Stability of Equilibria with a Condensate

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Abstract

We consider a quantum system composed of a small part, having finitely many degrees of freedom, interacting with a free, spatially infinitely extended Bose gas. An equilibrium state for the uncoupled system is given by the product state where the small part is in the Gibbs state at some temperature $T > 0$, and the Bose gas is in a state where a spatially homogeneous Bose-Einstein Condensate is immersed in black body radiation at the same temperature T.

An interaction between the two subsystems is specified by a coupled dynamics. The interaction strength is measured by the size of a coupling constant. We show that the equilibrium state of the uncoupled system is stable: any initial state close to it, evolving according to the interacting dynamics, converges to it, in the successive limits of large time and small coupling constant.

We deduce the stability result from properties of structure and regularity of eigenvectors of the generator of the dynamics, called the Liouville operator. Among our technical results is a Virial Theorem for Liouville type operators which has new applications to systems with and without a condensate.

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1 Introduction

This paper is devoted to the study of the dynamics of a class of quantum systems consisting of a small part in interaction with a large heat reservoir, modelled by an infinitely extended ideal gas of Bosons. We further develop spectral methods in the framework of algebraic quantum field theory and apply them to the class of systems at hand, for which the already existing techniques have not been applicable.

Our main physical interest is the long-time behaviour of initial states close to an equilibrium state of the uncoupled system, describing a Bose gas that is so dense (for fixed temperature) or so cold (for fixed density) that it has a Bose-Einstein condensate. One of our goals is to prove that this equilibrium state is *stable* (*attractive*) in the sense that any initial condition close to it, when evolving under the coupled dynamics, converges to the equilibrium state when one takes first the limit of large time and then the limit of small coupling. The analysis given in this paper shows that any initial condition as specified above has a limiting state, as time alone tends to infinity. This limiting state is close to the *interacting* equilibrium state (or equivalently, close to the uncoupled equilibrium state), provided the coupling is small. A stronger result, called *Return to Equilibrium*, saying that the limiting state *equals* the interacting equilibrium state, has been obtained for systems without a condensate in a variety of recent papers, [JP1, BFS, M1, DJ, FM2]. It is surprising that none of the methods developed in these references can be applied to the present case. This is due to the fact that the *form factor* of the interaction, a coupling function $g \in L^2(\mathbb{R}^3, d^3k)$, whose properties are dictated by physics, exhibits the infrared behaviour $0 < |g(0)| < \infty$. It lies in between the two "extreme" behaviours $g(0) = 0$ (more precisely, $g(k) \sim |k|^p$, some $p > 0$, as $|k| \sim 0$) and $|g(0)| = \infty$ (more precisely, $g(k) \sim |k|^{-1/2}$ as $|k| \sim 0$), which are the only ones that can be treated using the approaches developed in the above references. We give in this paper a partial remedy to this situation by establishing a "positive commutator theory" (a first step in a Mourre theory) which is applicable to a wide variety of interactions, including the case where $g(0)$ is a nonzero, finite constant. Our remedy is only partial in that so far, we show that the equilibrium state is stable, in the sense mentioned above, but we cannot prove return to equilibrium. The obstruction seems to be of technical nature, see Section 2.2.1 for a discussion of this point.

Our analysis consists of two main steps. The first one is a reduction of the system with a condensate to a family of systems without condensate: the equilibrium state with a condensate is *not a factor state*, i.e. the von Neumann algebra of observables, represented in the Hilbert space of this state, is not a factor. The state has thus a natural decomposition into a superposition (an integral) of factor states, called the central decomposition of the state. Accordingly, the Hilbert space and the von Neumann algebra of observables are decomposed into a direct integral of Hilbert spaces and a direct integral of factor von Neumann algebras. It turns out that the dynamics of both the non-interacting and the interacting system is *reduced* by this decomposition. We can thus view each component as an independent system without condensate, equipped with its own dynamics (varying with each component).

The second step in our analysis, which is the main technical part of this paper, consists in analyzing the time asymptotic behaviour of each independent component. We do this by examining the spectrum of the Liouville operators generating the dynamics. Our approach gives an extension of the positive commutator method, including a new virial theorem which has useful applications to related problems for systems without a condensate.

Here is a presentation of our main results which we give without entering into technical elaborations, referring to Section 2 for more detail.

