Resonant Perturbation Theory of Decoherence, Relaxation and Evolution of Entanglement for Quantum Bits

Marco Merkli
Department of Mathematics, Memorial University, St. John’s, Canada

Collaborators:
Gennady Berman
Theoretical Division, Los Alamos National Laboratory, Los Alamos, USA
Fausto Borgonovi
Dipartimento di Matematica, Università Cattolica, Brescia, Italy
Michael Sigal
Department of Mathematics, University of Toronto, Toronto, Canada

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Outline

• Open quantum systems:
  Superconducting qubits in thermal environment

• New method: **Resonance approach**

• New results:
  – Expression for dynamics valid for all times
  – **Clustering** of matrix elements: classification of decoherence times
  – Application to non-integrable systems: decoherence-, entanglement survival/death/revival times

• Resolves some problems of master equation approach:
  – Incorrect results for times $t > (\text{coupling})^{-2}$
  – Incorrect final state due to $O(\text{coupling}^2)$ corrections long-lived metastable states
Open Quantum Systems

• Total system: “system S” + “reservoir R” + “interaction”
• S: superconducting qubit, atom, molecule, oscillator; few degrees of freedom
• R: collection of spins or oscillators; many degrees of freedom, in thermal equilibrium at temperature \( T \geq 0 \)
• Total system: Hamiltonian \( H = H_S + H_R + H_I \), dynamics of total density matrix \( \rho_{SR} \)

\[
\rho_{SR}(t) = e^{-i t H / \hbar} \rho_{SR}(0) e^{i t H / \hbar}
\]

• Reduced density matrix: \( \rho(t) = \text{Tr}_R \rho_{SR}(t) \) partial trace over R

• Time-scales:
  \( \tau_S \), isolated S \( \leftrightarrow \omega_S = (E - E')/\hbar \)
  \( \tau_{\text{relax}} \), relaxation time of S \( \leftrightarrow H_I \)
  \( \tau_R = \frac{\hbar}{k_B T} \), thermal reservoir correlation time
Quantum Optical Master Equation

[Legget et al. ‘81, Palma et. al. ‘96, Gardiner-Zoller ‘04, Weiss ‘99]

• Finite system coupled to bosonic reservoir

\[ H = H_S + \sum_k \hbar \omega_k a_k^\dagger a_k + G \sum_k g_k (a_k^\dagger + a_k) \]

\( H_S, G: N \times N \) matrices, \( g_k \): coupling function; reduced evolution

\[ \frac{d}{dt} \rho(t) = -\frac{i}{\hbar} \int_0^t \text{Tr}_R \left[ H_I(t), [H_I(s), \rho_{RS}(s)] \right] ds \]

• Born-Markov approximation: system relaxation much slower than decay of reservoir correlations (memory effects weak) + Rotating wave approximation: syst. relax. much slower than free system dynamics

⇒ Quantum Optical Regime: \( \max\{\tau_R, \tau_S\} \ll \tau_{\text{relax}} \)

→ Lindblad form of Master Equation: \( \rho(t) = e^{t\mathcal{L}} \rho(0), \) markovian
Quantum Brownian Motion Master Equation

Damped harmonic oscillator \cite{Caldeira-Leggett '83, Haake-Reibold '84, Unruh-Zurek '89, Hu-Paz-Zhang '92}

\[
H = \frac{p_0^2}{2m_0} + \frac{1}{2}m_0\omega_0^2q_0^2 + \sum_{n=1}^{N} \left[ \frac{p_n^2}{2m_n} + \frac{1}{2}m_n\omega_n^2q_n^2 \right] + q_0 \sum_{n=1}^{N} g_n q_n
\]

Quadratic hamiltonian $\Rightarrow$ exact master equation (position representation):

\[
i\hbar\frac{\partial}{\partial t} \rho(x, x', t) = F \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial x'}, t \right) \rho(x, x', t)
\]

$F$: complicated function encoding effects of reservoir

Quantum Brownian Motion Regime: $\tau_R \ll \tau_{\text{relax}}$ and $\tau_R \ll \tau_S$

$\rightarrow$ Caldeira-Leggett master equation; $\rho(x, x, t)$ follows classical BM
Spectral density

