ENGEL'S THEOREM FOR GENERALIZED LIE ALGEBRAS

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Abstract. We prove analogues of the classical Engel's Theorem for Lie algebras in the category of comodules over a cotriangular Hopf algebra, generalizing the known result for Lie coloralgebras.

1. Introduction

There are several well-known generalizations of the classical notion of Lie algebra, beginning with Lie superalgebras and coloralgebras. The class of generalized Lie algebras studied in this paper was first introduced by D. Gurevich in [4] and appeared later in geometric context in a paper by Yu. Manin [9]. These are algebras that satisfy the anticommutativity and Jacobi identities with respect to some symmetric braiding operator. More recently there appeared a number of publications that study these objects in the context of the theory of Hopf algebras [6, 7, 10] and their actions on rings [3, 1]. In particular, A. Masuoka extends the classical category equivalence between finite-dimensional nilpotent Lie algebras and unipotent algebraic affine groups to generalized Lie algebras [10, Theorems 4.4 and 6.7].

The classical Engel's Theorem states that any Lie algebra that consists of nilpotent matrices has a common eigenvector. Applying this fact inductively, one finds a basis relative to which the elements of the Lie algebra are represented by strictly upper triangular matrices, and hence the Lie algebra is nilpotent. Engel's Theorem extends easily to Lie superalgebras [12, p. 236] and coloralgebras [2]. In this paper we extend it to generalized Lie algebras whose braiding operator comes from a coaction by a cotriangular Hopf algebra: see the definition of an (H,β)-Lie algebra below. It is worth mentioning that any finite-dimensional generalized Lie algebra in the sense of Gurevich is an (H,β)-Lie algebra [8] for a suitable cotriangular bialgebra (H,β) (which is not necessarily a Hopf algebra).

We fix a ground field $\Bbbk$ of characteristic not 2. Recall the definition of a Lie coloralgebra:

Definition 1.1. Let G be an abelian group and $\beta : G\times G\to \Bbbk^\times$ a skew-symmetric bicharacter. A G-graded algebra $L=\bigoplus_{g\in G} L_g$ over $\Bbbk$ with operation $[\ ,\ ]$ is called a *Lie coloralgebra with commutation factor* β, or a *β-Lie coloralgebra* for short, if the following identities hold for homogeneous elements of $a,b,c\in L$:

$$[a,b]+\beta(d(a),d(b))[b,a]=0$$

and

$$[[a,b],c]+\beta(d(ab),d(c))[[c,a],b]+\beta(d(a),d(bc))[[b,c],a]=0,$$

where $d(a)$ stands for the degree of a, etc.

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The above two identities are referred to as *β-anticommutativity* and *β-Jacobi identity*. Any graded associative algebra $A = \bigoplus_{g\in G} A_g$ becomes a β-Lie coloralgebra under the *β-commutator*

$$[a,b]_\beta = ab - \beta(d(a), d(b))ba$$

for homogeneous $a, b \in A$. In particular, for any finite-dimensional G-graded vector space V, the algebra $\mathrm{End}\,(V)$ is G-graded in a natural way. Namely, $T : V \to V$ is graded of degree $g \in G$ if $T(V_h) \subset V_{gh}$ for all $h \in G$. Thus $\mathrm{End}\,(V)$ becomes a β-Lie coloralgebra, which we denote by $\mathfrak{gl}_\beta(V)$. (Note that if V is infinite-dimensional, then $\mathrm{End}\,(V)$ may not be G-graded, so one has to consider the subalgebra $\mathrm{End}^{\,gr}(V)$ spanned by the graded endomorphisms of various degrees.)

Much of the theory of Lie algebras carries over to Lie coloralgebras (see e.g. [2]). The following is the analogue of Engel's Theorem:

Theorem 1.2 (Engel's Theorem for Lie coloralgebras)**.** *Let G be an abelian group, V a nonzero G-graded vector space, $\beta : G \times G \to \Bbbk^\times$ a bicharacter. Let L be a finite-dimensional graded subalgebra of $\mathfrak{gl}_\beta(V)$ whose homogeneous elements are nilpotent. Then there exists a nonzero homogeneous $v \in V$ such that $xv = 0$ for all $x \in L$.* □

The usual proof of Engel's Theorem for Lie algebras goes through with minor modifications. Theorem 1.2 also follows from Jacobson's result on weakly closed sets [5, Chapter II, Theorem 1].