The small quantum system has finitely many degrees of freedom, its Hilbert space is $\mathbb{C}^d$, and the dynamics of observables $A \in \mathcal{B}(\mathbb{C}^d)$ (the von Neumann algebra of all bounded operators on $\mathbb{C}^d$) is generated by a Hamiltonian H_1, according to $A \mapsto \alpha_1^t(A) = e^{itH_1} A e^{-itH_1}$. The kinematical algebra describing the Bose gas is the *Weyl algebra* $\mathfrak{W}(\mathcal{D})$ over a suitably chosen test-function space of one-particle wave functions $\mathcal{D} \subset L^2(\mathbb{R}^3, d^3k)$. $\mathfrak{W}(\mathcal{D})$ is generated by Weyl operators $W(f)$, $f \in \mathcal{D}$, satisfying the canonical commutation relations (CCR)

$$W(f)W(g) = e^{-\frac{i}{2}\mathrm{Im}\langle f,g\rangle} W(f+g), \tag{1}$$

where $\langle \cdot, \cdot \rangle$ is the inner product induced by $L^2(\mathbb{R}^3, d^3k)$. The dynamics of the Bose gas is given by the Bogoliubov transformation

$$W(f) \mapsto \alpha_2^t(W(f)) = W(e^{it\omega} f),$$

where

$$\omega(k) = |k|, \quad \text{or} \quad \omega(k) = |k|^2. \tag{2}$$

The first choice in (2) describes massless relativistic Bosons, while the second one describes massless non-relativistic Bosons. The observable algebra of the combined system is the C^*-algebra

$$\mathfrak{A} = \mathcal{B}(\mathbb{C}^d) \otimes \mathfrak{W}(\mathcal{D}), \tag{3}$$

and the non-interacting dynamics is the $*$automorphism group of $\mathfrak{A}$ given by

$$\alpha_0^t = \alpha_1^t \otimes \alpha_2^t. \tag{4}$$

The equilibrium state of the uncoupled system which we are interested in is the (β, α_0^t)-KMS state

$$\omega_{\beta,0}^{\rm con} = \omega_{1,\beta} \otimes \omega_{2,\beta}, \tag{5}$$

where $\omega_{1,\beta}$ is the (β, α_1^t)-KMS state (Gibbs state) of the small system, and $\omega_{2,\beta}$ is a (β, α_2^t)-KMS state of the Weyl algebra which has a Bose-Einstein condensate. The latter is obtained by taking the thermodynamic limit of Gibbs states of the Bose gas in a finite volume, and it needs to be described in a more precise way.

To understand the construction (definition) of the equilibrium state $\omega_{2,\beta}$ we first remind the reader that any state ω on the Weyl algebra $\mathfrak{W}(\mathcal{D})$ is uniquely determined by its so-called *generating* (or expectation) *functional* $E : \mathcal{D} \to \mathbb{C}$, given by

$$\omega(W(f)) = E(f), \tag{6}$$

and that conversely, if $E : \mathcal{D} \to \mathbb{C}$ is a (non linear) function satisfying certain compatibility conditions then it defines uniquely a state on $\mathfrak{W}(\mathcal{D})$, see e.g. [A, M2].

Let $\mathbb{R}^3 \ni k \mapsto \rho(k) > 0$ be a given function (the "continuous momentum-density distribution"), and $\rho_0 \geq 0$ a fixed number (the "condensate density"). Araki and Woods [AW] obtain a generating functional E_{ρ,ρ_0} of the infinite Bose gas by the following procedure. First restrict the gas to a finite box of volume V in $\mathbb{R}^3$ and putting $V\rho_0$ particles in the ground state of the one particle Hamiltonian $H_V = -\Delta$ (or $H_V = \sqrt{-\Delta}$), and a discrete distribution of particles in excited states. Then take the limit $V \to \infty$ while keeping ρ_0 fixed and letting the discrete distribution of excited states tend to $\rho(k)$.

Like this [AW] obtain a family of generating functionals E_{ρ,ρ_0} (whose explicit form is given in (35)), each member of which defines uniquely a state of the infinitely extended Bose gas according to (6). The physical interpretation is that the resulting state describes a free Bose gas where a sea of particles, all being in the same state (corresponding to the ground state of the finite-volume Hamiltonian), form a condensate with density ρ_0, which is immersed in a gas of particles where $\rho(k)$ particles per unit volume have momentum in the infinitesimal volume d^3k around $k \in \mathbb{R}^3$. Since the Hamiltonian in the finite box is taken with periodic boundary conditions the condensate is homogeneous in space (the ground state wave function is a constant in position space).