- Effect of reservoir characterized by **spectral density**

\[ J(\omega) = \sum_n \delta(\omega - \omega_n) \frac{g_n^2}{\omega_n} \]

- Limit of continuous modes of reservoir:

\[ J(\omega) = \gamma \omega \left( \frac{\omega}{\Lambda} \right)^{n-1} e^{-\omega^2/\Lambda^2} \]

\( \gamma > 0 \): measures overall size of coupling \( g_n \)
\( \Lambda \): UV cutoff parameter (other forms of cutoff possible)

- \( n = 1 \) **ohmic**, \( n > 1 \) **superohmic**, \( n < 1 \) **subohmic** reservoir
Resonance Theory

$N$-level system coupled to reservoir(s)

$$H = H_S + \sum_k \hbar \omega_k a_k^\dagger a_k + \lambda G \sum_k g_k (a_k^\dagger + a_k)$$

$\lambda$: coupling constant; free dynamics ($\lambda = 0$)

$$[\rho_t]_{mn} = e^{it(E_n - E_m)/\hbar}[\rho_0]_{mn}$$

Effects of coupling to reservoirs:

- **Irreversibility, energies become complex:**
  $$E_n - E_m \Rightarrow E_n - E_m + \lambda^2 \delta_{E_n - E_m} + O(\lambda^4)$$
  with $\text{Im} \delta_{E_n - E_m} \geq 0$ (decay!)

- **Clustering, $[\rho_t]_{mn}$ determined by $[\rho_0]_{kl}$ with $(k, l) \sim (m, n)$:**
  $$[\rho_t]_{mn} = F_t([\rho_0]_{kl} : E_k - E_l = E_m - E_n) + O(\lambda^2)$$
Method

- **Complex scaling**, complex spectral deformation” à la Balslev-Combes ‘71 (Schrödinger operators), Feshbach resonance method
- $H$ replaced by “**non-hermitian Hamiltonian**” $K$:
  - complex eigenvalues $=$ resonance energies
  - eigenvectors $=$ metastable states

- **Time-scales**:
  $$\tau_S = \max_{E \neq E'} \frac{\hbar}{E - E'}, \quad \tau_R = \frac{\hbar}{k_B T}, \quad \tau_{\text{relax}} \propto \lambda^{-2}$$

- **Assumptions**:
  - **infra-red**: $J(\omega) \sim \omega^n$ for $\omega \to 0$, $n = -1, 1, 3, 5, \ldots$
  - **ultra-violet**: $J(\omega) \sim e^{-\omega/\Lambda}, e^{-\omega^2/\Lambda^2}$ (or similar) for $\omega \to \infty$
  - $\lambda$ small: $\max\{\tau_S, \tau_R\} \ll \tau_{\text{relax}}$
  - $\Lambda^3 \ll \lambda^{-2}$
Result

For all times \( t \geq 0 \),

\[
[\rho_t]_{mn} = \sum_{(k,l) \in C(E_m - E_n)} A_t(m,n;k,l)[\rho_0]_{kl} + O(\lambda^2)
\]

\( O(\lambda^2) \) independent of \( t \); different matrix element clusters

\[
C(E_m - E_n) := \{(k,l) : E_k - E_l = E_m - E_n\}
\]

evolve independently; each cluster \textbf{markovian} evolution; Chapman-Kolmogorov relation

\[
A_{t+s}(m,n;k,l) = \sum_{(p,q) \in C(E_m - E_n)} A_t(m,n;p,q)A_s(p,q;k,l)
\]
• Markov transition amplitudes $A_t$ given by resonance data:

$$A_t(m, n; k, l) = \sum_{s=1}^{\text{mult}(E_n - E_m)} e^{it\epsilon_{E_n - E_m}^{(s)}} C_{k,l;m,n}(s)$$

$C_{k,l;m,n}(s)$: overlap coefficient between resonance- and energy states of S

References:
Merkli-Berman-Borgonovi-Gebresellasie, *submitted* 2010 (entanglement)
Comparison: Master Equation and Resonance Approach

