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COMPLEX SPECTRAL DEFORMATION & OPEN QUANTUM SYSTEMS

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- *Decoherence and Thermalization*, Phys. Rev. Lett. **98**, 130401 (2007), quant-ph/0608181 (2006)
- *Resonance theory of decoherence and thermalization*, to appear in Ann. Phys. (2007), quant-ph/0702207.

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1 OPEN QUANTUM SYSTEMS

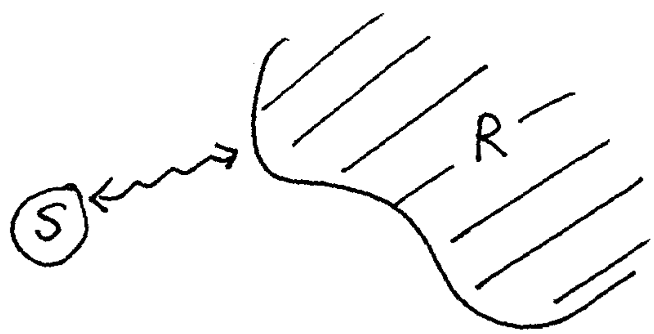
Open system S: connected to environment R
S = system of interest, e.g. a few spins

Environment R ("reservoir"): large compared to S
- characterized by macroscopic quantities ($T, \mu, \rho, ...$)
- dissipation, irreversible processes
irreversibility $\leftrightarrow$ size of R $\leftrightarrow$ large times

Coupling S $\leftrightarrow$ R: induces irreversible processes of S
e.g. S approaches temperature of R

Three classes of systems built from R, S
1) systems close to equilibrium $\longleftarrow$ DECOHERENCE
2) systems far from equilibrium
3) repeated interaction systems

1) S + R: **systems close to equilibrium**



Example: array of qubits (quantum register) interacting with a substrate

Effects: thermalization and decoherence

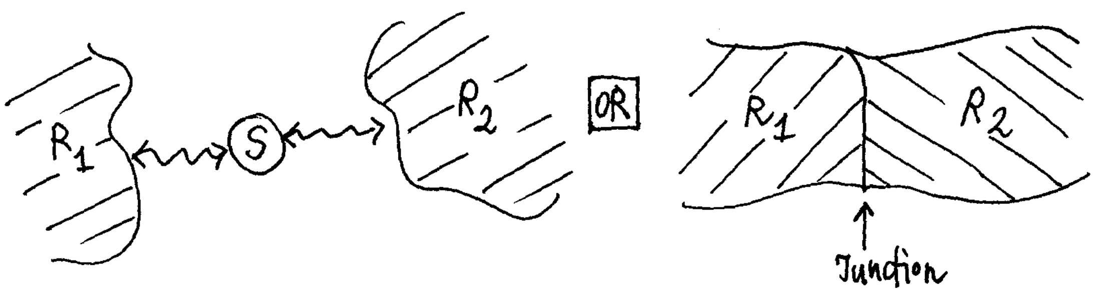
Thermalization: S + R $\longrightarrow$ equilibrium of coupled system, as $t \to \infty$

Decoherence: disappearence of phase relations

$$\sum_{j,k} c_{j,k} |\psi_j\rangle\langle\psi_k| \longrightarrow \sum_n p_n |\psi_n\rangle\langle\psi_n|, \quad \text{as } t \to \infty$$

$\Rightarrow$ Suppression of quantum effects

2) $S + R_1 + R_2$: systems far from equilibrium



Example: junction of two pieces of metal

Phenomena:

- approach of Non-Equilibrium Stationary State (NESS)

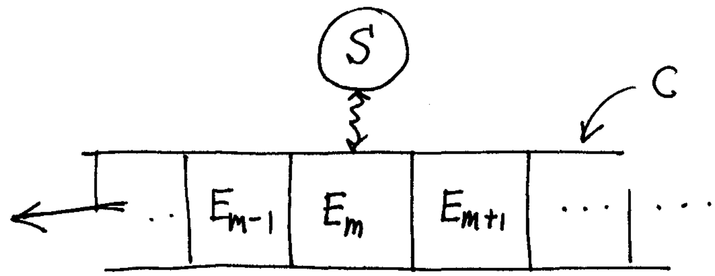
$$S + R_1 + R_2 \longrightarrow \text{NESS}, \quad \text{as } t \to \infty$$

