Quantum Measurements of Scattered Particles

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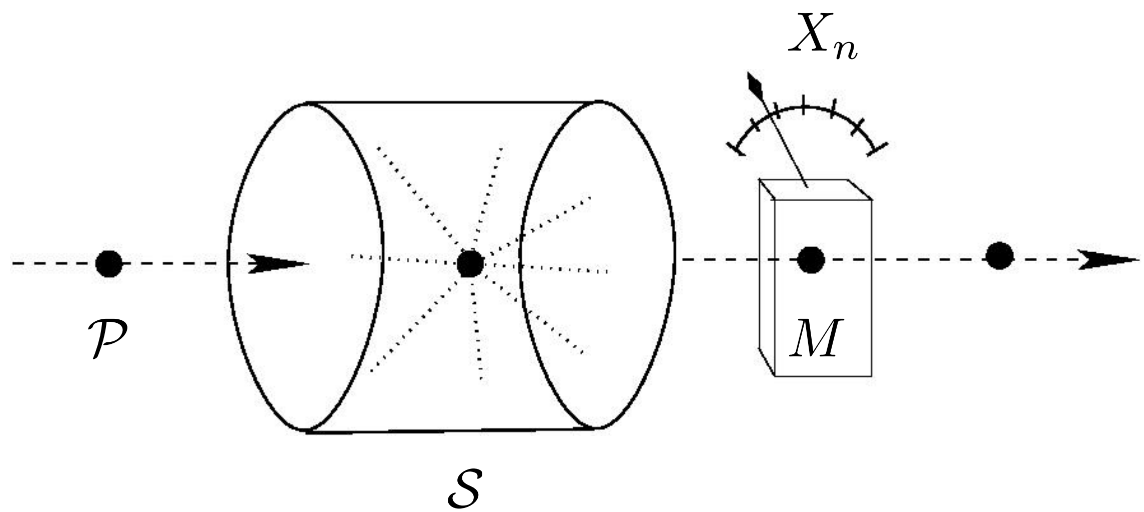
Measurement of scattered probes



$$\left.\begin{array}{lcl} \mathcal{P} & : & \text{probe} \\ \mathcal{S} & : & \text{scatterer} \\ M & : & \text{measurement} \end{array}\right\} \quad \{X_n\}_{n\geq 1} \quad \text{measurement history}$$

Incoming probe states: $\omega_{\rm in}$ independent and identical

Single probe-scatterer interaction: duration τ, operator V

Projective von Neumann measurement: operator M, $X_n \in \mathrm{spec}(M)$

- Both $\mathcal{S}$ and $\mathcal{P}$ are finite-dimensional quantum systems.
- The single probe - scatterer dynamics is generated by the Hamiltonian

$$H = H_{\mathcal{S}} + H_{\mathcal{P}} + V.$$

- The incoming probe states are stationary.
- During scattering, a new probe becomes entangled with $\mathcal{S}$, which is entangled with all previous probes $\Rightarrow X_n$ are dependent random variables.
- **Ergodicity assumption**

Without measurements the scattering process drives $\mathcal{S}$ to an asymptotic state (independent of the initial condition). The convergence is exponentially quick in time.

$\Rightarrow$ The scatterer loses memory. Correlations between X_k and X_m decrease for growing time difference $|k-m|$, because $\mathcal{S}$ initiates convergence to asymptotic state during time span $|k-m|$.

Decay of correlations

$\sigma(X_{k_1},\ldots,X_{k_N})$: Sigma-algebra generated by N random variables $X_{k_1},\ldots,X_{k_N}$

Example: $\big\{X_5=m, X_7\in\{m',m'',m'''\}\big\}\in\sigma(X_5,X_7)$

Theorem (Correlation decay). *There are constants $c>0$, $\gamma>0$ such that, for all $A\in\sigma(X_k,\ldots,X_l)$, $B\in\sigma(X_m,\ldots,X_n)$, $1\le k\le l<m\le n\le\infty$, we have*

$$\big|P(A\cap B)-P(A)P(B)\big| \ \le\ c\,\mathrm{e}^{-\gamma(m-l)}P(A).$$

- Decaying correlations $\Rightarrow$ **Kolmogorov Zero-One Law**:
  Any event A in the tail sigma-algebra ("tail event")

$$\mathcal{T} = \bigcap_{k\geq 1} \sigma(X_k, X_{k+1}, \ldots)$$

  satisfies $P(A) = 0$ or $P(A) = 1$.

