

CLASSIFICATION OF GROUP GRADINGS ON SIMPLE LIE ALGEBRAS OF TYPES A, B, C AND D

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ABSTRACT. For a given abelian group G , we classify the isomorphism classes of G -gradings on the simple Lie algebras of types \mathcal{A}_n ($n \geq 1$), \mathcal{B}_n ($n \geq 2$), \mathcal{C}_n ($n \geq 3$) and \mathcal{D}_n ($n > 4$), in terms of numerical and group-theoretical invariants. The ground field is assumed to be algebraically closed of characteristic different from 2.

1. INTRODUCTION

Let U be an algebra (not necessarily associative) over a field \mathbb{F} and let G be an abelian group, written multiplicatively.

Definition 1.1. A G -grading on U is a vector space decomposition

$$U = \bigoplus_{g \in G} U_g$$

such that

$$U_g U_h \subset U_{gh} \text{ for all } g, h \in G.$$

U_g is called the *homogeneous component* of degree g . The *support* of the G -grading is the set

$$\{g \in G \mid U_g \neq 0\}.$$

Definition 1.2. We say that two G -gradings, $U = \bigoplus_{g \in G} U_g$ and $U = \bigoplus_{g \in G} U'_g$, are *isomorphic* if there exists an algebra automorphism $\psi : U \rightarrow U$ such that

$$\psi(U_g) = U'_g \text{ for all } g \in G,$$

i.e., $U = \bigoplus_{g \in G} U_g$ and $U = \bigoplus_{g \in G} U'_g$ are isomorphic as G -graded algebras.

The purpose of this paper is to classify, for a given abelian group G , the isomorphism classes of G -gradings on the classical simple Lie algebras of types \mathcal{A}_n ($n \geq 1$), \mathcal{B}_n ($n \geq 2$), \mathcal{C}_n ($n \geq 3$) and \mathcal{D}_n ($n > 4$), in terms of numerical and group-theoretical invariants. Descriptions of such gradings were obtained in [4, 8, 5, 2, 1], but the question of distinguishing non-isomorphic gradings was not addressed in those papers. Also, A. Elduque [13] has recently found a counterexample to [8, Proposition 6.4], which was used in the description of gradings on Lie algebras of type \mathcal{A} . The fine gradings (i.e., those that cannot be refined) on Lie algebras of types \mathcal{A} , \mathcal{B} , \mathcal{C} and \mathcal{D} (including \mathcal{D}_4) have been classified, up to equivalence, in [13]

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over algebraically closed fields of characteristic zero. For a discussion of the difference between classification up to equivalence and classification up to isomorphism see [16]. The two kinds of classification cannot be easily obtained from each other.

We will assume throughout this paper that the ground field \mathbb{F} is algebraically closed. We will usually assume that $\text{char } \mathbb{F} \neq 2$ and in one case also $\text{char } \mathbb{F} \neq 3$. We obtain a description of gradings in type \mathcal{A} without using [8, Proposition 6.4] and with methods simpler than those in [2, 1]. We also obtain invariants that allow us to distinguish among non-isomorphic gradings in types \mathcal{A} , \mathcal{B} , \mathcal{C} and \mathcal{D} .

The paper is structured as follows. In Section 2 we recall the description of G -gradings on a matrix algebra $R = M_n(\mathbb{F})$ and determine when two such gradings are isomorphic (Theorem 2.6). We also obtain a canonical form for an anti-automorphism of R that preserves the grading and restricts to an involution on the identity component R_e (Theorem 2.10). In particular, this allows us to classify (up to isomorphism) the pairs (R, φ) where $R = M_n(\mathbb{F})$ is G -graded and φ is an involution that preserves the grading (Corollary 2.15). In Section 3 we use affine group schemes to show how one can reduce the classification of G -gradings on classical simple Lie algebras to the classification of G -gradings on $R = M_n(\mathbb{F})$ and of the pairs (R, φ) where φ is an involution or an anti-automorphism satisfying certain properties. In Section 4 we obtain a classification of G -gradings on simple Lie algebras of type \mathcal{A} — see Theorem 4.9. Finally, in Section 5 we state a classification of G -gradings on simple Lie algebras of types \mathcal{B} , \mathcal{C} and \mathcal{D} (except \mathcal{D}_4) — see Theorem 5.2, which is an immediate consequence of Corollary 2.15.

2. GRADINGS ON MATRIX ALGEBRAS

Let $R = M_n(\mathbb{F})$ where \mathbb{F} is an algebraically closed field of arbitrary characteristic. Let G be an abelian group. A description of G -gradings on R was obtained in [3, 7, 6]. In this section we restate that description in a slightly different form and obtain invariants that allow us to distinguish among non-isomorphic gradings. Criteria for isomorphism of the so-called “elementary” gradings (see below) on matrix algebras $M_n(\mathbb{F})$ and on the algebra of finitary matrices were obtained in [11] and [9], respectively.

We start with gradings $R = \bigoplus_{g \in G} R_g$ with the property $\dim R_g \leq 1$ for all $g \in G$. As shown in the proof of [3, Theorem 5], R is then a graded division algebra, i.e., any nonzero homogeneous element is invertible in R . Consequently, the support $T \subset G$ of the grading is a subgroup. Following [13], we will call such $R = \bigoplus_{g \in G} R_g$ a *division grading* (the terms used in [3, 7, 6] and in [14] are “fine gradings” and “Pauli gradings”, respectively). Note that since $R \cong \mathbb{F}^\sigma T$ is semisimple, $\text{char } \mathbb{F}$ does not divide $n^2 = |T|$.

For each $t \in T$, let X_t be a nonzero element in the component R_t . Then

$$X_u X_v = \sigma(u, v) X_{uv}$$

for some nonzero scalar $\sigma(u, v)$. Clearly, the function $\sigma : T \times T \rightarrow \mathbb{F}^\times$ is a 2-cocycle, and the G -graded algebra R is isomorphic to the twisted group algebra $\mathbb{F}^\sigma T$ (with its natural T -grading regarded as a G -grading). Rescaling the elements X_t corresponds to replacing σ with a cohomologous cocycle. Let

$$\beta_\sigma(u, v) := \frac{\sigma(u, v)}{\sigma(v, u)}.$$

Then $\beta = \beta_\sigma$ depends only on the class of σ in $H^2(T, \mathbb{F}^\times)$ and $\beta : T \times T \rightarrow \mathbb{F}^\times$ is an *alternating bicharacter*, i.e., it is multiplicative in each variable and has the property $\beta(t, t) = 1$ for all $t \in T$.

Clearly, $X_u X_v = \beta(u, v) X_v X_u$. Since the centre $Z(R)$ is spanned by the identity element, β is *nondegenerate* in the sense that $\beta(u, t) = 1$ for all $u \in T$ implies $t = e$. Conversely, if σ is a 2-cocycle such that β_σ is nondegenerate, then $\mathbb{F}^\sigma T$ is a semisimple associative algebra whose centre is spanned by the identity element, so $\mathbb{F}^\sigma T$ is isomorphic to R . Therefore, the isomorphism classes of division G -gradings on $R = M_n(\mathbb{F})$ with support $T \subset G$ are in one-to-one correspondence with the classes $[\sigma] \in H^2(T, \mathbb{F}^\times)$ such that β_σ is nondegenerate.

The classes $[\sigma]$ and the corresponding gradings on R can be found explicitly as follows. As shown in the proof of [3, Theorem 5], there exists a decomposition of T into the direct product of cyclic subgroups:

$$(1) \quad T = H'_1 \times H''_1 \times \cdots \times H'_r \times H''_r$$

such that $H'_i \times H''_i$ and $H'_j \times H''_j$ are β -orthogonal for $i \neq j$, and H'_i and H''_i are in duality by β . Denote by ℓ_i the order of H'_i and H''_i . If we pick generators a_i and b_i for H'_i and H''_i , respectively, then $\varepsilon_i := \beta(a_i, b_i)$ is a primitive ℓ_i -th root of unity, and all other values of β on the elements $a_1, b_1, \dots, a_r, b_r$ are 1. Pick elements $X_{a_i} \in R_{a_i}$ and $X_{b_i} \in R_{b_i}$ such that $X_{a_i}^{\ell_i} = X_{b_i}^{\ell_i} = 1$. Then we obtain an isomorphism $\mathbb{F}^\sigma T \rightarrow M_{\ell_1}(\mathbb{F}) \otimes \cdots \otimes M_{\ell_r}(\mathbb{F})$ defined by

$$(2) \quad X_{a_i} \mapsto I \otimes \cdots I \otimes X_i \otimes I \otimes \cdots I \quad \text{and} \quad X_{b_i} \mapsto I \otimes \cdots I \otimes Y_i \otimes I \otimes \cdots I,$$

where

$$(3) \quad X_i = \begin{bmatrix} \varepsilon_i^{n-1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & \varepsilon_i^{n-2} & 0 & \cdots & 0 & 0 \\ \cdots & & & & & \\ 0 & 0 & 0 & \cdots & \varepsilon_i & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \quad \text{and} \quad Y_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & & & & & \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

are in the i -th factor, $M_{\ell_i}(\mathbb{F})$.

It follows that the class $[\sigma] \in H^2(T, \mathbb{F}^\times)$, and hence the isomorphism class of the G -graded algebra $\mathbb{F}^\sigma T$, is uniquely determined by $\beta = \beta_\sigma$. Conversely, since the relation $X_u X_v = \beta(u, v) X_v X_u$ does not change when we rescale X_u and X_v , the values of β are determined by the G -grading. We summarize our discussion in the following

Proposition 2.1. *There exist division G -gradings on $R = M_n(\mathbb{F})$ with support $T \subset G$ if and only if $\text{char } \mathbb{F}$ does not divide n and $T \cong \mathbb{Z}_{\ell_1}^2 \times \cdots \times \mathbb{Z}_{\ell_r}^2$ where $\ell_1 \cdots \ell_r = n$. The isomorphism classes of division G -gradings with support T are in one-to-one correspondence with nondegenerate alternating bicharacters $\beta : T \times T \rightarrow \mathbb{F}^\times$. \square*

We also note that taking

$$X_{(a_1^{i_1}, b_1^{j_1}, \dots, a_r^{i_r}, b_r^{j_r})} = X_{a_1}^{i_1} X_{b_1}^{j_1} \cdots X_{a_r}^{i_r} X_{b_r}^{j_r},$$

we obtain a representative of the cohomology class $[\sigma]$ that is multiplicative in each variable, i.e., it is a bicharacter (not alternating unless T is the trivial subgroup). In what follows, we will always assume that σ is chosen in this way.

Definition 2.2. A concrete representative of the isomorphism class of division G -graded algebras with support T and bicharacter β can be obtained as follows.

First decompose T as in (1) and pick generators $a_1, b_1, \dots, a_r, b_r$. Then define a grading on $M_{\ell_i}(\mathbb{F})$ by declaring that X_i has degree a_i and Y_i has degree b_i , where X_i and Y_i are given by (3) and $\varepsilon_i = \beta(a_i, b_i)$. Then $M_{\ell_1}(\mathbb{F}) \otimes \dots \otimes M_{\ell_r}(\mathbb{F})$ with tensor product grading is a representative of the desired class. We will call any representative obtained in this way a *standard realization*.

If R has a division grading, then its structure is quite rigid. Any automorphism of the graded algebra R must send X_t to a scalar multiple of itself, hence it is given by $X_t \mapsto \lambda(t)X_t$ where $\lambda : T \rightarrow \mathbb{F}^\times$ is a character of T . Since β is nondegenerate, it establishes an isomorphism between T and \widehat{T} . It follows that the automorphism of R corresponding to λ is given by $X \mapsto X_t^{-1}XX_t$ where $t \in T$ is determined by $\beta(u, t) = \lambda(u)$ for all $u \in T$.

