=\bigoplus_{g\in G}\mathcal{A}_g \text{ and } \Gamma':\ \mathcal{B}=\bigoplus_{h\in H}\mathcal{B}_h$$

be two gradings on algebras, with supports S and T, respectively.

Definition 2.2. We say that Γ and Γ' are *equivalent* if there exists an isomorphism of algebras $\psi\colon\mathcal{A}\to\mathcal{B}$ and a bijection $\alpha\colon S\to T$ such that $\psi(\mathcal{A}_s)=\mathcal{B}_{\alpha(s)}$ for all $s\in S$. Any such ψ will be called an *equivalence* of Γ and Γ' (or of $\mathcal{A}$ and $\mathcal{B}$ if the gradings are clear from the context).

The algebras graded by a fixed group G form a category where the morphisms are the *homomorphisms of G-graded algebras*, i.e., algebra homomorphisms $\psi\colon\mathcal{A}\to\mathcal{B}$ such that $\psi(\mathcal{A}_g)\subset\mathcal{B}_g$ for all $g\in G$.

Definition 2.3. In the case $G=H$, we say that Γ and Γ' are *isomorphic* if $\mathcal{A}$ and $\mathcal{B}$ are isomorphic as G-graded algebras, i.e., there exists an isomorphism of algebras $\psi\colon\mathcal{A}\to\mathcal{B}$ such that $\psi(\mathcal{A}_g)=\mathcal{B}_g$ for all $g\in G$.

It is known that if Γ is a grading on a simple Lie algebra, then $\operatorname{Supp}\Gamma$ generates an abelian group (see e.g. [Koc09, Proposition 3.3]). From now on, we will assume that our grading groups are *abelian*. Given a group grading Γ on an algebra $\mathcal{A}$, there are many groups G such that Γ can be realized as a G-grading, but there is one distinguished group among them [PZ89].

Definition 2.4. Suppose that Γ admits a realization as a G_0-grading for some group G_0. We will say that G_0 is a *universal group of* Γ if, for any other realization of Γ as a G-grading, there exists a unique homomorphism $G_0 \to G$ that restricts to identity on $\operatorname{Supp}\Gamma$.

One shows that the universal group, which we denote by $U(\Gamma)$, exists and depends only on the equivalence class of Γ. Indeed, $U(\Gamma)$ is generated by $S = \operatorname{Supp}\Gamma$ with defining relations $s_1 s_2 = s_3$ whenever $0 \neq \mathcal{A}_{s_1}\mathcal{A}_{s_2} \subset \mathcal{A}_{s_3}$ ($s_i \in S$).

As in [PZ89], we associate to Γ three subgroups of the automorphism group $\operatorname{Aut}(\mathcal{A})$ as follows.

Definition 2.5. The *automorphism group of* Γ, denoted $\operatorname{Aut}(\Gamma)$, consists of all automorphisms of $\mathcal{A}$ that permute the components of Γ. Each $\psi \in \operatorname{Aut}(\Gamma)$ determines a self-bijection $\alpha = \alpha(\psi)$ of the support S such that $\psi(\mathcal{A}_s) = \mathcal{A}_{\alpha(s)}$ for all $s \in S$. The *stabilizer of* Γ, denoted $\operatorname{Stab}(\Gamma)$, is the kernel of the homomorphism $\operatorname{Aut}(\Gamma) \to \operatorname{Sym}(S)$ given by $\psi \mapsto \alpha(\psi)$. Finally, the *diagonal group of* Γ, denoted $\operatorname{Diag}(\Gamma)$, is the subgroup of the stabilizer consisting of all automorphisms ψ such that the restriction of ψ to any homogeneous component of Γ is the multiplication by a (nonzero) scalar.

Thus $\operatorname{Aut}(\Gamma)$ is the group of self-equivalences of the graded algebra $\mathcal{A}$ and $\operatorname{Stab}(\Gamma)$ is the group of automorphisms of the graded algebra $\mathcal{A}$. Also, $\operatorname{Diag}(\Gamma)$ is isomorphic to the group of characters of $U(\Gamma)$ via the usual action of characters on $\mathcal{A}$: if Γ is a G-grading (in particular, we may take $G = U(\Gamma)$), then any character $\chi \in \widehat{G}$ acts as an automorphism of $\mathcal{A}$ by setting $\chi * a = \chi(g)a$ for all $a \in \mathcal{A}_g$ and $g \in G$. If $\dim \mathcal{A} < \infty$, then $\operatorname{Diag}(\Gamma)$ is a diagonalizable algebraic group (quasitorus). If, in addition, $\mathbb{F}$ is algebraically closed and $\operatorname{char}\mathbb{F} = 0$, then Γ is the eigenspace decomposition of $\mathcal{A}$ relative to $\operatorname{Diag}(\Gamma)$ (see e.g. [Koc09]), the group $\operatorname{Stab}(\Gamma)$ is the centralizer of $\operatorname{Diag}(\Gamma)$, and $\operatorname{Aut}(\Gamma)$ is its normalizer. If we want to work over an arbitrary field $\mathbb{F}$, we can define the subgroupscheme $\mathbf{Diag}(\Gamma)$ of the automorphism group scheme $\mathbf{Aut}(\mathcal{A})$ as follows:

$$\mathbf{Diag}(\Gamma)(\mathcal{S}) := \{f \in \operatorname{Aut}_{\mathcal{S}}(\mathcal{A} \otimes \mathcal{S}) \mid f|_{\mathcal{A}_g \otimes \mathcal{S}} \in \mathcal{S}^\times \mathrm{id}_{\mathcal{A}_g \otimes \mathcal{S}} \text{ for all } g \in G\}$$

for any unital commutative associative algebra $\mathcal{S}$ over $\mathbb{F}$. Thus $\operatorname{Diag}(\Gamma)$ is the group of $\mathbb{F}$-points of $\mathbf{Diag}(\Gamma)$. One checks that $\mathbf{Diag}(\Gamma) = U(\Gamma)^D$, the Cartier dual of $U(\Gamma)$, also $\operatorname{Stab}(\Gamma)$ is the centralizer of $\mathbf{Diag}(\Gamma)$ and $\operatorname{Aut}(\Gamma)$ is its normalizer with respect to the action of $\operatorname{Aut}(\mathcal{A})$ on $\mathbf{Aut}(\mathcal{A})$ by conjugation (see e.g. [EKa, §2.2]).

Definition 2.6. The quotient group $\operatorname{Aut}(\Gamma)/\operatorname{Stab}(\Gamma)$, which is a subgroup of $\operatorname{Sym}(S)$, will be called the *Weyl group of* Γ and denoted by $W(\Gamma)$.

It follows from the universal property of $U(\Gamma)$ that, for any $\psi \in \operatorname{Aut}(\Gamma)$, the bijection $\alpha(\psi)\colon \operatorname{Supp}\Gamma \to \operatorname{Supp}\Gamma$ extends to a unique automorphism of $U(\Gamma)$. This gives an action of $\operatorname{Aut}(\Gamma)$ by automorphisms of $U(\Gamma)$. Since the kernel of this action is $\operatorname{Stab}(\Gamma)$, we may regard $W(\Gamma) = \operatorname{Aut}(\Gamma)/\operatorname{Stab}(\Gamma)$ as a subgroup of $\operatorname{Aut}(U(\Gamma))$. Given a G-grading $\Gamma : \mathcal{A} = \bigoplus_{g\in G}\mathcal{A}_g$ and a group homomorphism $\alpha\colon G \to H$, we obtain the induced H-grading ${}^\alpha\Gamma : \mathcal{A} = \bigoplus_{h\in H}\mathcal{A}'_h$ by setting $\mathcal{A}'_h = \bigoplus_{g\in\alpha^{-1}(h)}\mathcal{A}_g$. Clearly, an automorphism α of $U(\Gamma)$ belongs to $W(\Gamma)$ if and only if the $U(\Gamma)$-gradings ${}^\alpha\Gamma$ and Γ are isomorphic.

Given gradings $\Gamma : \mathcal{A} = \bigoplus_{g\in G}\mathcal{A}_g$ and $\Gamma' : \mathcal{A} = \bigoplus_{h\in H}\mathcal{A}'_h$, we say that Γ' is a *coarsening* of Γ, or that Γ is a *refinement* of Γ', if for any $g \in G$ there exists

$h \in H$ such that $\mathcal{A}_g \subset \mathcal{A}'_h$. The coarsening (or refinement) is said to be *proper* if the inclusion is proper for some g. (In particular, ${}^{\alpha}\Gamma$ is a coarsening of Γ, which is not necessarily proper.) A grading Γ is said to be *fine* if it does not admit a proper refinement in the class of (abelian) group gradings. Any G-grading on a finite-dimensional algebra $\mathcal{A}$ is induced from some fine grading Γ by a homomorphism $\alpha\colon U(\Gamma) \to G$. The classification of fine gradings on $\mathcal{A}$ up to equivalence is the same as the classification of maximal diagonalizable subgroupschemes of $\mathbf{Aut}(\mathcal{A})$ up to conjugation by $\mathrm{Aut}(\mathcal{A})$ (see e.g. [EKa, §2.2]). Fine gradings on simple Lie algebras belonging to the series A, B, C and D (including D_4) were classified in [Eld10] assuming $\mathbb{F}$ algebraically closed of characteristic 0. If we replace automorphism groups by automorphism group schemes, as was done in [BK10], then the arguments of [Eld10] for all cases except D_4 (which required a completely different method) work under the much weaker assumption — which we adopt from now on — that $\mathbb{F}$ is *algebraically closed of characteristic different from* 2.

3. Fine φ-gradings on matrix algebras

The goal of this section is to determine certain automorphism groups of fine φ-gradings on matrix algebras. These groups will be used in the next two sections to compute the Weyl groups of fine gradings on simple Lie algebras of series A, B, C and D.

3.1. **Classification of fine φ-gradings on matrix algebras.**

Here we present the results of [Eld10, §3] in a more explicit form. We also introduce certain objects that will appear throughout the paper.

Definition 3.1. Let $\mathcal{A}$ be an algebra and let φ be an anti-automorphism of $\mathcal{A}$. A G-grading $\Gamma :\ \mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$ is said to be a *φ-grading* if $\varphi(\mathcal{A}_g) = \mathcal{A}_g$ for all $g \in G$ (i.e., φ is an anti-automorphism of the G-graded algebra $\mathcal{A}$) and $\varphi^2 \in \mathrm{Diag}(\Gamma)$. The universal group of a φ-grading is defined disregarding φ.

We have natural concepts of isomorphism and equivalence for φ-gradings. In addition, we will need another relation, which is weaker than equivalence.

Definition 3.2. If Γ_1 is a φ_1-grading on $\mathcal{A}$ and Γ_2 is a φ_2-gradings on $\mathcal{B}$, we will say that (Γ_1, φ_1) is *isomorphic* (respectively, *equivalent*) to (Γ_2, φ_2) if there exists an isomorphism (respectively, equivalence) of graded algebras $\psi\colon \mathcal{A} \to \mathcal{B}$ such that $\varphi_2 = \psi\varphi_1\psi^{-1}$. In the special case $\mathcal{A} = \mathcal{B}$ and $\varphi_1 = \varphi_2$, we will simply say that Γ_1 is *isomorphic* (respectively, *equivalent*) to Γ_2. We will say that (Γ_1, φ_1) is *weakly equivalent* to (Γ_2, φ_2) if there exists an equivalence of graded algebras $\psi\colon \mathcal{A} \to \mathcal{B}$ such that $\xi\varphi_2 = \psi\varphi_1\psi^{-1}$ for some $\xi \in \mathrm{Diag}(\Gamma_2)$.

Note that if φ is an involution, then the condition $\varphi^2 \in \mathrm{Diag}(\Gamma)$ is satisfied for any Γ. Also, any φ-grading Γ on $\mathcal{A}$ restricts to the space of skew-symmetric elements $\mathcal{K}(\mathcal{A}, \varphi)$.

Suppose $\mathcal{R}$ is a matrix algebra equipped with a G-grading Γ. Then $\mathcal{R}$ is isomorphic to $\mathrm{End}_{\mathcal{D}}(V)$ where $\mathcal{D}$ is a matrix algebra with a division grading (i.e., a grading that makes it a graded division algebra) and V is a graded right $\mathcal{D}$-module (which is necessarily free of finite rank). Let $T \subset G$ be the support of $\mathcal{D}$. Then T is a group and $\mathcal{D}$ can be identified with a twisted group algebra $\mathbb{F}^{\sigma}T$ for some 2-cocycle $\sigma\colon T \times T \to \mathbb{F}^{\times}$, i.e., $\mathcal{D}$ has a basis X_t, $t \in T$, such that $X_uX_v = \sigma(u, v)X_{uv}$ for all

$u,v\in T$ (we may assume $X_e=I$, the identity element of $\mathcal{D}$). Let $\beta(u,v)=\frac{\sigma(u,v)}{\sigma(v,u)}$, so

$$X_uX_v=\beta(u,v)X_vX_u.$$

Then $\beta\colon T\times T\to\mathbb{F}^\times$ is a nondegenerate alternating bicharacter — see e.g. [BK10, §2]. A division grading on a matrix algebra with a given support T and bicharacter β can be constructed as follows. Since β is nondegenerate and alternating, T admits a "symplectic basis", i.e., there exists a decomposition of T into the direct product of cyclic subgroups:

$$T=(H_1'\times H_1'')\times\cdots\times(H_r'\times H_r'')$$

such that $H_i'\times H_i''$ and $H_j'\times H_j''$ are β-orthogonal for $i\neq j$, and H_i' and H_i'' are in duality by β. Denote by ℓ_i the order of H_i' and H_i''. (We may assume without loss of generality that ℓ_i are prime powers.) If we pick generators a_i and b_i for H_i' and H_i'', respectively, then $\varepsilon_i:=\beta(a_i,b_i)$ is a primitive ℓ_i-th root of unity, and all other values of β on the elements $a_1,b_1,\dots,a_r,b_r$ are 1. Define the following elements of the algebra $M_{\ell_1}(\mathbb{F})\otimes\cdots\otimes M_{\ell_r}(\mathbb{F})$:

$$X_{a_i}=I\otimes\cdots I\otimes X_i\otimes I\otimes\cdots I\quad\text{and}\quad X_{b_i}=I\otimes\cdots I\otimes Y_i\otimes I\otimes\cdots I,$$

where

$$X_i=\begin{bmatrix}\varepsilon_i^{n-1}&0&0&\dots&0&0\\0&\varepsilon_i^{n-2}&0&\dots&0&0\\\cdots&&&&&\\0&0&0&\dots&\varepsilon_i&0\\0&0&0&\dots&0&1\end{bmatrix}\text{ and }Y_i=\begin{bmatrix}0&1&0&\dots&0&0\\0&0&1&\dots&0&0\\\cdots&&&&&\\0&0&0&\dots&0&1\\1&0&0&\dots&0&0\end{bmatrix}$$

are the generalized Pauli matrices in the i-th factor, $M_{\ell_i}(\mathbb{F})$. Finally, set

$$X_{(a_1^{i_1},b_1^{j_1},\dots,a_r^{i_r},b_r^{j_r})}=X_{a_1}^{i_1}X_{b_1}^{j_1}\cdots X_{a_r}^{i_r}X_{b_r}^{j_r}.$$

Identify $M_{\ell_1}(\mathbb{F})\otimes\cdots\otimes M_{\ell_r}(\mathbb{F})$ with $M_\ell(\mathbb{F})$, $\ell=\ell_1\cdots\ell_r=\sqrt{|T|}$, via Kronecker product. Then

$$M_\ell(\mathbb{F})=\bigoplus_{t\in T}\mathbb{F}X_t$$

is a division grading with support T and bicharacter β.