- fluxes of energy/matter, entropy production

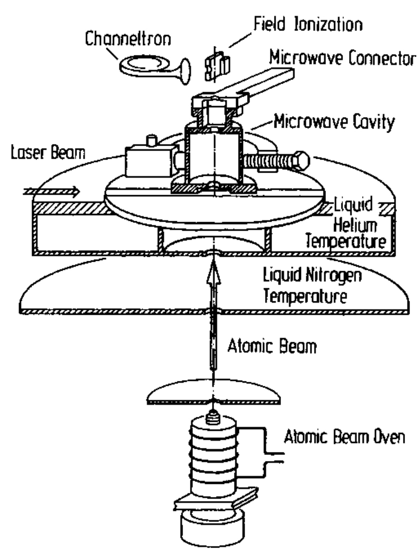
$$\frac{d}{dt}\langle \text{energy of } R_1 \rangle_{\text{NESS}} \propto T_2 - T_1$$

$$\langle \text{entropy production} \rangle_{\text{NESS}} > 0$$

3) $\mathrm{S} + \mathrm{C}$, $\mathrm{C} = \mathrm{E}_1 + \mathrm{E}_2 + \cdots$: **Repeated interaction systems**



Example: One-Atom Maser[1]



Phenomena & applications:

- approach of asymptotic state (periodic, RIAS)
- control of S by variation of interaction
- monitoring of S

[1]Meschede et al, PRL **54**, 551 (1985)

2 DECOHERENCE

Open quantum system S + R:

−Hilbert space	$\mathfrak{H} = \mathfrak{H}_{\rm S} \otimes \mathfrak{H}_{\rm R}$
−pure state	$\psi \in \mathfrak{H},\ \|\psi\| = 1$
−observables	self-adjoint operators on $\mathfrak{H}$
−average	$\langle A \rangle = \langle \psi, A\psi \rangle$
−Hamiltonian	$H = H_{\rm S} + H_{\rm R} + \lambda v$

($\lambda \in \mathbb{R}$: coupling constant, v: interaction S $\leftrightarrow$ R)

Evolution: $\psi_t = {\rm e}^{-{\rm i}tH}\psi$ (Schrödinger equation)

General state: density matrix $\rho = \sum_n p_n |\psi_n\rangle\langle\psi_n|$,

$$\rho_t = {\rm e}^{-{\rm i}tH} \rho {\rm e}^{{\rm i}tH}$$

Average of A in state ρ at time t: $\langle A_{\rm S}\rangle_t = {\rm Tr}_{\rm R+S}(\rho_t A)$

Reduction to system S: $A = A_{\rm S} \otimes \mathbb{1}_{\rm R} \ \Rightarrow$

$$\langle A_{\rm S}\rangle_t = {\rm Tr}_{\rm S+R}(\rho_t(A_{\rm S} \otimes \mathbb{1}_{\rm R})\,) = {\rm Tr}_{\rm S}(\overline{\rho}_t A_{\rm S})$$

Reduced density matrix of S: $\overline{\rho}_t = {\rm Tr}_{\rm R}(\rho_t)$
(trace taken over $\mathfrak{H}_{\rm R}$)

Matrix representation in fixed basis $\{\varphi\}_{j=1}^{N}$ of $\mathfrak{H}_{\mathrm{S}}$

$$[\overline{\rho}_t]_{m,n} := \langle \varphi_m, \overline{\rho}_t \varphi_n \rangle$$

A definition of decoherence: vanishing of off-diagonals as $t \to \infty$,

$$\lim_{t\to\infty} [\overline{\rho}_t]_{m,n} = 0, \quad \forall m \neq n.$$

Decoherence = basis dependent notion of disappearance of correlations,

$$\overline{\rho}_t = \sum_{m,n} c_{m,n}(t) |\varphi_m\rangle\langle\varphi_n| \longrightarrow \sum_m p_m(t) |\varphi_m\rangle\langle\varphi_m|,$$

as $t \to \infty$.