A rigorous argument linking [AW]'s results to the equilibrium states of the infinite Bose gas has been given in [C] (see also [LP]), and can be summarized as follows. Let $\rho_{\rm tot} > 0$ be the "total" density of the Bose gas (i.e., $\rho_{\rm tot}$ is the number of particles per unit volume). For a fixed inverse temperature $0 < \beta < \infty$ define the *critical density* by

$$\rho_{\rm crit}(\beta) = (2\pi)^{-3} \int \frac{d^3k}{e^{\beta\omega} - 1}. \tag{7}$$

Let V be the box defined by $-L/2 \leq x_j \leq L/2$ $(j = 1, 2, 3)$ and define the canonical state at inverse temperature β and density $\rho_{\rm tot}$ by

$$\langle A \rangle^{\rm c}_{\beta,\rho_{\rm tot},V} = \frac{{\rm tr}\, A P_{\rho_{\rm tot}V} e^{-\beta H_V}}{{\rm tr}\, P_{\rho_{\rm tot}V} e^{-\beta H_V}}, \tag{8}$$

where the trace is over Fock space over $L^3(V, d^3x)$, $P_{\rho_{\rm tot}V}$ is the projection onto the subspace of Fock space with $\rho_{\rm tot}V$ particles (if $\rho_{\rm tot}V$ is not an integer take a convex combination of canonical states with integer values $\rho_1 V$ and $\rho_2 V$ extrapolating $\rho_{\rm tot}V$). The Hamiltonian H_V is negative the Laplacian with periodic boundary conditions. The observable A in (8) belongs to the Weyl algebra over the test function space C_0^∞, realized as a C^*-algebra acting on Fock space. Cannon shows that for any $\beta, \rho_{\rm tot} > 0$ and $f \in C_0^\infty$,

$$\langle W(f) \rangle^{\rm c}_{\beta,\rho_{\rm tot},V} \longrightarrow \begin{cases} e^{-\frac{1}{4}\|f\|^2} e^{-\frac{1}{2}\left\langle f, \frac{z_\infty}{e^{\beta\omega} - z_\infty} f \right\rangle}, & \rho_{\rm tot} \leq \rho_{\rm crit}(\beta) \\ E_{\rho,\rho_0}(f), & \rho_{\rm tot} \geq \rho_{\rm crit}(\beta) \end{cases} \tag{9}$$

for any sequence $L \to \infty$. Here, $z_\infty \in [0, 1]$ is such that for subcritical density, the momentum density distribution of the gas is given by

$$\rho(k) = (2\pi)^{-3} \frac{z_\infty}{e^{\beta\omega} - z_\infty}, \tag{10}$$

so that z_∞ is the solution of

$$\rho_{\text{tot}} = (2\pi)^{-3} \int \frac{z}{e^{\beta\omega} - z} d^3k. \tag{11}$$

The generating functional E_{ρ,ρ_0} in (9) is the one obtained by Araki and Woods, where ρ is the continuous momentum density distribution prescribed by Planck's law of black body radiation (compare with (7)),

$$\rho(k) = (2\pi)^{-3} \frac{1}{e^{\beta\omega} - 1}, \tag{12}$$

and where

$$\rho_0 = \rho_{\text{tot}} - \rho_{\text{crit}}. \tag{13}$$

This gives the following picture: if the system has density $\rho_{\text{tot}} \leq \rho_{\text{crit}}$ then the particle momentum distribution of the equilibrium state is purely continuous, meaning that below critical density there is no condensate. As ρ_{tot} increases and surpasses the critical value, $\rho_{\text{tot}} > \rho_{\text{crit}}$, the "excess" particles form a condensate which is immersed in a gas of particles radiating according to Planck's law.

We shall from now on, in this section, concentrate on the supercritical case and denote the corresponding equilibrium state of the Weyl algebra by $\omega_{2,\beta}$ (see (5)).

Let $\mathcal{H}$ denote the (GNS-) Hilbert space of state vectors obtained from the algebra $\mathfrak{A}$, (3), and the equilibrium state $\omega_{\beta,0}^{\text{con}}$, (5). Furthermore, let $\Omega_{\beta,0}^{\text{con}} \in \mathcal{H}$ denote the cyclic vector in $\mathcal{H}$ corresponding to the state $\omega_{\beta,0}^{\text{con}}$, and let π be the GNS representation of $\mathfrak{A}$ on $\mathcal{H}$. Since $\omega_{\beta,0}^{\text{con}}$ is invariant under α_0^t (see (4)), meaning that $\omega_{\beta,0}^{\text{con}} \circ \alpha_0^t = \omega_{\beta,0}^{\text{con}}$ for all t, there is a selfadjoint operator $\mathcal{L}_0$ acting on $\mathcal{H}$, called the thermal Hamiltonian or *Liouvillian*, satisfying