Let φ be an anti-automorphism of $\mathcal{R}$ such that Γ is a φ-grading. It is shown in [Eld10, §3] that there exists an involution φ_0 of the graded algebra $\mathcal{D}$ and a φ_0-sesquilinear form $B\colon V\times V\to\mathcal{D}$, which is nondegenerate, homogeneous and balanced, such that, for all $r\in\mathcal{R}$, $\varphi(r)$ is the adjoint of r with respect to B, i.e., $B(x,\varphi(r)y)=B(rx,y)$ for all $x,y\in V$ and $r\in\mathcal{R}$. By *φ_0-sesquilinear* we mean that B is $\mathbb{F}$-bilinear and, for all $x,y\in V$ and $d\in\mathcal{D}$, we have $B(xd,y)=\varphi_0(d)B(x,y)$ and $B(x,yd)=B(x,y)d$; by *balanced* we mean that, for all homogeneous $x,y\in V$, $B(x,y)=0$ is equivalent to $B(y,x)=0$. Moreover, the existence of φ_0 forces T to be an elementary 2-group. The pair (φ_0,B) is uniquely determined by φ up to the following transformations: for any nonzero homogeneous $d\in\mathcal{D}$, we may simultaneously replace φ_0 by $\varphi_0'\colon a\mapsto d\varphi_0(a)d^{-1}$ and B by $B'=dB$. Using Pauli matrices (of order 2) as above to construct a realization of $\mathcal{D}$, we see that matrix transpose $X\mapsto {}^tX$ preserves the grading: for any $u\in T$, the transpose of X_u equals $\pm X_u$. It follows from [BK10, Proposition 2.3] that (φ_0,B) can be adjusted so that φ_0 coincides with the matrix transpose. We will always assume that (φ_0,B)

is adjusted in this way, which makes B unique up to a scalar in $\mathbb{F}$. Also, we may write

$$\varphi_0(X_u) = \beta(u)X_u$$

where $\beta(u) \in \{\pm 1\}$ for all $u \in T$. If we regard T as a vector space over the field of two elements, then the function $\beta\colon T \to \{\pm 1\}$ is a quadratic form whose polar form is the bicharacter $\beta\colon T \times T \to \{\pm 1\}$.

We will say that a φ-grading is *fine* if it is not a proper coarsening of another φ-grading. The following construction of fine φ-gradings on matrix algebras was given in [Eld10] starting from $\mathcal{D}$. We start from T, an elementary 2-group of even dimension, i.e., $T = \mathbb{Z}_2^{\dim T}$, which we continue to write multiplicatively. Let β be a nondegenerate alternating bicharacter on T. Fix a realization, $\mathcal{D}$, of the matrix algebra endowed with a division grading with support T and bicharacter β, and let φ_0 be the matrix transpose on $\mathcal{D}$. Let $q \geq 0$ and $s \geq 0$ be two integers. Let

$$\tau = (t_1, \ldots, t_q), \quad t_i \in T. \tag{1}$$

Denote by $\widetilde{G} = \widetilde{G}(T, q, s, \tau)$ the abelian group generated by T and the symbols $\widetilde{g}_1, \ldots, \widetilde{g}_{q+2s}$ with defining relations

$$\widetilde{g}_1^2 t_1 = \ldots = \widetilde{g}_q^2 t_q = \widetilde{g}_{q+1}\widetilde{g}_{q+2} = \ldots = \widetilde{g}_{q+2s-1}\widetilde{g}_{q+2s}. \tag{2}$$

Definition 3.3. Let $\mathcal{M}(\mathcal{D}, q, s, \tau)$ be the $\widetilde{G}$-graded algebra $\mathrm{End}_{\mathcal{D}}(V)$ where V has a $\mathcal{D}$-basis $\{v_1, \ldots, v_{q+2s}\}$ with $\deg v_i = \widetilde{g}_i$. Let $n = (q+2s)2^{\frac{1}{2}\dim T}$ and $\mathcal{R} = M_n(\mathbb{F})$. The grading Γ on $\mathcal{R}$ obtained by identifying $\mathcal{R}$ with $\mathcal{M}(\mathcal{D}, q, s, \tau)$ will be denoted by $\Gamma_{\mathcal{M}}(\mathcal{D}, q, s, \tau)$. In other words, we define this grading by identifying $\mathcal{R} = M_{q+2s}(\mathcal{D})$ and setting $\deg(E_{ij} \otimes X_t) := \widetilde{g}_i t \widetilde{g}_j^{-1}$. By abuse of notation, we will also write $\Gamma_{\mathcal{M}}(T, q, s, \tau)$.

Let $\widetilde{G}^0$ be the subgroup of $\widetilde{G}$ generated by $\mathrm{Supp}\,\Gamma$, which consists of the elements $z_{i,j,t} := \widetilde{g}_i t \widetilde{g}_j^{-1}$, $t \in T$ (so $z_{i,i,t} = t$ for all $t \in T$). Set $z_i := z_{i,i+1,e}$ for $i = 1, \ldots, q$ ($i \neq q$ if $s = 0$), $z_{q+i} = z_{q+2i-1,q+2i+1,e}$ for $i = 1, \ldots, s-1$, and $z_{q+s} = z_{q+1,q+2,e}$ (if $s > 0$). If $s = 0$, then $\widetilde{G}^0$ is generated by T and the elements $z_1, \ldots, z_{q-1}$. If $s = 1$, then $\widetilde{G}^0$ is generated by T and $z_1, \ldots, z_{q+1}$. If $s > 1$, then relations (2) imply that $z_{q+2i,q+2i+2,e} = z_{q+i}^{-1}$ for $i = 1, \ldots, s-1$, hence $\widetilde{G}^0$ is generated by T and $z_1, \ldots, z_{q+s}$. Moreover, relations (2) are equivalent to the following:

$$z_i^2 = t_i t_{i+1} \ (1 \leq i < q), \quad z_q^2 z_{q+s} = t_q \ (\text{if } q > 0 \text{ and } s > 0).$$

Let $\widetilde{G}^1$ be the subgroup generated by T and $z_1, \ldots, z_{q-1}$. Let $\widetilde{G}^2$ be the subgroup generated by $z_1, \ldots, z_s$ if $q = 0$ and by $z_q, \ldots, z_{q+s-1}$ if $q > 0$. Then it is clear from the above relations that $\widetilde{G}^0 = \widetilde{G}^1 \times \widetilde{G}^2$, $\widetilde{G}^2 \cong \mathbb{Z}^s$, while $\widetilde{G}^1 = T$ if $q = 0$ and $\widetilde{G}^1 \cong \mathbb{Z}_2^{\dim T + q - 1 - 2\dim T_0} \times \mathbb{Z}_4^{\dim T_0}$ if $q > 0$, where T_0 is the subgroup of T generated by the elements $t_i t_{i+1}$, $i = 1, \ldots, q-1$. To summarize:

$$\widetilde{G}^0 \cong \mathbb{Z}_2^{\dim T - 2\dim T_0 + \max(0, q-1)} \times \mathbb{Z}_4^{\dim T_0} \times \mathbb{Z}^s. \tag{3}$$

Note that relations (2) are also equivalent to the following:

$$\begin{array}{ll}
z_{i,j,t_i t} = z_{j,i,t_j t}, & i, j \leq q, \quad t \in T; \\
z_{i,q+2j-1,t_i t} = z_{q+2j,i,t}, \quad z_{i,q+2j,t_i t} = z_{q+2j-1,i,t}, & i \leq q, \quad j \leq s, \quad t \in T; \\
z_{q+2i-1,q+2j-1,t} = z_{q+2j,q+2i,t}, & i, j \leq s, \quad t \in T; \\
z_{q+2i-1,q+2j,t} = z_{q+2j-1,q+2i,t}, & i, j \leq s, \quad i \neq j, \quad t \in T.
\end{array}$$

One verifies that, apart from the above equalities and $z_{i,i,t} = t$, the elements $z_{i,j,t}$ are distinct, so the support of $\Gamma = \Gamma_{\mathcal{M}}(\widetilde{G}, \mathcal{D}, \kappa, \widetilde{\gamma})$ is given by

$$\begin{aligned}\operatorname{Supp}\Gamma = &\{z_{i,j,t} \mid i < j \le q,\ t \in T\} \cup \{z_{i,q+j,t} \mid i \le q,\ j \le 2s,\ t \in T\}\\ &\cup \{z_{q+2i-1,q+2j-1,t} \mid i < j \le s,\ t \in T\} \cup \{z_{q+2i,q+2j,t} \mid i < j \le s,\ t \in T\}\\ &\cup \{z_{q+2i-1,q+2j,t} \mid i, j \le s,\ i \ne j,\ t \in T\}\\ &\cup \{z_{q+2i-1,q+2i,t} \mid i \le s,\ t \in T\} \cup \{z_{q+2i,q+2i-1,t} \mid i \le s,\ t \in T\} \cup T,\end{aligned}$$

where the union is disjoint and all homogeneous components except those that appear in the last line have dimension 2, the components of degrees $z_{q+2i-1,q+2i,t}$ and $z_{q+2i,q+2i-1,t}$ have dimension 1, and the components of degree t have dimension $q + 2s$.

Proposition 3.4. *Let $\Gamma = \Gamma_{\mathcal{M}}(\mathcal{D}, q, s, \tau)$. Then $\widetilde{G}^0 = \widetilde{G}^0(T, q, s, \tau)$ is the universal group of Γ, and* $\operatorname{Diag}(\Gamma)$ *consists of all automorphisms of the form $X \mapsto DXD^{-1}$, $X \in \mathcal{R}$, where*

$$D = \operatorname{diag}(\lambda_1, \dots, \lambda_{q+2s}) \otimes X_t, \quad \lambda_i \in \mathbb{F}^\times,\ t \in T, \tag{4}$$

satisfying the relation

$$\lambda_1^2 \beta(t, t_1) = \ldots = \lambda_q^2 \beta(t, t_q) = \lambda_{q+1}\lambda_{q+2} = \ldots = \lambda_{q+2s-1}\lambda_{q+2s}. \tag{5}$$

Proof. The relations $z_{i,\ell,u} z_{\ell,j,v} = z_{i,j,uv}$, $u, v \in T$, can be rewritten in terms of the elements of $\operatorname{Supp}\Gamma$, producing a set of defining relations for $\widetilde{G}^0$. It follows that $\widetilde{G}^0$ is the universal group of Γ.

Since $\widetilde{G}^0$ is the universal group of Γ, $\operatorname{Diag}(\Gamma)$ consists of all automorphisms of the form $X \mapsto \chi * X$ where χ is a character of $\widetilde{G}^0$. Since $\mathbb{F}^\times$ is a divisible group, we can assume that χ is a character of $\widetilde{G}$. Let $\lambda_i = \chi(\widetilde{g}_i)$, $i = 1, \dots, q + 2s$. Let t be the element of T such that $\chi(u) = \beta(t, u)$ for all $u \in T$. Looking at relations (2), we see that (5) must hold. Conversely, any $t \in T$ and a set of $\lambda_i \in \mathbb{F}^\times$ satisfying (5) will determine a character χ of $\widetilde{G}$. It remains to observe that the action of χ on $\mathcal{R}$ coincides with the conjugation by D as in (4). □

The following is Proposition 3.3 from [Eld10].

Theorem 3.5. *Consider the grading $\Gamma = \Gamma_{\mathcal{M}}(\mathcal{D}, q, s, \tau)$ on $\mathcal{R} = M_{q+2s}(\mathcal{D})$ by $\widetilde{G}^0 = \widetilde{G}^0(T, q, s, \tau)$ where τ is given by* (1)*. Let $\mu = (\mu_1, \dots, \mu_s)$ where μ_i are scalars in $\mathbb{F}^\times$. Let $\varphi = \varphi_{\tau,\mu}$ be the anti-automorphism of $\mathcal{R}$ defined by $\varphi(X) = \Phi^{-1}({}^tX)\Phi$, $X \in \mathcal{R}$, where Φ is the block-diagonal matrix given by*

$$\Phi = \operatorname{diag}\left(X_{t_1}, \dots, X_{t_q}, \begin{bmatrix} 0 & I \\ \mu_1 I & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & I \\ \mu_s I & 0 \end{bmatrix}\right) \tag{6}$$

and I is the identity element of $\mathcal{D}$. Then Γ is a fine φ-grading unless $q = 2$, $s = 0$ and $t_1 = t_2$. In the latter case, Γ can be refined to a φ-grading that makes $\mathcal{R}$ a graded division algebra. □

This result and the discussion preceding Proposition 3.8 in [Eld10] yield

Theorem 3.6. *Let Γ be a fine φ-grading on the matrix algebra $\mathcal{R} = M_n(\mathbb{F})$ over an algebraically closed field $\mathbb{F}$, $\operatorname{char}\mathbb{F} \ne 2$. Then (Γ, φ) is equivalent to some $(\Gamma_{\mathcal{M}}(T, q, s, \tau), \varphi_{\tau,\mu})$ as in Theorem 3.5 where $(q + 2s)2^{\frac{1}{2}\dim T} = n$.* □

In [Eld10], in order to obtain the classification of fine gradings on simple Lie algebras of series A, one classifies, *up to weak equivalence*, all pairs (Γ, φ) where Γ is a fine φ-grading on a matrix algebra. At the same time, for series B, C and D, one classifies, *up to equivalence*, such pairs where φ is an *involution* of appropriate type: orthogonal for series B and D (we write $\operatorname{sgn}(\varphi) = 1$) and symplectic for series C (we write $\operatorname{sgn}(\varphi) = -1$). The classifications involve equivalences $\mathcal{D} \to \mathcal{D}'$ satisfying certain conditions, where $\mathcal{D}$ and $\mathcal{D}'$ are matrix algebras with division gradings. If T is the support of $\mathcal{D}$ and T' is the support of $\mathcal{D}'$, then the graded algebras $\mathcal{D}$ and $\mathcal{D}'$ are equivalent if and only if the groups T and T' are isomorphic. Identifying T and T', we may assume that $\mathcal{D} = \mathcal{D}'$ and look at self-equivalences of $\mathcal{D}$, i.e., the elements of $\operatorname{Aut}(\Gamma_0)$ where Γ_0 is the grading on $\mathcal{D}$. By [EKb, Proposition 2.7], the Weyl group $W(\Gamma_0)$ is isomorphic to $\operatorname{Aut}(T, \beta)$, the group of automorphisms of T that preserve the bicharacter β. Explicitly, if $\psi_0 \in \operatorname{Aut}(\Gamma_0)$, then $\psi_0(X_t) \in \mathbb{F}X_{\alpha(t)}$, for all $t \in T$, where $\alpha \in \operatorname{Aut}(T, \beta)$, and the mapping $\psi_0 \mapsto \alpha$ yields an isomorphism $\operatorname{Aut}(\Gamma_0)/\operatorname{Stab}(\Gamma_0) \to \operatorname{Aut}(T, \beta)$. Hence the conditions in [Eld10] can be rewritten in terms of the group T rather than the graded division algebra $\mathcal{D}$. Note that $\operatorname{Aut}(T, \beta)$ can be regarded as a sort of symplectic group; in particular, if T is an elementary 2-group, then $\operatorname{Aut}(T, \beta) \cong \operatorname{Sp}_m(2)$ where $m = \dim T$.