- Tail event $=$ does not depend on any finite collection of the X_k
- *Examples:*
  - $\{X_k \in S$ eventually$\} = \bigcup_{n\geq 1} \{X_k \in S\ \forall k \geq n\}$
    $= \bigcup_{n\geq 1} \bigcap_{l\geq 1} \{X_k \in S,\ k = n, \ldots, n+l\} \in \mathcal{T}$
  - $P(X_k \in S$ ev.$) = \lim_{n\to\infty} \lim_{l\to\infty} P(X_k \in S,\ k = n, \ldots, n+l) \in \{0, 1\}$
  - $P(X_k$ converges$) = P(X_{k+1} = X_k$ eventually$) \in \{0, 1\}$

Weak interaction

- $$\begin{cases} P(X_n = m) \;=\; p_{\rm in}(m) + O(V), \text{ where } \; p_{\rm in}(m) \;=\; \omega_{\rm in}(E_{M=m}) \\ P(X_k = m_k, X_l = m_l) \;=\; P(X_k = m_k)P(X_l = m_l) + O(V) \end{cases}$$

$$\Rightarrow \; P(X_{n+1} = X_n) \;=\; \sum_m p_{\rm in}(m)^2 + O(V)$$

- $P(X_n \text{ converges}) \;\leq\; \liminf_{n\to\infty} P(X_{n+1} = X_n)$
  $= \; \sum_m p_{\rm in}(m)^2 + O(V)$

- $\sum_m p_{\rm in}(m)^2 \leq 1$. Equality $\Leftrightarrow \omega_{\rm in}(E_m) = \delta_{m,m^*}$ for exactly one m^*
  $\Rightarrow \quad {\rm Var}_{\rm in}(M) \equiv \omega_{\rm in}(M^2) - \omega_{\rm in}(M)^2 = 0$

- *Conclusion:* If ${\rm Var}_{\rm in}(M) > 0$ and V is small, then $P(X_n \text{ converges}) = 0$.

Proposition. *There is a constant C such that, for any $S \subset {\rm spec}(M)$ with $\omega_{\rm in}(E_S) \neq 1$, if $\|V\| \leq C(1 - \omega_{\rm in}(E_S))$, then*

$$P(X_n \in S \textit{ eventually}) = 0.$$

Frequencies

- Frequency of possible measurement outcome m:

$$f_m \equiv \lim_{n\to\infty} \frac{1}{n} \{ \#k \in \{1,\dots,n\} : \; X_k = m \}$$

  - ω_+: asymptotic state of the scatterer (no measurement dynamics)
  - τ : probe - scatterer interaction time
  - $H = H_{\mathcal{S}} + H_{\mathcal{P}} + V$: single probe-scatterer Hamiltonian
  - E_m : spectral projection of M associated to the eigenvalue m

Theorem. *The frequency f_m exists as an almost everywhere limit and takes the deterministic value*

$$f_m = \omega_+ \otimes \omega_{\rm in} \left(\mathrm{e}^{\mathrm{i}\tau H} E_m \, \mathrm{e}^{-\mathrm{i}\tau H} \right).$$

- More generally, for m fixed,

$$\lim_{n\to\infty} \frac{1}{n}\big\{\#j \leq n+m \ : \ X_j \in S_1, \dots, X_{j+m} \in S_m\big\}$$
$$= \quad \omega_+ \otimes \omega_{\rm in} \cdots \otimes \omega_{\rm in}\big(\mathrm{e}^{\mathrm{i}\tau H_1} \dots \mathrm{e}^{\mathrm{i}\tau H_m} E_{S_1} \cdots E_{S_m} \mathrm{e}^{-\mathrm{i}\tau H_m} \dots \mathrm{e}^{-\mathrm{i}\tau H_1}\big)$$