It follows from [8, Lemma 6.1] that the graded algebra R admits anti-automorphisms only when T is an elementary 2-group (and hence $\text{char } \mathbb{F} \neq 2$ or T is trivial). In this case, we can regard T as a vector space over the field of order 2 and think of $\sigma(u, v)$ as a bilinear form on T (recall that σ is chosen so that it is a bicharacter). Hence $\sigma(t, t)$ is a quadratic form, and $\beta(u, v)$ is the polar bilinear form for $\sigma(t, t)$. Note that $\sigma(t, t)$ depends on the choice of σ , so it is not an invariant of the graded algebra R . In fact, any quadratic form with polar form $\beta(u, v)$ can be achieved by changing generators a_i, b_i in the i -th copy of \mathbb{Z}_2^2 . However, once we fix a standard realization of R , $\sigma(t, t)$ is uniquely determined. Following the usual convention regarding quadratic forms, we will denote $\sigma(t, t)$ by $\beta(t)$ so that $\beta(u, v) = \beta(uv)\beta(u)\beta(v)$. Note that

$$(4) \quad X^\beta = \beta(u)X \quad \text{for all } X \in R_u, u \in T$$

is an involution of the graded algebra R . Hence any anti-automorphism of the graded algebra R is given by $X \mapsto X_t^{-1}X^\beta X_t$ for a suitable $t \in T$. In the standard realization of R as $M_2(\mathbb{F})^{\otimes r}$, the involution β is given by matrix transpose on each slot of the tensor power. We summarize the above discussion for future reference:

Proposition 2.3. *Suppose $R = M_n(\mathbb{F})$ has a division G -grading with support $T \subset G$ and bicharacter β . Then the mapping that sends $t \in T$ to the inner automorphism $X \mapsto X_t^{-1}XX_t$ is an isomorphism between T and the group of automorphisms $\text{Aut}_G(R)$ of the graded algebra R . The graded algebra R admits anti-automorphisms if and only if T is an elementary 2-group. If this is the case, then, in any standard realization of R , the mapping $X \mapsto {}^tX$ is an involution of the graded algebra R . This involution can be written in the form (4), where $\beta : T \rightarrow \{\pm 1\}$ is a quadratic form. The bicharacter $\beta(u, v)$ is the polar bilinear form associated to β . The group $\overline{\text{Aut}}_G(R)$ of automorphisms and anti-automorphisms of the graded algebra R is equal to $\text{Aut}_G(R) \times \langle \beta \rangle$. In particular, any anti-automorphism of the graded algebra R is an involution, given by $X \mapsto X_t^{-1}X^\beta X_t$ for a uniquely determined $t \in T$. \square*

We now turn to general G -gradings on R . As shown in [3, 7, 6], there exist graded unital subalgebras C and D in R such that $D \cong M_\ell(\mathbb{F})$ has a division grading, $C \cong M_k(\mathbb{F})$ has an *elementary grading* given by a k -tuple (g_1, \dots, g_k) of elements of G :

$$C_g = \text{Span} \{E_{ij} \mid g_i^{-1}g_j = g\} \quad \text{for all } g \in G,$$

where E_{ij} is a basis of matrix units in C , and we have an isomorphism $C \otimes D \rightarrow R$ given by $c \otimes d \mapsto cd$. Moreover, the intersection of the support $\{g_i^{-1}g_j\}$ of the grading on C and the support T of the grading on D is equal to $\{e\}$.

Without loss of generality, we may assume that the k -tuple has the form

$$(g_1^{(k_1)}, \dots, g_s^{(k_s)})$$

where the elements g_1, \dots, g_s are pairwise distinct and we write $g^{(q)}$ for $\underbrace{g, \dots, g}_{q \text{ times}}$.

It is important to note that the subalgebras C and D are not uniquely determined. We are now going to obtain invariants of the graded algebra R . The partition $k = k_1 + \dots + k_s$ gives a block decomposition of C . Let e_i be the block-diagonal matrix $\text{diag}(0, \dots, I_{k_i}, \dots, 0)$ where I_{k_i} is in the i -th position, $i = 1, \dots, s$. Consider the Peirce decomposition of C corresponding to the orthogonal idempotents e_1, \dots, e_s : $C_{ij} = e_i C e_j$. We will write C_i instead of C_{ii} for brevity. Then the identity component is

$$R_e = C_1 \otimes I \oplus \dots \oplus C_s \otimes I.$$

It follows that the idempotents e_1, \dots, e_s and the (non-unital) subalgebras C_1, \dots, C_s of R are uniquely determined (up to permutation). It is easy to verify that the centralizer of R_e in R is equal to $e_1 \otimes D \oplus \dots \oplus e_s \otimes D$. Hence the (non-unital) subalgebras $D_i := e_i \otimes D$ of R are uniquely determined (up to permutation). All D_i are isomorphic to D as G -graded algebras, so the isomorphism class of D is uniquely determined. This gives us invariants T and β according to Proposition 2.1. However, there is no canonical way to choose the isomorphisms of D with D_i . According to Proposition 2.3, the possible choices are parameterized by $t_i \in T$, $i = 1, \dots, s$. If we fix isomorphisms $\eta_i : D \rightarrow D_i$, then each Peirce component $R_{ij} = e_i R e_j$ becomes a D -bimodule by setting $d \cdot r = \eta_i(d)r$ and $r \cdot d = r\eta_j(d)$ for all $d \in D$ and $r \in R_{ij}$. Taking $\eta_i(d) = e_i \otimes d$ for all $d \in D$, we recover the subspaces C_{ij} for $i \neq j$ as the centres of these bimodules:

$$C_{ij} = \{r \in R_{ij} \mid d \cdot r = r \cdot d \text{ for all } d \in D\}.$$

Also, the subalgebra D of R can be identified:

$$D = \{\eta_1(d) + \dots + \eta_s(d) \mid d \in D\}.$$

If we replace η_i by $\eta'_i(d) = \eta_i(X_{t_i}^{-1} d X_{t_i})$, then we get $C'_{ij} = \eta_i(X_{t_i}^{-1}) C_{ij} \eta_j(X_{t_j})$. Let $C' = C_1 \oplus \dots \oplus C_s \oplus \bigoplus_{i \neq j} C'_{ij}$ and $D' = \{\eta'_1(d) + \dots + \eta'_s(d) \mid d \in D\}$. Then C' and D' are graded unital subalgebras of R . Let $\Psi = e_1 \otimes X_{t_1} + \dots + e_s \otimes X_{t_s}$. Then Ψ is an invertible matrix and the mapping $\psi(X) = \Psi^{-1} X \Psi$ is an automorphism of the (ungraded) algebra R that sends C to C' and D to D' . The restriction of ψ to D preserves the grading, whereas the restriction of ψ to C sends homogeneous elements of degree $g_i^{-1} g_j$ to homogeneous elements of degree $t_i^{-1} g_i^{-1} g_j t_j$ (i.e., “shifts” the grading in the (i, j) -th Peirce components by $t_i^{-1} t_j$). We conclude that the G -grading of R associated to the k -tuple $(g_1^{(k_1)}, \dots, g_s^{(k_s)})$ is isomorphic to the G -grading associated to the k -tuple $((g_1 t_1)^{(k_1)}, \dots, (g_s t_s)^{(k_s)})$. Finally, we note that the cosets $g_i^{-1} g_j T$ are uniquely determined by the G -graded algebra R , because they are the supports of the grading on the Peirce components R_{ij} ($i \neq j$). We have obtained an irredundant classification of G -gradings on R .

To state the result precisely, we introduce some notation. Let

$$\kappa = (k_1, \dots, k_s) \quad \text{where } k_i \text{ are positive integers.}$$

We will write $|\kappa|$ for $k_1 + \dots + k_s$ and e_i , $i = 1, \dots, s$, for the orthogonal idempotents in $M_{|\kappa|}(\mathbb{F})$ associated to the block decomposition determined by κ . Let

$$\gamma = (g_1, \dots, g_s) \quad \text{where } g_i \in G \text{ are such that } g_i^{-1}g_j \notin T \text{ for all } i \neq j.$$

Definition 2.4. We will write $(\kappa, \gamma) \sim (\tilde{\kappa}, \tilde{\gamma})$ if κ and $\tilde{\kappa}$ have the same number of components s and there exist an element $g \in G$ and a permutation π of the symbols $\{1, \dots, s\}$ such that $\tilde{k}_i = k_{\pi(i)}$ and $\tilde{g}_i \equiv g_{\pi(i)}g \pmod{T}$, for all $i = 1, \dots, s$.

Definition 2.5. Let D be a standard realization of division G -graded algebra with support $T \subset G$ and bicharacter β . Let κ and γ be as above. Let $C = M_{|\kappa|}(\mathbb{F})$. We endow the algebra $M_{|\kappa|}(D) = C \otimes D$ with a G -grading by declaring the degree of $U \otimes d$ to be $g_i^{-1}tg_j$ for all $U \in e_iCe_j$ and $d \in D_t$. We will denote this G -graded algebra by $\mathcal{M}(G, T, D, \kappa, \gamma)$. By abuse of notation, we will also write $\mathcal{M}(G, T, \beta, \kappa, \gamma)$, since the isomorphism class of D is uniquely determined by β .

Theorem 2.6. Let \mathbb{F} be an algebraically closed field of arbitrary characteristic. Let G be an abelian group. Let $R = \bigoplus_{g \in G} R_g$ be a grading of the matrix algebra $R = M_n(\mathbb{F})$. Then the G -graded algebra R is isomorphic to some $\mathcal{M}(G, T, \beta, \kappa, \gamma)$ where $T \subset G$ is a subgroup, $\beta : T \times T \rightarrow \mathbb{F}^\times$ is a nondegenerate alternating bicharacter, κ and γ are as above with $|\kappa|\sqrt{|T|} = n$. Two G -graded algebras $\mathcal{M}(G, T_1, \beta_1, \kappa_1, \gamma_1)$ and $\mathcal{M}(G, T_2, \beta_2, \kappa_2, \gamma_2)$ are isomorphic if and only if $T_1 = T_2$, $\beta_1 = \beta_2$ and $(\kappa_1, \gamma_1) \sim (\kappa_2, \gamma_2)$. \square

Remark 2.7. In fact, it follows from the above discussion that, for any permutation π as in Definition 2.4, there exists an isomorphism from $\mathcal{M}(G, T, \beta, \tilde{\kappa}, \tilde{\gamma})$ to $\mathcal{M}(G, T, \beta, \kappa, \gamma)$ that sends \tilde{e}_i to $e_{\pi(i)}$. We can construct such an isomorphism explicitly in the following way. Let $P = P_\pi$ be the block matrix with I_{k_i} in the $(i, \pi(i))$ -th positions and 0 elsewhere (i.e., the block-permutation matrix corresponding to π). Pick $t_i \in T$ such that $\tilde{g}_i = g_{\pi(i)}t_{\pi(i)}g$ and let B be the block-diagonal matrix $e_1 \otimes X_{t_1} + \dots + e_s \otimes X_{t_s}$. Then the map $X \mapsto (BP)X(BP)^{-1}$ has the desired properties. We will refer to isomorphisms of this type as *monomial*.

Let $\text{Sym}(s)$ be the group of permutations on $\{1, \dots, s\}$. Let $\text{Aut}(\kappa, \gamma)$ be the subgroup of $\text{Sym}(s)$ that consists of all π such that, for some $g \in G$, we have $k_i = k_{\pi(i)}$ and $g_i \equiv g_{\pi(i)}g \pmod{T}$ for all $i = 1, \dots, s$.

Proposition 2.8. The group of automorphisms $\text{Aut}_G(R)$ of the graded algebra $R = \mathcal{M}(G, T, \beta, \kappa, \gamma)$ is an extension of $\text{Aut}(\kappa, \gamma)$ by $\text{PGL}_\kappa(\mathbb{F}) \times \text{Aut}_G(D)$ where

$$\text{PGL}_\kappa(\mathbb{F}) = (\text{GL}_{\kappa_1}(\mathbb{F}) \times \dots \times \text{GL}_{\kappa_s}(\mathbb{F})) / \mathbb{F}^\times,$$

where \mathbb{F}^\times is identified with nonzero scalar matrices.