Advantages of RA

- **Extended time-range**
  RA valid for $t \geq 0$, while ME resolves only times $t < \lambda^{-2}$:
  - even for single qubit: ME predicts asymptotically Gibbs state $\propto e^{-\beta H_S}$, but true final state has corrections $O(\lambda^2)$ to Gibbs state
  - $H_S$ degenerate levels $\Rightarrow$ metastable states with lifetimes $\propto \lambda^{-n}$, $n > 2$; ME predicts wrong stationary states

- **Cluster Classification**
  - different time-scales: each cluster has own decay = decoherence time
  - cluster containing diagonal relaxes to thermal values
  - initially not populated clusters stay small $O(\lambda^2)$ forever
  - for given quantum algorithm only a few clusters may be important
    $\Rightarrow$ only a few decoherence rates need analyzing

- **Applicability and Rigor**
  RA applies to not exactly solvable systems, rigorous error control homogeneously in time, coincides with ME results where latter applicable
Limitations of RA

- RA (and MA) does not generally resolve variations of quantities of $O(\lambda^2)$
- RA assumes finite number $N$ of degrees of freedom of $S$, due to condition $\tau_S \ll \tau_{\text{relax}}$, i.e., $\lambda^2 \ll \min(E - E') \sim 2^{-N}$
- Exact models show: for short times $t < \tau_\beta$, true dynamics can deviate significantly from markovian approximation (“initial slip”): both ME and RA may produce density matrices having negative eigenvalues (however RA correct up to $O(\lambda^2)$)

Possible extensions of RA

- Non-markovian corrections: matrix element clusters start to interact, time-homogeneous error reduced to $O(\lambda^4)$, or smaller
- Overlapping resonances: $\max(E - E'), \lambda^2 \ll \min\{1, k_B T\}$
- Time-dependent Hamiltonians: e.g. $H_S(t), H_I(t)$ (slow variation and sudden jumps in two-level $H_S$: [Merkli-Starr ‘09])
Resonance Theory: Decoherence

\(N\)-qubit register \textit{collectively} coupled to single bosonic reservoir \((\hbar = 1)\)

\[
H_S = \sum_{j=1}^{N} B_j S_j^z + \sum_{i,j=1}^{N} J_{ij} S_i^z S_j^z, \quad H_R = \sum_k \omega_k a_k^\dagger a_k
\]

\(B_j\): magnetic field at the location of spin \(j\), \(J_{ij}\): pair interaction constants

- Interaction: collective energy conserving and energy exchange

\[
H_I = \lambda_1 \sum_{j=1}^{N} S_j^z \otimes \phi(g_1) + \lambda_2 \sum_{j=1}^{N} S_j^x \otimes \phi(g_2).
\]

- \(\phi(g_{1,2}) = \sum_k g_{1,2}(k)[a_k^\dagger + a_k]\)
- Energy basis: \(H_S \varphi_\sigma = E(\sigma) \varphi_\sigma, E(\sigma) = \sum_{j=1}^{N} B_j \sigma_j\)
- Bohr energies: \(e(\sigma, \tau) = E(\sigma) - E(\tau)\)
- Matrix element clusters: \(C(\sigma, \tau) = \{(\sigma', \tau') : e(\sigma, \tau) = e(\sigma', \tau')\}\)
Resonance representation of dynamics

\[
[rho_t]_{\sigma,\tau} = \sum_{(\sigma',\tau') \in C(\sigma,\tau)} \sum_{s=1}^{\text{mult}(e(\sigma,\tau))} \exp\{it\varepsilon_e^{(s)}_{e(\sigma',\tau')}\} \ C'(\sigma,\tau;\sigma',\tau') \ [\bar{\rho}_0]_{\sigma',\tau'} \\
+ O(\lambda^2_1 + \lambda^2_2)
\]

- Perturbation expansion: \( \varepsilon_e^{(s)} = e + \delta_e^{(s)} + O(\lambda^4_1 + \lambda^4_2) \)
Cluster decoherence rates

\[ \gamma_e = \min \left\{ \text{Im} \varepsilon_e^{(s)} : s = 1, \ldots, \text{mult}(e) \text{ s.t. } \varepsilon_e^{(s)} \neq 0 \right\} \]