Class of explicitly solvable models:
Non-demolition models, H_{S} conserved: processes of absorption and emission of quanta of the reservoir by the system S are suppressed. To enable such processes, need $[H_{\mathrm{S}}, v] \neq 0$. But then will also have **thermalization**!

$\rho(\beta, \lambda)$: equilibrium state of total system at temperature $T = 1/\beta$

Thermalization: for any observable A of total system,

$$\mathrm{Tr}_{\mathrm{S+R}}(\rho_t A) \longrightarrow \mathrm{Tr}_{\mathrm{S+R}}(\rho(\beta,\lambda) A), \qquad \text{as } t \to \infty$$

This implies

$$\overline{\rho}_t \to \overline{\rho}_\infty(\beta, \lambda) := \mathrm{Tr}_{\mathrm{R}}(\overline{\rho}(\beta, \lambda)), \qquad \text{as } t \to \infty$$

Expansion of $\overline{\rho}_\infty(\beta, \lambda)$ in coupling constant:

$$\overline{\rho}_\infty(\beta, \lambda) = \overline{\rho}_\infty(\beta, 0) + O(\lambda)$$

where $\overline{\rho}_\infty(\beta, 0)$ is **Gibbs state** of system S. Now Gibbs state (density matrix) is *diagonal* in energy basis (H_{S}), but correction term $O(\lambda)$ is *not*, in general.

$\Rightarrow$ *Even if* S *is initially in incoherent superposition of energy eigenstates, it will acquire some* "residual coherence" *of order* $O(\lambda)$ *during the process of thermalization.*

$\Rightarrow$ Define decoherence as decay of off-diagonals of $\overline{\rho}_t$ to limit values (= off-diagonals of $\overline{\rho}_\infty(\beta, \lambda)$)

In (vast) literature on this topic we have encountered only

- models with energy-conserving interactions (which are explicitly solvable)
- models with markovian approximations (master equations, Lindblad dynamics, with uncontrolled errors)

Our goal:
Describe decoherence for systems which may also exhibit thermalization, in a rigorous fashion (controlled perturbation expansions)

Main tool: dynamical resonance theory based on complex deformations and recent progress in theory of open quantum systems

3 RESULTS ON DECOHERENCE

S: N-level system, energies $\{E_j\}_{j=1}^N$

R: free massless Bose field ($\omega(k) = |k|$, spatially ∞ extended)

Standard coupling: $\lambda v = \lambda\, G \otimes \varphi(g)$

For observables A of S we set

$$\begin{aligned} \langle A \rangle_t &:= \mathrm{Tr}_{\mathrm{S}}(\overline{\rho}_t A) \\ \langle\!\langle A \rangle\!\rangle_\infty &:= \lim_{T\to\infty} \frac{1}{T} \int_0^T \langle A \rangle_t \mathrm{d}t \end{aligned}$$

Theorem 1. *There is a $\lambda_0 > 0$ s.t. the following statements hold for $|\lambda| < \lambda_0$.*

1. *$\langle\!\langle A \rangle\!\rangle_\infty$ exists for all A*
2. *We have*

$$\langle A \rangle_t - \langle\!\langle A \rangle\!\rangle_\infty = \sum_{\varepsilon \neq 0} \mathrm{e}^{\mathrm{i}t\varepsilon} R_\varepsilon(A) + O(\lambda^2 \mathrm{e}^{-\omega t}),$$