Definition 3.7. Given τ as in (1), we will denote by $\Sigma(\tau)$ the multiset in T determined by τ, i.e., the underlying set of $\Sigma(\tau)$ consists of the elements that occur in $(t_1, \ldots, t_q)$, and the multiplicity of each element is the number of times it occurs there.

The group $\operatorname{Aut}(T, \beta)$ acts naturally on T, so we can form the semidirect product $T \rtimes \operatorname{Aut}(T, \beta)$, which also acts on T: a pair (u, α) sends $t \in T$ to $\alpha(t)u$. Clearly, if $\dim T = 2r$, then $T \rtimes \operatorname{Aut}(T, \beta)$ is isomorphic to $\operatorname{ASp}_{2r}(2)$, the affine symplectic group of order $2r$ over the field of two elements ("rigid motions" of the symplectic space of dimension $2r$).

Using this notation, Theorem 3.17 of [Eld10] can be recast as follows:

Theorem 3.8. *Consider two pairs, (Γ, φ) and (Γ', φ'), as in Theorem 3.5, namely, $\Gamma = \Gamma_{\mathcal{M}}(T, q, s, \tau)$, $\varphi = \varphi_{\tau,\mu}$ and $\Gamma' = \Gamma_{\mathcal{M}}(T', q', s', \tau')$, $\varphi' = \varphi_{\tau',\mu'}$, where $T = \mathbb{Z}_2^{2r}$ and $T' = \mathbb{Z}_2^{2r'}$. Then (Γ, φ) and (Γ', φ') are weakly equivalent if and only if $r = r'$, $q = q'$, $s = s'$, and $\Sigma(\tau)$ is conjugate to $\Sigma(\tau')$ by the natural action of $T \rtimes \operatorname{Aut}(T, \beta) \cong \operatorname{ASp}_{2r}(2)$.* $\square$

Let $\psi_0 \colon \mathcal{D} \to \mathcal{D}$ be an equivalence. Then the map $\psi_0^{-1}\varphi_0\psi_0$ is an involution of the graded algebra $\mathcal{D}$, which has the same type as φ_0 (orthogonal). Hence there exists a nonzero homogeneous element $d_0 \in \mathcal{D}$ such that

$$d_0\varphi_0(d)d_0^{-1} = (\psi_0^{-1}\varphi_0\psi_0)(d) \quad \text{for all} \quad d \in \mathcal{D}. \tag{7}$$

Note that d_0 is determined up to a scalar in $\mathbb{F}$. Moreover, d_0 is symmetric with respect to φ_0. By a similar argument, $\psi_0(d_0)$ is also symmetric. Let α be the element of $\operatorname{Aut}(T, \beta)$ corresponding to ψ_0 and let t_0 be the degree of d_0. Then (7) is equivalent to the following:

$$\beta(t_0, t)\beta(t) = \beta(\alpha(t)) \quad \text{for all} \quad t \in T, \tag{8}$$

so t_0 depends only on α. Moreover, $\beta(t_0) = \beta(\alpha(t_0)) = 1$.

Definition 3.9. For any $\alpha \in \mathrm{Aut}(T,\beta)$, the map $t \mapsto \beta(\alpha^{-1}(t))\beta(t)$ is a character of T, so there exists a unique element $t_\alpha \in T$ such that $\beta(t_\alpha, t) = \beta(\alpha^{-1}(t))\beta(t)$ for all $t \in T$. We define a new action of the group $\mathrm{Aut}(T,\beta)$ on T by setting

$$\alpha \cdot t := \alpha(t)t_\alpha \quad \text{for all} \quad \alpha \in \mathrm{Aut}(T,\beta) \text{ and } t \in T.$$

In other words, $\mathrm{Aut}(T,\beta)$ acts through the (injective) homomorphism to $T \rtimes \mathrm{Aut}(T,\beta)$, $\alpha \mapsto (t_\alpha, \alpha)$, and the natural action of $T \rtimes \mathrm{Aut}(T,\beta)$ on T.

Comparing this definition with equation (8), which defines the element t_0 associated to α, we see that $t_\alpha = \alpha(t_0)$. In particular, $\beta(t_\alpha) = 1$. This implies that $\beta(\alpha \cdot t) = \beta(t)$ for all $t \in T$, so the sets

$$T_+ := \{t \in T \mid \beta(t) = 1\} \quad \text{and} \quad T_- := \{t \in T \mid \beta(t) = -1\},$$

which correspond, respectively, to symmetric and skew-symmetric homogeneous components of $\mathcal{D}$ (relative to φ_0), are invariant under the twisted action of $\mathrm{Aut}(T,\beta)$.

Now Proposition 3.8(2) and Theorem 3.22 of [Eld10] can be recast as follows:

Theorem 3.10. *Let $\varphi = \varphi_{\tau,\mu}$ be as in Theorem 3.5. Then φ is an involution with $\mathrm{sgn}(\varphi) = \delta$ if and only if*

$$\delta = \beta(t_1) = \ldots = \beta(t_q) = \mu_1 = \ldots = \mu_s.$$

For gradings $\Gamma = \Gamma_{\mathcal{M}}(T,q,s,\tau)$ with $T = \mathbb{Z}_2^{2r}$ and $\Gamma' = \Gamma_{\mathcal{M}}(T',q',s',\tau')$ with $T' = \mathbb{Z}_2^{2r'}$ and for involutions $\varphi = \varphi_{\tau,\mu}$ and $\varphi' = \varphi_{\tau',\mu'}$, the pairs (Γ,φ) and (Γ',φ') are equivalent if and only if $r = r'$, $q = q'$, $s = s'$, $\mathrm{sgn}(\varphi) = \mathrm{sgn}(\varphi')$, and $\Sigma(\tau)$ is conjugate to $\Sigma(\tau')$ by the twisted action of $\mathrm{Aut}(T,\beta) \cong \mathrm{Sp}_{2r}(2)$ as in Definition 3.9. □

3.2. **Automorphism groups of fine φ-gradings on matrix algebras.** We are now going to study automorphisms of the fine φ-gradings $\Gamma_{\mathcal{M}}(T,q,s,\tau)$. We begin with some general observations. Let $\mathcal{D}$ and $\mathcal{D}'$ be graded division algebras, with the same grading group G. Let V be a graded right $\mathcal{D}$-module and V' a graded right $\mathcal{D}'$-module, both of nonzero finite rank. By an *isomorphism from $(\mathcal{D},V)$ to $(\mathcal{D}',V')$* we mean a pair (ψ_0,ψ_1) where $\psi_0\colon \mathcal{D} \to \mathcal{D}'$ is an isomorphism of graded algebras, $\psi_1\colon V \to V'$ is an isomorphism of graded vector spaces over $\mathbb{F}$, and $\psi_1(vd) = \psi_1(v)\psi_0(d)$ for all $v \in V$ and $d \in \mathcal{D}$.

Let $\mathcal{R} = \mathrm{End}_{\mathcal{D}}(V)$ and $\mathcal{R}' = \mathrm{End}_{\mathcal{D}'}(V')$. If $\psi\colon \mathcal{R} \to \mathcal{R}'$ is an isomorphism of graded algebras, then there exist $g \in G$ and an isomorphism (ψ_0,ψ_1) from $(\mathcal{D},V^{[g]})$ to $(\mathcal{D}',V')$ such that $\psi_1(rv) = \psi(r)\psi_1(v)$ for all $r \in \mathcal{R}$ and $v \in V$ (see e.g. [Eld10, Proposition 2.5]). Here $V^{[g]}$ denotes a shift of grading: the $(\mathcal{R},\mathcal{D})$-bimodule structure of $V^{[g]}$ is the same as that of V, but we set $V^{[g]}_h = V_{hg^{-1}}$ for all $h \in G$. Conversely, given an isomorphism (ψ_0,ψ_1) of the above pairs, there exists a unique isomorphism $\psi\colon \mathcal{R} \to \mathcal{R}'$ of graded algebras such that $\psi_1(rv) = \psi(r)\psi_1(v)$ for all $r \in \mathcal{R}$ and $v \in V$. Two isomorphisms (ψ_0,ψ_1) and (ψ'_0,ψ'_1) determine the same isomorphism $\mathcal{R} \to \mathcal{R}'$ if and only if there exists a nonzero homogeneous $d \in \mathcal{D}'$ such that $\psi'_0(x) = d^{-1}\psi_0(x)d$ and $\psi'_1(v) = \psi_1(v)d$ for all $x \in \mathcal{D}$ and $v \in V$.

Lemma 3.11. *Let $\psi\colon \mathcal{R} \to \mathcal{R}'$ be the isomorphism of graded algebras determined by an isomorphism (ψ_0,ψ_1) from $(\mathcal{D},V^{[g]})$ to $(\mathcal{D}',V')$. Suppose that the graded algebras $\mathcal{R}$ and $\mathcal{R}'$ admit anti-automorphisms φ and φ', respectively, determined by a φ_0-sesquilinear form $B\colon V \times V \to \mathcal{D}$ and a φ'_0-sesquilinear form $B'\colon V' \times V' \to \mathcal{D}'$.*

Then $\varphi' = \psi\varphi\psi^{-1}$ if and only if there exists a nonzero homogeneous $d_0 \in \mathcal{D}$ such that

$$B'(\psi_1(v), \psi_1(w)) = \psi_0\big(d_0 B(v, w)\big) \quad \text{for all} \quad v, w \in V. \tag{9}$$

Moreover, $d_0\varphi_0(d)d_0^{-1} = (\psi_0^{-1}\varphi_0'\psi_0)(d)$ for all $d \in \mathcal{D}$.

Proof. Set $\varphi'' := \psi^{-1}\varphi'\psi$ and $B''(v, w) := \psi_0^{-1}\big(B'(\psi_1(v), \psi_1(w))\big)$ for all $v, w \in V$. Then we compute:

$$\begin{aligned}
B''(v, wd) &= \psi_0^{-1}\big(B'(\psi_1(v), \psi_1(w)\psi_0(d))\big) \\
&= \psi_0^{-1}\big(B'(\psi_1(v), \psi_1(w))\psi_0(d)\big) = B''(v, w)d; \\
B''(vd, w) &= \psi_0^{-1}\big(B'(\psi_1(v)\psi_0(d), \psi_1(w))\big) \\
&= \psi_0^{-1}\big(\varphi_0'(\psi_0(d))B'(\psi_1(v), \psi_1(w))\big) = (\psi_0^{-1}\varphi_0'\psi_0)(d)B''(v, w); \\
B''(v, \varphi''(r)w) &= \psi_0^{-1}\big(B'(\psi_1(v), \psi(\varphi''(r))\psi_1(w))\big) \\
&= \psi_0^{-1}\big(B'(\psi_1(v), \varphi'(\psi(r))\psi_1(w))\big) \\
&= \psi_0^{-1}\big(B'(\psi(r)\psi_1(v), \psi_1(w))\big) = B''(rv, w).
\end{aligned}$$

We have shown that B'' is a $(\psi_0^{-1}\varphi_0'\psi_0)$-sesquilinear form corresponding to φ''. Hence $\varphi'' = \varphi$ if and only if there exists a nonzero homogeneous element $d_0 \in \mathcal{D}$ such that $B'' = d_0 B$, i.e., equation (9) holds. $\square$

Now consider $\Gamma = \Gamma_{\mathcal{M}}(T, q, s, \tau)$ and $\varphi = \varphi_{\tau,\mu}$ as in Theorem 3.5. There are two kinds of automorphism groups that we will need. Namely, there is

$$\mathrm{Aut}^*(\Gamma, \varphi) := \{\psi \in \mathrm{Aut}(\Gamma) \mid \psi\varphi\psi^{-1} = \xi\varphi \ \text{ for some } \ \xi \in \mathrm{Diag}(\Gamma)\},$$

which will be relevant to computing the Weyl group of the corresponding fine grading on the simple Lie algebra of type A, and there is

$$\mathrm{Aut}(\Gamma, \varphi) := \{\psi \in \mathrm{Aut}(\Gamma) \mid \psi\varphi\psi^{-1} = \varphi\},$$

which will be relevant to computing the Weyl groups of fine gradings on the simple Lie algebras of types B, C and D. Hence, we are intersted in $\mathrm{Aut}(\Gamma, \varphi)$ only if φ is an involution. Similarly, define

$$\mathrm{Stab}(\Gamma, \varphi) := \{\psi \in \mathrm{Stab}(\Gamma) \mid \psi\varphi\psi^{-1} = \varphi\}.$$

(We could also define $\mathrm{Stab}^*(\Gamma, \varphi)$, but we will not need it.)