Statistical average

- Statistical average of $\{X_n\}$:

$$\overline{X}_n \equiv \frac{1}{n} \sum_{j=1}^{n} X_j$$

Theorem (Strong law of large numbers). *As $n \to \infty$, the sequence $\overline{X}_n$ converges almost everywhere to the deterministic value*

$$\mu \equiv \lim_{n\to\infty} \overline{X}_n = \omega_+ \otimes \omega_{\rm in}\left(\mathrm{e}^{\mathrm{i}\tau H} M \mathrm{e}^{-\mathrm{i}\tau H} \right).$$

Repeated interactions setup

At time step n, the first $n-1$ probes have scattered and the n-th one is interacting with the scatterer.

- Hilbert space: $\mathcal{H} = \mathcal{H}_\mathcal{S} \otimes \mathcal{H}_\mathcal{P} \otimes \mathcal{H}_\mathcal{P} \otimes \cdots \otimes \mathcal{H}_\mathcal{P}$
- Initial state: $\rho_0 = \rho_\mathcal{S} \otimes \rho_{\rm in} \otimes \rho_{\rm in} \otimes \cdots \otimes \rho_{\rm in}$
- Define the Hamiltonian

$$H_j = \sum_{k=1}^{n} H_{\mathcal{P},k} + H_\mathcal{S} + V_j$$

  where V_j is a fixed interaction operator V acting on $\mathcal{S}$ and the j-th $\mathcal{P}$
- Dynamics (no measurement)

$$\rho_n = \mathrm{e}^{-\mathrm{i}\tau H_n} \cdots \mathrm{e}^{-\mathrm{i}\tau H_1}\ \rho_0\ \mathrm{e}^{\mathrm{i}\tau H_1} \cdots \mathrm{e}^{\mathrm{i}\tau H_n}$$

• Measurement observable: self-adjoint M on $\mathcal{H}_{\mathcal{P}}$, eigenvalues m_j, spectral projections E_{m_j}

• Suppose M is measured on each probe exiting the scattering process, and that the measurement results are $m_1, \ldots, m_n$. Then the (full) state after the last measurement is

$$\rho_n = \frac{E_{m_n} \mathrm{e}^{-\mathrm{i}\tau H_n} \ldots E_{m_1} \mathrm{e}^{-\mathrm{i}\tau H_1} \ \rho_0 \ \mathrm{e}^{\mathrm{i}\tau H_1} E_{m_1} \cdots \mathrm{e}^{\mathrm{i}\tau H_n} E_{m_n}}{P(m_1, \ldots, m_n)},$$

where

$$P(m_1, \ldots, m_n) \\ = \mathrm{Tr}\Big(E_{m_n} \mathrm{e}^{-\mathrm{i}\tau H_n} \ldots E_{m_1} \mathrm{e}^{-\mathrm{i}\tau H_1} \ \rho_0 \ \mathrm{e}^{\mathrm{i}\tau H_1} E_{m_1} \cdots \mathrm{e}^{\mathrm{i}\tau H_n} E_{m_n} \Big)$$

is the probability of the measurement history $m_1, \ldots, m_n$.

- Stochastic process of measurement outcomes $\{X_n\}$:

$$\Omega = (\mathrm{spec}(M))^{\mathbf{N}} = \big\{\, \omega = (\omega_1, \omega_2, \ldots) \ : \ \omega_j \in \mathrm{spec}(M) \,\big\}$$

$\mathcal{F}$: σ-algebra of subsets of Ω generated by cylinder sets

$$\big\{\, \omega \in \Omega \ : \ \omega_1 \in S_1, \ldots, \omega_n \in S_n, \ n \in \mathbf{N}, \, S_j \subseteq \mathrm{spec}(M) \,\big\}$$

On $(\Omega, \mathcal{F})$ define random variable $X_n : \Omega \to \mathrm{spec}(M)$, representing the measurement outcome on probe n, by

$$X_n(\omega) = \omega_n, \quad n = 1, 2, \ldots.$$

- Finite-dimensional distribution of $\{X_n\}$:

$$P(X_1 \in S_1, \ldots, X_n \in S_n) \\ \equiv \mathrm{Tr}\big(E_{S_n} \mathrm{e}^{-\mathrm{i}\tau H_n} \cdots E_{S_1} \mathrm{e}^{-\mathrm{i}\tau H_1} \ \rho_0 \ \mathrm{e}^{\mathrm{i}\tau H_1} E_{S_1} \cdots \mathrm{e}^{\mathrm{i}\tau H_n} E_{S_n} \big)$$

extends to probability measure on $(\Omega, \mathcal{F})$.