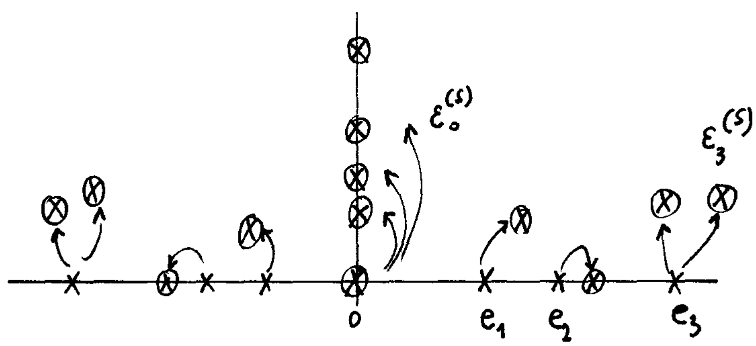
   where the ε are resonance energies, *$0 \leq \mathrm{Im}\varepsilon < \omega$, and $R_\varepsilon(A)$ are linear functionals of A which depend on the initial state $\rho_{t=0}$.*

3. *Let e be an eigenvalue of the operator $H_{\mathrm{S}} \otimes \mathbb{1}_{\mathrm{S}} - \mathbb{1}_{\mathrm{S}} \otimes H_{\mathrm{S}}$ (acting on $\mathfrak{H}_{\mathrm{S}} \otimes \mathfrak{H}_{\mathrm{S}}$). For $\lambda = 0$ each*

ε coincides with one of the e and we have the following expansion for small λ

$$\varepsilon \equiv \varepsilon_e^{(s)} = e - \lambda^2 \delta_e^{(s)} + O(\lambda^4).$$

The $\delta_e^{(s)} \in \mathbb{C}$ are eigenvalues of explicit matrices, satisfying $\mathrm{Im}(\delta_e^{(s)}) \leq 0$.



Furthermore, we have

$$R_\varepsilon(A) = \sum_{(m,n)\in I_e} \varkappa_{m,n} A_{m,n} + O(\lambda^2),$$

with $I_e = \{(m,n) \mid E_m - E_n = e\}$, and where $A_{m,n}$ is the (m,n)-matrix element of A and the numbers $\varkappa_{m,n}$ depend on the initial state.

Discussion.

• Detailed picture of dynamics: resonance energies ε and functionals R_ε can be calculated for concrete

• In absence of interaction ($\lambda = 0$) we have $\varepsilon = e \in \mathbb{R}$. Depending on interaction, each resonance energy ε may migrate into upper complex plane, or it may stay on real axis, as $\lambda \neq 0$.

• Averages $\langle A \rangle_t$ approach their ergodic means $\langle\langle A \rangle\rangle_\infty$ if and only if $\text{Im}\varepsilon > 0$ for all $\varepsilon \neq 0$. In this case, convergence is on time scale $[\text{Im}\varepsilon]^{-1}$. Otherwise $\langle A \rangle_t$ oscillates.

• Sufficient condition for decay: $\text{Im}\delta_e^{(s)} < 0$ (and λ small).

4 APPLICATION TO QUBIT (SPIN 1/2)

$$\mathfrak{H}_{\rm S} = \mathbb{C}^2, \qquad H_{\rm S} = \mathrm{diag}(E_1, E_2)$$

Let

$$\Delta = E_2 - E_1 > 0, \qquad \varphi_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \varphi_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Coupling operator

$$v = \begin{bmatrix} a & c \\ \overline{c} & b \end{bmatrix} \otimes \varphi(g)$$

Theorem 1 $\Longrightarrow$ For all $t \geq 0$,

$$\begin{aligned}[\overline{\rho}_t]_{1,1} - \langle\!\langle |\varphi_1\rangle\langle\varphi_1| \rangle\!\rangle_\infty &= \mathrm{e}^{\mathrm{i}t\varepsilon_0(\lambda)}[C_0 + O(\lambda^2)] \\ &\quad + \mathrm{e}^{\mathrm{i}t\varepsilon_\Delta(\lambda)}O(\lambda^2) + \mathrm{e}^{\mathrm{i}t\varepsilon_{-\Delta}(\lambda)}O(\lambda^2) \\ &\quad + O(\lambda^2 \mathrm{e}^{-t\omega})\end{aligned}$$