Recall that Γ is the grading on $\mathcal{R} = \mathrm{End}_{\mathcal{D}}(V)$ where $\mathcal{D}$ is a matrix algebra equipped with a division grading with support $T = \mathbb{Z}_2^{2r}$ and bicharacter β, and V has a $\mathcal{D}$-basis $\{v_1, \ldots, v_k\}$ with $\deg v_i = \widetilde{g}_i$ and $k = q + 2s$. We will use the universal group $\widetilde{G}^0$ for the grading Γ. If $\psi\colon \mathcal{R} \to \mathcal{R}$ is an equivalence, then there exists an automorphism α of the group $\widetilde{G}^0$ such that ψ sends ${}^{\alpha}\Gamma$ to Γ. In other words, $\psi\colon \mathcal{R}' \to \mathcal{R}$ is an isomorphism of graded algebras where $\mathcal{R}'$ is $\mathcal{R}$ as an algebra, but equipped with the grading ${}^{\alpha}\Gamma$. Define $\mathcal{D}'$ similarly to $\mathcal{R}'$, using the restriction of α to $T \subset \widetilde{G}^0$. The support of $\mathcal{D}'$ is $T' = \alpha(T)$. Since $V^{[\widetilde{g}_1^{-1}]}$ is $\widetilde{G}^0$-graded, we can also define V' so that $\mathcal{R}' = \mathrm{End}_{\mathcal{D}'}(V')$ as a graded algebra. Therefore, ψ is determined by (ψ_0, ψ_1) where $\psi_0\colon \mathcal{D}' \to \mathcal{D}$ is an isomorphism of graded algebras and $\psi_1\colon V' \to V$ is an isomorphism up to a shift of grading. Hence $T' = T$ and $\psi_0 \in \mathrm{Aut}(\Gamma_0)$, so $\psi_0(X_t) \in \mathbb{F}X_{\alpha(t)}$, for all $t \in T$, and the map $\alpha\colon T \to T$ belongs to $\mathrm{Aut}(T,\beta) \cong \mathrm{Sp}_{2r}(2)$. Also, if Ψ is the matrix of ψ_1 relative to $\{v_1,\dots,v_k\}$, we have

$$\psi(X) = \Psi\psi_0(X)\Psi^{-1} \quad \text{for all} \quad X \in \mathcal{R}.$$

Since all $\widetilde{g}_i$ are distinct modulo T, matrix Ψ necessarily has the form $\Psi = PD$ where P is a permutation matrix and $D = \mathrm{diag}(d_1,\dots,d_k)$ where d_i are nonzero homogeneous elements of $\mathcal{D}$. Moreover, the permutation $\pi \in \mathrm{Sym}(k)$ corresponding to P and the coset of ψ_0 modulo $\mathrm{Stab}(\Gamma_0)$ are uniquely determined by ψ. Hence, we have a well-defined homomorphism

$$\mathrm{Aut}(\Gamma) \to \mathrm{Sym}(k) \times \mathrm{Aut}(T,\beta)$$

that sends ψ to the corresponding (π,α).

Now we turn to the anti-automorphism $\varphi\colon \mathcal{R} \to \mathcal{R}$, which is given by the adjoint with respect to a φ_0-sesquilinear form B on V where $\varphi_0\colon \mathcal{D} \to \mathcal{D}$ is given by matrix transpose, $X_t \mapsto \beta(t)X_t$ for all $t \in T$. Recall that such B is determined up to a scalar in $\mathbb{F}$. We can take for B the φ_0-sesquilinear form whose matrix with respect to $\{v_1,\dots,v_k\}$ is Φ displayed in Theorem 3.5. Pick $\xi \in \mathrm{Diag}(\Gamma)$ and let B' be a φ_0-sesquilinear form on V corresponding to $\xi\varphi$. By Lemma 3.11, ψ satisfies $\psi\varphi\psi^{-1} = \xi\varphi$ if and only if condition (9) holds for some nonzero homogeneous $d_0 \in \mathcal{D}$. Clearly, (9) is equivalent to (7) and

$$\widehat{\Phi} = \psi_0(d_0\Phi), \tag{10}$$

where $\widehat{\Phi}$ is the matrix of B' relative to $\{\psi_1(v_1),\dots,\psi_1(v_k)\}$. Recall that (7) is equivalent to condition (8) on $t_0 := \deg d_0$. To summarize, ψ satisfies $\psi\varphi\psi^{-1} = \xi\varphi$ if and only if

$$\widehat{\Phi} = d_0\psi_0(\Phi) \tag{11}$$

for some $d_0 \in \mathcal{D}$ of degree t_α as in Definition 3.9 (we have replaced $\psi_0(d_0)$ in (10) by d_0 to simplify notation).

The matrix of B' relative to $\{v_1,\dots,v_k\}$ is $\Phi(D')^{-1}$ where $\xi(X) = D'X(D')^{-1}$, for all $X \in \mathcal{R}$, with D' of the form given by Proposition 3.4: $D' = \mathrm{diag}(\nu_1X_u,\dots,\nu_kX_u)$ for some $u \in T$ and $\nu_i \in \mathbb{F}^\times$ satisfying

$$\nu_1^2\beta(u,t_1) = \ldots = \nu_q^2\beta(u,t_q) = \nu_{q+1}\nu_{q+2} = \ldots = \nu_{q+2s-1}\nu_{q+2s}. \tag{12}$$

It follows at once that, for $\psi \in \mathrm{Aut}^*(\Gamma,\varphi)$, the permutation π must preserve the set $\{1,\dots,q\}$ and the pairing of $q+2i-1$ with $q+2i$, for $i = 1,\dots,s$. It is convenient to introduce the group $W(s) := \mathbb{Z}_2^s \rtimes \mathrm{Sym}(s)$ (i.e., the wreath product of $\mathrm{Sym}(s)$ and $\mathbb{Z}_2$), which will be regarded as the group of permutations on $\{q+1,\dots,q+2s\}$ that respect the block decomposition $\{q+1,q+2\} \cup \ldots \cup \{q+2s-1,q+2s\}$. The reason for the notation $W(s)$ is that $\mathbb{Z}_2^s \rtimes \mathrm{Sym}(s)$ is the classical Weyl group of type B_s or C_s (and also the extended Weyl group of type D_s if $s > 4$). By the above discussion, we have a homomorphism:

$$\mathrm{Aut}^*(\Gamma,\varphi) \to \mathrm{Sym}(q) \times W(s) \times \mathrm{Aut}(T,\beta). \tag{13}$$

We need some more notation to state the main result of this section. Let Σ be a multiset of cardinality q and let $m_1,\dots,m_\ell$ be the multiplicities of the elements of Σ, written in some order. Thus, m_i are positive integers whose sum is q. We will denote by $\mathrm{Sym}\Sigma$ the subgroup $\mathrm{Sym}(m_1) \times \cdots \times \mathrm{Sym}(m_\ell)$ of $\mathrm{Sym}(q)$, which may be thought of as "interior symmetries" of Σ. For a multiset Σ in T, let $\mathrm{Aut}^*\Sigma$ be the stabilizer of Σ under the natural action of $T \rtimes \mathrm{Aut}(T,\beta)$ on T, i.e., $\mathrm{Aut}^*\Sigma$

is the set of "rigid motions" of the symplectic space T that permute the elements of Σ preserving multiplicity. These are "exterior symmetries" of Σ. Note that each bijection $\theta\colon T\to T$ that stabilizes Σ determines an element of $\mathrm{Sym}(q)$ that permutes the blocks of sizes $m_1,\dots,m_\ell$ in the same way θ permutes the elements of Σ (thus, only blocks of equal size may be permuted) and preserves the order within each block; we will call this permutation the *restriction of θ to Σ*. Hence, we obtain a restriction homomorphism $\mathrm{Aut}^*\Sigma\to\mathrm{Sym}(q)$. In particular, $\mathrm{Aut}^*\Sigma$ acts naturally on $\mathrm{Sym}\Sigma$ by permuting factors (of equal order). Finally, let $\mathrm{Aut}\,\Sigma$ be the stabilizer of Σ under the twisted action of $\mathrm{Aut}(T,\beta)$ on T as in Definition 3.9. Note that $\mathrm{Aut}\,\Sigma$ may be regarded as a subgroup of $\mathrm{Aut}^*\Sigma$.

Theorem 3.12. *Let $\Gamma=\Gamma_{\mathcal{M}}(T,q,s,\tau)$ and let φ be as in Theorem 3.5 such that Γ is a fine φ-grading. Let $\Sigma=\Sigma(\tau)$, so $|\Sigma|=q$.*

1) $\mathrm{Stab}(\Gamma,\varphi)=\mathrm{Diag}(\Gamma)$.
2) $\mathrm{Aut}^*(\Gamma,\varphi)/\,\mathrm{Stab}(\Gamma,\varphi)$ *is isomorphic to an extension of the group* $\big((T^{q+s-1}\times\mathbb{Z}_2^s)\rtimes(\mathrm{Sym}\Sigma\times\mathrm{Sym}(s)\big)\rtimes\mathrm{Aut}^*\Sigma$ *by* $\mathbb{Z}_2^{q+s-1}$*, with the following actions: T^{q+s-1} is identified with T^{q+s}/T and $\mathbb{Z}_2^{q+s-1}$ is identified with $\mathbb{Z}_2^{q+s}/\mathbb{Z}_2$, where T and $\mathbb{Z}_2$ are imbedded diagonally, then*
   - $\mathrm{Sym}\Sigma\subset\mathrm{Sym}(q)$ *acts on T^{q+s}/T and $\mathbb{Z}_2^{q+s}/\mathbb{Z}_2$ by permuting the first q components and trivially on $\mathbb{Z}_2^s$;*
   - $\mathrm{Sym}(s)$ *acts on T^{q+s}/T and $\mathbb{Z}_2^{q+s}/\mathbb{Z}_2$ by permuting the last s components and naturally on $\mathbb{Z}_2^s$;*
   - $\mathrm{Aut}^*\Sigma$ *acts on* $\mathrm{Sym}\Sigma$ *and $\mathbb{Z}_2^{q+s}/\mathbb{Z}_2$ through the restriction homomorphism* $\mathrm{Aut}^*\Sigma\to\mathrm{Sym}(q)$*, trivially on* $\mathrm{Sym}(s)$*, and as follows on $(T^{q+s}/T)\times\mathbb{Z}_2^s$: an element $(u,\alpha)\in\mathrm{Aut}^*\Sigma\subset T\rtimes\mathrm{Aut}(T,\beta)$ sends a pair $\big((u_1,\dots,u_q,u_{q+1},\dots,u_{q+s})T,\underline{x}\big)\in(T^{q+s}/T)\times\mathbb{Z}_2^s$ to $\big((\alpha(u_{\pi^{-1}(1)}),\dots,\alpha(u_{\pi^{-1}(q)}),\alpha(u_{q+1})u^{x_1},\dots,\alpha(u_{q+s})u^{x_s})T,\underline{x}\big)$, where π is the image of (u,α) under the restriction homomorphism;*
   - $T^{q+s-1}\times\mathbb{Z}_2^s$ *acts trivially on $\mathbb{Z}_2^{q+s-1}$.*
3) *If φ is an involution, then* $\mathrm{Aut}(\Gamma,\varphi)/\,\mathrm{Stab}(\Gamma,\varphi)$ *is isomorphic to* $\big((T^{q+s-1}\times\mathbb{Z}_2^s)\rtimes(\mathrm{Sym}\Sigma\times\mathrm{Sym}(s)\big)\rtimes\mathrm{Aut}\,\Sigma$*, with the following actions: T^{q+s-1} is identified with T^{q+s}/T, where T is imbedded diagonally, then*
   - $\mathrm{Sym}\Sigma\subset\mathrm{Sym}(q)$ *acts on T^{q+s}/T by permuting the first q components and trivially on $\mathbb{Z}_2^s$;*
   - $\mathrm{Sym}(s)$ *acts on T^{q+s}/T by permuting the last s components and naturally on $\mathbb{Z}_2^s$;*
   - $\mathrm{Aut}\,\Sigma$ *acts on* $\mathrm{Sym}\Sigma$ *as a subgroup of* $\mathrm{Aut}^*\Sigma$*, i.e., through the twisted action on T (Definition 3.9) and restriction to Σ, trivially on* $\mathrm{Sym}(s)$*, and as follows on $(T^{q+s}/T)\times\mathbb{Z}_2^s$: an element $\alpha\in\mathrm{Aut}\,\Sigma\subset\mathrm{Aut}(T,\beta)$ sends a pair $\big((u_1,\dots,u_q,u_{q+1},\dots,u_{q+s})T,\underline{x}\big)\in(T^{q+s}/T)\times\mathbb{Z}_2^s$ to $\big((\alpha(u_{\pi^{-1}(1)}),\dots,\alpha(u_{\pi^{-1}(q)}),\alpha(u_{q+1})t_\alpha^{x_1},\dots,\alpha(u_{q+s})t_\alpha^{x_s})T,\underline{x}\big)$, where π is the image of (t_α,α) under the restriction to Σ.*

Proof. 1) If $\psi\in\mathrm{Stab}(\Gamma,\varphi)$, then $\Psi=PD$ where P corresponds to $\pi\in\mathrm{Sym}(q)\times W(s)$, and $\psi_0\in\mathrm{Stab}(\Gamma_0)$. Adjusting D if necessary, we may assume $\psi_0=\mathrm{id}$. We claim that π is the trivial permutation. Since ψ does not permute the homogeneous components of Γ, π must act trivially on $\widetilde{G}^0/T$. So, we consider the action of $\mathrm{Sym}(q)\times W(s)$ on $\widetilde{G}^0/T$ in terms of the generators z_i ($i=1,\dots,q-1$ if $s=0$ and $i=1,\dots,q+s$ if $s>0$) that were introduced after Definition 3.3.

$\mathrm{Sym}(q)$ acts trivially on the subgroup $\langle z_{q+1},\dots,z_{q+s}\rangle$ and via the action of the classical Weyl group of type A_{q-1}, taken modulo 2, on the subgroup $\langle z_1,\dots,z_{q-1}\rangle\cong\mathbb{Z}_2^{q-1}$ where z_i is identified with the element $\varepsilon_i-\varepsilon_{i+1}$, with $\{\varepsilon_1,\dots,\varepsilon_q\}$ being the standard basis of $\mathbb{Z}_2^q$, on which $\mathrm{Sym}(q)$ acts naturally.