Lie coloralgebras are a special case of the so-called (H, β)-Lie algebras [1].

Definition 1.3. Let (H, β) be a cotriangular bialgebra. A *(right) (H, β)-Lie algebra* is a (right) H-comodule L together with a bracket operation $[\, ,\,] : L \otimes L \to L$ that is an H-comodule map and satisfies, for all $a, b, c \in L$, *β-anticommutativity*:

$$[a,b] + \beta(a_{(1)}, b_{(1)})[b_{(0)}, a_{(0)}] = 0 \tag{1}$$

and *β-Jacobi identity*:

$$[[a,b],c] + \beta(a_{(1)}b_{(1)}, c_{(1)})[[c_{(0)}, a_{(0)}], b_{(0)}] + \beta(a_{(1)}, b_{(1)}c_{(1)})[[b_{(0)}, c_{(0)}], a_{(0)}] = 0. \tag{2}$$

Here we are using the standard *sigma notation* for coalgebras and comodules [11]. Namely, if C is a coalgebra with comultiplication $\Delta : C \to C \otimes C$, we write $\Delta c = c_{(1)} \otimes c_{(2)}$ for $c \in C$. If M is a right C-comodule with coaction $\rho : M \to M \otimes C$, we write $\rho(m) = m_{(0)} \otimes m_{(1)}$ for $m \in M$. Similarly, if M is a left C-comodule via $\rho : M \to C \otimes M$, then we write $\rho(m) = m_{(-1)} \otimes m_{(0)}$. The subcomodule generated by $m \in M$ will be denoted by $\langle m \rangle$.

Let H be a bialgebra and $\mathcal{M}^H$ the category of right H-comodules. Then $\mathcal{M}^H$ is endowed with tensor product in the usual way. If (H, β) is a cotriangular bialgebra, then $\mathcal{M}^H$ is a symmetric category via

$$M \otimes N \to N \otimes M : m \otimes n \mapsto \beta(m_{(1)}, n_{(1)}) n_{(0)} \otimes m_{(0)}.$$

Then an (H, β)-Lie algebra is simply a Lie algebra in the symmetric category $\mathcal{M}^H$. Clearly, G-graded β-Lie coloralgebras are obtained as a special case of (H, β)-Lie algebras when $H = \Bbbk G$. As in that special case, so in general any associative algebra A in $\mathcal{M}^H$ becomes an (H, β)-Lie algebra, denoted $[A]_\beta$, under the *β-commutator*:

$$[a,b]_\beta = ab - \beta(a_{(1)}, b_{(1)}) b_{(0)} a_{(0)}.$$

Remark 1.4. Similarly to the case of ordinary Lie algebras, one can use (1) and the properties of β to rewrite the β-Jacobi identity (2) in several equivalent forms, e.g.

$$[[a,b],c] = [a,[b,c]] - \beta(a_{(1)}, b_{(1)})[b_{(0)}, [a_{(0)}, c]]. \tag{3}$$

We will prove the following analogue of Engel's Theorem for (H,β)-Lie algebras:

Theorem 1.5. *Let (H,β) be a cotriangular Hopf algebra and V a nonzero finite-dimensional H-comodule. Suppose that we have a representation φ of an (H,β)-Lie algebra L on V. If for every $x \in L$ the subcomodule $\langle\varphi(x)\rangle$ of $\mathfrak{gl}_\beta(V)$ is nilpotent, then there exists a nonzero $v \in V$ such that $\varphi(x)\langle v\rangle = 0$ for all $x \in L$.* □

We also obtain a stronger version in the case when H is cosemisimple:

Theorem 1.6. *Let (H,β) be a cosemisimple cotriangular Hopf algebra and V a nonzero finite-dimensional H-comodule. Suppose that we have a representation φ of a (H,β)-Lie algebra L. If every simple subcomodule in the subalgebra $\varphi(L) \subset \mathfrak{gl}_\beta(V)$ is nilpotent, then there exists a nonzero $v \in V$ such that $\varphi(x)\langle v\rangle = 0$ for all $x \in L$.* □

As in the classical case, one obtains the following corollary.