Representation of joint probabilities

Liouville space (GNS): density matrices are viewed as vectors in an ("enlarged") Hilbert space.

- ρ a density matrix on $\mathcal{H}$, dynamics $\mathrm{e}^{-\mathrm{i}tH}\rho\,\mathrm{e}^{\mathrm{i}tH}$
  Represent ρ in $\mathcal{H}\otimes\mathcal{H}$ as

$$\rho = \sum_k p_k|\chi_k\rangle\langle\chi_k| \quad \mapsto \quad \Psi = \sum_k \sqrt{p_k}\;\chi_k\otimes\chi_k$$

- $\mathrm{Tr}\rho A \;=\; \langle\Psi,(A\otimes\mathbb{1})\,\Psi\rangle_{\mathcal{H}\otimes\mathcal{H}}$, so observables are identified as $A\otimes\mathbb{1}$.
- Dynamics is implemented as

$$(\mathrm{e}^{\mathrm{i}tH}A\mathrm{e}^{-\mathrm{i}tH})\otimes\mathbb{1} \;=\; \mathrm{e}^{\mathrm{i}t(H\otimes\mathbb{1}+\mathbb{1}\otimes H')}\,(A\otimes\mathbb{1})\,\mathrm{e}^{-\mathrm{i}t(H\otimes\mathbb{1}+\mathbb{1}\otimes H')}$$

  for an *arbitrary self-adjoint* H'

- Dynamics generator: **Liouville operator**

$$L = H \otimes \mathbb{1} + \mathbb{1} \otimes H'$$

- Schrödinger dynamics: $\Psi_t = \mathrm{e}^{-\mathrm{i}tL}\Psi$
- **Reference state**: trace state, $\Psi_{\mathrm{ref}} = \frac{1}{\sqrt{\dim \mathcal{H}}}\sum_j \chi_j \otimes \chi_j$, where χ_j is arbitrary ONB of $\mathcal{H}$
- $\mathcal{C}$ = complex conjugation in basis $\{\chi_j\}$, X an arbitrary operator:

$$(X \otimes \mathbb{1})\,\Psi_{\mathrm{ref}} \;=\; (\mathbb{1} \otimes \mathcal{C}X^*\mathcal{C})\,\Psi_{\mathrm{ref}}$$

$\Longrightarrow\;\; K \equiv H \otimes \mathbb{1} - \mathbb{1} \otimes \mathcal{C}H\mathcal{C}$ satisfies $K\Psi_{\mathrm{ref}} = 0$.

- Trace state "generates" any state:
  For an arbitrary $\Psi \in \mathcal{H} \otimes \mathcal{H}$, $\exists!$ operator B s.t.

$$\Psi = (\mathbb{1} \otimes B)\,\Psi_{\mathrm{ref}}, \qquad \text{we set } B' \equiv \mathbb{1} \otimes B$$

- Putting things together:

$$\begin{aligned}\mathrm{Tr}\big(\rho\,\mathrm{e}^{\mathrm{i}tH}A\mathrm{e}^{-\mathrm{i}tH}\big) &= \big\langle\Psi,\mathrm{e}^{\mathrm{i}tL}(A\otimes\mathbb{1})\mathrm{e}^{-\mathrm{i}tL}\Psi\big\rangle\\ &= \big\langle\Psi_{\mathrm{ref}},\,(B')^*B'\ \mathrm{e}^{\mathrm{i}tL}(A\otimes\mathbb{1})\mathrm{e}^{-\mathrm{i}tL}\Psi_{\mathrm{ref}}\big\rangle\\ &= \big\langle\Psi_{\mathrm{ref}},\,(B')^*B'\ \mathrm{e}^{\mathrm{i}tK}(A\otimes\mathbb{1})\Psi_{\mathrm{ref}}\big\rangle\end{aligned}$$