Proof. Any $\psi \in \text{Aut}_G(R)$ leaves the identity component R_e invariant and hence permutes the idempotents e_1, \dots, e_s . This gives a homomorphism $f : \text{Aut}_G(R) \rightarrow \text{Sym}(s)$. Looking at the supports of the Peirce components, we see that $f(\psi) \in \text{Aut}(\kappa, \gamma)$. Conversely, any element of $\text{Aut}(\kappa, \gamma)$ is in $\text{im } f$ by Remark 2.7, since it comes from a monomial automorphism of the graded algebra R . Finally, any $\psi \in \ker f$ leaves C_i and D_i invariant and hence is given by $\psi(X) = \Psi^{-1}X\Psi$ where $\Psi = B_1 \otimes Q_1 \oplus \dots \oplus B_s \otimes Q_s$ for some $B_i \in \text{GL}_{\kappa_i}(\mathbb{F})$ and $Q_i \in D$. In view of Proposition 2.3, we may assume that $Q_i = X_{t_i}$ for some $t_i \in T$. It is easy to see that ψ preserves the grading if and only if $t_1 = \dots = t_s$. The result follows. \square

In order to classify gradings on Lie algebras of types \mathcal{B} , \mathcal{C} and \mathcal{D} , we will need to study involutions on G -graded matrix algebras. A description of such involutions was given in [5]. Here we will slightly simplify that description and obtain invariants that will allow us to distinguish among isomorphism classes. We start with a more general situation, which we will need for the classification of gradings in type \mathcal{A} .

Definition 2.9. Let G be an abelian group and let $U = \bigoplus_{g \in G} U_g$ be a G -graded algebra. We will say that an anti-automorphism φ of U is *compatible* with the grading if $\varphi(U_g) = U_g$ for all $g \in G$. If U_1 and U_2 are G -graded algebras and φ_1 and φ_2 are anti-automorphisms on U_1 and U_2 , respectively, compatible with the grading, then we will say that (U, φ_1) and (U, φ_2) are *isomorphic* if there exists an isomorphism $\psi : U_1 \rightarrow U_2$ of G -graded algebras such that $\varphi_1 = \psi^{-1} \varphi_2 \psi$.

Suppose $R = M_n(\mathbb{F})$ is G -graded and there exists an anti-automorphism φ compatible with the grading and such that $\varphi^2|_{R_e} = \text{id}$. Then φ leaves some of the components of R_e invariant and swaps the remaining components in pairs. Without loss of generality, we may assume that e_i are φ -invariant for $i = 1, \dots, m$ and not φ -invariant for $i > m$. It will be convenient to change the notation and write $e'_{m+1}, e''_{m+1}, \dots, e'_k, e''_k$ so that φ swaps e'_i and e''_i for $i > m$. (Thus the total number of orthogonal idempotents in question is $2k - m$.) It will also be convenient to distinguish φ -invariant idempotents of even and odd rank. Thus we assume that e_1, \dots, e_ℓ have odd rank and $e_{\ell+1}, \dots, e_m$ have even rank. We will change the notation for κ and γ accordingly:

$$(5) \quad \kappa = (q_1, \dots, q_\ell, 2q_{\ell+1}, \dots, 2q_m, q_{m+1}, q_{m+1} \dots, q_k, q_k)$$

where q_i are positive integers with q_1, \dots, q_ℓ odd, and

$$(6) \quad \gamma = (g_1, \dots, g_\ell, g_{\ell+1}, \dots, g_m, g'_{m+1}, g''_{m+1}, \dots, g'_k, g''_k)$$

where $g_i \in G$ are such that $g_i^{-1} g_j \notin T$ for all $i \neq j$.

As shown in [8, 5], the existence of the anti-automorphism φ places strong restrictions on the G -grading. First of all, note that the centralizer of R_e in R , which is equal to $D_1 \oplus \dots \oplus D_m \oplus D'_{m+1} \oplus D''_{m+1} \oplus \dots \oplus D'_k \oplus D''_k$, is φ -invariant. Since e_1, \dots, e_m are φ -invariant and belong to D_1, \dots, D_m , respectively, we see that D_1, \dots, D_m are also φ -invariant. By a similar argument, φ swaps D'_i and D''_i for $i > m$. Each of the D_i , D'_i and D''_i is an isomorphic copy of D , so we see that D admits an anti-automorphism. By Proposition 2.3, T must be an elementary 2-group and we have a standard realization $D \cong M_2(\mathbb{F})^{\otimes r}$.

Since φ preserves the G -grading and $\varphi(e_i R e_j) = e_j R e_i$ for $i, j \leq m$, the supports of these two Peirce components must be equal, which gives $g_i^{-1} g_j \equiv g_j^{-1} g_i \pmod{T}$ for $i, j \leq m$. Similarly, $\varphi(e'_i R e''_j) = e''_j R e'_i$ implies $(g'_i)^{-1} g''_j \equiv (g''_j)^{-1} g'_i \pmod{T}$ for $i, j > m$. Also, $\varphi(e_i R e'_j) = e''_j R e_i$ implies $g_i^{-1} g'_j \equiv (g''_j)^{-1} g_i \pmod{T}$ for $i \leq m$ and $j > m$. These conditions can be summarized as follows:

$$(7) \quad g_1^2 \equiv \dots \equiv g_m^2 \equiv g'_{m+1} g''_{m+1} \equiv \dots \equiv g'_k g''_k \pmod{T}.$$

If γ satisfies (7), then we have

$$g_1^2 t_1 = \dots = g_m^2 t_m = g'_{m+1} g''_{m+1} t_{m+1} = \dots = g'_k g''_k t_k$$

for some $t_1, \dots, t_k \in T$. We can replace the G -grading by an isomorphic one so that γ satisfies

$$(8) \quad g_1^2 t_1 = \dots = g_m^2 t_m = g'_{m+1} g''_{m+1} = \dots = g'_k g''_k.$$

Indeed, it suffices to replace g_i'' by $g_i''t_i$, $i = m+1, \dots, k$ (which does not change the cosets mod T).

Theorem 2.10. *Let \mathbb{F} be an algebraically closed field, $\text{char } \mathbb{F} \neq 2$. Let G be an abelian group. Let $R = \mathcal{M}(G, T, \beta, \kappa, \gamma)$. Assume that R admits an anti-automorphism φ that is compatible with the grading and satisfies $\varphi^2|_{R_e} = \text{id}$. Write κ and γ in the form (5) and (6), respectively. Then T is an elementary 2-group and γ satisfies (7). Up to an isomorphism of the pair (R, φ) , γ satisfies (8) for some $t_1, \dots, t_m \in T$ and φ is given by $\varphi(X) = \Phi^{-1}({}^tX)\Phi$ for all $X \in R$, where matrix Φ has the following block-diagonal form:*

$$(9) \quad \Phi = \sum_{i=1}^{\ell} I_{q_i} \otimes X_{t_i} \oplus \sum_{i=\ell+1}^m S_i \otimes X_{t_i} \oplus \sum_{i=m+1}^k \begin{pmatrix} 0 & I_{q_i} \\ \mu_i I_{q_i} & 0 \end{pmatrix} \otimes I$$

where, for $i = \ell+1, \dots, m$, each S_i is either I_{2q_i} or $\begin{pmatrix} 0 & I_{q_i} \\ -I_{q_i} & 0 \end{pmatrix}$, and μ_{m+1}, \dots, μ_k are nonzero scalars.

Proof. There exists an invertible matrix Φ such that φ is given by $\varphi(X) = \Phi^{-1}({}^tX)\Phi$ for all $X \in R$. Recall that conjugating φ by the automorphism $\psi(X) = \Psi^{-1}X\Psi$ replaces matrix Φ by ${}^t\Psi\Phi\Psi$, i.e., Φ is transformed as the matrix of a bilinear form.

Recall that we fixed the idempotents

$$(10) \quad e_1, \dots, e_{\ell}, e_{\ell+1}, \dots, e_m, e'_{m+1}, e''_{m+1}, \dots, e'_k, e''_k.$$

It is also convenient to introduce $e_i = e'_i + e''_i$ for $i = m+1, \dots, k$.

Following the proof of [5, Lemma 6 and Proposition 1], we see that, up to an automorphism of the G -graded algebra R , Φ has the following block-diagonal form — in agreement with the idempotents given by (10):

$$\Phi = \sum_{i=1}^{\ell} S_i Y_i \otimes Q_i \oplus \sum_{i=\ell+1}^m S_i Y_i \otimes Q_i \oplus \sum_{i=m+1}^k S_i Y_i \otimes Q_i.$$

(This is formula (20) of just cited paper, rewritten according to our present notation.) For $i = 1, \dots, m$, the matrix Y_i is in the centralizer of the simple algebra C_i , i.e., has the form $Y_i = \xi_i I_{q_i}$. For $i = m+1, \dots, k$, the matrix Y_i is in the centralizer of the semisimple algebra $C'_i \oplus C''_i$, i.e., has the form $Y_i = \text{diag}(\eta_i I_{q_i}, \xi_i I_{q_i})$. Each Q_i is in D_i , and the map $X \mapsto Q_i^{-1}({}^tX)Q_i$ is an anti-automorphism of D . Hence, by Proposition 2.3, each Q_i is, up to a scalar multiple, of the form X_{t_i} , for an appropriate choice of $t_i \in T$. The scalar can be absorbed in Y_i . Finally, the matrix S_i is I_{q_i} for $i = 1, \dots, \ell$, either I_{2q_i} or $\begin{pmatrix} 0 & I_{q_i} \\ -I_{q_i} & 0 \end{pmatrix}$ for $i = \ell+1, \dots, m$, and $\begin{pmatrix} 0 & I_{q_i} \\ I_{q_i} & 0 \end{pmatrix}$ for $i = m+1, \dots, k$. This allows us to rewrite the above formula as follows:

$$\Phi = \sum_{i=1}^{\ell} \xi_i I_{q_i} \otimes X_{t_i} \oplus \sum_{i=\ell+1}^m \xi_i S_i \otimes X_{t_i} \oplus \sum_{i=m+1}^k \begin{pmatrix} 0 & \xi_i I_{q_i} \\ \eta_i I_{q_i} & 0 \end{pmatrix} \otimes X_{t_i}.$$

Here ξ_i, η_i are some nonzero scalars. If we now apply the inner automorphism of the graded algebra R given by the matrix $P = \frac{1}{\sqrt{\xi_1}} e_1 \otimes I + \dots + \frac{1}{\sqrt{\xi_k}} e_k \otimes I$, then φ is transformed to the anti-automorphism given by the following matrix (which we again denote by Φ):

$$\Phi = \sum_{i=1}^{\ell} I_{q_i} \otimes X_{t_i} \oplus \sum_{i=\ell+1}^m S_i \otimes X_{t_i} \oplus \sum_{i=m+1}^k \begin{pmatrix} 0 & I_{q_i} \\ \mu_i I_{q_i} & 0 \end{pmatrix} \otimes X_{t_i},$$

for an appropriate set of nonzero scalars μ_{m+1}, \dots, μ_k . It can be easily verified (and is shown in the proof of [5, Theorem 3]) that t_1, \dots, t_k satisfy the following condition: $g_1^2 t_1 = \dots = g_m^2 t_m = g'_{m+1} g''_{m+1} t_{m+1} = \dots = g'_s g''_s t_s$.