Corollary 1.7. *Under the assumptions of Theorem 1.5 or 1.6, the (H,β)-Lie algebra $\varphi(L)$ is nilpotent.*

Proof. Passing from V to $W = V/\langle v\rangle$, one shows by induction on $n = \dim V$ that $\varphi(L)^n V = 0$. □

Corollary 1.8. *Let (H,β) be a cotriangular Hopf algebra and L a finite-dimensional (H,β)-Lie algebra. Assume that either each cyclic H-subcomodule of L is ad-nilpotent or that H is cosemisimple and each simple H-subcomodule of L is ad-nilpotent. Then the algebra L is nilpotent.* □

In Section 2 we define representations of an (H,β)-Lie algebra and prove some preliminary facts that may be of interest in their own right. Section 3 is devoted to the proof of Theorems 1.5 and 1.6.

2. Representations of (H,β)-Lie algebras

Fix a Hopf algebra H. We will assume that H-comodules are *right*, unless stated otherwise. It is known that if V is a finite-dimensional H-comodule, then $\mathrm{End}\,(V)$ becomes an H-comodule algebra via the identification $\mathrm{End}\,(V) \cong V \otimes V^*$, and the natural $\mathrm{End}\,(V)$-module structure on V agrees with the H-comodule structure — see e.g. [3, Lemma 2.10]. However, the precise statement is sensitive to the convention regarding the side on which endomorphisms are written when applied to vectors: in [3], for example, H-comodules are left and the endomorphisms are also written on the left, which necessitates the use of $\bar{S}$, the antipode of H^{cop}. For clarity we sketch the proof of the facts mentioned above in our situation. Some of the formulas in the proof will be referred to later.

Lemma 2.1. *Let $V \in \mathcal{M}^H$ be finite-dimensional. Then $\mathrm{End}\,(V)$ is an algebra in $\mathcal{M}^H$ and the evaluation map $ev : \mathrm{End}\,(V) \otimes V \to V : T \otimes v \mapsto Tv$ is a morphism in $\mathcal{M}^H$.*

Proof. We first define a right H-coaction on V^*. Since V is finite-dimentional, $\rho : V \to V \otimes H$ has its image in $V \otimes C$, where C is a finite-dimensional subcoalgebra of H. Thus we may define a left H-coaction on V^* as follows. The coaction $\rho : V \to V \otimes C$ gives rise to the dual action $C^* \otimes V \to V$ defined by $f \otimes v \mapsto f(v_{(1)})v_{(0)}$ for all $v \in V$, $f \in C^*$. The transpose operator $\rho' : V^* \to (C^* \otimes V)^* \cong C \otimes V^*$ then gives a left H-coaction on V^*, namely

$$\rho'(v^*)(f \otimes v) = v^*(f(v_{(1)})v_{(0)}) = f(v_{(1)})v^*(v_{(0)}).$$

for all $v^* \in V^*$, $v \in V$ and $f \in C^*$. Since C is a subcoalgebra of H, ρ' can also be considered as a left H-coaction. Writing ρ' in the sigma notation, $\rho'(v^*) = v^*_{(-1)} \otimes v^*_{(0)}$, we obtain:

$$v^*_{(-1)}(f)v^*_{(0)}(v) = f(v_{(1)})v^*(v_{(0)}).$$

Rewriting the above as $f\big(v^*_{(-1)}v^*_{(0)}(v)\big) = f\big(v^*(v_{(0)})v_{(1)}\big)$ and using the fact that $f \in C^*$ is arbitrary, we obtain

$$v^*_{(-1)}v^*_{(0)}(v) = v^*(v_{(0)})v_{(1)}. \tag{4}$$

Now V^* becomes a right H-comodule via the coaction $v^* \mapsto v^*_{(0)} \otimes S(v^*_{(-1)})$ and so $V \otimes V^*$ becomes a right H-comodule via

$$v \otimes v^* \mapsto v_{(0)} \otimes v^*_{(0)} \otimes v_{(1)}S(v^*_{(-1)}). \tag{5}$$

Finally, $\mathrm{End}\,(V)$ becomes a right H-comodule via the identification $\mathrm{End}\,(V) \cong V \otimes V^*$, and one checks using (4) and (5) that the composition map $\mathrm{End}\,(V) \otimes \mathrm{End}\,(V) \to \mathrm{End}\,(V)$ and the evaluation map $\mathrm{End}\,(V) \otimes V \to V$ are morphisms in $\mathcal{M}^H$. □

Suppose now that (H, β) is a cotriangular Hopf algebra. Then we can define representations of an (H, β)-Lie algebra on a finite-dimensional H-comodule V using the structure of an H-comodule algebra on $\mathrm{End}\,(V)$ as in Lemma 2.1.