$W(s)$ acts trivially on the subgroup $\langle z_1,\dots,z_{q-1}\rangle$ and via the action of the classical Weyl group of type B_s or C_s on the subgroup $\langle z_{q+1},\dots,z_{q+s}\rangle\cong\mathbb{Z}^s$ where z_{q+i} is identified with the element $\varepsilon_i-\varepsilon_{i+1}$ for $i\neq s$ and z_{q+s} is identified with the element $2\varepsilon_1$, with $\{\varepsilon_1,\dots,\varepsilon_s\}$ being the standard basis of $\mathbb{Z}^s$. The easiest way to see this is to extend $\widetilde{G}$ by adding a new element $\widehat{g}_0$ satisfying $(\widehat{g}_0)^{-2}=\widetilde{g}_1\widetilde{g}_2$ and set $\widehat{g}_i=\widetilde{g}_i\widehat{g}_0$. The elements of the subgroup $\widetilde{G}^0$ are not affected if we replace $\widetilde{g}_i$ by $\widehat{g}_i$, but then we have $\widehat{g}_{q+2j}=\widehat{g}_{q+2j-1}^{-1}$ for $j=1,\dots,s$, so we can map $\widehat{g}_{q+2j-1}$ to ε_j and $\widehat{g}_{q+2j}$ to $-\varepsilon_j$.

Note that the action of $W(s)$ on $\langle z_{q+1},\dots,z_{q+s}\rangle$ is always faithful, while the action of $\mathrm{Sym}(q)$ on $\langle z_1,\dots,z_{q-1}\rangle$ is faithful unless $q=2$. If $q>0$ and $s>0$, then we also have the generator z_q, on which $\pi\in\mathrm{Sym}(q)\times W(s)$ acts in this way (note that $\pi(q)\le q$ and $\pi(q+1)>q$):

$$z_q\mapsto\begin{cases} z_{\pi(q)}\cdots z_q z_{q+1}\cdots z_{q+j} & \text{if } \pi(q+1)=q+2j+1;\\ z_{\pi(q)}\cdots z_q z_{q+1}^{-1}\cdots z_{q+j}^{-1}z_{q+s} & \text{if } \pi(q+1)=q+2j+2.\end{cases}$$

If π acts trivially on $\langle z_{q+1},\dots,z_{q+s}\rangle$, then $\pi(q+1)=q+1$. Hence, if π also acts trivially on z_q, then $\pi(q)=q$. It follows that the action of $\mathrm{Sym}(q)\times W(s)$ on $\widetilde{G}^0/T$ is faithful unless $q=2$ and $s=0$. In this remaining case, we have $\tau=(t_1,t_2)$ where $t_1\neq t_2$ (otherwise Γ is not a fine φ-grading). If ψ_1 yields $\pi=(12)$, then $\psi_1(v_1)=v_2d_1$ and $\psi(v_2)=v_1d_2$ for some nonzero homogeneous $d_1,d_2\in\mathcal{D}$, but then $B(\psi_1(v_1),\psi_1(v_1))$ has degree t_2, while $B(v_1,v_1)$ has degree t_1. This contradicts (11), because here we have $\psi_0=\mathrm{id}$, $d_0\in\mathbb{F}^\times$ and $B'=B$. The proof of the claim is complete.

Since $P=I$, we have $\Psi=\mathrm{diag}(d_1,\dots,d_k)$, where the d_i must necessarily have the same degree, say, t, so $\Psi=\mathrm{diag}(\lambda_1,\dots,\lambda_k)\otimes X_t$, but then (11) implies that (5) must hold, hence $\psi\in\mathrm{Diag}(\Gamma)$. We have proved that $\mathrm{Stab}(\Gamma,\varphi)\subset\mathrm{Diag}(\Gamma)$. The opposite inclusion is obvious.

2) We can extract more information about an element $\psi\in\mathrm{Aut}^*(\Gamma,\varphi)$ than given by its image under the homomorphism (13) if we look at the action of ψ on φ. Write $\psi\varphi\psi^{-1}=\xi_\psi\varphi$ where ξ_ψ is a uniquely determined element of $\mathrm{Diag}(\Gamma)$. Clearly, we have $\xi_{\psi\psi'}=\xi_\psi(\psi\xi_{\psi'}\psi^{-1})$. Since ξ_ψ is the conjugation by $\mathrm{diag}(\nu_1,\dots,\nu_k)\otimes X_{u_\psi}$, for a uniquely determined $u_\psi\in T$, we obtain $u_{\psi\psi'}=u_\psi\alpha_\psi(u_{\psi'})$ where α_ψ is the element of $\mathrm{Aut}(T,\beta)$ corresponding to ψ under (13). Hence, we can construct a homomorphism

$$\mathrm{Aut}^*(\Gamma,\varphi)\to\mathrm{Sym}(q)\times W(s)\times(T\rtimes\mathrm{Aut}(T,\beta)),\tag{14}$$

where the first two components are as in (13) and the third is $\psi\mapsto(u_\psi,\alpha_\psi)$.

Theorem 3.8 implies that we may assume without loss of generality that

$$\Phi=\mathrm{diag}\left(X_{t_1},\dots,X_{t_q},\begin{bmatrix}0&I\\I&0\end{bmatrix},\dots,\begin{bmatrix}0&I\\I&0\end{bmatrix}\right).$$

(In other words, the scalars μ_i are all equal to 1.) Then, for ψ given by $\Psi=PD$ and $\psi_0\in\mathrm{Aut}(\Gamma_0)$, with P corresponding to $\pi\in\mathrm{Sym}(q)\times W(s)$, condition (11) is equivalent to the following, with $u = u_\psi$:

$$\varphi_0(d_i)X_{t_{\pi(i)}}\nu_{\pi(i)}^{-1}X_u^{-1}d_i = d_0\psi_0(X_{t_i}), \quad i = 1,\dots,q, \tag{15}$$

and, for each $j = 1,\dots,s$, one of the following depending on whether $\pi(q+2j-1) < \pi(q+2j)$ or $\pi(q+2j-1) > \pi(q+2j)$:

$$\varphi_0(d_{q+2j-1})\nu_{\pi(q+2j)}^{-1}X_u^{-1}d_{q+2j} = \varphi_0(d_{q+2j})\nu_{\pi(q+2j-1)}^{-1}X_u^{-1}d_{q+2j-1} = d_0 \tag{16}$$

in the first case, and

$$\varphi_0(d_{q+2j-1})\nu_{\pi(q+2j-1)}^{-1}X_u^{-1}d_{q+2j} = \varphi_0(d_{q+2j})\nu_{\pi(q+2j)}^{-1}X_u^{-1}d_{q+2j-1} = d_0 \tag{17}$$

in the second case.

If $\psi \in \mathrm{Aut}^*(\Gamma,\varphi)$, then, looking at the degrees in (15), we obtain

$$t_{\pi(i)} = \alpha_\psi(t_i)t_{\alpha_\psi}u_\psi, \quad i = 1,\dots,q, \tag{18}$$

which implies that $(t_{\alpha_\psi}u_\psi, \alpha_\psi)$ belongs to $\mathrm{Aut}^*\Sigma$. Composing the third component of the homomorphism (14) with the automorphism $(u,\alpha) \mapsto (t_\alpha u, \alpha)$ of the group $T \rtimes \mathrm{Aut}(T,\beta)$, we obtain a homomorphism

$$\mathrm{Aut}^*(\Gamma,\varphi) \to \mathrm{Sym}(q) \times W(s) \times \mathrm{Aut}^*\Sigma. \tag{19}$$

For any element $(t_\alpha u, \alpha) \in \mathrm{Aut}^*\Sigma$, let $\pi_{u,\alpha} \in \mathrm{Sym}(q)$ be its restriction to Σ. Then (18) implies that the permutation $\pi\pi_{u_\psi,\alpha_\psi}^{-1}$ does not move the elements of the underlying set of Σ, so it belongs to $\mathrm{Sym}\Sigma$. It follows that (19) can be rearranged as follows:

$$f\colon \mathrm{Aut}^*(\Gamma,\varphi) \to W(s) \times (\mathrm{Sym}\Sigma \rtimes \mathrm{Aut}^*\Sigma).$$

We claim that f is surjective. We will construct representatives in $\mathrm{Aut}^*(\Gamma,\varphi)$ for the elements of each of the subgroups $W(s)$, $\mathrm{Sym}\Sigma$ and $\mathrm{Aut}^*\Sigma$.

For any $\pi \in W(s)$, let P be the corresponding permutation matrix and let ψ_π be given by $\Psi = P$ and $\psi_0 = \mathrm{id}$. Let α be the automorphism of $\widetilde{G}$ that restricts to identity on T and sends $\widetilde{g}_i$ to $\widetilde{g}_{\pi(i)}$ (in particular, $\widetilde{g}_i$ are fixed for $i = 1,\dots,q$). Then ψ_π sends ${}^\alpha\Gamma$ to Γ, so $\psi_\pi \in \mathrm{Aut}(\Gamma)$. Also, conditions (15) through (17) are satisfied with $d_0 = I$, $u = e$ and $\nu_i = 1$, so $\psi_\pi \in \mathrm{Aut}(\Gamma,\varphi)$.

For any $\pi \in \mathrm{Sym}\Sigma$, let P be the corresponding permutation matrix and let ψ_π be given by $\Psi = P$ and $\psi_0 = \mathrm{id}$. Since we have $t_{\pi(i)} = t_i$ for all $i = 1,\dots,q$, we can define the automorphism α of $\widetilde{G}$ in the same way as above (this time, $\widetilde{g}_i$ are fixed for $i = q+1,\dots,q+2s$). Then ψ_π sends ${}^\alpha\Gamma$ to Γ, so $\psi_\pi \in \mathrm{Aut}(\Gamma)$. Also, conditions (15) and (16) are satisfied with $d_0 = I$, $u = e$ and $\nu_i = 1$, so $\psi_\pi \in \mathrm{Aut}(\Gamma,\varphi)$.

Now, for any $(t_\alpha u, \alpha) \in \mathrm{Aut}^*\Sigma$, let $\pi = \pi_{u,\alpha}$. Then $t_{\pi(i)} = \alpha(t_i)t_\alpha u$ for $i = 1,\dots,q$ and hence we can extend $\alpha\colon T \to T$ to an automorphism of $\widetilde{G}$ by setting $\alpha(\widetilde{g}_i) = \widetilde{g}_{\pi(i)}$ for $i = 1,\dots,q$, $\alpha(\widetilde{g}_{q+2j-1}) = \widetilde{g}_{q+2j-1}$ and $\alpha(\widetilde{g}_{q+2j}) = \widetilde{g}_{q+2j}t_\alpha u$ for $j = 1,\dots,s$. Choose $\nu_i \in \mathbb{F}^\times$ such that $\nu_i^2 = \beta(u,t_i)\beta(u)$, $i = 1,\dots,q$, and set $\nu_{q+2j} = 1$ and $\nu_{q+2j-1} = \beta(u)$, $j = 1,\dots,s$. Then (12) holds, so the conjugation by $\mathrm{diag}(\nu_1 X_u,\dots,\nu_k X_u)$ is an element $\xi \in \mathrm{Diag}(\Gamma)$. Choose ψ_0 such that $\psi_0(X_t) \in \mathbb{F}X_{\alpha(t)}$. Let P be the permutation matrix corresponding to π and let

$$D = \mathrm{diag}(\lambda_1 I,\dots,\lambda_q I, I, X_u X_{t_\alpha},\dots,I, X_u X_{t_\alpha}),$$

where $\lambda_i \in \mathbb{F}^\times$ are selected in such a way that condition (15) holds with $d_0 = X_{t_\alpha}$ (the degrees of both sides match, so it is indeed possible to find such λ_i). Since $\beta(t_\alpha) = 1$, condition (16) also holds. Finally, let $\psi_{u,\alpha}$ be given by $\Psi = PD$ and ψ_0.

Then $\psi_{u,\alpha}$ sends ${}^{\alpha}\Gamma$ to Γ and φ to $\xi\varphi$, with α and ξ indicated above. Therefore, $\psi_{u,\alpha}$ belongs to $\mathrm{Aut}^*(\Gamma,\varphi)$.

We have proved that the homomorphism f is surjective. Let K be the kernel of f. It consists of the conjugations by matrices of the form $D = \mathrm{diag}(d_1,\dots,d_k)$ such that (15) and (16) are satisfied with $\pi = \mathrm{id}$, $\psi_0 = \mathrm{id}$, $d_0 \in \mathbb{F}^\times$ and $u = e$. Hence $\deg d_{q+2j-1} = \deg d_{q+2j}$ for all $j = 1,\dots,s$. Conversely, given $(u_1,\dots,u_k) \in T^k$ with $u_{q+2j-1} = u_{q+2j}$ for $j = 1,\dots,s$, we can find elements d_i with $\deg d_i = u_i$ such that the conjugation by D belongs to $\mathrm{Aut}(\Gamma,\varphi)$.

According to 1), the subgroup

$$N = \{\psi \in K \mid \deg d_1 = \dots = \deg d_k\}$$

contains $\mathrm{Stab}(\Gamma,\varphi)$. Clearly, N is normal in $\mathrm{Aut}^*(\Gamma,\varphi)$. From the previous paragraph it follows that $K/N \cong T^{q+s}/T$ where T is imbedded into T^{q+s} diagonally. The representatives ψ_π that we constructed above for $\pi \in W(s)$ and for $\pi \in \mathrm{Sym}\Sigma$ form subgroups of $\mathrm{Aut}(\Gamma,\varphi)$ that commute with one another. But observe also that the representatives $\psi_{u,\alpha}$ for $(t_\alpha u,\alpha) \in \mathrm{Aut}^*\Sigma$ form a subgroup modulo N. Moreover, for $\pi \in \mathrm{Sym}(s) \subset W(s)$ the elements $\psi_{u,\alpha}$ and ψ_π commute modulo N, while for $\pi \in \mathrm{Sym}\Sigma$ we have $\psi_{u,\alpha}\psi_\pi\psi_{u,\alpha}^{-1} \in \psi_{\pi_{u,\alpha}\pi\pi_{u,\alpha}^{-1}}N$. Finally, for the transposition $\pi = (q+2j-1,q+2j)$, we have $\psi_\pi\psi_{u,\alpha}\psi_\pi\psi_{u,\alpha}^{-1} \in \psi N$ where ψ is the conjugation by $\mathrm{diag}(d_1,\dots,d_k)$ with $d_{q+2j-1} = d_{q+2j} = X_{t_\alpha u}$ and all other $d_i = I$. It follows that $\mathrm{Aut}^*(\Gamma,\varphi)/N$ is isomorphic to $\big((T^{q+s-1} \times \mathbb{Z}_2^s) \rtimes (\mathrm{Sym}\Sigma \times \mathrm{Sym}(s)\big) \rtimes \mathrm{Aut}^*\Sigma$, with the stated actions.