- Apply this to the joint probability:
  - Scalar product of $(\mathcal{H}_{\mathcal{S}}\otimes\mathcal{H}_{\mathcal{S}})\otimes(\mathcal{H}_{\mathcal{P}}\otimes\mathcal{H}_{\mathcal{P}})\otimes\cdots\otimes(\mathcal{H}_{\mathcal{P}}\otimes\mathcal{H}_{\mathcal{P}})$
  - Reference state is product of trace states, $\Psi_{\mathrm{ref}}=\Psi_{\mathcal{S}}\otimes\Psi_{\mathcal{P}}\otimes\cdots\otimes\Psi_{\mathcal{P}}$

$$\begin{aligned}&P(X_1\in S_1,\ldots,X_n\in S_n)\\ &\quad\equiv\ \mathrm{Tr}\big(\rho_0\ \mathrm{e}^{\mathrm{i}\tau H_1}\ E_{S_1}\cdots\mathrm{e}^{\mathrm{i}\tau H_n}\ E_{S_n}\ \mathrm{e}^{-\mathrm{i}\tau H_n}\cdots E_{S_1}\mathrm{e}^{-\mathrm{i}\tau H_1}\big)\\ &\quad=\ \big\langle\Psi_{\mathrm{ref}}\,,\,(B'_{\mathcal{S}})^*B'_{\mathcal{S}}\ \big[(B'_1)^*B'_1\mathrm{e}^{\mathrm{i}\tau K_1}(E_{S_1}\otimes\mathbb{1}_{\mathcal{P}})\big]\cdots\\ &\qquad\qquad\cdots\big[(B'_n)^*B'_n\mathrm{e}^{\mathrm{i}\tau K_n}(E_{S_n}\otimes\mathbb{1}_{\mathcal{P}})\big]\ \Psi_{\mathrm{ref}}\big\rangle\end{aligned}$$

- Each $(B'_j)^* B'_j \, \mathrm{e}^{\mathrm{i}\tau K_j} \, E_{S_j}$ acts as an operator $(B')^* B' \mathrm{e}^{\mathrm{i}\tau K} E_{S_j}$ on the scatterer and a single probe, $(\mathcal{H}_\mathcal{S} \otimes \mathcal{H}_\mathcal{S}) \otimes (\mathcal{H}_\mathcal{P} \otimes \mathcal{H}_\mathcal{P})$
- Let $P = |\Psi_\mathcal{P}\rangle\langle\Psi_\mathcal{P}|$ and identify

$$T_S = P(B')^* B' \, \mathrm{e}^{\mathrm{i}\tau K} E_S P$$

as acting on $\mathcal{H}_\mathcal{S} \otimes \mathcal{H}_\mathcal{S}$.

- Then we have the representation

$$\boxed{P(X_1 \in S_1, \ldots, X_n \in S_n) \;=\; \langle \Psi_\mathcal{S}, (B'_\mathcal{S})^* B'_\mathcal{S} \, T_{S_1} \cdots T_{S_n} \Psi_\mathcal{S} \rangle}$$

- $\mathrm{spec}(T_S) \;\subset\; \{\, |z| \leq 1 \,\}$
- No measurement: $T \equiv T_{\mathrm{spec}(M)}$, $T\Psi_\mathcal{S} = \Psi_\mathcal{S}$
- **Ergodicity assumption:** The only eigenvalue of T on the unit circle is 1 and it is simple. Riesz projection: $|\Psi_\mathcal{S}\rangle\langle\Psi^*_\mathcal{S}|$

Showing decay of correlations

- We show that $|P(A\cap B)-P(A)P(B)|\le c\,\mathrm{e}^{-\gamma(m-l)}$ for events $A=\{\omega: X_l\in S_l\}$, $B=\{\omega: X_m\in S_m\}$
- We have

$$P(A\cap B) \;=\; \left\langle \Psi_{\mathcal{S}},\, T^{l-1}\, T_{S_l}\, T^{m-l-1}\, T_{S_m}\, \Psi_{\mathcal{S}}\right\rangle$$