$$\begin{aligned}[\overline{\rho}_t]_{1,2} - \langle\!\langle |\varphi_2\rangle\langle\varphi_1| \rangle\!\rangle_\infty &= \mathrm{e}^{\mathrm{i}t\varepsilon_\Delta(\lambda)}[C_0 + O(\lambda^2)] \\ &\quad + \mathrm{e}^{\mathrm{i}t\varepsilon_0(\lambda)}O(\lambda^2) + \mathrm{e}^{\mathrm{i}t\varepsilon_{-\Delta}(\lambda)}O(\lambda^2) \\ &\quad + O(\lambda^2 \mathrm{e}^{-t\omega})\end{aligned}$$

C_0, C_Δ: explicit constants, depend on initial state $\rho_{t=0}$

Have explicit expansion of resonance energies ε.

Thermalization time: $\omega_{\rm th} := [{\rm Im}\varepsilon_0(\lambda)]^{-1}$
Decoherence time: $\omega_{\rm dec} := [{\rm Im}\varepsilon_\Delta(\lambda)]^{-1}$

$$\frac{\omega_{\rm dec}}{\omega_{\rm th}} = \frac{1}{2}\left[1 + \frac{(b-a)^2}{|c|^2}\, C(T)\right] + O(\lambda^2),$$

where $C(T) \sim T$ for small T

5 DYNAMICAL RESONANCE THEORY

1. *Resolvent representation*

Observable A of system S:

$$\begin{aligned}\langle A\rangle_t &= \mathrm{Tr}_{\mathrm{S}}[\overline{\rho}_t A] \\ &= \mathrm{Tr}_{\mathrm{S+R}}[\rho_t A] \\ &= \left\langle \psi_0, \mathrm{e}^{\mathrm{i}tK} A\psi_0\right\rangle\end{aligned}$$

In last step, we pass to the *representation Hilbert space* of system (the GNS Hilbert space), where initial density matrix is represented by a *vector* ψ_0.

Resolvent representation

$$\mathrm{e}^{\mathrm{i}tK} = \frac{-1}{2\pi\mathrm{i}}\int_{\mathbb{R}-\mathrm{i}} (K-z)^{-1}\mathrm{e}^{\mathrm{i}tz}\mathrm{d}z$$

$$\Rightarrow \langle A\rangle_t = \frac{-1}{2\pi\mathrm{i}}\int_{\mathbb{R}-\mathrm{i}} \left\langle \psi_0, (K_\lambda - z)^{-1}A\,\psi_0\right\rangle \mathrm{e}^{\mathrm{i}tz}\mathrm{d}z \qquad (1)$$

2. *Uncovering resonances*

Deformation transformation: $U(\omega) = \mathrm{e}^{-\mathrm{i}\omega D}$, "generator of translations D" (explicit)

Transformed generator of dynamics

$$K_\lambda(\omega) = U(\omega)K_\lambda U(\omega)^{-1} = L_0 + \omega N + \lambda I(\omega)$$

$U(\omega)$ unitary for $\omega \in \mathbb{R} \Rightarrow \quad \mathrm{spec}(K_\lambda) = \mathrm{spec}(K_\lambda(\omega))$

$K_\lambda(\omega)$ analytic for $\omega \in \mathbb{C}$, $|\mathrm{Im}\ \omega| < 2\pi T$