Definition 2.2. Let L be an (H, β)-Lie algebra and $V \in \mathcal{M}^H$ with $\dim V < \infty$. Then a map $\varphi : L \to \mathrm{End}\,(V)$ is called a *representation* of L if φ is a morphism in $\mathcal{M}^H$ and, for all $a, b \in L$,

$$\varphi([a, b]) = \varphi(a)\varphi(b) - \beta(a_{(1)}, b_{(1)})\varphi(b_{(0)})\varphi(a_{(0)}). \tag{6}$$

In other words, $\varphi : L \to \mathfrak{gl}_\beta(V)$ is a morphism of H-comodule algebras, where $\mathfrak{gl}_\beta(V) = [\mathrm{End}\,(V)]_\beta$.

We can also take the point of view of modules rather than that of representations. Note that here we do not need to assume V finite-dimensional.

Definition 2.3. Let L be an (H, β)-Lie algebra and $V \in \mathcal{M}^H$. An *L-module structure* on V is an $\mathcal{M}^H$-morphism $L \otimes V \to V : a \otimes v \mapsto a \cdot v$ such that, for all $a, b \in L$ and $v \in V$,

$$[a, b] \cdot v = a \cdot (b \cdot v) - \beta(a_{(1)}, b_{(1)})\, b_{(0)} \cdot (a_{(0)} \cdot v). \tag{7}$$

To show that the above two definitions are equivalent in the case $\dim V < \infty$, we will need the following lemma, which essentially says that the H-comodule structure on $\mathrm{End}\,(V)$ is determined uniquely by the property that $ev : \mathrm{End}\,(V) \otimes V \to V$ is an H-comodule map.

Lemma 2.4. *Let $V \in \mathcal{M}^H$ be finite-dimensional. Then* $\operatorname{End}(V)$ *is an H-comodule as in Lemma 2.1. Let $T, T_i \in \operatorname{End}(V)$ and $h_i \in H$ where i ranges over a finite set. Then $\rho(T) = \sum_i T_i \otimes h_i$ if and only if*

$$\sum_i T_i u_{(0)} \otimes h_i u_{(1)} = (Tu)_{(0)} \otimes (Tu)_{(1)} \qquad \forall u \in V. \tag{8}$$

Proof. First suppose that $T = v \otimes v^*$. Then (5) is equivalent to the equation:

$$T_{(0)} u \otimes T_{(1)} = v_{(0)} v^*_{(0)}(u) \otimes v_{(1)} S(v^*_{(-1)}) \qquad \forall u \in V.$$

Using (4), the right-hand side can be rewritten as

$$v_{(0)} \otimes v_{(1)} S\big(v^*_{(-1)} v^*_{(0)}(u)\big) = v_{(0)} \otimes v_{(1)} S\big(v^*(u_{(0)}) u_{(1)}\big) = v_{(0)} v^*(u_{(0)}) \otimes v_{(1)} S u_{(1)}.$$

Recalling that $T = v \otimes v^*$, we conclude that (5) is equivalent to the equation:

$$T_{(0)} u \otimes T_{(1)} = (Tu_{(0)})_{(0)} \otimes (Tu_{(0)})_{(1)} S u_{(1)} \qquad \forall u \in V.$$

Since the operators $v \otimes v^*$ span $\operatorname{End}(V)$, the above holds for any $T \in \operatorname{End}(V)$. Therefore, $\rho(T) = \sum_i T_i \otimes h_i$ if and only if

$$\sum_i T_i u \otimes h_i = (Tu_{(0)})_{(0)} \otimes (Tu_{(0)})_{(1)} S u_{(1)} \quad \forall u \in V. \tag{9}$$

Now (9) implies $\sum_i T_i u_{(0)} \otimes h_i u_{(1)} = (Tu_{(0)})_{(0)} \otimes (Tu_{(0)})_{(1)} (Su_{(1)}) u_{(2)}$, yielding (8). Conversely, (8) implies $\sum_i T_i u_{(0)} \otimes h_i u_{(1)} S u_{(2)} = (Tu_{(0)})_{(0)} \otimes (Tu_{(0)})_{(1)} S u_{(1)}$, yielding (9). □