- By the ergodicity assumption,

$$\left\|\, T^k - |\Psi_{\mathcal{S}}\rangle\langle\Psi^*_{\mathcal{S}}|\,\right\| \le C\mathrm{e}^{-\gamma k}$$

and so

$$P(A\cap B) \;=\; \underbrace{\left\langle \Psi_{\mathcal{S}}, T^{l-1} T_{S_l}\Psi_{\mathcal{S}}\right\rangle}_{P(A)}\; \langle\Psi^*_{\mathcal{S}}, T_{S_m}\Psi_{\mathcal{S}}\rangle + O\big(\mathrm{e}^{-\gamma(m-l)}\big)$$

Next,

$$\begin{aligned}\langle \Psi_{\mathcal{S}}^*, T_{S_m}\Psi_{\mathcal{S}}\rangle &= \Big\langle \Psi_{\mathcal{S}}, \big(|\Psi_{\mathcal{S}}\rangle\langle\Psi_{\mathcal{S}}^*|\big) T_{S_m}\Psi_{\mathcal{S}}\Big\rangle \\ &= \Big\langle \Psi_{\mathcal{S}}, T^{m-1} T_{S_m}\Psi_{\mathcal{S}}\Big\rangle + O\big(\mathrm{e}^{-\gamma m}\big) \\ &= P(B) + O\big(\mathrm{e}^{-\gamma m}\big)\end{aligned}$$

This shows that $|P(A\cap B) - P(A)P(B)| \le c\,\mathrm{e}^{-\gamma(m-l)}$.

The frequencies

We first show convergence of the mean.

$$\frac{1}{n}\mathbf{E}\big[\,\#k \in \{1,\ldots,n\} \text{ such that } X_k = m\,\big]$$

$$= \quad \frac{1}{n}\sum_{m_1,\ldots,m_n}\left(\sum_{j=1}^{n}\chi(m_j = m)\right) P(X_1 = m_1,\ldots,X_n = m_n)$$

$$= \quad \frac{1}{n}\sum_{j=1}^{n} P(X_j = m)$$

$$= \quad \frac{1}{n}\sum_{j=1}^{n}\big\langle \Psi_{\mathcal{S}}, T^{j-1}\, T_m\, \Psi_{\mathcal{S}}\big\rangle$$

$$\longrightarrow \quad \langle \Psi_{\mathcal{S}},\ (|\Psi_{\mathcal{S}}\rangle\langle\Psi^*_{\mathcal{S}}|)\ T_m \Psi_{\mathcal{S}}\rangle = \langle \Psi^*_{\mathcal{S}},\, T_m\, \Psi_{\mathcal{S}}\rangle$$

- Next, since $T_m = P(B')^* B' \mathrm{e}^{\mathrm{i}\tau K} E_m P$,

$$
\begin{aligned}
\langle \Psi_{\mathcal{S}}^*, T_m \Psi_{\mathcal{S}} \rangle &= \Big\langle \Psi_{\mathcal{S}}^* \otimes \Psi_{\mathcal{P}},\, (B')^* B' \mathrm{e}^{\mathrm{i}\tau K} E_m \Psi_{\mathcal{S}} \otimes \Psi_{\mathcal{P}} \Big\rangle \\
&= \Big\langle \Psi_{\mathcal{S}}^* \otimes \Psi_{\mathcal{P}},\, (B')^* B' \mathrm{e}^{\mathrm{i}\tau L} E_m \, \mathrm{e}^{-\mathrm{i}\tau L} \Psi_{\mathcal{S}} \otimes \Psi_{\mathcal{P}} \Big\rangle \\
&= \Big\langle \Psi_{\mathcal{S}}^* \otimes \Psi_{\mathrm{in}},\, \mathrm{e}^{\mathrm{i}\tau L} E_m \, \mathrm{e}^{-\mathrm{i}\tau L} \Psi_{\mathcal{S}} \otimes \Psi_{\mathrm{in}} \Big\rangle \\
&= \omega_+ \otimes \omega_{\mathrm{in}} \big(\mathrm{e}^{\mathrm{i}\tau H} E_m \, \mathrm{e}^{-\mathrm{i}\tau H} \big).
\end{aligned}
$$