Finally, the inner automorphism $\psi(X) = \Psi^{-1} X \Psi$ of R where

$$\Psi^{-1} = e_1 \otimes I + \dots + e_m \otimes I + e'_{m+1} \otimes I + e''_{m+1} \otimes X_{t_{m+1}} + \dots + e'_k \otimes I + e''_k \otimes X_{t_k}$$

sends the G -grading to the one given by

$$(g_1, \dots, g_m, g'_{m+1}, g''_{m+1} t_{m+1}, \dots, g'_k, g''_k t_k)$$

and transforms φ to the anti-automorphism given by a matrix of form (9). \square

If φ is an involution on R , then one can get rid of the parameters μ_{m+1}, \dots, μ_s , and the selection of $S_{\ell+1}, \dots, S_m$ is uniquely determined. Indeed, the matrix Φ is then either symmetric or skew-symmetric. In the first case, φ is called an *orthogonal* (or *transpose*) involution. In the second case, φ is called a *symplectic* involution. Set $\text{sgn}(\varphi) = 1$ if φ is orthogonal and $\text{sgn}(\varphi) = -1$ if φ is symplectic. Similarly, set $\text{sgn}(S_i) = 1$ if ${}^t S_i = S_i$ and $\text{sgn}(S_i) = -1$ if ${}^t S_i = -S_i$. We restate the main result of [5] in our notation (and setting $t_{m+1} = \dots = t_k = e$):

Theorem 2.11. [5, Theorem 3] *Under the conditions of Theorem 2.10, assume that $\varphi^2 = \text{id}$. Then, up to an isomorphism of the pair (R, φ) , γ satisfies (8) for some $t_1, \dots, t_m \in T$ and φ is given by $\varphi(X) = \Phi^{-1}({}^t X) \Phi$ for all $X \in R$, where matrix Φ has the following block-diagonal form:*

$$(11) \quad \Phi = \sum_{i=1}^{\ell} I_{q_i} \otimes X_{t_i} \oplus \sum_{i=\ell+1}^m S_i \otimes X_{t_i} \oplus \sum_{i=m+1}^k S_i \otimes I$$

where

- for $i = \ell + 1, \dots, m$, each S_i is either I_{2q_i} or $\begin{pmatrix} 0 & I_{q_i} \\ -I_{q_i} & 0 \end{pmatrix}$, and
- for $i = m + 1, \dots, k$, all S_i are either $\begin{pmatrix} 0 & I_{q_i} \\ I_{q_i} & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & I_{q_i} \\ -I_{q_i} & 0 \end{pmatrix}$

such that the following condition is satisfied:

$$(12) \quad \begin{aligned} \text{sgn}(\varphi) &= \beta(t_1) = \dots = \beta(t_\ell) \\ &= \beta(t_{\ell+1}) \text{sgn}(S_{\ell+1}) = \dots = \beta(t_m) \text{sgn}(S_m) \\ &= \text{sgn}(S_{m+1}) = \dots = \text{sgn}(S_k). \end{aligned}$$

Conversely, if γ satisfies (8) and condition (12) holds, then Φ defines an involution of the type indicated by $\text{sgn}(\varphi)$ on the G -graded algebra R . \square

It is convenient to introduce the following notation (for $m > 0$):

$$(13) \quad \tau = (t_1, \dots, t_m).$$

Note that for the elements t_1, \dots, t_m in (8), the ratios $t_i^{-1} t_j$ are uniquely determined by the cosets of $g_1, \dots, g_m \bmod T$, so it is sufficient to specify only one t_i to find τ .

Definition 2.12. We will say that γ is **-admissible* if it satisfies (7) and, for some $t_1, \dots, t_\ell \in T$, we have $g_1^2 t_1 = \dots = g_\ell^2 t_\ell$ and

$$(14) \quad \beta(t_1) = \dots = \beta(t_\ell).$$

(If $\ell \leq 1$, then condition (14) is automatically satisfied.)

Definition 2.13. Let $T \subset G$ be an elementary 2-group (of even rank) with a nondegenerate alternating bicharacter β . Suppose γ is $*$ -admissible, and γ and τ satisfy (8) and (14). If $\ell > 0$, let δ be the common value of $\beta(t_1), \dots, \beta(t_\ell)$. If $\ell = 0$, select $\delta \in \{\pm 1\}$ arbitrarily. Consider $R = \mathcal{M}(G, T, \beta, \kappa, \gamma)$. Let Φ be the matrix given by (11) where the matrices S_i are selected so that equation (12) holds with $\text{sgn}(\varphi) = \delta$. Then, by Theorem 2.11, $\varphi(X) = \Phi^{-1}(^tX)\Phi$ is an involution on R that is compatible with the grading. We will denote (R, φ) defined in this way by $\mathcal{M}^*(G, T, \beta, \kappa, \gamma, \tau, \delta)$. (Here τ is empty if $m = 0$.)

Definition 2.14. Referring to Definition 2.13, we will write $(\kappa, \gamma, \tau) \approx (\tilde{\kappa}, \tilde{\gamma}, \tilde{\tau})$ if κ and $\tilde{\kappa}$ have the same number of components of each type, i.e., the same values of ℓ , m and k , and there exist an element $g \in G$ and a permutation π of the symbols $\{1, \dots, k\}$ preserving the sets $\{1, \dots, \ell\}$, $\{\ell + 1, \dots, m\}$ and $\{m + 1, \dots, k\}$ such that $\tilde{q}_i = q_{\pi(i)}$ for all i , $\tilde{g}_i \equiv g_{\pi(i)}g \pmod{T}$ for all $i = 1, \dots, m$, $\{\tilde{g}'_i, \tilde{g}''_i\} \equiv \{g'_{\pi(i)}g, g''_{\pi(i)}g\} \pmod{T}$ for all $i = m + 1, \dots, k$, and

- if $m > 0$, then $\tilde{t}_i = t_{\pi(i)}$ for all $i = 1, \dots, m$;
- if $m = 0$, then $\tilde{g}'_i \tilde{g}''_i = g'_{\pi(i)} g''_{\pi(i)} g^2$ for some (and hence all) $i = 1, \dots, k$.

In the case $m = 0$, τ is empty, so we may write $(\kappa, \gamma) \approx (\tilde{\kappa}, \tilde{\gamma})$.

Corollary 2.15. Let $\text{char } \mathbb{F} \neq 2$ and $R = \mathcal{M}(G, T, \beta, \kappa, \gamma)$. Then the G -graded algebra R admits an involution if and only if T is an elementary 2-group and γ is $*$ -admissible. If φ is an involution on R , then (R, φ) is isomorphic to some $\mathcal{M}^*(G, T, \beta, \kappa, \gamma, \tau, \delta)$ where $\delta = \text{sgn}(\varphi)$. Two G -graded algebras with involution, $\mathcal{M}^*(G, T_1, \beta_1, \kappa_1, \gamma_1, \tau_1, \delta_1)$ and $\mathcal{M}^*(G, T_2, \beta_2, \kappa_2, \gamma_2, \tau_2, \delta_2)$, are isomorphic if and only if $T_1 = T_2$, $\beta_1 = \beta_2$, $(\kappa_1, \gamma_1, \tau_1) \approx (\kappa_2, \gamma_2, \tau_2)$ and $\delta_1 = \delta_2$.

Proof. The first two statements are a combination of Theorems 2.10 and 2.11. It remains to prove the last statement.

Let $R_1 = \mathcal{M}(G, T_1, \beta_1, \kappa_1, \gamma_1)$, $R_2 = \mathcal{M}(G, T_2, \beta_2, \kappa_2, \gamma_2)$ and let φ_1 and φ_2 be the corresponding involutions. Suppose $T_1 = T_2$, $\beta_1 = \beta_2$, and $(\kappa_1, \gamma_1, \tau_1) \approx (\kappa_2, \gamma_2, \tau_2)$. Then, by Theorem 2.6, there exists an isomorphism of G -graded algebras $\psi : R_1 \rightarrow R_2$. By Remark 2.7, ψ can be chosen to be a monomial isomorphism associated to the permutation π in Definition 2.14. The matrix of the involution $\psi^{-1}\varphi_2\psi$ on R_1 is then obtained from the matrix of φ_2 by permuting the blocks on the diagonal so that they align with the corresponding blocks of φ_1 and possibly multiplying some of the blocks by -1 (the extra condition for the case $m = 0$ in Definition 2.14 guarantees that the second tensor factor in each block remains I). If $\delta_1 = \delta_2$, then $\psi^{-1}\varphi_2\psi$ can be transformed to φ_1 by an automorphism of the G -graded algebra R_1 (see the proof of Theorem 2.10).

Conversely, suppose there exists an isomorphism $\psi : (R_1, \varphi_1) \rightarrow (R_2, \varphi_2)$. First of all, δ_1 and δ_2 are determined by the type of involution (orthogonal or symplectic), so $\delta_1 = \delta_2$. By Theorem 2.6, we also have $T_1 = T_2$, $\beta_1 = \beta_2$, $(\kappa_1, \gamma_1) \sim (\kappa_2, \gamma_2)$. The partitions of κ_1 and κ_2 according to $\{1, \dots, \ell\}$, $\{\ell + 1, \dots, m\}$ and $\{m + 1, \dots, k\}$ are determined by φ_1 and φ_2 , hence they must correspond under ψ . At the same time, for some $g \in G$, the cosets of $\gamma_1 g T$ and $\gamma_2 T$ must correspond under ψ up to switching g'_i with g''_i ($i > m$). In the case $m > 0$, by Proposition 2.3, τ_1 and τ_2 are uniquely determined by the restrictions of φ_1 and φ_2 to D_1, \dots, D_m and hence must match under the permutation determined by ψ . Therefore, in this case $(\kappa_1, \gamma_1, \tau_1) \approx (\kappa_2, \gamma_2, \tau_2)$. It remains to consider the case $m = 0$. Looking at

the description of the automorphism group given by Proposition 2.8, we see that $\psi = \psi_0 \alpha$ where ψ_0 is a monomial isomorphism and α is in $\mathrm{PGL}_{\kappa_1}(\mathbb{F}) \times \mathrm{Aut}_G(D)$. The action of ψ_0 on φ_2 leads to the permutation of blocks and the replacement of the second tensor factor I by X_{t_0} for some $t_0 \in T$. Then α must transform $\psi_0^{-1} \varphi_2 \psi_0$ to φ_1 . The effect of α on one block is the following (we omit subscripts to simplify notation):

$$\begin{aligned} & \left(\begin{pmatrix} {}^t A & 0 \\ 0 & {}^t B \end{pmatrix} \otimes {}^t X_u \right) \left(\begin{pmatrix} 0 & I \\ \varepsilon I & 0 \end{pmatrix} \otimes X_{t_0} \right) \left(\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \otimes X_u \right) \\ &= \pm \begin{pmatrix} 0 & {}^t AB \\ \varepsilon {}^t BA & 0 \end{pmatrix} \otimes X_{t_0}. \end{aligned}$$

We see that α cannot change t_0 . It follows that $t_0 = e$ and $(\kappa_1, \gamma_1) \approx (\kappa_2, \gamma_2)$. \square

3. CORRESPONDENCE BETWEEN LIE GRADINGS AND ASSOCIATIVE GRADINGS

Let U be an algebra and let G be a group. Then a G -grading on U is equivalent to a structure of an $\mathbb{F}G$ -comodule algebra (see e.g. [17] for background). If we assume that U is finite-dimensional and G is abelian and finitely generated, then the comodule structure can be regarded as a morphism of (affine) algebraic group schemes $G^D \rightarrow \mathbf{Aut}(U)$ where G^D is the Cartier dual of G and $\mathbf{Aut}(G)$ is the automorphism group scheme of U (see e.g. [19] for background). Two G -gradings are isomorphic if and only if the corresponding morphisms $G^D \rightarrow \mathbf{Aut}(U)$ are conjugate by an automorphism of U . Note also that, if U is finite-dimensional, then we may always assume without loss of generality that G is finitely generated (just replace G by the subgroup generated by the support of the grading).

If $\mathrm{char} \mathbb{F} = 0$, then $G^D = \widehat{G}$, the algebraic group of characters on G , and $\mathbf{Aut}(G) = \mathrm{Aut}(G)$, the algebraic group of automorphisms. If $\mathrm{char} \mathbb{F} = p > 0$, then we can write $G = G_0 \times G_1$ where G_0 has no p -torsion and G_1 is a p -group. Hence $G^D = \widehat{G}_0 \times G_1^D$, where \widehat{G}_0 is smooth and G_1^D is finite and connected. The algebraic group \widehat{G}_0 (which is equal to \widehat{G}) acts on U as follows:

$$\chi * X = \chi(g)X \quad \text{for all } X \in U_g \text{ and } g \in G.$$

The group scheme $\mathbf{Aut}(U)$ contains the group $\mathrm{Aut}(U)$ as the largest smooth subgroup scheme. The tangent Lie algebra of $\mathbf{Aut}(U)$ is $\mathrm{Der}(U)$, so $\mathbf{Aut}(U)$ is smooth if and only if $\mathrm{Der}(U)$ equals the tangent Lie algebra of the group $\mathrm{Aut}(U)$.