$$\pi(\alpha_0^t(A)) = e^{it\mathcal{L}_0} \pi(A) e^{-it\mathcal{L}_0}, \tag{14}$$

for all $A \in \mathfrak{A}$, and

$$\mathcal{L}_0 \Omega_{\beta,0}^{\text{con}} = 0. \tag{15}$$

In order to describe interactions between the small system and the Bose gas one replaces the (non-interacting) Liouvillian $\mathcal{L}_0$ by an (interacting) Liouvillian $\mathcal{L}_\lambda$, which is a selfadjoint operator on $\mathcal{H}$ given by

$$\mathcal{L}_\lambda = \mathcal{L}_0 + \lambda \mathcal{I}, \tag{16}$$

where $\lambda \in \mathbb{R}$ is a coupling constant and $\mathcal{I}$ is an operator on $\mathcal{H}$ determined by the formal interaction Hamiltonian

$$\lambda G \otimes (a^*(g) + a(g)) \tag{17}$$

(or a finite sum of such terms). Here, G is a selfadjoint matrix on $\mathbb{C}^d$ and $a^{\#}(g)$ are creation and annihilation operators of the heat bath, smeared out with a function $g \in \mathcal{D}$, called a *form factor*. Of course, (17) has a meaning only in a regular representation of the Weyl algebra, e.g. the representation π above, see Subsection 2.1.2. The interaction $\mathcal{I}$ has the property that the dynamics generated by $\mathcal{L}_\lambda$ defines a $*$automorphism group σ_λ^t of the von Neumann algebra $\mathfrak{M}_\beta^{\rm con} \subset \mathcal{B}(\mathcal{H})$ obtained by taking the weak closure of the algebra $\pi(\mathfrak{A})$. One can show that, for a large class of interactions $\mathcal{I}$, there exists a vector $\Omega_{\beta,\lambda}^{\rm con} \in \mathcal{H}$ defining a $(\beta, \sigma_\lambda^t)$-KMS state on $\mathfrak{M}_\beta^{\rm con}$. We call $\Omega_{\beta,\lambda}^{\rm con}$ the perturbed KMS state, it satisfies

$$\mathcal{L}_\lambda \Omega_{\beta,\lambda}^{\rm con} = 0. \tag{18}$$

Our stability result, Theorem 2.1, can be formulated as follows. Let ω be any state represented by a density matrix on $\mathfrak{M}_\beta^{\rm con}$. If some regularity and effectiveness conditions on the interaction are satisfied (see the next paragraph and also Section 2.2), then we have, for any $A \in \mathfrak{M}_\beta^{\rm con}$,

$$\lim_{\lambda\to 0} \lim_{t\to\infty} \frac{1}{t} \int_0^t ds\ \omega(\sigma_\lambda^s(A)) = \omega_{\beta,0}^{\rm con}(A). \tag{19}$$

We expect that relation (19) holds if the small coupling limit is removed on the l.h.s. and $\omega_{\beta,0}^{\rm con}$ is replaced by the perturbed KMS state $\omega_{\beta,\lambda}^{\rm con}$ (represented by $\Omega_{\beta,\lambda}^{\rm con}$), provided λ is small enough ("return to equilibrium" in the sense of ergodic means). See Subsection 2.2.1 for a discussion of this point.

The "effectiveness condition" we impose on the interaction, determined by the operator G and the form factor g (see (16), (17)) has the following physical meaning. The interaction describes processes where field quanta (Bosons) are absorbed and emitted by inducing transitions of the small system. In particular, it is instructive to calculate the transition probability of the system corresponding to an initial state $\varphi_1 \otimes \Phi_V$ and a final state $\varphi_2 \otimes a^*(f)\Phi_V$, where $\varphi_{1,2}$ are eigenstates of the Hamiltonian H_1 of the small system, with energies $E_{1,2}$, and where Φ_V is the "ground state" (in Fock space) of the Bose gas in a box with volume V, describing $\rho_0 = n/V$ particles in the ground state (constant function $V^{-1/2}$) of the one-particle Hamiltonian $h_V = -\Delta$ (or $h_V = \sqrt{-\Delta}$), with periodic boundary conditions. In the limit $V \to \infty$, and to second order in λ, the transition probability

$$\left|\left\langle \varphi_2 \otimes a^*(f)\Phi_V, e^{-itH_\lambda}\varphi_1 \otimes \Phi_V\right\rangle\right|^2, \tag{20}$$

where $H_\lambda = H_0 + \lambda G \otimes (a^*(g) + a(g))$, $H_0 = H_1 + \mathrm{d}\Gamma(h_V)$, is calculated to be

$$\mathcal{P}_2(t) = \lambda^2 \left|\langle \varphi_2, G\varphi_1\rangle\right|^2 \left| \int_0^t ds\ e^{is(E_1-E_2)} \Big(\left\langle f, e^{-is\omega} g\right\rangle + \rho_0 \overline{f(0)} g(0) \Big) \right|^2. \tag{21}$$