- Thermalization rate: \( \gamma_{\text{therm}} = \gamma_0 \)
- Assume generic magnetic fields: given any \( n_j \in \{0, \pm 1, \pm 2\} \), the relation \( \sum_{j=1}^{N} B_j n_j = 0 \) implies \( n_j = 0 \) for all \( j \) (facilitates enumeration of register energies and eigenstates)
- Results

\[ \gamma_e = \left\{ \begin{array}{ll}
\lambda_1^2 y_0, & e = 0 \\
\lambda_1^2 y_1(e) + \lambda_2^2 y_2(e) + y_{12}(e), & e \neq 0
\end{array} \right\} + O(\lambda_1^4 + \lambda_2^4) \]

- \( y_1 \): due to energy conserving interaction; \( y_0, y_2 \): due to energy exchange interaction; \( y_{12} \): due to both interactions, \( O(\lambda_1^2 + \lambda_2^2) \).
- \( y_0 = 4\pi \min_{1 \leq j \leq N} \{ B_j^2 G_2(2B_j) \coth(\beta B_j) \} \) \( (G_2(x) \propto g_2(x)) \)
- \( y_1(e) = \frac{\pi}{2\beta} \gamma_+ e_0^2(e) \) \( (e_0(e) = \sum_{j=1}^{N} (\sigma_j - \tau_j), \gamma_+ = \lim_{|k| \to 0} |k| g_1(k)) \)
- \( y_2(e) = 2\pi \sum_{j: \sigma_j \neq \tau_j} B_j^2 G_2(2B_j) \coth(\beta B_j) \)
- $y_{12}(e) \geq 0$: more complicated expression; $> 0$ unless $\lambda_1$ or $\lambda_2$ or $e_0(e)$ or $\gamma_+$ vanish; $y_{12}(e)$ approaches constant values as $T \to 0, \infty$

- **Full decoherence** $\gamma_e > 0$ for all $e \neq 0$: occurs for $\lambda_2 \neq 0$ and $g_2(2B_j) \neq 0$ for all $j$ (provided $\lambda_1, \lambda_2$ small enough)

- **Dependence on register size $N$**
  - Thermalization rate $\gamma_0$ independent of $N$
  - Assume distribution of magnetic field $\langle \rangle$;

  $$
  \langle y_1 \rangle = y_1 \propto e_0^2, \quad \langle y_2 \rangle \propto D(\sigma - \tau), \quad \langle y_{12} \rangle \propto N_0(e),
  $$

  where $N_0(e) = \{ \#j : \sigma_j = \tau_j \}$, $D(\sigma - \tau) := \sum_{j=1}^{N} |\sigma_j - \tau_j|$ is **Hamming distance** ($N_0, D$ depend on $e$ only)

- Decoherence rates:
  - Pure energy-conserving interaction: $\gamma_e \propto \lambda_1^2 \left[ \sum_{j=1}^{N} (\sigma_j - \tau_j) \right]^2$, can be as large as $O(\lambda_1^2 N^2)$
  - Pure energy exchange interaction: $\gamma_e \propto \lambda_2^2 D(\sigma - \tau) \leq O(\lambda_2^2 N)$
Both interactions: additional term $\langle y_{12} \rangle = O((\lambda_1^2 + \lambda_2^2)N)$

Fastest decay rate of reduced off-diagonal density matrix elements:
- due to the energy conserving interaction alone $O(\lambda_1^2 N^2)$
- due to energy exchange interaction alone $O(\lambda_2^2 N)$
- relaxation of diagonal matrix elements $O(\lambda_1^2)$

Remarks:
- Local, energy-conserving interaction $\Rightarrow$ fastest decoherence rate $O(\lambda_1^2 N)$
- Assumption $\tau_S \ll \tau_{\text{relax}} \Leftrightarrow \lambda_{1,2}^2 \ll \Delta_N := \min^*_{\sigma,\tau} |E(\sigma) - E(\tau)|$
- Magnetic field roughly constant $B_j \sim B \Rightarrow \Delta_N \sim B$ indep. of $N$
Resonance Theory: Evolution of Entanglement

\[ H = H_{S1} + H_{S2} + H_{R1} + H_{R2} + H_{R0} + W \]