Proposition 2.5. *Let L be an (H, β)-Lie algebra and V a finite-dimensional H-comodule. Then $\varphi : L \to \operatorname{End}(V)$ is a representation if and only if the map $a \cdot v = \varphi(a)v$, for $a \in L$ and $v \in V$, is a structure of an L-module on V.*

Proof. Clearly, (6) is equivalent to (7), so it remains to check that $\varphi : L \to \operatorname{End}(V)$ is a morphism in $\mathcal{M}^H$ if and only if $L \otimes V \to L : a \otimes v \mapsto \varphi(a)v$ is a morphism in $\mathcal{M}^H$. Applying Lemma 2.4 to $T = \varphi(a)$ and $\sum_i T_i \otimes h_i = \varphi(a_{(0)}) \otimes a_{(1)}$, we see that φ is a morphism in $\mathcal{M}^H$ iff $\varphi(a_{(0)}) v_{(0)} \otimes a_{(1)} v_{(1)} = (\varphi(a)v)_{(0)} \otimes (\varphi(a)v)_{(1)}$ for all $a \in L$ and $v \in V$, which precisely says that $a \otimes v \mapsto \varphi(a)v$ is a morphism in $\mathcal{M}^H$. □

Remark 2.6. In order to define representations of L on an *infinite-dimensional* H-comodule V, one has to consider only a part of $\operatorname{End}(V)$ so that an H-comodule structure can be defined. This can be done as follows. Let $V, W \in \mathcal{M}^H$. Set

$$\begin{aligned} \operatorname{Hom}^f(V, W) \quad = \quad & \big\{ T : V \to W \mid \exists \text{ finitely many } T_i : V \to W \text{ and } h_i \in H \\ & \text{such that } \rho(Tv) = \sum_i T_i v_{(0)} \otimes h_i v_{(1)} \quad \forall v \in V \big\}. \end{aligned}$$

Clearly, given $T \in \operatorname{Hom}^f(V, W)$, the tensor $\sum_i T_i \otimes h_i$ is uniquely defined. One can show that T_i can be chosen in $\operatorname{Hom}^f(V, W)$ and $\rho(T) := \sum_i T_i \otimes h_i$ is an H-comodule structure on $\operatorname{Hom}^f(V, W)$. One can also show that the composition of two operators in Hom^f is again in Hom^f and that composition is a morphism in $\mathcal{M}^H$. In particular, $\operatorname{End}^f(V) := \operatorname{Hom}^f(V, V)$ is an algebra in $\mathcal{M}^H$ and thus it makes sense to define a representation of L on V as a morphism of H-comodule algebras $\varphi : L \to [\operatorname{End}^f(V)]$. Furthermore, it follows immediately from the definition of Hom^f that if $M \in \mathcal{M}^H$ and $\varphi : M \to \operatorname{Hom}(V, W)$ is a linear map such that $\rho(\varphi(m)v) = \varphi(m_{(0)}) v_{(0)} \otimes m_{(1)} v_{(1)}$ for all $m \in M$ and $v \in V$, then in fact $\varphi(M) \subset$

$\mathrm{Hom}^f(V,W)$ and φ is a morphism in $\mathcal{M}^H$. This establishes an equivalence between representations of L and L-module structures on arbitrary $V \in \mathcal{M}^H$.

Definition 2.7. Let L be any (H,β)-Lie algebra. Then $a \cdot x = [a,x]$, $a, x \in L$, is called the *adjoint action* of L on itself. Using (3) and the fact that $[\,,\,]$ is a morphism in $\mathcal{M}^H$, we see that this action makes L an L-module. The corresponding representation $L \to \mathfrak{gl}_\beta(L)$ (if $\dim L < \infty$) is denoted by ad, so $(\mathrm{ad}\, a)x = [a,x]$ for $a, x \in L$.

Let L be an (H,β)-Lie algebra and V an L-module.

Definition 2.8. An element $x \in V$ is called *L-invariant* if $a \cdot x = 0$ for all $a \in L$. The set of invariants in V is denoted by V^L.