The function $\omega(k)$ is either $|k|$ or $|k|^2$, c.f. (2). Expression (21) is a good approximation to (20) provided $|t\lambda| \ll 1$. We recognize two contributions to $\mathcal{P}_2(t)$, one for $\rho_0 = 0$, which is the same one would get by replacing Φ_V by the Fock vacuum Ω in (20), and the contribution coming from the interaction of the small system with the modes of the condensate. We see from (21) that if $g(0) = 0$ then there is no coupling to the modes of the condensate: a physically trivial situation where the condensate evolves freely and the small system coupled to the "excited modes" undergo return to equilibrium. In this paper we develop a theory which includes the case $g(0) \neq 0$.

If $E_1 = E_2$ then, for large values of t (and small values of λ, as to preserve $|t\lambda| \ll 1$), we have

$$\mathcal{P}_2(t) \sim (t\lambda)^2 \rho_0^2\ \left|\langle \varphi_2, G\varphi_1\rangle\right|^2 |f(0)g(0)|^2, \tag{22}$$

and only the zero mode $k = 0$ is involved in the emission process (the non-interacting energy is conserved to this order in the perturbation). If $E_1 \neq E_2$ then we have, for large times,

$$\mathcal{P}_2(t) \sim \lambda^2 \left|\langle \varphi_2, G\varphi_1\rangle\right|^2 \left|\langle f, \delta(E_1 - E_2 - \omega) g\rangle\right|^2, \tag{23}$$

so the mode determined by $\omega(k) = E_1 - E_2$ is engaged in the emission process (and the condensate does not participate). Our physical assumptions on the interaction is that the process described by (20) is not suppressed at second order in the perturbation, i.e., that (22), (23) are nonzero (see Condition (A2) in Section 2.2).

We conclude the introduction by outlining the spirit of the proof of (19) and by explaining the structure of this paper. In the central decomposition of the equilibrium state we have

$$
\begin{aligned}
\mathcal{L}_\lambda &= \int_{S^1}^{\oplus} d\theta \; L_{\lambda,\theta}, \\
\mathfrak{M}_\beta^{\rm con} &= \int_{S^1}^{\oplus} d\theta \; \mathfrak{M}_\theta, \\
\Omega_{\beta,0}^{\rm con} &= \int_{S^1}^{\oplus} d\theta \; \Omega_{\beta,0}^\theta, \\
\Omega_{\beta,\lambda}^{\rm con} &= \int_{S^1}^{\oplus} d\theta \; \Omega_{\beta,\lambda}^\theta
\end{aligned}
$$

and it suffices to prove (19) on each fixed fiber (labelled by θ). The von Neumann ergodic theorem tells us that the limit $t \to \infty$ in (19) is essentially determined by the projection onto the kernel of $\mathcal{L}_\lambda$, or, for a fixed fiber, by the projection onto the kernel of $L_{\lambda,\theta}$ (note that $\dim\ker\mathcal{L}_0 = \infty$, while $\dim\ker L_{0,\theta} = d$, the dimension of the small system). Our Theorem 2.3 describes the structure of elements in the kernel of $L_{\lambda,\theta}$ and shows in particular that all of them, except the perturbed KMS state $\Omega_{\beta,\lambda}^\theta$, converge to zero in the weak sense, as $\lambda \to 0$, see Corollary 2.4. Thus, the projection onto the kernel of $L_{\lambda,\theta}$ reduces to the projection $|\Omega_{\beta,0}^\theta\rangle\langle\Omega_{\beta,0}^\theta|$ when we take $\lambda \to 0$, and this leads to (19).

In order to prove Theorem 2.3 we develop a general virial theorem in a new setting, see Section 3, Theorem 3.2. In the particular case of the systems with a condensate considered in this paper the general virial theorem reduces to Theorem 2.2. We point out that Theorem 3.2 will be applied to give an improvement of the results on return to equilibrium and thermal ionization presented in [M1, FM1, FMS, FM2]. We will explain this in [FM3] (see also the discussion after Corollary 2.4 in Section 2).