\[ W = \left\{ \begin{array}{l}
\lambda (S_1^x + S_2^x) \otimes \varphi_0(g) \\
+ \kappa (S_1^z + S_2^z) \otimes \varphi_0(f) \\
+ \mu (S_1^x \otimes \varphi_1(g) + S_2^x \otimes \varphi_2(g)) \\
+ \nu (S_1^z \otimes \varphi_1(f) + S_2^z \otimes \varphi_2(f))
\end{array} \right\} \text{collective} \]

\[ W = \left\{ \begin{array}{l}
\lambda (S_1^x + S_2^x) \otimes \varphi_0(g) \\
+ \kappa (S_1^z + S_2^z) \otimes \varphi_0(f) \\
+ \mu (S_1^x \otimes \varphi_1(g) + S_2^x \otimes \varphi_2(g)) \\
+ \nu (S_1^z \otimes \varphi_1(f) + S_2^z \otimes \varphi_2(f))
\end{array} \right\} \text{local} \]

energy exchange terms \( \lambda, \mu \), energy conserving terms \( \kappa, \nu \)

\[ H_{Sj} = B_j S_j^z, \quad B_j > 0 \text{ magnetic fields, } S_j^z \text{ Pauli matrix, energies } \pm B_j \]

\[ H_{Rj} = \sum_k \omega_k a_{j,k}^\dagger a_{j,k}, \quad R_j \text{ at temperature } T = 1/\beta \]

\[ \varphi_j(f) = \sum_k f_k a_{j,k}^\dagger + h.c. \]
• Magnetic fields: $0 < B_1 < B_2$ s.t. $\frac{B_2}{B_1} \neq 2$ (avoids degeneracies)
• Transition energies: $\{0, \pm 2B_1, \pm 2B_2, \pm 2(B_2 - B_1), \pm 2(B_1 + B_2)\}$
• Matrix element clusters: $\mathcal{C}_1, \ldots, \mathcal{C}_5$

\[
\begin{bmatrix}
* & \bullet & \bullet & \diamond \\
* & \diamond & \bullet \\
* & \bullet & \diamond \\
* & \bullet & \diamond \\
* & \bullet & \diamond
\end{bmatrix}
\quad (& \text{hermitian})
\]

\[
\sigma_f(\omega) = \coth(\beta\omega/2)J_f(\omega), \quad J_f(\omega) = \sum_k f_k^2 \delta(\omega - \omega_k) \quad \text{spectral density}
\]

Coupling functions $f = \text{energy exchange}, \ g = \text{energy conserving}$

\[
Y_2 = \frac{1}{2} \left| \text{Im} \left[ 16\kappa_1^2\kappa_2^2 r^2 - (\lambda_2^2 + \mu_2^2)^2 \sigma_g^2(2B_2) - 8i\kappa_1\kappa_2 (\lambda_2^2 + \mu_2^2) rr' \right]^{1/2} \right|
\]

\[
Y_3 = \frac{1}{2} \left| \text{Im} \left[ 16\kappa_1^2\kappa_2^2 r^2 - (\lambda_1^2 + \mu_1^2)^2 \sigma_g^2(2B_1) - 8i\kappa_1\kappa_2 (\lambda_1^2 + \mu_1^2) rr' \right]^{1/2} \right|
\]

where \[ r = \text{P.V.} \int_{\mathbb{R}^3} \frac{|f|^2}{|k|} d^3k, \quad rr' = 4\pi B_j^2 \int_{S^2} |g(2B_j, \Sigma)|^2 d\Sigma \]
Cluster decoherence rates