- Use a probabilistic *4th moment method* to upgrade the convergence in expectation to almost sure convergence, *i.e.*,

$$
\lim_{n\to\infty} \frac{1}{n} \big[\# k \in \{1, \ldots, n\} \ : \ X_k = m \big] \ = \ \omega_+ \otimes \omega_{\mathrm{in}} \big(\mathrm{e}^{\mathrm{i}\tau H} E_m \, \mathrm{e}^{-\mathrm{i}\tau H} \big) \quad a.s.
$$

Evolution of the scatterer

- ω_n: state of scatterer at time step n
- ω_n is random variable – determined by random measurement history

Lemma. *The expectation* $\mathbf{E}[\omega_n]$ *is the state obtained by evolving the initial state according to the repeated interaction dynamics without measurement.*

Proof. For a given measurement path $m_1, \ldots, m_n$,

$$\omega_n(A) = \frac{\langle \Psi_{\mathcal{S}}, T_{m_1} \cdots T_{m_n} A \Psi_{\mathcal{S}} \rangle}{\langle \Psi_{\mathcal{S}}, T_{m_1} \cdots T_{m_n} \Psi_{\mathcal{S}} \rangle}$$

So

$$\mathbf{E}[\omega_n(A)] \ = \sum_{m_1,\ldots,m_n} \langle \Psi_{\mathcal{S}}, T_{m_1} \cdots T_{m_n} A \Psi_{\mathcal{S}} \rangle = \langle \Psi_{\mathcal{S}}, T^n A \Psi_{\mathcal{S}} \rangle \, .$$

A spin-spin example

- Both $\mathcal{S}$ and $\mathcal{P}$ are spins,

$$H_{\mathcal{S}} = H_{\mathcal{P}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Energy-exchange interaction

$$V = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \mathrm{h.c.} \qquad \in \ \mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{P}}$$

- Take incoming probes to be in state up,

$$\omega_{\mathrm{in}} \ \leftrightarrow \ \rho_{\mathrm{in}} = \left| \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle\left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right|$$

- Final state ω_+ of scatterer (under dynamics without measurement) is spin up.
- Here, $\omega_+ \otimes \omega_{\rm in}$ is invariant under probe-scatterer dynamics (Hamilt. H). $\Rightarrow$ the frequencies and mean are those of incoming states,

$$f_m = \omega_{\rm in}(E_m), \quad \mu = \omega_{\rm in}(M)$$

  So scatterer becomes 'transparent' after many interactions.
- Measurement of outcoming spin along the direction given by an angle $\theta \in [0, \pi/2]$ in $x - z$ plane; $\theta = 0$ is spin up direction (Azimuthal angle plays no role, as Hamiltonian is invariant under rotation about z-axis)
- Measurement operator

$$M = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

- Possible measurement outcomes: $m = 1, -1$
- The operators T, T_m can be calculated explicitly. One shows

$$P(X_n = 1 \text{ eventually }) \;=\; \begin{cases} 1 & \text{if } \theta = 0 \\ 0 & \text{if } \theta \neq 0 \end{cases}$$

- Frequencies: $f_{+1} = \cos^2(\theta/2)$, $f_{-1} = \sin^2(\theta/2)$; average: $\mu = \cos\theta$.
- Large deviation analysis: *e.g.* logarithmic moment-generating function for $\overline{X}_n$, $\lim_{n\to\infty} n^{-1} \log \mathbf{E}[\mathrm{e}^{n\alpha \overline{X}_n}]$, can be analyzed via spectral properties of operators T_S. For example $(0 < \epsilon < \epsilon' \ll 1)$

$$P\big(\epsilon \leq |\overline{X}_n - \cos\theta| \leq \epsilon'\big) \sim \exp\left[-n\left\{\frac{\epsilon^2}{2\sin^2\theta} + O((\epsilon')^4)\right\}\right], \quad n \to \infty$$

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et

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