Lemma 2.9. *V^L is an H-subcomodule of V.*

Proof. Let $x \in V^L$. We have to show that $\rho(x) \in V^L \otimes H$, i.e., that $a \cdot x_{(0)} \otimes x_{(1)} = 0$ for all $a \in L$. We know that $a_{(0)} \cdot x \otimes a_{(1)} = 0$ for all $a \in L$. Applying $\rho \otimes id$ to this equation, we obtain $(a_{(0)} \cdot x)_{(0)} \otimes (a_{(0)}x)_{(1)} \otimes a_{(1)} = 0$, which can be rewritten as $a_{(0)} \cdot x_{(0)} \otimes a_{(1)}x_{(1)} \otimes a_{(2)} = 0$. This implies $a_{(0)} \cdot x_{(0)} \otimes S(a_{(1)}x_{(1)})a_{(2)} = 0$, which yields $a \cdot x_{(0)} \otimes Sx_{(1)} = 0$. Since S is bijective, the proof is complete. □

3. Engel's Theorem

In this section we prove (simultaneously) Theorems 1.5 and 1.6. The proof follows the general outline of the standard proof of Engel's Theorem for Lie algebras.

Definition 3.1. Let A be an algebra. A subspace $U \subset A$ is said to be *nilpotent of degree n* if $U^n = 0$, but $U^{n-1} \neq 0$. (Here we use the standard notation $UW = \{\sum_i u_i w_i \mid u_i \in U,\, w_i \in W\}$ for subspaces $U, W \subset A$.)

The following lemma is crucial for the proof.

Lemma 3.2. *Let A be a (finite-dimensional) associative algebra in $\mathcal{M}^H$. Let $[A]_\beta$ be the corresponding (H,β)-Lie algebra and* ad *its adjoint representation. Let $U \subset A$ be an H-subcomodule. If U is nilpotent of degree n, then* $\mathrm{ad}\, U \subset \mathrm{End}\,(A)$ *is nilpotent of degree at most $2n-1$.*

Proof. Let $a, b, \ldots \in U$ and $x \in A$. Then we compute

$$\begin{aligned}
(\mathrm{ad}\, a)x &= ax - \beta(a_{(1)}, x_{(1)})x_{(0)}a_{(0)},\\
(\mathrm{ad}\, b)(\mathrm{ad}\, a)x &= (\mathrm{ad}\, b)\big(ax - \beta(a_{(1)}, x_{(1)})x_{(0)}a_{(0)}\big)\\
&= bax - \beta(b_{(1)}, a_{(1)}x_{(1)})a_{(0)}x_{(0)}b_{(0)}\\
&\quad -\beta(a_{(1)}, x_{(1)})bx_{(0)}a_{(0)} + \beta(a_{(2)}, x_{(2)})\beta(b_{(1)}, x_{(1)}a_{(1)})x_{(0)}a_{(0)}b_{(0)},
\end{aligned}$$

and so on. Since U is a subcomodule, we see by induction that the expression for $\ldots(\mathrm{ad}\, b)(\mathrm{ad}\, a)(x)$, with k operators applied to x, is a linear combination of terms each of which is the product of one element of A and k elements of U, written in some order. Since U is nilpotent of degree n, we observe that, for $k = 2n-1$, every term vanishes, which proves that $\mathrm{ad}\, U$ is nilpotent of degree at most $2n-1$. □

Now we are ready to prove Theorems 1.5 and 1.6. Let (H,β) be a cotriangular Hopf algebra and $V \neq 0$ a finite-dimensional H-comodule. Suppose that we have a representation φ of an (H,β)-Lie algebra L on V. We assume either that each cyclic (i.e., generated by a single element) subcomodule of $\varphi(L)$ is nilpotent or that

H is cosemisimple and each simple subcomodule of $\varphi(L)$ is nilpotent. We have to show that $V^L \neq 0$ (recall that by Lemma 2.9 it is a subcomodule).

Replacing L with $\varphi(L)$, we may assume without loss of generality that $L \subset \mathfrak{gl}_\beta(V)$ and φ is the imbedding. In particular, $\dim L < \infty$. We proceed by induction on $\dim L$. For $\dim L = 0$, there is nothing to prove, so assume that $\dim L > 0$ and the conclusion of the theorems holds for (H, β)-Lie algebras of smaller dimension.