$2B_1, 2B_2$: qubit transition energies

\[
\begin{align*}
\gamma_{\text{therm}} &= \min_{j=1,2} \left\{ (\lambda_j^2 + \mu_j^2)\sigma_g(2B_j) \right\} + O(\alpha^4) \\
\gamma_2 &= \frac{1}{2}(\lambda_1^2 + \mu_1^2)\sigma_g(2B_1) + \frac{1}{2}(\lambda_2^2 + \mu_2^2)\sigma_g(2B_2) - Y_2 + (\kappa_1^2 + \nu_1^2)\sigma_f(0) + O(\alpha^4) \\
\gamma_3 &= \frac{1}{2}(\lambda_1^2 + \mu_1^2)\sigma_g(2B_1) + \frac{1}{2}(\lambda_2^2 + \mu_2^2)\sigma_g(2B_2) - Y_3 + (\kappa_2^2 + \nu_2^2)\sigma_f(0) + O(\alpha^4) \\
\gamma_4 &= (\lambda_1^2 + \mu_1^2)\sigma_g(2B_1) + (\lambda_2^2 + \mu_2^2)\sigma_g(2B_2) + \left[ (\kappa_1 - \kappa_2)^2 + \nu_1^2 + \nu_2^2 \right] \sigma_f(0) + O(\alpha^4) \\
\gamma_5 &= (\lambda_1^2 + \mu_1^2)\sigma_g(2B_1) + (\lambda_2^2 + \mu_2^2)\sigma_g(2B_2) + \left[ (\kappa_1 + \kappa_2)^2 + \nu_1^2 + \nu_2^2 \right] \sigma_f(0) + O(\alpha^4)
\end{align*}
\]
Discussion: decoherence rates

• Thermalization rate depends on energy-exchange coupling only.

• Purely energy-exchange interactions: \( \kappa_j = \nu_j = 0 \Rightarrow \) rates depend symmetrically on local and collective influence through \( \lambda_j^2 + \mu_j^2 \).

• Purely energy-conserving interactions: \( \lambda_j = \mu_j = 0 \Rightarrow \) rates do not depend symmetrically on local and collective terms. E.g. \( \gamma_4 \) may depend on local interaction only \( (\kappa_1 = \kappa_2) \).

• \( Y_1 \) and \( Y_2 \) contain products of exchange and conserving terms.
Entanglement evolution

• Entanglement of formation [Bennet et al ‘96] of two qubits ↔ concurrence [Wootters ‘97]:

\[
C(\rho) = \max\{0, D(\rho)\}, \quad D(\rho) = \sqrt{\nu_1} - [\sqrt{\nu_2} - \sqrt{\nu_3} - \sqrt{\nu_4}]
\]

\(\nu_1 \geq \nu_2 \geq \nu_3 \geq \nu_4 \geq 0\) eigenvalues of matrix \(\xi := \rho(\sigma^y \otimes \sigma^y)\bar{\rho}(\sigma^y \otimes \sigma^y)\)

• Dominant dynamics: only initially populated clusters have nontrivial dynamics

• Example: pure initial state \(\psi_0 = a|++\rangle + b|--\rangle\)

\[
\rho_0 = \begin{bmatrix}
p & 0 & 0 & u \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\bar{u} & 0 & 0 & 1 - p
\end{bmatrix} \Rightarrow \rho_t = \begin{bmatrix}
x_1(t) & 0 & 0 & u(t) \\
0 & x_2(t) & 0 & 0 \\
0 & 0 & x_3(t) & 0 \\
\bar{u}(t) & 0 & 0 & x_4(t)
\end{bmatrix} + O(\alpha^2)
\]
- Initial concurrence:  $C(\rho_0) = 2\sqrt{p(1-p)}$

- Dynamics

$$x_1(t) = pA_t(11; 11) + (1 - p)A_t(11; 44)$$
$$x_2(t) = pA_t(22; 11) + (1 - p)A_t(22; 44)$$
$$\vdots$$
$$u(t) = e^{it\varepsilon_2(B_1+B_2)}u(0)$$

$A_t(kk; ll) \leftarrow$ resonance energies bifurcating out of $e = 0$. Leading terms:

$$\delta_2 = (\lambda_1^2 + \mu_1^2)\sigma_g(B_1), \quad \delta_3 = (\lambda_2^2 + \mu_2^2)\sigma_g(B_2), \quad \delta_4 = \delta_2 + \delta_3$$

Leading term of $\text{Im} \varepsilon_2(B_1+B_2)$:

$$\delta_5 = \delta_2 + \delta_3 + [(\kappa_1 + \kappa_2)^2 + \nu_1^2 + \nu_2^2]\sigma_f(0)$$
Entanglement death/survival times

Take coupling s.t. $\delta_2, \delta_3 > 0$ (thermalization). There is a positive constant $\alpha_0$ (independent of $p$) s.t. if $0 < \alpha \leq \alpha_0 \sqrt{p(1-p)}$, then we have the following.