It remains to compute the quotient $N/\mathrm{Stab}(\Gamma,\varphi)$. Since any element $\psi \in N$ belongs to $\mathrm{Stab}(\Gamma)$, the mapping $\psi \mapsto \xi_\psi$ is a homomorphism $N \to \mathrm{Diag}(\Gamma)$ whose kernel is exactly $\mathrm{Stab}(\Gamma,\varphi)$. Hence, it suffices to compute the image. Since here $u = e$ and $\deg d_{q+2j-1} = \deg d_{q+2j}$, condition (16) implies that $\nu_{q+2j-1} = \nu_{q+2j}$ for $j = 1,\dots,s$. But then (12) implies that all ν_i^2 are equal to each other. Since multiplying all ν_i by the same scalar in $\mathbb{F}^\times$ does not change ξ, we may assume that $\nu_i \in \{\pm 1\}$. In fact, for $D = \mathrm{diag}(\lambda_1 I,\dots,\lambda_k I)$, conditions (15) and (16) reduce to the following: up to a common scalar multiple, $\nu_i = \lambda_i^2$ for $i = 1,\dots,q$, and $\nu_{q+2j-1} = \nu_{q+2j} = \lambda_{q+2j-1}\lambda_{q+2j}$ for $j = 1,\dots,s$. Hence every $(\nu_1,\dots,\nu_k)$ with $\nu_i \in \{\pm 1\}$ and $\nu_{q+2j-1} = \nu_{q+2j}$ indeed appears in ξ_ψ for some $\psi \in N$. Therefore, the quotient $N/\mathrm{Stab}(\Gamma,\varphi)$ is isomorphic to $\mathbb{Z}_2^{q+s}/\mathbb{Z}_2$ where $\mathbb{Z}_2$ is imbedded into $\mathbb{Z}_2^{q+s}$ diagonally.

3) The proof is similar to 2), so we will merely point out the differences. According to Theorem 3.10, here we have

$$\Phi = \mathrm{diag}\left(X_{t_1},\dots,X_{t_q},\begin{bmatrix}0 & I\\ \delta I & 0\end{bmatrix},\dots,\begin{bmatrix}0 & I\\ \delta I & 0\end{bmatrix}\right),$$

where $\delta = \mathrm{sgn}(\varphi)$ and $\beta(t_i) = \delta$ for $i = 1,\dots,q$. Also, B' equals B and hence, for ψ given by $\Psi = PD$ and $\psi_0 \in \mathrm{Aut}(\Gamma_0)$, with P corresponding to $\pi \in \mathrm{Sym}(q) \times W(s)$, condition (11) is equivalent to the following:

$$\varphi_0(d_i)X_{t_{\pi(i)}}d_i = d_0\psi_0(X_{t_i}), \quad i = 1,\dots,q, \tag{20}$$

and, for each $j = 1,\dots,s$, one of the following depending on whether $\pi(q+2j-1) < \pi(q+2j)$ or $\pi(q+2j-1) > \pi(q+2j)$:

$$\varphi_0(d_{q+2j-1})d_{q+2j} = d_0 \tag{21}$$

in the first case, and

$$\varphi_0(d_{q+2j-1})d_{q+2j} = \delta d_0 \tag{22}$$

in the second case. Here we took into account that, since $\varphi_0(d_0) = d_0$, either (21) or (22) implies $\varphi_0(d_{q+2j-1})d_{q+2j} = \varphi_0(d_{q+2j})d_{q+2j-1}$.

If $\psi \in \operatorname{Aut}(\Gamma, \varphi)$, then, looking at the degrees in (20), we obtain

$$t_{\pi(i)} = \alpha_\psi(t_i)t_{\alpha_\psi}, \quad i = 1, \ldots, q, \tag{23}$$

which implies that $(t_{\alpha_\psi}, \alpha_\psi)$ stabilizes Σ, i.e., α_ψ belongs to $\operatorname{Aut}\Sigma$. Hence we obtain a homomorphism

$$\operatorname{Aut}(\Gamma, \varphi) \to \operatorname{Sym}(q) \times W(s) \times \operatorname{Aut}\Sigma. \tag{24}$$

For any element $\alpha \in \operatorname{Aut}\Sigma$, let $\pi_\alpha \in \operatorname{Sym}(q)$ be the restriction of its twisted action to Σ. Then (23) implies that the permutation $\pi\pi_{\alpha_\psi}^{-1}$ does not move the elements of the underlying set of Σ, so it belongs to $\operatorname{Sym}\Sigma$. It follows that (24) can be rearranged as follows:

$$f\colon \operatorname{Aut}(\Gamma, \varphi) \to W(s) \times (\operatorname{Sym}\Sigma \rtimes \operatorname{Aut}\Sigma).$$

To prove that f is surjective, we construct representatives in $\operatorname{Aut}(\Gamma, \varphi)$ for the elements of each of the subgroups $W(s)$, $\operatorname{Sym}\Sigma$ and $\operatorname{Aut}\Sigma$.

For π in $\operatorname{Sym}\Sigma$ or in $\operatorname{Sym}(s) \subset W(s)$, we take the same representatives as in the proof of 2). For $\pi = (q + 2j - 1, q + 2j) \in W(s)$, a slight modification is needed: we take $\Psi = PD$ rather than just P, where $d_{q+2j} = \delta I$ and all other $d_i = I$. For any $\alpha \in \operatorname{Aut}\Sigma$, let $\pi = \pi_\alpha$. Then $t_{\pi(i)} = \alpha(t_i)t_\alpha$ for $i = 1, \ldots, q$ and hence we can extend $\alpha\colon T \to T$ to an automorphism of $\widetilde{G}$ by setting $\alpha(\widetilde{g}_i) = \widetilde{g}_{\pi(i)}$ for $i = 1, \ldots, q$, $\alpha(\widetilde{g}_{q+2j-1}) = \widetilde{g}_{q+2j-1}$ and $\alpha(\widetilde{g}_{q+2j}) = \widetilde{g}_{q+2j}t_\alpha$ for $j = 1, \ldots, s$. Choose ψ_0 such that $\psi_0(X_t) \in \mathbb{F}X_{\alpha(t)}$. Let P be the permutation matrix corresponding to π and let

$$D = \operatorname{diag}(\lambda_1 I, \ldots, \lambda_q I, I, X_{t_\alpha}, \ldots, I, X_{t_\alpha}),$$

where $\lambda_i \in \mathbb{F}^\times$ are selected in such a way that condition (20) holds with $d_0 = X_{t_\alpha}$. Clearly, condition (21) also holds. Finally, let ψ_α be given by $\Psi = PD$ and ψ_0. Then ψ_α sends ${}^\alpha\Gamma$ to Γ and fixes φ, so ψ_α belongs to $\operatorname{Aut}(\Gamma, \varphi)$.

Let K be the kernel of f and let

$$N = \{\psi \in K \mid \deg d_1 = \cdots = \deg d_k\}.$$

The same arguments as in 2) show that $K/N \cong T^{q+s}/T$ and $\operatorname{Aut}(\Gamma, \varphi)/N$ is isomorphic to $\big((T^{q+s-1} \times \mathbb{Z}_2^s) \rtimes (\operatorname{Sym}\Sigma \times \operatorname{Sym}(s)\big) \rtimes \operatorname{Aut}\Sigma$, with the stated actions. But here we have $N = \operatorname{Stab}(\Gamma, \varphi)$, which completes the proof. □

4. Series A

In this section we describe the Weyl groups of fine gradings on the simple Lie algebras of series A. Thus, we take $\mathcal{R} = M_n(\mathbb{F})$, $n \geq 2$, and $\mathcal{L} = \mathfrak{psl}_n(\mathbb{F}) = [\mathcal{R}, \mathcal{R}]/(Z(\mathcal{R}) \cap [\mathcal{R}, \mathcal{R}])$. First we review the classification of fine gradings on $\mathcal{L}$ from [Eld10] (extended to positive characteristic using automorphism group schemes) and then derive the Weyl groups for $\mathcal{L}$ from what we already know about automorphisms of fine gradings ([EKb]) and fine φ-gradings (Section 3) on $\mathcal{R}$.

4.1. **Classification of fine gradings.** The case $n = 2$ is easy, because the restriction from $\mathcal{R}$ to $\mathcal{L}$ yields an isomorphism $\mathbf{Aut}(\mathcal{R}) \to \mathbf{Aut}(\mathcal{L})$. It follows that the classification of fine gradings on $\mathcal{L}$ is the same as that on $\mathcal{R}$. Namely, there are two fine gradings on $\mathfrak{sl}_2(\mathbb{F})$, up to equivalence: the Cartan grading, whose universal group is $\mathbb{Z}$, and the Pauli grading, whose universal group is $\mathbb{Z}_2^2$.

Now assume $n \geq 3$. Then the restriction and passing modulo the center yields a closed imbedding $\mathbf{Aut}(\mathcal{R}) \to \mathbf{Aut}(\mathcal{L})$, which is not an isomorphism. To rectify this, one introduces the affine group scheme $\overline{\mathbf{Aut}}(\mathcal{R})$ corresponding to the algebraic group of automorphisms and anti-automorphisms of $\mathcal{R}$ (see [BK10, §3]). Unless $n = \operatorname{char}\mathbb{F} = 3$, we obtain an isomorphism $\overline{\mathbf{Aut}}(\mathcal{R}) \to \mathbf{Aut}(\mathcal{L})$. It is convenient to divide gradings on $\mathcal{L}$ into two types: for Type I the corresponding diagonalizable subgroupscheme of $\mathbf{Aut}(\mathcal{L})$ is contained in the image of the closed imbedding $\mathbf{Aut}(\mathcal{R}) \to \mathbf{Aut}(\mathcal{L})$, while for Type II it is not. In other words, a grading on $\mathcal{L}$ is of Type I if and only if it is induced from a (unique) grading on $\mathcal{R}$ by restriction and passing modulo the center.

In [BK10], the *distinguished element* of a Type II grading Γ is introduced. It can be characterized as the unique element h of order 2 in the grading group G such that the coarsening $\overline{\Gamma}$ induced from Γ by the quotient map $G \to \overline{G} := G/\langle h\rangle$ is a Type I grading. The original grading Γ can be recovered from $\overline{\Gamma}$ if we know the action of some character χ of G with $\chi(h) = -1$. Indeed, we just have to split each component of $\overline{\Gamma}$ into eigenspaces with respect to the action of χ. We can transfer this procedure to $\mathcal{R}$ in the following way. The action of χ on $\mathcal{L}$ is induced by $-\varphi$ where φ is an anti-automorphism of $\mathcal{R}$. The Type I grading $\overline{\Gamma}$ on $\mathcal{L}$ comes from a grading $\overline{\Gamma}'$ on $\mathcal{R}$. Since $-\varphi$ is an automorphism of $\mathcal{R}^{(-)}$ (the Lie algebra $\mathcal{R}$ under commutator) and φ^2 acts as a scalar on each component of $\overline{\Gamma}'$, we can refine the $\overline{G}$-grading $\overline{\Gamma}' : \ \mathcal{R} = \bigoplus_{\overline{g}\in\overline{G}} \mathcal{R}_{\overline{g}}$ to a G-grading $\Gamma' : \ \mathcal{R}^{(-)} = \bigoplus_{g\in G} \mathcal{R}_g$ by splitting each component $\mathcal{R}_{\overline{g}}$ into eigenspaces of φ. In detail, φ^2 acts on $\mathcal{R}_{\overline{g}}$ as multiplication by $\chi^2(\overline{g})$ (where we regard χ^2 as a character of $\overline{G}$, since $\chi^2(h) = 1$), so we set

$$\mathcal{R}_g = \{X \in \mathcal{R}_{\overline{g}} \mid \varphi(X) = -\chi(g)X\} = \{\varphi(X) - \chi(g)X \mid X \in \mathcal{R}_{\overline{g}}\}. \tag{25}$$

Then Γ' induces the original Type II grading Γ on $\mathcal{L}$ by restriction and passing modulo the center.

Now we apply the above to fine gradings on $\mathcal{L}$. The fine gradings of Type I come from the fine gradings on $\mathcal{R}$ that do not admit an anti-automorphism φ making them φ-gradings. All fine gradings on $\mathcal{R}$ are obtained as follows. We start from T, a finite abelian group that admits a nondegenerate alternating bicharacter β (hence $|T|$ is a square). Fix a realization, $\mathcal{D}$, of the matrix algebra endowed with a division grading with support T and bicharacter β. Let $k \geq 1$ be an integer. Denote by $\widetilde{G} = \widetilde{G}(T, k)$ the abelian group freely generated by T and the symbols $\widetilde{g}_1, \ldots, \widetilde{g}_k$.

Definition 4.1. Let $\mathcal{M}(\mathcal{D}, k)$ be the $\widetilde{G}$-graded algebra $\operatorname{End}_{\mathcal{D}}(V)$ where V has a $\mathcal{D}$-basis $\{v_1, \ldots, v_k\}$ with $\deg v_i = \widetilde{g}_i$. Let $n = k\sqrt{|T|}$ and $\mathcal{R} = M_n(\mathbb{F})$. The grading on $\mathcal{R}$ obtained by identifying $\mathcal{R}$ with $\mathcal{M}(\mathcal{D}, k)$ will be denoted by $\Gamma_{\mathcal{M}}(\mathcal{D}, k)$. In other words, we define this grading by identifying $\mathcal{R} = M_k(\mathcal{D})$ and setting $\deg(E_{ij} \otimes X_t) := \widetilde{g}_i t \widetilde{g}_j^{-1}$. By abuse of notation, we will also write $\Gamma_{\mathcal{M}}(T, k)$.

The universal group of $\Gamma_{\mathcal{M}}(T, k)$ is the subgroup $\widetilde{G}^0 = \widetilde{G}(T, k)^0$ of $\widetilde{G}$ generated by the support, i.e., by the elements $z_{i,j,t} := \widetilde{g}_i t \widetilde{g}_j^{-1}$, $t \in T$. Clearly, $\widetilde{G}^0 \cong T \times \mathbb{Z}^{k-1}$. By [Eld10, Proposition 3.24], $\Gamma_{\mathcal{M}}(T, k)$ is a φ-grading for some φ if and only if T

is an elementary 2-group and $k \leq 2$. Two gradings, $\Gamma_{\mathcal{M}}(T,k)$ and $\Gamma_{\mathcal{M}}(T',k')$, are equivalent if and only if $T \cong T'$ and $k = k'$.