2 Main results

In Section 2.1 we introduce the class of systems considered in this paper and we explain the central decomposition of the equilibrium state with a condensate (references we find useful for this are [AW] and also [H]). Our main results are presented in Section 2.2, at the end of which we also give the quite short proof of the stability theorem, Theorem 2.1.

2.1 Definition of model

We introduce the uncoupled system in Subsection 2.1.1 and present its Hilbert space (GNS) description (see (56), (57)) including the uncoupled standard Liouvillian $\mathcal{L}_0$, see (65). The interaction is defined by an interacting standard Liouvillian $\mathcal{L}_\lambda$, introduced in Subsection 2.1.2, see (81).

2.1.1 Non-interacting system

The *states* of the small system are determined by density matrices ρ on the finite dimensional Hilbert space $\mathbb{C}^d$. A density matrix is a positive trace-class operator, normalized as $\mathrm{tr}\,\rho = 1$, and the corresponding state

$$\omega_\rho(A) = \mathrm{tr}\,(\rho A), \quad A \in \mathcal{B}(\mathbb{C}^d) \tag{24}$$

is a normalized positive linear functional on the C^*-algebra $\mathcal{B}(\mathbb{C}^d)$ of all bounded operators on $\mathbb{C}^d$, which we call the algebra of *observables*. The (Heisenberg-) *dynamics* of the small system is given by the group of $*$automorphisms of $\mathcal{B}(\mathbb{C}^d)$ generated by a *Hamiltonian* H_1,

$$\alpha_1^t(A) = e^{itH_1} A e^{-itH_1}, \quad t \in \mathbb{R}, \tag{25}$$

where we take H_1 to be a selfadjoint diagonal matrix on $\mathbb{C}^d$ with *simple* spectrum

$$\mathrm{spec}(H_1) = \{E_0 < E_1 < \ldots < E_{d-1}\}. \tag{26}$$

(We would find it interesting to investigate also the case where H_1 has some degenerate eigenvalues, but do not address this here). Denote the normalized eigenvector of H_1 corresponding to E_j by φ_j. Given any inverse temperature $0 < \beta < \infty$ the *Gibbs state* $\omega_{1,\beta}$ is the unique β-KMS state on $\mathcal{B}(\mathbb{C}^d)$

associated to the dynamics (25). It corresponds to the density matrix

$$\rho_\beta = \frac{e^{-\beta H_1}}{\operatorname{tr} e^{-\beta H_1}}. \tag{27}$$

Let ρ be a density matrix of rank d (equivalently, $\rho > 0$) and let $\{\varphi_j\}_{j=0}^{d-1}$ be an orthonormal basis of eigenvectors of ρ, corresponding to eigenvalues $0 < p_j < 1$, $\sum_j p_j = 1$. The GNS representation of the pair $(\mathcal{B}(\mathbb{C}^d), \omega_\rho)$ is given by $(\mathcal{H}_1, \pi_1, \Omega_1)$, where the Hilbert space $\mathcal{H}_1$ and the cyclic (and separating) vector Ω_1 are

$$\mathcal{H}_1 = \mathbb{C}^d \otimes \mathbb{C}^d, \tag{28}$$

$$\Omega_1 = \sum_j \sqrt{p_j}\, \varphi_j \otimes \varphi_j \in \mathbb{C}^d \otimes \mathbb{C}^d, \tag{29}$$

and the representation map $\pi_1 : \mathcal{B}(\mathbb{C}^d) \to \mathcal{B}(\mathcal{H}_1)$ is

$$\pi_1(A) = A \otimes \mathbb{1}. \tag{30}$$

We introduce the von Neumann algebra

$$\mathfrak{M}_1 = \mathcal{B}(\mathbb{C}^d) \otimes \mathbb{1}_{\mathbb{C}^d} \subset \mathcal{B}(\mathcal{H}_1). \tag{31}$$

The modular conjugation operator J_1 associated to the pair $(\mathfrak{M}_1, \Omega_1)$ is given by

$$J_1 \psi_\ell \otimes \psi_r = \mathcal{C}_1 \psi_r \otimes \mathcal{C}_1 \psi_\ell, \tag{32}$$

where $\mathcal{C}_1$ is the antilinear involution $\mathcal{C}_1 \sum_j z_j \varphi_j = \sum_j \overline{z_j} \varphi_j$ (complex conjugate). According to (29) and (27) the vector $\Omega_{1,\beta}$ representing the Gibbs state $\omega_{1,\beta}$ is given by

$$\Omega_{1,\beta} = \frac{1}{\sqrt{\operatorname{tr} e^{-\beta H_1}}} \sum_j e^{-\beta E_j/2} \varphi_j \otimes \varphi_j \in \mathcal{H}_1. \tag{33}$$