**Entanglement death time.** There is a constant $C_A > 0$ (independent of $p$, $\alpha$) such that concurrence $C(\rho_t) = 0$ for all $t \geq t_A$, where

$$t_A := \max \left\{ \frac{1}{\delta_5} \ln \left[ C_A \frac{\sqrt{p(1-p)}}{\alpha^2} \right], \frac{1}{\delta_2 + \delta_3} \ln \left[ C_A \frac{p(1-p)}{\alpha^2} \right] \right\}.$$

**Entanglement survival time.** There is a constant $C_B > 0$ (independent of $p$, $\alpha$) such that concurrence $C(\rho_t) > 0$ for all $t \leq t_B$, where

$$t_B := \frac{1}{\max\{\delta_2, \delta_3\}} \ln \left[ 1 + C_B \alpha^2 \right].$$
Discussion: entanglement evolution

• Result gives disentanglement bounds for the true dynamics of the qubits for non-integrable interactions
• Disentanglement time is finite since $\delta_2, \delta_3 > 0$ (which implies thermalization). If system does not thermalize then it may happen that entanglement stays nonzero for all times (it may decay or even stay constant)
• Rates $\delta_j$ are of order $\alpha^2$. Both $t_A$ and $t_B$ increase with decreasing coupling strength
Entanglement creation

Braun ‘02: energy conserving collective coupling, initial product state
\( \frac{1}{\sqrt{2}} (|+\rangle - |--\rangle) \otimes \frac{1}{\sqrt{2}} (|+\rangle + |--\rangle) \Rightarrow \) concurrence creation, death and revival

Dynamics in resonance approximation:

• Purely energy-exchange coupling
  
  \([\rho_t]_{mn}\) depends on \(\lambda^2 + \mu^2\) only \(\Rightarrow\) Creation of entanglement under purely collective and purely local energy-exchange dynamics is the same

• Purely energy-conserving coupling
  
  Evolution of the density matrix is not symmetric as function of \(\kappa\) (collective) and \(\nu\) (local). Absence of collective coupling (\(\kappa = 0\)): concurrence evolution independent of local coupling; however for \(\kappa \neq 0\) concurrence depends on \(\nu\) (numerical results).

• Full coupling
  
  Matrix elements evolve as complicated functions of all coupling parameters, showing that the effects of different interactions are correlated.
Numerical results: concurrence creation

Amount of entanglement created is independent of coupling $\kappa$; peak at $t_0 \approx 0.5\kappa^{-2}$; revival of entanglement $t_1 \approx 2.1\kappa^{-2}$.
Switching on local (energy conserving) coupling:

- creation of entanglement reduced (and delayed, $t_0 \propto (\kappa^2 + \nu^2)^{-1}$)
- local coupling exceeds collective one $\Rightarrow$ no concurrence is created
Energy-exchange collective and local interactions: $\lambda = \mu$ (symmetric); $\kappa = 0.02$ (collective, conserving), $\nu = 0$ fixed

- entanglement creation is reduced and peak time $t_0$ slightly reduced
- revival suppressed for increasing $\lambda$
- small times: density matrix in resonance approx. has partly negative eigenvalues (as Caldeira-Legget, Unruh-Zurek); numerics not reliable (non-smooth behavior in $\lambda$)
Conclusion

• New **resonance approach** to dynamics of open quantum systems:
  – Valid for all times $t \geq 0 \Rightarrow$ correct large-time behaviour
  – Cluster-wise independent markovian evolution $\Rightarrow$ different time scales
    $\Rightarrow$ simplification of analysis of quantum algorithms

• New results:
  – **Decoherence:**
    $N$ qubits, collective energy conserving $+$ exchange coupling
    Decoherence rates: cons. $\propto N^2$, exch. $\propto N$, both: $+$ interference term
  – **Entanglement:**
    Two qubits, collective $+$ local, energy conserving $+$ exchange coupling
    Concurrence survival/death times in terms of cluster deco. times
    Numerical analysis of concurrence creation, sudden death, revival