Definition 4.2. Consider the grading $\Gamma_{\mathcal{M}}(T,k)$ on $\mathcal{R}$ by the group $\widetilde{G}(T,k)^0$ where $k \geq 3$ if T is an elementary 2-group. The $\widetilde{G}(T,k)^0$-grading on $\mathcal{L}$ obtained by restriction and passing modulo the center will be denoted by $\Gamma_A^{(\mathrm{I})}(T,k)$.

The grading $\Gamma_A^{(\mathrm{I})}(T,k)$ is fine, and $\widetilde{G}(T,k)^0$ is its universal group. To deal with fine gradings of Type II, we will need the following general observation:

Lemma 4.3. *Let $\overline{\Gamma}$ be a φ-grading on an algebra $\mathcal{A}$ and let $\overline{G}$ be its universal group. Then there exist an abelian group G, an element $h \in G$ of order 2, a character χ of G with $\chi(h) = -1$ such that $\overline{G} = G/\langle h\rangle$ and the action of χ^2 on the $\overline{G}$-graded algebra $\mathcal{A}$ (regarding χ^2 as a character of the group $\overline{G}$) coincides with φ^2. The pair (G,h) is determined uniquely up to isomorphism over $\overline{G}$ (i.e., $\langle h\rangle \to G \to \overline{G}$ is unique up to equivalence of extensions).*

Proof. For each $\overline{g} \in \overline{G}$, φ^2 acts on $\mathcal{A}_{\overline{g}}$ as multiplication by some $\lambda(\overline{g}) \in \mathbb{F}^\times$. Since $\overline{G}$ is the universal group of $\overline{\Gamma}$, $\lambda\colon \overline{G} \to \mathbb{F}^\times$ is a homomorphism. For each $\overline{g} \in \overline{G}$, we select $\mu(\overline{g}) \in \mathbb{F}^\times$ such that $\mu(\overline{g})^2 = \lambda(\overline{g})$ (there are two choices). It will be convenient to choose $\mu(\overline{e}) = 1$. It follows that

$$\mu(\overline{x}\,\overline{y}) = \varepsilon(\overline{x},\overline{y})\mu(\overline{x})\mu(\overline{y}) \quad \text{for all} \quad \overline{x},\overline{y} \in \overline{G} \tag{26}$$

where $\varepsilon(\overline{x},\overline{y}) \in \{\pm 1\}$. One immediately verifies that ε is a symmetric 2-cocycle on $\overline{G}$ with $\varepsilon(\overline{g},\overline{e}) = 1$ for all $\overline{g} \in \overline{G}$ and, moreover, the class of ε in $H^2(\overline{G},\mathbb{Z}_2)$ (where we identified $\{\pm 1\}$ with $\mathbb{Z}_2$) does not depend on the choices of $\mu(\overline{g})$. Let G be the central extension of $\overline{G}$ by $\mathbb{Z}_2$ determined by ε, i.e., G consists of the pairs $(\overline{g},\delta)$, $\overline{g} \in \overline{G}$, $\delta \in \{\pm 1\}$, with multiplication given by

$$(\overline{x},\delta_1)(\overline{y},\delta_2) = (\overline{x}\,\overline{y},\, \varepsilon(\overline{x},\overline{y})\delta_1\delta_2) \quad \text{for all} \quad \overline{x},\overline{y} \in \overline{G} \quad \text{and} \quad \delta_1,\delta_2 \in \{\pm 1\}. \tag{27}$$

Define $\chi\colon G \to \mathbb{F}^\times$ by $(\overline{g},\delta) \mapsto \mu(\overline{g})\delta$. Comparing (26) and (27), we see that χ is a homomorphism. Set $h = (\overline{e},-1) \in G$. Then h has order 2 and $\chi(h) = -1$. By construction, the action of χ^2 on $\mathcal{A}$ determined by $\overline{\Gamma}$ coincides with φ^2. □

Let T be an elementary 2-group of even dimension. Recall the group $\widetilde{G}(T,q,s,\tau)$, which was introduced before Definition 3.3, and its subgroup $\widetilde{G}(T,q,s,\tau)^0$.

Definition 4.4. Consider the grading $\overline{\Gamma} = \Gamma_{\mathcal{M}}(T,q,s,\tau)$ on $\mathcal{R}$ by the group $\overline{G} = \widetilde{G}(T,q,s,\tau)^0$ where $t_1 \neq t_2$ if $q = 2$ and $s = 0$. Let Φ be the matrix given by

$$\Phi = \operatorname{diag}\left(X_{t_1},\ldots,X_{t_q},\begin{bmatrix}0 & I\\ I & 0\end{bmatrix},\ldots,\begin{bmatrix}0 & I\\ I & 0\end{bmatrix}\right).$$

Define $\varphi(X) = \Phi^{-1}({}^tX)\Phi$. Let G, h and χ be as in Lemma 4.3, so we obtain a G-grading on $\mathcal{R}^{(-)}$ defined by (25). The G-grading on $\mathcal{L}$ obtained by restriction and passing modulo the center will be denoted by $\Gamma_A^{(\mathrm{II})}(T,q,s,\tau)$.

The grading $\Gamma_A^{(\mathrm{II})}(T,q,s,\tau)$ is fine, and G is its universal group. Note that $\varphi^4 = \mathrm{id}$. It can be shown (cf. [Eld10, Example 3.21]) that the extension $\langle h\rangle \to G \to \overline{G}$ is split if and only if there exists $t \in T$ such that t_it are in T_+ for all i or in T_- for all i. Taking into account (3), we see that G is isomorphic to

$$\begin{cases} \mathbb{Z}_2^{\dim T-2\dim T_0+\max(0,q-1)+1}\times\mathbb{Z}_4^{\dim T_0}\times\mathbb{Z}^s & \text{if}\quad \exists t\in T\quad \beta(t_1t)=\ldots=\beta(t_qt);\\ \mathbb{Z}_2^{\dim T-2\dim T_0+\max(0,q-1)}\times\mathbb{Z}_4^{\dim T_0+1}\times\mathbb{Z}^s & \text{otherwise,}\end{cases}$$

where T_0 is the subgroup of T generated by the elements t_it_{i+1}, $i=1,\ldots,q-1$.

Now Theorem 4.2 of [Eld10] can be extended to positive characteristic and recast as follows:

Theorem 4.5. *Let* $\mathbb{F}$ *be an algebraically closed field,* $\operatorname{char}\mathbb{F}\neq 2$. *Let* $n\geq 3$ *if* $\operatorname{char}\mathbb{F}\neq 3$ *and* $n\geq 4$ *if* $\operatorname{char}\mathbb{F}=3$. *Then any fine grading on* $\mathfrak{psl}_n(\mathbb{F})$ *is equivalent to one of the following:*

- $\Gamma_A^{(\mathrm{I})}(T,k)$ *as in Definition 4.2 with* $k\sqrt{|T|}=n$,
- $\Gamma_A^{(\mathrm{II})}(T,q,s,\tau)$ *as in Definition 4.4 with* $(q+2s)\sqrt{|T|}=n$.

Gradings belonging to different types listed above are not equivalent. Within each type, we have the following:

- $\Gamma_A^{(\mathrm{I})}(T_1,k_1)$ *and* $\Gamma_A^{(\mathrm{I})}(T_2,k_2)$ *are equivalent if and only if* $T_1\cong T_2$ *and* $k_1=k_2$;
- $\Gamma_A^{(\mathrm{II})}(T_1,q_1,s_1,\tau_1)$ *and* $\Gamma_A^{(\mathrm{II})}(T_2,q_2,s_2,\tau_2)$ *are equivalent if and only if* $T_1\cong T_2$, $q_1=q_2$, $s_1=s_2$ *and, identifying* $T_1=T_2=\mathbb{Z}_2^{2r}$, $\Sigma(\tau_1)$ *is conjugate to* $\Sigma(\tau_2)$ *by the natural action of* $\mathrm{ASp}_{2r}(2)$. □

The missing case $n=\operatorname{char}\mathbb{F}=3$ can be treated using octonions, because in characteristic 3 the algebra of traceless octonions under commutator is a Lie algebra isomorphic to $\mathfrak{psl}_3(\mathbb{F})$ (cf. [BK10, Remark 4.11]).

4.2. **Weyl groups of fine gradings.** By [EKb, Theorem 2.8], the Weyl group of $\Gamma_{\mathcal{M}}(T,k)$ is isomorphic to $T^{k-1}\rtimes(\mathrm{Sym}(k)\times\mathrm{Aut}(T,\beta))$, with $\mathrm{Sym}(k)$ and $\mathrm{Aut}(T,\beta)$ acting on T^{k-1} through their natural action on T^k and identification of T^{k-1} with T^k/T where T is imbedded into T^k diagonally. Thanks to the isomorphism $\mathbf{Aut}(M_2(\mathbb{F}))\to\mathbf{Aut}(\mathfrak{sl}_2(\mathbb{F}))$, it follows that the Weyl group of the Cartan grading on $\mathfrak{sl}_2(\mathbb{F})$ is $\mathrm{Sym}(2)$ (the classical Weyl group of type A_1) and the Weyl group of the Pauli grading on $\mathfrak{sl}_2(\mathbb{F})$ is $\mathrm{Sp}_2(2)=\mathrm{GL}_2(2)$ (this is known in the case $\operatorname{char}\mathbb{F}=0$ — see [HPPT02]).

To state our result for $\mathfrak{psl}_n(\mathbb{F})$, $n\geq 3$, it is convenient to introduce the following notation:

$$\overline{\mathrm{Aut}}(T,\beta):=\mathrm{Aut}(T,\beta)\rtimes\langle\sigma\rangle,$$

where σ is an element of order 2 acting as the automorphism of T that sends a_i to a_i^{-1} and b_i to b_i, where a_i and b_i are the generators of T used for the chosen realization of $\mathcal{D}$ (a "symplectic basis" of T with respect to β). We observe that $\beta(\sigma\cdot u,\sigma\cdot v)=\beta(u,v)^{-1}$, for all $u,v\in T$, and hence we obtain an induced action of σ on $\mathrm{Aut}(T,\beta)$ by setting $(\sigma\cdot\alpha)(t):=\sigma\cdot\alpha(\sigma\cdot t)$ for all $\alpha\in\mathrm{Aut}(T,\beta)$ and $t\in T$. The elements of $\overline{\mathrm{Aut}}(T,\beta)$ act as automorphisms of T that send β to $\beta^{\pm1}$. However, this action is not faithful if T is an elementary 2-group.

Theorem 4.6. *Let* $\mathbb{F}$ *be an algebraically closed field,* $\operatorname{char}\mathbb{F}\neq 2$. *Let* $n\geq 3$ *if* $\operatorname{char}\mathbb{F}\neq 3$ *and* $n\geq 4$ *if* $\operatorname{char}\mathbb{F}=3$. *Consider the fine grading* $\Gamma=\Gamma_A^{(\mathrm{I})}(T,k)$ *on* $\mathfrak{psl}_n(\mathbb{F})$ *as in Definition 4.2,* $k\sqrt{|T|}=n$. *Then*

$$W(\Gamma)\cong T^{k-1}\rtimes(\mathrm{Sym}(k)\times\overline{\mathrm{Aut}}(T,\beta)),$$

with $\mathrm{Sym}(k)$ *and* $\overline{\mathrm{Aut}}(T,\beta)$ *acting on* T^{k-1} *through their natural action on* T^k *and identification of* T^{k-1} *with* T^k/T *where* T *is imbedded into* T^k *diagonally.*

Proof. The grading Γ on $\mathcal{L}=\mathfrak{psl}_n(\mathbb{F})$ is induced by the grading $\Gamma'=\Gamma_{\mathcal{M}}(T,k)$ on $\mathcal{R}=M_n(\mathbb{F})$. The universal group of both gradings is $G=\widetilde{G}(T,k)^0$. Since restriction is a bijection between gradings on $\mathcal{R}$ and Type I gradings on $\mathcal{L}$, an automorphism ψ' of $\mathcal{R}$ sends ${}^\alpha\Gamma'$ to Γ', for some automorphism α of G, if and only if the induced automorphism ψ of $\mathcal{L}$ sends ${}^\alpha\Gamma$ to Γ. The automorphism group of $\mathcal{L}$ is the semidirect product of $\mathrm{Aut}(\mathcal{R})$, in its induced action on $\mathcal{L}$, and $\langle\sigma\rangle$, where σ is given by the negative of matrix transpose. To compute the action of σ, recall that $(u_1,\ldots,u_k)T\in T^k/T$ can be represented by the automorphism $X\mapsto DXD^{-1}$ where $D=\mathrm{diag}(X_{u_1},\ldots,X_{u_k})$, $\pi\in\mathrm{Sym}(k)$ can be represented by $X\mapsto PXP^{-1}$ where P is the permutation matrix corresponding to π, and $\alpha\in\mathrm{Aut}(T,\beta)$ can be represented by $X\mapsto\psi_0(X)$ where ψ_0 is an automorphism of $\mathcal{D}$ such that $\psi_0(X_t)\in\mathbb{F}X_{\alpha(t)}$ for all $t\in T$. The conjugation by σ sends the automorphism $X\mapsto\Psi X\Psi^{-1}$ to the automorphism $X\mapsto({}^t\Psi^{-1})X({}^t\Psi)$, i.e., replaces Ψ by ${}^t\Psi^{-1}$. Hence, σ commutes with $\mathrm{Sym}(k)$, while the conjugation by σ sends $(u_1,\ldots,u_k)T$ to $(\sigma\cdot u_1,\ldots,\sigma\cdot u_k)T$, where the action of σ on T is as indicated above. Also, the action of σ on G sends $z_{i,j,t}:=\widetilde{g}_it\widetilde{g}_j^{-1}$ to $z_{i,j,\sigma\cdot t}^{-1}$, so σ belongs to $\mathrm{Aut}(\Gamma)$, but not to $\mathrm{Stab}(\Gamma)$. Hence we obtain $\mathrm{Aut}(\Gamma)=\mathrm{Aut}(\Gamma')\rtimes\langle\sigma\rangle$ and $\mathrm{Stab}(\Gamma)=\mathrm{Stab}(\Gamma')$. The result follows. □

Theorem 4.7. *Let* $\mathbb{F}$ *be an algebraically closed field,* $\mathrm{char}\,\mathbb{F}\neq 2$. *Let* $n\geq 3$ *if* $\mathrm{char}\,\mathbb{F}\neq 3$ *and* $n\geq 4$ *if* $\mathrm{char}\,\mathbb{F}=3$. *Consider the fine grading* $\Gamma=\Gamma_A^{(\mathrm{II})}(T,q,s,\tau)$ *on* $\mathfrak{psl}_n(\mathbb{F})$ *as in Definition 4.4,* $(q+2s)\sqrt{|T|}=n$. *Let* $\Sigma=\Sigma(\tau)$. *Then* $W(\Gamma)$ *contains a normal subgroup* N *isomorphic to* $\mathbb{Z}_2^{q+s-1}$ *such that*

$$W(\Gamma)/N\cong\big((T^{q+s-1}\times\mathbb{Z}_2^s)\rtimes(\mathrm{Sym}\Sigma\times\mathrm{Sym}(s)\big)\rtimes\mathrm{Aut}^*\Sigma,$$

where the actions are described naturally if we identify T^{q+s-1} *with* T^{q+s}/T *and* $\mathbb{Z}_2^{q+s-1}$ *with* $\mathbb{Z}_2^{q+s}/\mathbb{Z}_2$ *(diagonal imbeddings). Moreover,* $W(\Gamma)$ *contains a subgroup isomorphic to* $\big((T^{q+s-1}\times\mathbb{Z}_2^s)\rtimes(\mathrm{Sym}\Sigma\times\mathrm{Sym}(s)\big)\rtimes\mathrm{Aut}\,\Sigma$ *that is disjoint from* N.