Let M be a maximal proper H-comodule subalgebra of L. Restricting the adjoint action of L, we obtain a representation $M \to \mathfrak{gl}_\beta(L)$. Since $M \subset L$ is an M-submodule, we can pass to the quotient and obtain a representation $\sigma : M \to \mathfrak{gl}_\beta(L/M)$ defined by $\sigma(a)(x + M) = [a, x] + M$ for $a \in M$ and $x \in L$. Now by Lemma 3.2 applied to $A = \mathrm{End}\,(V)$, we have that $\mathrm{ad}\,(U) \subset \mathrm{End}\,(\mathrm{End}\,(V))$ is nilpotent either for every cyclic subcomodule $U \subset L$ or, in the case of cosemisimple H, for every simple subcomodule $U \subset L$. In particular, this holds for $U \subset M$. Since the operators in $\sigma(U)$ are obtained from the operators in $\mathrm{ad}\,(U)$ by restricting to $L \subset \mathrm{End}\,(V)$ and then passing to the quotient L/M, we conclude that $\sigma(U)$ is nilpotent either for every cyclic subcomodule $U \subset M$ or, in the case of cosemisimple H, for every simple subcomodule $U \subset M$. Obviously, every cyclic subcomodule of $\sigma(M)$ is the image of a cyclic subcomodule of M and, in the case of cosemisimple H, every simple subcomodule of $\sigma(M)$ is the image of a simple subcomodule of M (by semisimplicity of H-comodules). Therefore, we can apply the induction hypothesis to M and $\sigma : M \to \mathfrak{gl}_\beta(L/M)$ to conclude that $(L/M)^M \neq 0$. Pick $0 \neq \bar{a} \in (L/M)^M$ and write $\bar{a} = a + M$, $a \in L$. Then $a \notin M$ and $[M, \langle a\rangle] \subset M$. Note that, in the case of cosemisimple H, $\bar{a}$ and a can be chosen so that $\langle a\rangle$ is a simple subcomodule of L. By the conditions on L, $\langle a\rangle$ is nilpotent.

Let $\mathrm{alg}\,\langle a\rangle$ be the subalgebra of L generated by $\langle a\rangle$, i.e., the span of all words, with all possible bracket arrangements, in the elements of $\langle a\rangle$. Let A_k be the span of the words of length k, $k \geq 1$, so $\mathrm{alg}\,\langle a\rangle = \sum_k A_k$. One checks by induction on k that A_k is a subcomodule of L. It follows that $\mathrm{alg}\,\langle a\rangle$ and $M + \mathrm{alg}\,\langle a\rangle$ are also subcomodules of L. Then one shows, by induction on k and using identities (1) and (3), that $[M, A_k] \subset M$ and $[A_k, M] \subset M$. It follows that $[M, \mathrm{alg}\,\langle a\rangle] \subset M$ and $[\mathrm{alg}\,\langle a\rangle, M] \subset M$. Hence $M + \mathrm{alg}\,\langle a\rangle$ is an H-comodule subalgebra of L and M is an H-comodule ideal of $M + \mathrm{alg}\,\langle a\rangle$. By maximality of M we conclude that $M + \mathrm{alg}\,\langle a\rangle = L$.

Let $W = V^M$. Applying the induction hypothesis to M and $\varphi : M \hookrightarrow \mathfrak{gl}_\beta(V)$, we conclude that $W \neq 0$. Since M is an H-comodule ideal of L, we see that W is an L-submodule. Indeed, for any $b \in L$ and $w \in W$, we have

$$m(bw) = [m, b]w + \beta(m_{(1)}, b_{(1)})b_{(0)}(m_{(0)}w) = 0 \qquad \forall m \in M,$$

which means that $bw \in W$. Now recall that $\langle a\rangle$ is a nilpotent subspace of $\mathrm{End}\,(V)$. Let n be the smallest positive integer such that $\langle a\rangle^n W = 0$. Set $W' = \langle a\rangle^{n-1}W$ (it is understood that $W' = W$ if $n = 1$). Then $W' \neq 0$ and $\langle a\rangle W' = 0$. It follows that $\mathrm{alg}\,\langle a\rangle W' = 0$. Since $L = M + \mathrm{alg}\,\langle a\rangle$, we conclude that $W' \subset V^L$.

The proof of Theorems 1.5 and 1.6 is complete. □

Remark 3.3. If we specialize to $H = \Bbbk G$, the group algebra of an abelian group, the Theorem 1.6 becomes Theorem 1.2 (since in this case any simple subcomodule is spanned by a homogeneous element) — at least for $\dim V < \infty$. In fact, one can use Remark 2.6 to replace the hypothesis $\dim V < \infty$ in Theorems 1.5 and 1.6 by a slightly weaker condition $\dim \varphi(L) < \infty$.

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