We now turn to the description of heat bath. Its algebra of observables is the *Weyl algebra* $\mathfrak{W}(\mathcal{D})$ over some linear subspace of test functions $\mathcal{D} \subset L^2(\mathbb{R}^3, d^3k)$. The elements of $\mathcal{D}$ represent the wave functions of a single quantum particle of the heat bath. The choice of $\mathcal{D}$ depends on the physics one wants to describe – in particular, it is not the same for a system of

Bosons with and without a condensate, as we will see shortly. For fixed $\mathcal{D}$, $\mathfrak{W}(\mathcal{D})$ is the C^*-algebra generated by elements $W(f)$, $f \in \mathcal{D}$, called the Weyl operators, which satisfy the CCR (1). The $*$operation of $\mathfrak{W}(\mathcal{D})$ is given by $W(f)^* = W(-f)$. The dynamics of the heat bath is described by the group of $*$automorphisms of $\mathfrak{W}(\mathcal{D})$

$$\alpha_2^t(W(f)) = W(e^{ith} f), \tag{34}$$

where h is a selfadjoint operator on $L^2(\mathbb{R}^3, d^3k)$. In the present paper, we choose h to be the operator of multiplication by the function $\omega(k)$, see (2). Our methods can be modified to accomodate for other dispersion relations than (2).

According to Araki and Woods, [AW], the expectation functional (6) describing the spatially infinitely extended Bose gas in a state where a condensate emerging from the macroscopic occupation of the ground state, with density $\rho_0 > 0$, is immersed in a gas of particles having a prescribed continuous momentum density distribution $\rho(k)$ (assumed to be > 0 a.e.), is given by

$$\begin{aligned} &E_{\rho,\rho_0}(f) \\ &= \exp\left\{-\frac{1}{4}\|f\|^2\right\} \exp\left\{-\frac{1}{2}\|\sqrt{(2\pi)^3\rho} f\|^2\right\} J_0\left(\sqrt{2(2\pi)^3\rho_0}|f(0)|\right), \end{aligned} \tag{35}$$

see also the discussion in the introduction. Here, J_0 is the Bessel function satisfying

$$J_0(\sqrt{\alpha^2+\beta^2}) = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{-i(\alpha\cos\theta + \beta\sin\theta)}, \quad \alpha, \beta \in \mathbb{R}, \tag{36}$$

and the test function space $\mathcal{D}$ consists of $f \in L^2(\mathbb{R}^3, (1+\rho)d^3k)$ which are continuous at zero. If $\rho_0 = 0$ the r.h.s. of (35) reduces to the product of the two exponentials (one may then extend $\mathcal{D}$ to all of $L^2(\mathbb{R}^3, (1+\rho)d^3k)$), and if in addition $\rho = 0$ then $E(f) = e^{-\frac{1}{4}\|f\|^2}$ is just the Fock generating functional corresponding to the zero temperature equilibrium state (in this case one may extend the test function space to all of $L^2(\mathbb{R}^3, d^3k)$).

Note that $E_{\rho,\rho_0}(e^{i\omega t} f) = E_{\rho,\rho_0}(f)$ for all $t \in \mathbb{R}$, so the corresponding state is invariant under the dynamics α_2^t, for any choice of $\rho(k)$, ρ_0. The generating functional of the *equilibrium state* of the heat bath at a given inverse temperature $0 < \beta < \infty$ is obtained as an infinite volume limit of the expectation functionals of the Gibbs states of the confined system; this fixes the densities $\rho(k), \rho_0$ as explained in (9)–(13).

Denote by ω_{ρ,ρ_0} the state on $\mathfrak{W}(\mathcal{D})$ whose generating functional is (35). The GNS representation of the pair $(\mathfrak{W}(\mathcal{D}), \omega_{\rho,\rho_0})$ has been given in [AW] as the triple $(\mathcal{H}_2, \pi_2, \Omega_2)$, where the representation Hilbert space is

$$\mathcal{H}_2 = \mathcal{F} \otimes \mathcal{F} \otimes L^2(S^1, d\sigma), \tag{37}$$