We will be interested in the following algebras: $M_n(\mathbb{F})$, $\mathfrak{psl}_n(\mathbb{F})$, $\mathfrak{so}_n(\mathbb{F})$ and $\mathfrak{sp}_n(\mathbb{F})$, where $\mathrm{char} \mathbb{F} \neq 2$. In all these cases the automorphism group scheme is smooth, i.e., coincides with the algebraic group of automorphisms (regarded as a group scheme). Indeed, for the associative algebra $R = M_n(\mathbb{F})$, it is well-known that $\mathrm{Aut}(R) = \mathrm{PGL}_n(\mathbb{F})$ and $\mathrm{Der}(R) = \mathfrak{pgl}_n(\mathbb{F})$. For the Lie algebra $L = \mathfrak{so}_n(\mathbb{F})$ ($n \geq 5$, $n \neq 8$) or $\mathfrak{sp}_n(\mathbb{F})$ ($n \geq 4$), it is known that every automorphism of L is the conjugation by an element of $\mathrm{O}_n(\mathbb{F})$ or $\mathrm{Sp}_n(\mathbb{F})$, respectively — see [15] for the case $\mathrm{char} \mathbb{F} = 0$ and [18] for the case $\mathrm{char} \mathbb{F} = p$ ($p \neq 2$). In particular, every automorphism of L is the restriction of an automorphism of R . Similarly, every derivation of L is the restriction of a derivation of R (see e.g. [10]).

Let φ be the involution of R such that $L = \mathcal{K}(R, \varphi)$, the space of skew-symmetric elements with respect to φ . Then the projectivizations of the groups $\mathrm{O}_n(\mathbb{F})$ and $\mathrm{Sp}_n(\mathbb{F})$ are equal to $\mathrm{Aut}(R, \varphi)$, and their tangent algebras are equal to $\mathrm{Der}(R, \varphi)$. Hence the restriction map $\theta : \mathrm{Aut}(R, \varphi) \rightarrow \mathrm{Aut}(L)$ is a surjective homomorphism

of algebraic groups such that $d\theta : \text{Der}(R, \varphi) \rightarrow \text{Der}(L)$ is also surjective. It follows that $\mathbf{Aut}(L)$ is smooth. Since L generates R as an associative algebra, both θ and $d\theta$ are also injective. Hence $\theta : \text{Aut}(R, \varphi) \rightarrow \text{Aut}(L)$ is an isomorphism of algebraic groups. For G -gradings this means the following. Clearly, if $R = \bigoplus_{g \in G} R_g$ is a grading that is compatible with φ , then the restriction $L_g = R_g \cap L$ is a grading of L . Since $\theta : \text{Aut}(R, \varphi) \rightarrow \text{Aut}(L)$ is an isomorphism and the automorphism groups are equal to the automorphism group schemes, the restriction map gives a bijection between the isomorphism classes of G -gradings on L and the $\text{Aut}(R, \varphi)$ -orbits on the set of φ -compatible G -gradings on R . The orbits correspond to isomorphism classes of pairs (R, φ) where $R = M_n(\mathbb{F})$ is G -graded and φ is an involution on R that is compatible with the grading.

The case of $L = \mathfrak{psl}_n(\mathbb{F})$ is more complicated. We have a homomorphism of algebraic groups $\theta : \text{Aut}(R) \rightarrow \text{Aut}(L)$ given by restriction and passing to cosets modulo the centre. It is well-known that this homomorphism is not surjective for $n \geq 3$, because the map $X \mapsto -{}^tX$ is not an automorphism of the associative algebra R , but it is an automorphism of the Lie algebra $R^{(-)}$ and hence induces an automorphism of L . Let $\overline{\text{Aut}}(R)$ be the group of automorphisms and anti-automorphisms of R . Then we can extend θ to a homomorphism $\overline{\text{Aut}}(R) \rightarrow \text{Aut}(L)$ by sending an anti-automorphism φ of R to the map induced on L by $-\varphi$. This extended θ is surjective for any $n \geq 3$ if $\text{char } \mathbb{F} \neq 2, 3$ (see [18]) and for any $n > 3$ if $\text{char } \mathbb{F} = 3$ (see [10]). It is easy to verify that θ and $d\theta$ are injective and hence θ is an isomorphism of algebraic groups (see e.g. [1, Lemma 5.3]). It is shown in [10] that, under the same assumptions on $\text{char } \mathbb{F}$, every derivation of L is induced by a derivation of R . It follows that $\mathbf{Aut}(L)$ is smooth, i.e., $\mathbf{Aut}(L) = \text{Aut}(L)$.

Now let $L = \bigoplus_{g \in G} L_g$ be a G -grading and let $\alpha : G^D \rightarrow \text{Aut}(L)$ be the corresponding morphism. Then we have a morphism $\tilde{\alpha} := \theta^{-1}\alpha : G^D \rightarrow \overline{\text{Aut}}(R)$, which gives a G -grading $R = \bigoplus_{g \in G} R_g$ on the Lie algebra $R^{(-)}$. The two gradings are related in the following way: $L_g = (R_g \cap [R, R]) \bmod Z(R)$.

Set $\Lambda = \tilde{\alpha}^{-1}(\text{Aut}(R))$. Then Λ is a subgroupscheme of G^D of index at most 2. Moreover, since G_1^D is connected, it is mapped by $\tilde{\alpha}$ to $\text{Aut}(R)$ and hence is contained in Λ . We have two possibilities: either $\Lambda = G^D$ or Λ has index 2. Following [8], we will say that the G -grading on L has *Type I* in the first case and has *Type II* in the second case. In Type I, the G -grading corresponding to $\tilde{\alpha}$ is a grading of R as an associative algebra. In Type II, we consider Λ^\perp , which is a subgroup of order 2 in G . Let h be the generator of this subgroup. Note that, since $\text{char } \mathbb{F} \neq 2$, the element h is in G_0 .

Remark 3.1. For the readers more familiar with the language of Hopf algebras, there is an alternative way to define the element h . The Hopf algebra $\mathbb{F}[\overline{\text{Aut}}(R)]$ of regular functions on the algebraic group $\overline{\text{Aut}}(R)$ has a group-like element f defined by $f(\psi) = 1$ if ψ is an automorphism and $f(\psi) = -1$ if ψ is an anti-automorphism. The morphism of group schemes $\tilde{\alpha} : G^D \rightarrow \overline{\text{Aut}}(R)$ corresponds to a homomorphism of Hopf algebras $\mathbb{F}[\overline{\text{Aut}}(R)] \rightarrow \mathbb{F}G$. The element h is the image of f under this homomorphism.

Let $\overline{G} = G/\langle h \rangle$. Then the restriction $\tilde{\alpha} : \Lambda \rightarrow \text{Aut}(R)$ corresponds to the coarsening of the G -grading on R given by the quotient map $G \rightarrow \overline{G}$:

$$R = \bigoplus_{\overline{g} \in \overline{G}} R_{\overline{g}} \quad \text{where } R_{\overline{g}} = R_g \oplus R_{gh}.$$

This \overline{G} -grading is a grading of R as an associative algebra. The G -grading on $R^{(-)}$ can be recovered as follows. Fix $\chi \in \widehat{G}_0 = \widehat{G}$ such that $\chi(h) = -1$. Then χ acts on R as $-\varphi$ where φ is an anti-automorphism preserving the \overline{G} -grading. Then we have

$$R_g = \{X \in R_{\overline{g}} \mid -\varphi(X) = \chi(g)X\} = \{-\varphi(X) + \chi(g)X \mid X \in R_{\overline{g}}\}.$$

Thus we obtain 1) a bijection between the isomorphism classes of G -gradings on L of Type I and the $\text{Aut}(R)$ -orbits on the set of G -gradings on R and 2) a bijection between the isomorphism classes of G -gradings on L of Type II and $\text{Aut}(R)$ -orbits on the set of pairs (R, φ) where $R = M_n(\mathbb{F})$ is \overline{G} -graded and φ is an anti-automorphism on R that is compatible with the \overline{G} -grading and has the property $\varphi^2(X) = \chi^2 * X$ for all $X \in R$.

Remark 3.2. If $n = 2$, then $\theta : \text{Aut}(R) \rightarrow \text{Aut}(L)$ is an isomorphism, so there are no gradings of Type II.

4. GRADINGS ON LIE ALGEBRAS OF TYPE \mathcal{A}

Let $L = \mathfrak{psl}_n(\mathbb{F})$ and $R = M_n(\mathbb{F})$, where $\text{char } \mathbb{F} \neq 2$ and, for $n = 3$, also $\text{char } \mathbb{F} \neq 3$. Let $L = \bigoplus_{g \in G} L_g$ be a grading of L by an abelian group G . As discussed in the previous section, this grading belongs to one of two types. Gradings of Type I are induced from G -gradings on the associative algebra R , which have been classified in Theorem 2.6.

Definition 4.1. Let $R = \mathcal{M}(G, T, \beta, \kappa, \gamma)$ and let $L_g = (R_g \cap [R, R]) \bmod Z(R)$. We will denote the G -graded algebra L obtained in this way as $\mathcal{A}^{(1)}(G, T, \beta, \kappa, \gamma)$.

Now assume that we have a grading of Type II. Then there is a distinguished element $h \in G$ of order 2. Let $\overline{G} = G/\langle h \rangle$. Then the G -grading on L is induced from a G -grading on the Lie algebra $R^{(-)}$ that is obtained by refining a \overline{G} -grading $R = \bigoplus_{\overline{g} \in \overline{G}} R_{\overline{g}}$ on the associative algebra R . Let $R = \mathcal{M}(\overline{G}, \overline{T}, \beta, \kappa, \gamma)$ as a \overline{G} -graded algebra. The refinement is obtained using the action of any character $\chi \in \widehat{G}$ with $\chi(h) = -1$, and the result does not depend on the choice of χ . So we fix $\chi \in \widehat{G}$ such that $\chi(h) = -1$.

Set $\varphi(X) = -\chi * X$ for all $X \in R$. Then φ is an anti-automorphism of the \overline{G} -graded algebra R . Moreover, $\varphi^2(X) = \chi^2 * X$. Since $\chi^2(h) = 1$, we can regard χ^2 as a character on \overline{G} and hence its action on $X \in R_{\overline{g}}$ is given by $\chi^2 * X = \chi^2(\overline{g})X$. In particular, $\varphi^2|_{R_{\overline{e}}} = \text{id}$. By Theorem 2.10, \overline{T} is an elementary 2-group, κ is given by (5) and γ is given by (6) with bars over the g 's. We may also assume that γ satisfies

$$(15) \quad \overline{g}_1^2 \overline{t}_1 = \dots = \overline{g}_m^2 \overline{t}_m = \overline{g}'_{m+1} \overline{g}''_{m+1} = \dots = \overline{g}'_s \overline{g}''_s$$

for some $\overline{t}_1, \dots, \overline{t}_m \in \overline{T}$, and φ is given by $\varphi(X) = \Phi^{-1}({}^t X) \Phi$ where

$$(16) \quad \Phi = \sum_{i=1}^{\ell} I_{q_i} \otimes X_{\overline{t}_i} \oplus \sum_{i=\ell+1}^m S_i \otimes X_{\overline{t}_i} \oplus \sum_{i=m+1}^k \begin{pmatrix} 0 & I_{q_i} \\ \mu_i I_{q_i} & 0 \end{pmatrix} \otimes I,$$

where μ_i are nonzero scalars. We will use the notation τ introduced in (13).

Our goal now is to determine the parameters $\mu_i \in \mathbb{F}^\times$ that appear in the above formula. On the one hand, the automorphism φ^2 is the conjugation by matrix

${}^t\Phi^{-1}\Phi$ given by

$${}^t\Phi^{-1}\Phi = \sum_{i=1}^{\ell} \beta(t_i) I_{q_i} \otimes I \oplus \sum_{i=\ell+1}^m \beta(t_i) \operatorname{sgn}(S_i) I_{2q_i} \otimes I \oplus \sum_{i=m+1}^k \begin{pmatrix} \mu_i I_{q_i} & 0 \\ 0 & \mu_i^{-1} I_{q_i} \end{pmatrix} \otimes I.$$

On the other hand, φ^2 acts as χ^2 . We now derive the conditions that are necessary and sufficient for $\chi^2 * X = ({}^t\Phi^{-1}\Phi)^{-1} X ({}^t\Phi^{-1}\Phi)$ to hold for all $X \in R$.