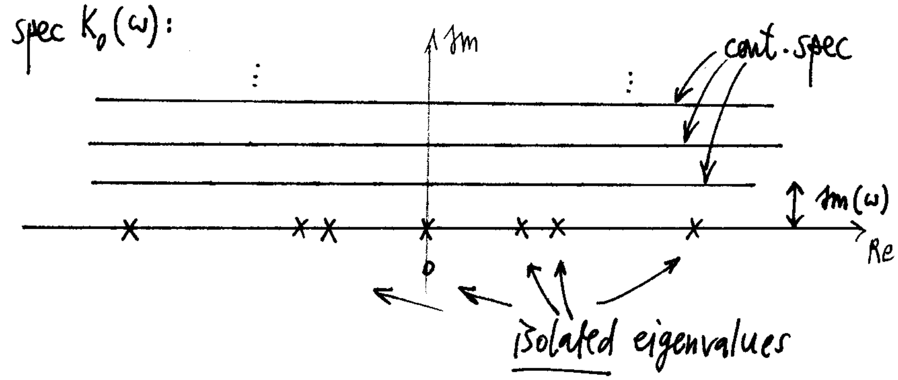
spec$(K_\lambda(\omega))$ varies as Im(ω) does $\Rightarrow$ **spectral deformation**

$U(\omega)\psi_0 = \psi_0$ & analyticity of $K_\lambda(\omega)$ & (1) $\Rightarrow$

$$\langle A \rangle_t = \frac{-1}{2\pi \mathrm{i}} \int_{\mathbb{R}-\mathrm{i}} \big\langle \psi_0, (K_\lambda(\omega) - z)^{-1} A\, \psi_0 \big\rangle\, \mathrm{e}^{\mathrm{i}tz} \mathrm{d}z$$

The point: spectrum of $K_\lambda(\omega)$ much easier to analyze than that of K_λ! $K_0(\mathrm{i}\omega') = L_0 + \mathrm{i}\omega' N$:

$$\mathrm{spec}(K_0(\mathrm{i}\omega')) = (\{E_i - E_j\}_{i,j=1,\ldots,N}) \cup_{n\geq 1} (\mathrm{i}\omega' n + \mathbb{R}).$$

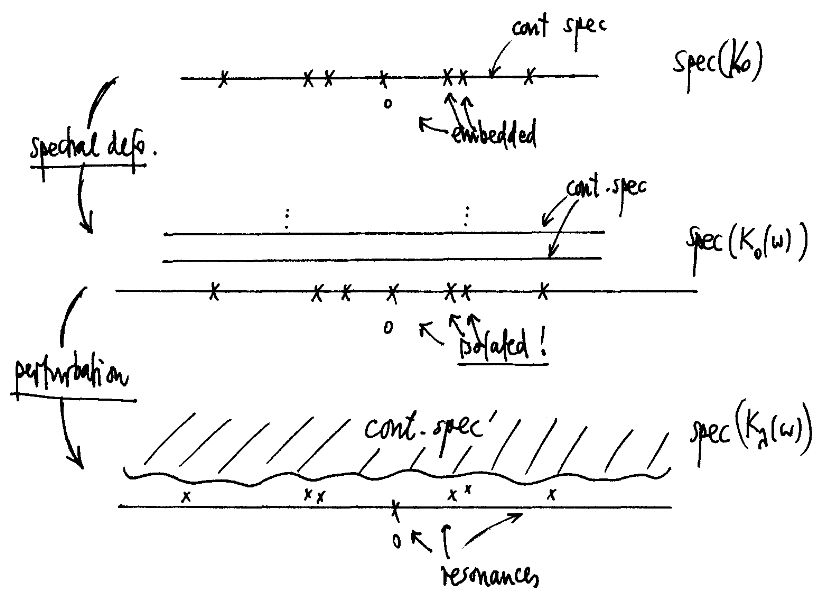


Gap of size ω' separating eigenvalues from the continuous spectrum of $K_0(\omega) \Rightarrow$ can follow location of eigenvalues by simple (analytic) perturbation theory, provided λ is small compared to ω'