Proof. The grading $\Gamma=\Gamma_A^{(\mathrm{II})}(T,q,s,\tau)$ on $\mathcal{L}=\mathfrak{psl}_n(\mathbb{F})$ is induced by the grading Γ' on $\mathcal{R}^{(-)}$, where $\mathcal{R}=M_n(\mathbb{F})$, obtained from $\overline{\Gamma}'=\Gamma_{\mathcal{M}}(T,q,s,\tau)$ and φ as in Definition 4.4. The universal group of $\overline{\Gamma}'$ is $\overline{G}=\widetilde{G}(T,q,s,\tau)^0$, while the universal group of Γ is the extension G of $\overline{G}$ as in Lemma 4.3. Similarly to Type I, an automorphism ψ' of $\mathcal{R}$ sends ${}^\alpha\Gamma'$ to Γ', for some automorphism α of G, if and only if the induced automorphism ψ of $\mathcal{L}$ sends ${}^\alpha\Gamma$ to Γ. Note that α fixes the distinguished element $h=(\bar{e},-1)$ and hence yields an automorphism $\overline{\alpha}$ of $\overline{G}$. It follows that ψ' sends ${}^{\overline{\alpha}}\overline{\Gamma}'$ to $\overline{\Gamma}'$. For any $g\in G$ and $X\in\mathcal{R}_g$, we have $\varphi(X)=-\chi(g)X$. Since $(\psi')^{-1}(X)\in\mathcal{R}_{\alpha^{-1}(g)}$, we also have $(\varphi(\psi')^{-1})(X)=-\chi(\alpha^{-1}(g))(\psi')^{-1}(X)$. It follows that $\psi'\varphi(\psi')^{-1}=\xi\varphi$ where ξ is the action of the character $(\chi\circ\alpha^{-1})\chi^{-1}$ on $\mathcal{R}$ determined by the G-grading Γ'. Since $\alpha(h)=h$, $(\chi\circ\alpha^{-1})\chi^{-1}$ can be regarded as a character of $\overline{G}$, hence ξ belongs to $\mathrm{Diag}(\overline{\Gamma}')$. Conversely, if ψ' sends ${}^{\overline{\alpha}}\overline{\Gamma}'$ to $\overline{\Gamma}'$ and $\psi'\varphi(\psi')^{-1}=\xi\varphi$ for some $\xi\in\mathrm{Diag}(\overline{\Gamma}')$, then for any $\overline{g}\in\overline{G}$ and $X\in\mathcal{R}_{\overline{g}}$, we have $\psi'(X)\in\mathcal{R}_{\overline{\alpha}(\overline{g})}$ and $\varphi(\psi'(X))=\nu\psi'(X)$ where $\nu\in\mathbb{F}^\times$ depends only on $\overline{g}$. It follows that ψ' permutes the components of Γ' and hence sends ${}^\alpha\Gamma'$ to Γ' where α is a lifting of $\overline{\alpha}$. We have proved that an automorphism ψ' of $\mathcal{R}$ belongs to $\mathrm{Aut}^*(\overline{\Gamma}', \varphi)$, respectively $\mathrm{Stab}(\overline{\Gamma}', \varphi)$, if and only if the induced automorphism ψ of $\mathcal{L}$ belongs to $\mathrm{Aut}(\Gamma)$, respectively $\mathrm{Stab}(\Gamma)$. Finally, note that $-\varphi$ induces an automorphism of $\mathcal{L}$ that belongs to $\mathrm{Stab}(\Gamma)$. It follows that the Weyl group of Γ is isomorphic to $\mathrm{Aut}^*(\overline{\Gamma}', \varphi)/\,\mathrm{Stab}(\overline{\Gamma}', \varphi)$. The latter group was described in Theorem 3.12. $\square$

If $\operatorname{char}\mathbb{F} = 3$, there are two fine gradings on $\mathfrak{psl}_3(\mathbb{F})$: the Cartan grading, whose universal group is $\mathbb{Z}^2$, and the grading induced by the Cayley–Dickson doubling process for octonions, whose universal group is $\mathbb{Z}_2^3$. The Weyl groups of these gradings are, respectively, the classical Weyl group of type G_2 [EKb, Theorem 3.3] and $\mathrm{GL}_3(2)$ [EKb, Theorem 3.5].

5. Series B, C and D

In this section we describe the Weyl groups of fine gradings on the simple Lie algebras of series B, C and D with exception of type D_4. Thus, we take $\mathcal{R} = M_n(\mathbb{F})$, $n \geq 4$, and $\mathcal{L} = \mathcal{K}(\mathcal{R}, \varphi)$ where φ is an involution on $\mathcal{R}$. If φ is symplectic, then, of course, n has to be even. If φ is orthogonal, we assume $n \geq 5$ and $n \neq 8$. First we review the classification of fine gradings on $\mathcal{L}$ from [Eld10] (extended to positive characteristic using automorphism group schemes) and then derive the Weyl groups for $\mathcal{L}$ from what we already know about automorphisms of fine φ-gradings (Section 3) on $\mathcal{R}$.

5.1. **Classification of fine gradings.** Under the stated assumptions on n, the restriction from $\mathcal{R}$ to $\mathcal{L}$ yields an isomorphism $\mathbf{Aut}(\mathcal{R}, \varphi) \to \mathbf{Aut}(\mathcal{L})$ (see [BK10, §3]). It follows that the classification of fine gradings on $\mathcal{L}$ is the same as the classification of fine φ-gradings on $\mathcal{R}$ (here φ is fixed).

The case of series B is quite easy, because n is odd and hence the elementary 2-group T must be trivial. Let $G = \widetilde{G}(\{e\}, q, s, \tau)^0$ where $\tau = (e, \ldots, e)$, so $G \cong \mathbb{Z}_2^{q-1} \times \mathbb{Z}^s$.

Definition 5.1. Consider the grading $\Gamma = \Gamma_{\mathcal{M}}(\{e\}, q, s, \tau)$ on $\mathcal{R}$ by G. Let Φ be the matrix given by

$$\Phi = \mathrm{diag}\left(\underbrace{1, \ldots, 1}_{q}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right).$$

Then Γ is a fine φ-grading for $\varphi(X) = \Phi^{-1}({}^tX)\Phi$ and hence its restriction is a fine grading on $\mathcal{L} \cong \mathfrak{so}_n(\mathbb{F})$. We will denote this grading by $\Gamma_B(q, s)$.

Now we turn to series C and D, where n is even and hence T may be nontrivial. So, let T be an elementary 2-group of even dimension. Choose τ as in (1) with all $t_i \in T_-$ in case of series C and all $t_i \in T_+$ in case of series D. Let $G = \widetilde{G}(T, q, s, \tau)^0$, so $G \cong \mathbb{Z}_2^{\dim T - 2\dim T_0 + \max(0, q-1)} \times \mathbb{Z}_4^{\dim T_0} \times \mathbb{Z}^s$ where T_0 is the subgroup of T generated by the elements $t_i t_{i+1}$, $i = 1, \ldots, q-1$.

Definition 5.2. Consider the grading $\Gamma = \Gamma_{\mathcal{M}}(\mathcal{D}, q, s, \tau)$ on $\mathcal{R}$ by G where $t_1 \neq t_2$ if $q = 2$ and $s = 0$. Let Φ be the matrix given by

$$\Phi = \mathrm{diag}\left(X_{t_1}, \ldots, X_{t_q}, \begin{bmatrix} 0 & I \\ \delta I & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & I \\ \delta I & 0 \end{bmatrix}\right),$$

where $\delta = -1$ for series C and $\delta = 1$ for series D. Then Γ is a fine φ-grading for $\varphi(X) = \Phi^{-1}({}^tX)\Phi$ and hence its restriction is a fine grading on $\mathcal{L} \cong \mathfrak{sp}_n(\mathbb{F})$ or $\mathfrak{so}_n(\mathbb{F})$. We will denote this grading by $\Gamma_C(T,q,s,\tau)$ or $\Gamma_D(T,q,s,\tau)$, respectively.

The following three results are Theorem 5.2 of [Eld10], stated separately for series B, C and D (and extended to positive characteristic).

Theorem 5.3. *Let $\mathbb{F}$ be an algebraically closed field,* $\operatorname{char}\mathbb{F} \neq 2$. *Let $n \geq 5$ be odd. Then any fine grading on $\mathfrak{so}_n(\mathbb{F})$ is equivalent to $\Gamma_B(q,s)$ where $q + 2s = n$. Also, $\Gamma_B(q_1,s_1)$ and $\Gamma_B(q_2,s_2)$ are equivalent if and only if $q_1 = q_2$ and $s_1 = s_2$.* $\square$

Theorem 5.4. *Let $\mathbb{F}$ be an algebraically closed field,* $\operatorname{char}\mathbb{F} \neq 2$. *Let $n \geq 4$ be even. Then any fine grading on $\mathfrak{sp}_n(\mathbb{F})$ is equivalent to $\Gamma_C(T,q,s,\tau)$ where $(q+2s)\sqrt{|T|} = n$. Moreover, $\Gamma_C(T_1,q_1,s_1,\tau_1)$ and $\Gamma_C(T_2,q_2,s_2,\tau_2)$ are equivalent if and only if $T_1 \cong T_2$, $q_1 = q_2$, $s_1 = s_2$ and, identifying $T_1 = T_2 = \mathbb{Z}_2^{2r}$, $\Sigma(\tau_1)$ is conjugate to $\Sigma(\tau_2)$ by the twisted action of* $\mathrm{Sp}_{2r}(2)$ *as in Definition* 3.9. $\square$

Theorem 5.5. *Let $\mathbb{F}$ be an algebraically closed field,* $\operatorname{char}\mathbb{F} \neq 2$. *Let $n \geq 6$ be even. Assume $n \neq 8$. Then any fine grading on $\mathfrak{so}_n(\mathbb{F})$ is equivalent to $\Gamma_D(T,q,s,\tau)$ where $(q+2s)\sqrt{|T|} = n$. Moreover, $\Gamma_D(T_1,q_1,s_1,\tau_1)$ and $\Gamma_D(T_2,q_2,s_2,\tau_2)$ are equivalent if and only if $T_1 \cong T_2$, $q_1 = q_2$, $s_1 = s_2$ and, identifying $T_1 = T_2 = \mathbb{Z}_2^{2r}$, $\Sigma(\tau_1)$ is conjugate to $\Sigma(\tau_2)$ by the twisted action of* $\mathrm{Sp}_{2r}(2)$ *as in Definition* 3.9. $\square$

5.2. **Weyl groups of fine gradings.** Let $\Gamma = \Gamma_B(q,s)$, $\Gamma_C(T,q,s,\tau)$ or $\Gamma_D(T,q,s,\tau)$, so Γ is the restriction of the grading $\Gamma' = \Gamma_{\mathcal{M}}(T,q,s,\tau)$ on $\mathcal{R}$ to $\mathcal{L} = \mathcal{K}(\mathcal{R},\varphi)$. By arguments similar to the proof of Theorem 4.7, one shows that the Weyl group of Γ is isomorphic to $\operatorname{Aut}(\Gamma',\varphi)/\operatorname{Stab}(\Gamma',\varphi)$, which was described in Theorem 3.12. For $\Gamma = \Gamma_B(q,s)$, T is trivial and Σ is a singleton of multiplicity q, so we obtain:

Theorem 5.6. *Let $\mathbb{F}$ be an algebraically closed field,* $\operatorname{char}\mathbb{F} \neq 2$. *Let $n \geq 5$ be odd. Consider the fine grading $\Gamma = \Gamma_B(q,s)$ on $\mathfrak{so}_n(\mathbb{F})$ as in Definition* 5.1*, where $q+2s = n$. Let $\Sigma = \Sigma(\tau)$. Then $W(\Gamma) \cong \mathrm{Sym}(q)\times W(s)$ where $W(s) = \mathbb{Z}_2^s \rtimes \mathrm{Sym}(s)$ (wreath product of* $\mathrm{Sym}(s)$ *and $\mathbb{Z}_2$).* $\square$

For $\Gamma_C(T,q,s,\tau)$ and $\Gamma_D(T,q,s,\tau)$, T may be nontrivial, so the answer is more complicated:

Theorem 5.7. *Let $\mathbb{F}$ be an algebraically closed field,* $\operatorname{char}\mathbb{F} \neq 2$. *Let $n \geq 4$ be even. Consider the fine grading $\Gamma = \Gamma_C(T,q,s,\tau)$ on $\mathfrak{sp}_n(\mathbb{F})$ or $\Gamma = \Gamma_D(T,q,s,\tau)$ on $\mathfrak{so}_n(\mathbb{F})$ as in Definition* 5.2*, where $(q+2s)\sqrt{|T|} = n$ and $n \neq 4, 8$ in the case of $\mathfrak{so}_n(\mathbb{F})$. Let $\Sigma = \Sigma(\tau)$. Then*

$$W(\Gamma) \cong \big((T^{q+s-1} \times \mathbb{Z}_2^s) \rtimes (\mathrm{Sym}\Sigma \times \mathrm{Sym}(s)\big) \rtimes \operatorname{Aut}\Sigma,$$

where the actions on T^{q+s-1} are via the identification with T^{q+s}/T (diagonal imbedding).

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