Recall the idempotents $e_1, \dots, e_k \in C$ defined earlier (where $e_i = e'_i + e''_i$ for $i > m$). We denote by U_{ij} any matrix in the Peirce component $e_i C e_j$. Then, for $1 \leq i, j \leq m$, we have, for all $\bar{t} \in \bar{T}$,

$$\chi^2 * (U_{ij} \otimes X_{\bar{t}}) = \chi^2(\bar{g}_i^{-1} \bar{g}_j \bar{t}) U_{ij} \otimes X_{\bar{t}}$$

while

$$\begin{aligned} \varphi^2(U_{ij} \otimes X_{\bar{t}}) &= ({}^t\Phi^{-1}\Phi)^{-1}(U_{ij} \otimes X_{\bar{t}})({}^t\Phi^{-1}\Phi) \\ &= \beta(\bar{t}_i) \operatorname{sgn}(S_i) \beta(\bar{t}_j) \operatorname{sgn}(S_j) (U_{ij} \otimes X_{\bar{t}}). \end{aligned}$$

For $m+1 \leq i, j \leq k$, we write $U_{ij} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ according to the decompositions $e_i = e'_i + e''_i$ and $e_j = e'_j + e''_j$. Then, for all $\bar{t} \in \bar{T}$,

$$\chi^2 * (U_{ij} \otimes X_{\bar{t}}) = \begin{pmatrix} \chi^2((\bar{g}'_i)^{-1} \bar{g}'_j \bar{t}) A & \chi^2((\bar{g}'_i)^{-1} \bar{g}''_j \bar{t}) B \\ \chi^2((\bar{g}''_i)^{-1} \bar{g}'_j \bar{t}) C & \chi^2((\bar{g}''_i)^{-1} \bar{g}''_j \bar{t}) D \end{pmatrix},$$

while

$$\varphi^2(U_{ij} \otimes X_{\bar{t}}) = \begin{pmatrix} \mu_i^{-1} \mu_j A & \mu_i^{-1} \mu_j^{-1} B \\ \mu_i \mu_j C & \mu_i \mu_j^{-1} D \end{pmatrix}.$$

For $1 \leq i \leq m$ and $m+1 \leq j \leq k$, we write $U_{ij} = \begin{pmatrix} A & B \end{pmatrix}$ according to the decomposition $e_j = e'_j + e''_j$. Then, for all $\bar{t} \in \bar{T}$,

$$\chi^2 * (U_{ij} \otimes X_{\bar{t}}) = \begin{pmatrix} \chi^2(\bar{g}_i^{-1} \bar{g}'_j \bar{t}) A & \chi^2(\bar{g}_i^{-1} \bar{g}''_j \bar{t}) B \end{pmatrix},$$

while

$$\varphi^2(U_{ij} \otimes X_{\bar{t}}) = \beta(\bar{t}_i) \operatorname{sgn}(S_i) \begin{pmatrix} \mu_j A & \mu_j^{-1} B \end{pmatrix}.$$

For $m+1 \leq i \leq k$ and $1 \leq j \leq m$, we have a similar calculation.

By way of comparison, we derive $\chi^2(\bar{t}) = \operatorname{const}$ for all $\bar{t} \in \bar{T}$, and so $\chi^2(\bar{T}) = 1$. Hence the natural epimorphism $\pi : G \rightarrow \bar{G}$ splits over \bar{T} , i.e., $\pi^{-1}(\bar{T}) = T \times \langle h \rangle$, where $T = \pi^{-1}(\bar{T}) \cap \ker \chi$. So we may identify T with \bar{T} and write t_i for the representative of the coset \bar{t}_i in T . Conversely, if $\pi : G \rightarrow \bar{G}$ splits over \bar{T} , then $\chi^2(\bar{T}) = 1$.

In the case $1 \leq i, j \leq m$, our relations are equivalent to $\beta(t_i) \operatorname{sgn}(S_i) \chi^2(\bar{g}_i) = \beta(t_j) \operatorname{sgn}(S_j) \chi^2(\bar{g}_j)$. Therefore, we have a fixed $\lambda \in \mathbb{F}^\times$ such that

$$(17) \quad \beta(t_i) \operatorname{sgn}(S_i) \chi^2(\bar{g}_i) = \lambda \quad \text{for all } i = 1, \dots, m.$$

In the case $m+1 \leq i, j \leq k$, our relations are equivalent to

$$\mu_i^{-1} \chi^2(\bar{g}'_i) = \mu_j^{-1} \chi^2(\bar{g}'_j)$$

and

$$\mu_i^{-1} \chi^2(\bar{g}''_i) = \mu_j \chi^2(\bar{g}''_j).$$

Therefore, we have a fixed $\mu \in \mathbb{F}^\times$ such that

$$(18) \quad \mu_i^{-1} \chi^2(\bar{g}'_i) = \mu_i \chi^2(\bar{g}''_i) = \mu \quad \text{for all } i = m+1, \dots, k.$$

In the case $1 \leq i \leq m$ and $m+1 \leq j \leq k$, our relations are equivalent to

$$(19) \quad \mu_j^{-1} \chi^2(\bar{g}'_j) = \beta(t_i) \operatorname{sgn}(S_i) \chi^2(\bar{g}_i) = \mu_j \chi^2(\bar{g}''_j).$$

If both (17) and (18) are present (i.e., $m \neq 0, k$), then (19) is equivalent to $\mu = \lambda$. We have proved that if the \bar{G} -grading on R is the coarsening a G -grading on $R^{(-)}$ induced by $\pi : G \rightarrow \bar{G}$, and χ acts on R as $-\varphi$, then $\pi^{-1}(\bar{T})$ splits and conditions (17) and (18) hold with $\lambda = \mu$. Conversely, if $R = \mathcal{M}(\bar{G}, \bar{T}, \beta, \kappa, \gamma)$ is such that $\pi^{-1}(\bar{T})$ splits, and an anti-automorphism φ is given by matrix (16) such that (17) and (18) hold with $\lambda = \mu$, then φ^2 acts as χ^2 on R and hence $-\varphi$ defines a refinement of the \bar{G} -grading on R to a G -grading (as a vector space). The latter is automatically a grading of the Lie algebra $R^{(-)}$, since $-\varphi$ is an automorphism of $R^{(-)}$.

To summarize, we state the following

Proposition 4.2. *Let $h \in G$ be an element of order 2 and let $\pi : G \rightarrow \bar{G} = G/\langle h \rangle$ be the quotient map. Fix $\chi \in \hat{G}$ with $\chi(h) = -1$. Let $R = \mathcal{M}(\bar{G}, \bar{T}, \beta, \kappa, \gamma)$ and let φ be the anti-automorphism of the \bar{G} -graded algebra R given by $\varphi(X) = \Phi^{-1}(^t X) \Phi$ with Φ as in (16). Set $H = \pi^{-1}(\bar{T})$. Then*

$$R_g = \{X \in R_{\bar{g}} \mid -\varphi(X) = \chi(g)X\} \quad \text{for all } g \in G$$

defines a G -grading on $R^{(-)}$ if and only if H splits as $T \times \langle h \rangle$ with $T = H \cap \ker \chi$ and the following condition holds (identifying \bar{T} with T):

$$(20) \quad \begin{aligned} & \beta(t_1) \chi^2(\bar{g}_1) = \dots = \beta(t_\ell) \chi^2(\bar{g}_\ell) \\ & = \beta(t_{\ell+1}) \operatorname{sgn}(S_{\ell+1}) \chi^2(\bar{g}_{\ell+1}) = \dots = \beta(t_m) \operatorname{sgn}(S_m) \chi^2(\bar{g}_m) \\ & = \mu_{m+1}^{-1} \chi^2(\bar{g}'_{m+1}) = \mu_{m+1} \chi^2(\bar{g}''_{m+1}) = \dots = \mu_k^{-1} \chi^2(\bar{g}'_k) = \mu_k \chi^2(\bar{g}''_k). \end{aligned}$$

□

It is convenient to distinguish the following three cases for a grading of Type II on L :

- The case with $\ell > 0$ will be referred to as Type II₁;
- The case with $\ell = 0$ but $m > 0$, will be referred to as Type II₂;
- The case with $m = 0$ will be referred to as Type II₃.

Definition 4.3. We will say that γ is *admissible* if it satisfies

$$(21) \quad \bar{g}_1^2 \equiv \dots \equiv \bar{g}_m^2 \equiv \bar{g}'_{m+1} \bar{g}''_{m+1} \equiv \dots \equiv \bar{g}'_k \bar{g}''_k \pmod{\bar{T}}$$

and, for some $\bar{t}_1, \dots, \bar{t}_\ell \in \bar{T}$, we have $\bar{g}_1^2 \bar{t}_1 = \dots = \bar{g}_\ell^2 \bar{t}_\ell$ and

$$(22) \quad \beta(\bar{t}_1) \chi^2(\bar{g}_1) = \dots = \beta(\bar{t}_\ell) \chi^2(\bar{g}_\ell).$$

(If $\ell \leq 1$, then condition (22) is automatically satisfied.)

Note that the above definition does not depend on the choice of $\chi \in \hat{G}$ with $\chi(h) = -1$. Indeed, if we replace χ by $\tilde{\chi} = \chi\psi$ where $\psi \in \hat{G}$ satisfies $\psi(h) = 1$, then ψ can be regarded as a character on \bar{G} and we can compute:

$$\tilde{\chi}^2(\bar{g}_i^{-1} \bar{g}_j) = \chi^2(\bar{g}_i^{-1} \bar{g}_j) \psi^2(\bar{g}_i^{-1} \bar{g}_j) = \chi^2(\bar{g}_i^{-1} \bar{g}_j) \psi(\bar{g}_i^{-2} \bar{g}_j^2) = \chi^2(\bar{g}_i^{-1} \bar{g}_j) \psi(\bar{t}_i \bar{t}_j)$$

for all $1 \leq i, j \leq \ell$. On the other hand, for $\bar{t} \in \bar{T}$, we have

$$\beta(\bar{t} \bar{t}_i) \beta(\bar{t} \bar{t}_j) = \beta(\bar{t}) \beta(\bar{t}_i) \beta(\bar{t}, \bar{t}_i) \beta(\bar{t}) \beta(\bar{t}_j) \beta(\bar{t}, \bar{t}_j) = \beta(\bar{t}_i) \beta(\bar{t}_j) \beta(\bar{t}, \bar{t}_i \bar{t}_j).$$

Therefore, if condition (22) holds for χ and $\bar{t}_1, \dots, \bar{t}_\ell$, then it holds for $\tilde{\chi}$ and $\tilde{t}_1, \dots, \tilde{t}_\ell$ where \bar{t} is the unique element of \bar{T} such that $\beta(\bar{t}, \bar{u}) = \psi(\bar{u})$ for all $\bar{u} \in \bar{T}$.

As pointed out earlier, for γ satisfying (21), we can replace \bar{g}_i'' , $i > m$, within their cosets mod \bar{T} so that γ satisfies (15).

We now give our standard realizations for gradings of Type II. Let $H \subset G$ be an elementary 2-group of odd rank containing h . Let β be a nondegenerate alternating bicharacter on $\bar{T} = H/\langle h \rangle$. Fix κ . Choose γ formed from elements of $\bar{G} = G/\langle h \rangle$ and τ formed from elements of $\bar{T} = H/\langle h \rangle$ so that they satisfy (15). Let $R = \mathcal{M}(\bar{G}, \bar{T}, \beta, \kappa, \gamma)$. Fix $\chi \in \hat{G}$ with $\chi(h) = -1$ and identify \bar{T} with $T = H \cap \ker \chi$.

Definition 4.4. Suppose $\ell > 0$ and γ is admissible. Let Φ be the matrix given by (16) where the scalars μ_i and matrices S_i are determined by equation (20). Then, by Proposition 4.2, the anti-automorphism $\varphi(X) = \Phi^{-1}(^t X)\Phi$ defines a refinement of the \bar{G} -grading on the associative algebra R to a G -grading $R = \bigoplus_{g \in G} R_g$ as a Lie algebra. Set $L_g = (R_g \cap [R, R]) \bmod Z(R)$. We will denote the G -graded algebra L obtained in this way as $\mathcal{A}^{(\Pi_1)}(G, H, h, \beta, \kappa, \gamma, \tau)$.