Theorem 1.1 *Fix* $\omega' > 0$. *There is a constant* $c_0 > 0$ *s.t. if* $|\lambda| \leq c_0/\beta$ *then, for all* ω *with* $\mathrm{Im}\omega > \omega'$, *the spectrum of* $K_\lambda(\omega)$ *in the complex half-plane* $\{\mathrm{Im}z < \omega'/2\}$ *is independent of* ω *and consists purely of the distinct eigenvalues*

$$\{\varepsilon_e^{(s)}(\lambda) \mid e \in \mathrm{spec}(L_{\mathrm{S}}), s = 1, \ldots, \nu(e)\},$$

where $1 \leq \nu(e) \leq \mathrm{mult}(e)$ *counts the splitting of the eigenvalue* e. *Moreover, we have* $\lim_{\lambda\to 0} |\varepsilon_e^{(s)}(\lambda) - e| = 0$ *for all* $s = 1, \ldots, \nu(e)$, *and we have* $\mathrm{Im}\varepsilon_e^{(s)}(\lambda) \geq 0$. *Also, the continuous spectrum of* $K_\lambda(\omega)$ *lies in the region* $\{\mathrm{Im}z \geq 3\omega'/4\}$.



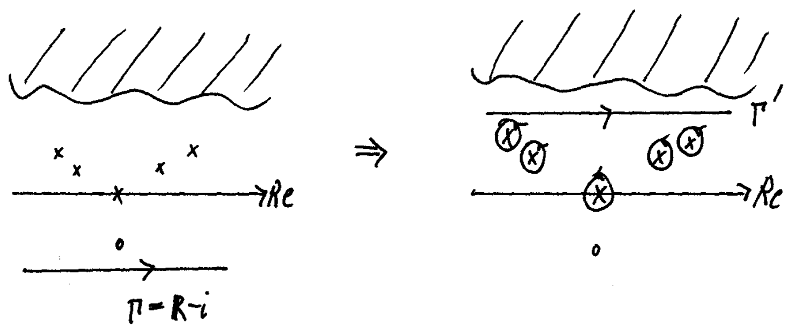
3. *Pole approximation*

Deform contour

$$z = \mathbb{R} - \mathrm{i} \;\mapsto\; z = \mathbb{R} + \mathrm{i}\omega'/2$$

$\Rightarrow$ pick up residues of poles of integrand, sitting at the resonance energies $\varepsilon_e^{(s)}(\lambda)$

$\mathcal{C}_e^{(s)}$: small circle around $\varepsilon_e^{(s)}$ not enclosing any other point of the spectrum of $K_\lambda(\omega)$



$$\int_\Gamma dz \longrightarrow \sum_j \int_{\mathcal{C}_j} dz + \int_{\Gamma'} dz$$

$$\Rightarrow \langle A \rangle_t = \sum_e \sum_{s=1}^{\nu(e)} \mathrm{e}^{\mathrm{i}t\varepsilon_e^{(s)}} \left\langle \psi_0, Q_e^{(s)} A\, \psi_0 \right\rangle + O(\lambda^2 \mathrm{e}^{-\omega' t/2})$$

$Q_e^{(s)}$: (non-orthogonal) Riesz projections

$$Q_e^{(s)} = Q_e^{(s)}(\omega, \lambda) = \frac{-1}{2\pi\mathrm{i}} \int_{\mathcal{C}_e^{(s)}} (K_\lambda(\omega) - z)^{-1} \mathrm{d}z$$

Finally

$$\begin{aligned} \langle\langle A \rangle\rangle_\infty &:= \lim_{T\to\infty} \frac{1}{T} \int_0^T \langle A \rangle_t \mathrm{d}t \\ &= \sum_{s':\ \varepsilon_0^{(s')}=0} \left\langle \psi_0, Q_0^{(s')} A\, \psi_0 \right\rangle \end{aligned}$$

All other terms vanish in the ergodic mean limit.

In specific models (like qubit), one can calculate (perturbatively in λ, to any order) resonance energies $\varepsilon_e^{(s)}$ and projection operators $Q_e^{(s)}$, and one obtains estimates on difference $\langle A \rangle_t - \langle\langle A \rangle\rangle_\infty$.

Evolution of reduced density matrix $[\overline{\rho}_t]_{m,n}$ is obtained from these formulas by using $A = |\varphi_n\rangle\langle\varphi_m|$.

——— **THE END** ———