Definition 4.5. Suppose $\ell = 0$ and $m > 0$. Choose $\delta = (\delta_1, \dots, \delta_m) \in \{\pm 1\}^m$ so that

$$\beta(t_1)\chi^2(\bar{g}_1)\delta_1 = \dots = \beta(t_m)\chi^2(\bar{g}_m)\delta_m.$$

(Note that there are exactly two such choices.) Let Φ be the matrix given by (16) where the matrices S_i are selected by the rule $\text{sgn}(S_i) = \delta_i$ and the scalars μ_i are determined by equation (20). Then, by Proposition 4.2, the anti-automorphism $\varphi(X) = \Phi^{-1}(^t X)\Phi$ defines a refinement of the \bar{G} -grading on the associative algebra R to a G -grading $R = \bigoplus_{g \in G} R_g$ as a Lie algebra. Set $L_g = (R_g \cap [R, R]) \bmod Z(R)$. We will denote the G -graded algebra L obtained in this way as $\mathcal{A}^{(\Pi_2)}(G, H, h, \beta, \kappa, \gamma, \tau, \delta)$.

Definition 4.6. Suppose $m = 0$. Then we have

$$\chi^2(\bar{g}'_1 \bar{g}''_1) = \dots = \chi^2(\bar{g}'_k \bar{g}''_k).$$

Let μ be a scalar such that μ^2 is equal to the common value of $\chi^2(\bar{g}'_i \bar{g}''_i)$. (There are two choices.) Let Φ be the matrix given by (16) where the scalars μ_i are determined by equation

$$\mu_1^{-1}\chi^2(\bar{g}'_1) = \mu_1\chi^2(\bar{g}''_1) = \dots = \mu_k^{-1}\chi^2(\bar{g}'_k) = \mu_k\chi^2(\bar{g}''_k) = \mu.$$

Then, by Proposition 4.2, the anti-automorphism $\varphi(X) = \Phi^{-1}(^t X)\Phi$ defines a refinement of the \bar{G} -grading on the associative algebra R to a G -grading $R = \bigoplus_{g \in G} R_g$ as a Lie algebra. Set $L_g = (R_g \cap [R, R]) \bmod Z(R)$. We will denote the G -graded algebra L obtained in this way as $\mathcal{A}^{(\Pi_3)}(G, H, h, \beta, \kappa, \gamma, \mu)$.

Definition 4.7. Referring to Definition 4.5, we will write $(\kappa, \gamma, \tau, \delta) \approx (\tilde{\kappa}, \tilde{\gamma}, \tilde{\tau}, \tilde{\delta})$ if κ and $\tilde{\kappa}$ have the same number of components of each type, i.e., the same values of m and k , and there exist an element $\bar{g} \in \bar{G}$ and a permutation π of the symbols $\{1, \dots, k\}$ preserving the sets $\{1, \dots, m\}$ and $\{m+1, \dots, k\}$ such that $\tilde{g}_i = q_{\pi(i)}$ for all i , $\tilde{t}_i = t_{\pi(i)}$, $\tilde{g}_i \equiv \bar{g}_{\pi(i)}\bar{g} \pmod{\bar{T}}$ and $\tilde{\delta}_i = \delta_{\pi(i)}$ for all $i = 1, \dots, m$, and $\{\tilde{g}'_i, \tilde{g}''_i\} \equiv \{\bar{g}'_{\pi(i)}\bar{g}, \bar{g}''_{\pi(i)}\bar{g}\} \pmod{\bar{T}}$ for all $i = m+1, \dots, k$.

Definition 4.8. Referring to Definition 4.6, we will write $(\kappa, \gamma, \mu) \approx (\tilde{\kappa}, \tilde{\gamma}, \tilde{\mu})$ if κ and $\tilde{\kappa}$ have the same number of components k and there exist an element $\bar{g} \in \bar{G}$ and a permutation π of the symbols $\{1, \dots, k\}$ such that $\tilde{q}_i = q_{\pi(i)}$, $\{\tilde{g}'_i, \tilde{g}''_i\} \equiv \{\bar{g}'_{\pi(i)}\bar{g}, \bar{g}''_{\pi(i)}\bar{g}\} \pmod{\bar{T}}$ and $\tilde{g}'_i\tilde{g}''_i = \bar{g}'_{\pi(i)}\bar{g}''_{\pi(i)}\bar{g}^2$ for all i , and, finally, $\tilde{\mu} = \mu\chi^2(\bar{g})$.

Theorem 4.9. Let \mathbb{F} be an algebraically closed field, $\text{char } \mathbb{F} \neq 2$. Let G be an abelian group. Let $L = \mathfrak{psl}_n(\mathbb{F})$ where $n \geq 3$. If $n = 3$, assume also that $\text{char } \mathbb{F} \neq 3$. Let $L = \bigoplus_{g \in G} L_g$ be a G -grading. Then the graded algebra L is isomorphic to one of the following:

- $\mathcal{A}^{(I)}(G, T, \beta, \kappa, \gamma)$,
- $\mathcal{A}^{(\text{II}_1)}(G, H, h, \beta, \kappa, \gamma, \tau)$,
- $\mathcal{A}^{(\text{II}_2)}(G, H, h, \beta, \kappa, \gamma, \tau, \delta)$,
- $\mathcal{A}^{(\text{II}_3)}(G, H, h, \beta, \kappa, \gamma, \mu)$,

as in Definitions 4.1, 4.4, 4.5 and 4.6, with $|\kappa|\sqrt{|T|} = n$ in Type I and $|\kappa|\sqrt{|H|/2} = n$ in Type II. Graded algebras belonging to different types listed above are not isomorphic. Within each type, we have the following:

- $\mathcal{A}^{(I)}(G, T_1, \beta_1, \kappa_1, \gamma_1) \cong \mathcal{A}^{(I)}(G, T_2, \beta_2, \kappa_2, \gamma_2)$ if and only if $T_1 = T_2$, $\beta_1 = \beta_2$, and $(\kappa_1, \gamma_1) \sim (\kappa_2, \gamma_2)$ or $(\kappa_1, \gamma_1) \sim (\kappa_2, \gamma_2^{-1})$;
- $\mathcal{A}^{(\text{II}_1)}(G, H_1, h_1, \beta_1, \kappa_1, \gamma_1, \tau_1) \cong \mathcal{A}^{(\text{II}_1)}(G, H_2, h_2, \beta_2, \kappa_2, \gamma_2, \tau_2)$ if and only if $H_1 = H_2$, $h_1 = h_2$, $\beta_1 = \beta_2$, and $(\kappa_1, \gamma_1, \tau_1) \approx (\kappa_2, \gamma_2, \tau_2)$ or $(\kappa_1, \gamma_1, \tau_1) \approx (\kappa_2, \gamma_2^{-1}, \tau_2)$;
- $\mathcal{A}^{(\text{II}_2)}(G, H_1, h_1, \beta_1, \kappa_1, \gamma_1, \tau_1, \delta_1) \cong \mathcal{A}^{(\text{II}_2)}(G, H_2, h_2, \beta_2, \kappa_2, \gamma_2, \tau_2, \delta_2)$ if and only if $H_1 = H_2$, $h_1 = h_2$, $\beta_1 = \beta_2$, and $(\kappa_1, \gamma_1, \tau_1, \delta_1) \approx (\kappa_2, \gamma_2, \tau_2, \delta_2)$ or $(\kappa_1, \gamma_1, \tau_1, \delta_1) \approx (\kappa_2, \gamma_2^{-1}, \tau_2, \delta_2)$;
- $\mathcal{A}^{(\text{II}_3)}(G, H_1, h_1, \beta_1, \kappa_1, \gamma_1, \mu_1) \cong \mathcal{A}^{(\text{II}_3)}(G, H_2, h_2, \beta_2, \kappa_2, \gamma_2, \mu_2)$ if and only if $H_1 = H_2$, $h_1 = h_2$, $\beta_1 = \beta_2$, and $(\kappa_1, \gamma_1, \mu_1) \approx (\kappa_2, \gamma_2, \mu_2)$ or $(\kappa_1, \gamma_1, \mu_1) \approx (\kappa_2, \gamma_2^{-1}, \mu_2^{-1})$.

Proof. The first statement is a combination of Theorem 2.10 and Proposition 4.2. The non-isomorphism of graded algebras belonging to different types is clear.

For Type I, let $R_1 = \mathcal{M}(G, T_1, \beta_1, \kappa_1, \gamma_1)$ and $R_2 = \mathcal{M}(G, T_2, \beta_2, \kappa_2, \gamma_2)$. By Theorem 2.6, $R_1 \cong R_2$ if and only if $T_1 = T_2$, $\beta_1 = \beta_2$, and $(\kappa_1, \gamma_1) \sim (\kappa_2, \gamma_2)$. It remains to observe that the outer automorphism $X \mapsto -^t X$ transforms $\mathcal{M}(G, T, \beta, \kappa, \gamma)$ to $\mathcal{M}(G, T, \beta, \kappa, \gamma^{-1})$.

For Type II, the element h , the subgroup H , and the bicharacter β on $\bar{T} = H/\langle h \rangle$ are uniquely determined by the grading, so we may assume $H_1 = H_2$, $h_1 = h_2$, and $\beta_1 = \beta_2$. Let $R_1 = \mathcal{M}(\bar{G}, \bar{T}, \beta, \kappa_1, \gamma_1)$ and $R_2 = \mathcal{M}(\bar{G}, \bar{T}, \beta, \kappa_2, \gamma_2)$. Fix $\chi \in \hat{G}$ with $\chi(h) = -1$. Let φ_1 and φ_2 be the corresponding anti-automorphisms. We have to check that $(R_1, \varphi_1) \cong (R_2, \varphi_2)$ if and only if

- II₁) $(\kappa_1, \gamma_1, \tau_1) \approx (\kappa_2, \gamma_2, \tau_2)$,
- II₂) $(\kappa_1, \gamma_1, \tau_1, \delta_1) \approx (\kappa_2, \gamma_2, \tau_2, \delta_2)$,
- II₃) $(\kappa_1, \gamma_1, \mu_1) \approx (\kappa_2, \gamma_2, \mu_2)$.

For Type II₁, the “only if” part is clear, since (κ, γ, τ) is an invariant of (R, φ) (up to transformations indicated in the definition of the equivalence relation \approx). Indeed, (κ, γ) is an invariant of the \bar{G} -grading, and τ corresponds to the restrictions of φ to D_1, \dots, D_m by Proposition 2.3. To prove the “if” part, assume $(\kappa_1, \gamma_1, \tau_1) \approx (\kappa_2, \gamma_2, \tau_2)$. Then, by Theorem 2.6, there exists an isomorphism of \bar{G} -graded algebras $\psi : R_1 \rightarrow R_2$. By Remark 2.7, we can take for ψ a monomial

isomorphism associated to the permutation π in Definition 2.14. The matrix of the anti-automorphism $\psi^{-1}\varphi_2\psi$ on R_1 is then obtained from the matrix of φ_2 by permuting the blocks on the diagonal so that they align with the corresponding blocks of φ_1 , and possibly multiplying some of the blocks by -1 . Hence, by Theorem 2.10, $\psi^{-1}\varphi_2\psi$ can be transformed to φ_1 by an automorphism of the \overline{G} -graded algebra R_1 .

For Type II_2 , the proof is similar, since δ corresponds to the restrictions of φ to C_1, \dots, C_m and thus is an invariant of (R, φ) .

For Type II_3 , we show in the same manner that if $(\kappa_1, \gamma_1, \mu_1) \approx (\kappa_2, \gamma_2, \mu_2)$, then $(R_1, \varphi_1) \cong (R_2, \varphi_2)$. Namely, we take a monomial isomorphism of \overline{G} -graded algebras $\psi : R_1 \rightarrow R_2$ associated to the permutation π in Definition 4.8. The effect of ψ on Φ_2 is just the permutation of blocks. The factor $\chi^2(\overline{g})$ in Definition 4.8 makes sure that the block with $\mu_i = \mu^{-1}\chi^2(\overline{g}'_i)$ in Φ_2 matches up with the block with $\mu_{\pi(i)} = \mu^{-1}\chi^2(\overline{g}'_{\pi(i)})$ in Φ_1 . Conversely, suppose there exists an isomorphism $\psi : (R_1, \varphi_1) \rightarrow (R_2, \varphi_2)$. As in the proof of Corollary 2.15, we write $\psi = \psi_0\alpha$ where ψ_0 is a monomial isomorphism and α is in $\text{PGL}_{\kappa_1}(\mathbb{F}) \times \text{Aut}_G(D)$. The action of ψ_0 on φ_2 permutes the blocks and replaces the second tensor factor I by X_{t_0} for some $t_0 \in T$. The action of α on $\psi_0^{-1}\varphi_2\psi_0$ cannot change t_0 or the values of the scalars. We conclude that $t_0 = e$ and $(\kappa_1, \gamma_1, \mu_1) \approx (\kappa_2, \gamma_2, \mu_2)$. \square

Remark 4.10. Let \mathbb{F} and G be as in Theorem 4.9. Let $L = \mathfrak{sl}_2(\mathbb{F})$. If $L = \bigoplus_{g \in G} L_g$ is a G -grading, then the graded algebra L is isomorphic to $\mathcal{A}^{(1)}(G, T, \beta, \kappa, \gamma)$ where $|\kappa|\sqrt{|T|} = 2$. This, of course, gives two possibilities: either $T = \{e\}$ or $T \cong \mathbb{Z}_2^2$. In the first case the G -grading is induced from a Cartan decomposition by a homomorphism $\mathbb{Z} \rightarrow G$. The isomorphism classes of such gradings are in one-to-one correspondence with unordered pairs of the form $\{g, g^{-1}\}$, $g \in G$. In the second case the G -grading is given by Pauli matrices. The isomorphism classes of such gradings are in one-to-one correspondence with subgroups $T \subset G$ such that $T \cong \mathbb{Z}_2^2$.

Remark 4.11. The remaining case $L = \mathfrak{psl}_3(\mathbb{F})$ where $\text{char } \mathbb{F} = 3$ can be handled using octonions. Let \mathbb{O} be the algebra of octonions over an algebraically closed field \mathbb{F} . Then the subspace \mathbb{O}' of zero trace octonions is a Malcev algebra with respect to the commutator $[x, y] = xy - yx$. If $\text{char } \mathbb{F} = 3$, then \mathbb{O}' is a Lie algebra isomorphic to L . Assuming $\text{char } \mathbb{F} \neq 2$, we have $xy = \frac{1}{2}([x, y] - n(x, y)1)$ for all $x, y \in \mathbb{O}'$, where n is the norm of \mathbb{O} . We also have $(\text{ad } x)^3 = -4n(x)(\text{ad } x)$ for all $x \in \mathbb{O}'$. It follows that if ψ is an automorphism of \mathbb{O}' , then ψ preserves n and, setting $\psi(1) = 1$, we obtain an automorphism of \mathbb{O} . Hence the restriction map $\text{Aut}(\mathbb{O}) \rightarrow \text{Aut}(\mathbb{O}')$ is an isomorphism of algebraic groups. Similarly, one shows that the restriction map $\text{Der}(\mathbb{O}) \rightarrow \text{Der}(\mathbb{O}')$ is an isomorphism of Lie algebras.¹ It follows that $\mathbf{Aut}(\mathbb{O}')$ is smooth and can be identified with the algebraic group $\text{Aut}(\mathbb{O})$. In particular, this means that the isomorphism classes of G -gradings on \mathbb{O} are in one-to-one correspondence (via restriction) with the isomorphism classes of G -gradings on \mathbb{O}' (cf. [12, Theorem 9]).

All gradings on \mathbb{O} (in any characteristic) were described in [12]. For $\text{char } \mathbb{F} \neq 2$, they are of two types:

¹This argument was communicated to us by A. Elduque.

- “elementary” gradings obtained by choosing $g_1, g_2, g_3 \in G$ with $g_1 g_2 g_3 = e$ and assigning degree e to e_1 and e_2 , degree g_i to u_i and degree g_i^{-1} to v_i , $i = 1, 2, 3$, where $\{e_1, e_2, u_1, u_2, u_3, v_1, v_2, v_3\}$ is a canonical basis for \mathbb{O} ;
- “division” gradings by \mathbb{Z}_2^3 obtained by iterating the Cayley-Dickson doubling process three times.

It is easy to see when two G -gradings on \mathbb{O} are isomorphic. The isomorphism classes of “elementary” gradings are in one-to-one correspondence with unordered pairs of the form $\{S, S^{-1}\}$ where S is an unordered triple $\{g_1, g_2, g_3\}$, $g_i \in G$ with $g_1 g_2 g_3 = e$. The isomorphism classes of “division” gradings are in one-to-one correspondence with subgroups $T \subset G$ such that $T \cong \mathbb{Z}_2^3$. An “elementary” grading is not isomorphic to a “division” grading.

If $\text{char } \mathbb{F} = 3$, then the above is also the classification of G -gradings on $L = \mathfrak{psl}_3(\mathbb{F})$. As shown in [16], up to isomorphism, any grading on L is induced from the matrix algebra $M_3(\mathbb{F})$. Namely, any “elementary” grading on L can be obtained as a Type I grading, and any “division” grading on L is isomorphic to a Type II gradings. The only difference with the case of $\mathfrak{sl}_3(\mathbb{F})$ where $\text{char } \mathbb{F} \neq 3$ is that there are fewer isomorphism classes of gradings in characteristic 3 (in particular, some “Type II” gradings are isomorphic to “Type I” gradings).

5. GRADINGS ON LIE ALGEBRAS OF TYPES \mathcal{B} , \mathcal{C} , \mathcal{D}

The classification of gradings for Lie algebras $\mathfrak{so}_n(\mathbb{F})$ and $\mathfrak{sp}_n(\mathbb{F})$ follows immediately from Corollary 2.15. We state the results here for completeness. Recall $\mathcal{M}^*(G, T, \beta, \kappa, \gamma, \tau, \delta)$ from Definition 2.13. Let $L = \mathcal{K}(R, \varphi) = \{X \in R \mid \varphi(X) = -X\}$. Then $L = \bigoplus_{g \in G} L_g$ where $L_g = R_g \cap L$.

Definition 5.1. Let $n = |\kappa| \sqrt{|T|}$.

- If $\delta = 1$ and n is odd, then necessarily $T = \{e\}$. We will denote the G -graded algebra L by $\mathcal{B}(G, \kappa, \gamma)$.
- If $\delta = -1$ (hence n is even), then we will denote the G -graded algebra L by $\mathcal{C}(G, T, \beta, \kappa, \gamma, \tau)$.
- If $\delta = 1$ and n is even, then we will denote the G -graded algebra L by $\mathcal{D}(G, T, \beta, \kappa, \gamma, \tau)$.

Theorem 5.2. Let \mathbb{F} be an algebraically closed field, $\text{char } \mathbb{F} \neq 2$. Let G be an abelian group.

- Let $L = \mathfrak{so}_n(\mathbb{F})$, with odd $n \geq 5$. Let $L = \bigoplus_{g \in G} L_g$ be a G -grading. Then the graded algebra L is isomorphic to $\mathcal{B}(G, \kappa, \gamma)$,
- Let $L = \mathfrak{sp}_n(\mathbb{F})$, with even $n \geq 6$. Let $L = \bigoplus_{g \in G} L_g$ be a G -grading. Then the graded algebra L is isomorphic to $\mathcal{C}(G, T, \beta, \kappa, \gamma, \tau)$,
- Let $L = \mathfrak{so}_n(\mathbb{F})$, with even $n \geq 10$. Let $L = \bigoplus_{g \in G} L_g$ be a G -grading. Then the graded algebra L is isomorphic to $\mathcal{D}(G, T, \beta, \kappa, \gamma, \tau)$,

as in Definition 5.1. Also, under the above restrictions on n , we have the following:

- $\mathcal{B}(G, \kappa_1, \gamma_1) \cong \mathcal{B}(G, \kappa_2, \gamma_2)$ if and only if $(\kappa_1, \gamma_1) \approx (\kappa_2, \gamma_2)$;
- $\mathcal{C}(G, T_1, \beta_1, \kappa_1, \gamma_1, \tau_1) \cong \mathcal{C}(G, T_2, \beta_2, \kappa_2, \gamma_2, \tau_2)$ if and only if $T_1 = T_2$, $\beta_1 = \beta_2$ and $(\kappa_1, \gamma_1, \tau_1) \approx (\kappa_2, \gamma_2, \tau_2)$;
- $\mathcal{D}(G, T_1, \beta_1, \kappa_1, \gamma_1, \tau_1) \cong \mathcal{D}(G, T_2, \beta_2, \kappa_2, \gamma_2, \tau_2)$ if and only if $T_1 = T_2$, $\beta_1 = \beta_2$ and $(\kappa_1, \gamma_1, \tau_1) \approx (\kappa_2, \gamma_2, \tau_2)$.

□

REFERENCES

- [1] Bahturin, Y.; Kochetov, M. *Group gradings on the Lie algebra \mathfrak{psl}_n in positive characteristic*. J. Pure and Applied Algebra, **213** (2009), no. 9, 1739–1749.
- [2] Bahturin, Y.; Kochetov, M.; Montgomery, S. *Group gradings on simple Lie algebras in positive characteristic*. Proc. Amer. Math. Soc., **137** (2009), no. 4, 1245–1254.
- [3] Bahturin, Y.; Sehgal, S.; and Zaicev, M. *Group gradings on associative algebras*. J. Algebra, **241** (2001), no. 2, 677–698.
- [4] Bahturin, Y.; Shestakov, I.; Zaicev, M. *Gradings on simple Jordan and Lie algebras*. J. Algebra, **283** (2005), no. 2, 849–868.
- [5] Bahturin, Yuri; Zaicev, M. *Involutions on graded matrix algebras*. J. Algebra, **315** (2007), no. 2, 527–540.
- [6] Bahturin, Y.; Zaicev, M. *Graded algebras and graded identities*. Polynomial identities and combinatorial methods (Pantelleria, 2001), 101–139, Lecture Notes in Pure and Appl. Math., **235**, Dekker, New York, 2003.
- [7] Bahturin, Y.; Zaicev, M. *Group gradings on matrix algebras*. Canad. Math. Bull., **45** (2002), no. 4, 499–508.
- [8] Bahturin, Y.; Zaicev, M. *Gradings on Simple Lie Algebras of Type “A”*. J. Lie Theory, **16** (2006), no. 4, 719–742.
- [9] Bahturin, Y.; Zaicev, M. *Gradings on simple algebras of finitary matrices*. To appear in J. Algebra. (Preprint arXiv:0906.4595v1 [math.RA].)
- [10] Beidar, K. I.; Brešar, M.; Chebotar, M. A.; Martindale, W. S., 3rd. *On Herstein’s Lie map conjectures. III*. J. Algebra, **249** (2002), no. 1, 59–94.
- [11] Chasov, A. A. *Isomorphisms of graded matrix algebras*. (Russian. Russian summary) Vestnik Moskov. Univ. Ser. I Mat. Mekh. 2004, , no. 5, 12–18, 70; translation in Moscow Univ. Math. Bull. **59** (2004), no. 5, 12–18 (2005)
- [12] Elduque, A. *Gradings on octonions*. J. Algebra, **207** (1998), no. 1, 342–354.
- [13] Elduque, A. *Fine gradings on simple classical Lie algebras*. arXiv:0906.0655v1 [math.RA].
- [14] Havlíček, M.; Patera, J.; Pelantová, E. *On Lie gradings. II*. Linear Algebra Appl., **277** (1998), no. 1-3, 97–125.
- [15] Jacobson, N. *Lie algebras*. Dover Publications, Inc., New York, 1979.
- [16] Kochetov, M. *Gradings on finite-dimensional simple Lie algebras*. Acta Appl. Math., **108** (2009), 101–127.
- [17] Montgomery, S. *Hopf Algebras and their Actions on Rings*. CBMS Regional Conference Series in Mathematics, **82**, American Math. Soc., Providence, RI, 1993.
- [18] Seligman, G. B. *Modular Lie algebras*. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 40 Springer-Verlag New York, Inc., New York, 1967.
- [19] Waterhouse, W. C. *Introduction to affine group schemes*. Graduate Texts in Mathematics, **66**, Springer-Verlag, New York-Berlin, 1979.

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