$\mathcal{F} = \mathcal{F}(L^2(\mathbb{R}^3, d^3k))$ is the Bosonic Fock space over $L^2(\mathbb{R}^3, d^3k)$ and $L^2(S^1, d\sigma)$ is the space of L^2-functions on the circle, with uniform normalized measure $d\sigma$ $(= (2\pi)^{-1} d\theta$, when viewed as the space of periodic functions of $\theta \in [-\pi, \pi])$. The cyclic vector is

$$\Omega_2 = \Omega_{\mathcal{F}} \otimes \Omega_{\mathcal{F}} \otimes \mathbf{1} \tag{38}$$

where $\Omega_{\mathcal{F}}$ is the vacuum in $\mathcal{F}$ and $\mathbf{1}$ is the normalized constant function in $L^2(S^1, d\sigma)$. The representation map $\pi_2 : \mathfrak{W}(\mathcal{D}) \to \mathcal{B}(\mathcal{H}_2)$ is given by

$$\pi_2(W(f)) = W_{\mathcal{F}}(\sqrt{1+\rho} f) \otimes W_{\mathcal{F}}(\sqrt{\rho}\overline{f}) \otimes e^{-i\Phi(f,\theta)}, \tag{39}$$

where

$$W_{\mathcal{F}} = e^{i\varphi_{\mathcal{F}}(f)}$$

is a Weyl operator in Fock representation and the field operator $\varphi_{\mathcal{F}}(f)$ is

$$\varphi_{\mathcal{F}}(f) = \frac{1}{\sqrt{2}}(a_{\mathcal{F}}{}^*(f) + a_{\mathcal{F}}(f)) \tag{40}$$

and $a_{\mathcal{F}}{}^*(f)$ and $a_{\mathcal{F}}(f)$ are the smeared out creation and annihilation operators satisfying the commutation relations

$$[a_{\mathcal{F}}(f), a_{\mathcal{F}}{}^*(g)] = \langle f, g \rangle\,,\ [a_{\mathcal{F}}(f), a_{\mathcal{F}}(g)] = [a_{\mathcal{F}}{}^*(f), a_{\mathcal{F}}{}^*(g)] = 0. \tag{41}$$

Our convention is that $f \mapsto a_{\mathcal{F}}(f)$ is an antilinear map. The phase $\Phi \in \mathbb{R}$ is given by

$$\Phi(f, \theta) = (2\pi)^{-3/2} \sqrt{2\rho_0} \big((\mathrm{Re} f(0)) \cos\theta + (\mathrm{Im} f(0)) \sin\theta\big). \tag{42}$$

It is not hard to check that E_{ρ,ρ_0} is the correct generating functional, i.e., that $E_{\rho,\rho_0}(f) = \langle \Omega_2, \pi_2(W(f))\Omega_2 \rangle$ (use (36)). In the absence of a condensate ($\rho_0 =$

$0 \Rightarrow \Phi = 0$) the third factor in (37)–(39) disappears and the representation reduces to the "Araki-Woods representation" in the form it has appeared in a variety of recent papers. We denote this representation by π_0. More precisely, the GNS representation of $(\mathfrak{W}(\mathcal{D}), \omega_{\rho=0,\rho_0=0})$ is given by $(\mathcal{F} \otimes \mathcal{F}, \pi_0, \Omega_0)$, where

$$\begin{aligned} \pi_0(W(f)) &= W_{\mathcal{F}}(\sqrt{1+\rho} f) \otimes W_{\mathcal{F}}(\sqrt{\rho} \bar{f}), & (43) \\ \Omega_0 &= \Omega_{\mathcal{F}} \otimes \Omega_{\mathcal{F}}. & (44) \end{aligned}$$

Let us introduce the von Neumann algebras

$$\begin{aligned} \mathfrak{M}_0 &= \pi_0(\mathfrak{W}(\mathcal{D}))'' \quad \subset \mathcal{B}(\mathcal{F} \otimes \mathcal{F}) & (45) \\ \mathfrak{M}_2 &= \pi_2(\mathfrak{W}(\mathcal{D}))'' \quad \subset \mathcal{B}(\mathcal{H}_2) & (46) \end{aligned}$$

which are the weak closures (double commutants) of the Weyl algebra represented as operators on the respective Hilbert spaces. $\mathfrak{M}_2$ splits into a product

$$\mathfrak{M}_2 = \mathfrak{M}_0 \otimes \mathcal{M} \subset \mathcal{B}(\mathcal{F} \otimes \mathcal{F}) \otimes \mathcal{B}(L^2(S^1, d\sigma)), \qquad (47)$$

where $\mathcal{M}$ is the abelian von Neumann algebra of all multiplication operators on $L^2(S^1, d\sigma)$. It satisfies $\mathcal{M}' = \mathcal{M}$. Relation (47) follows from this: clearly we have ${\mathfrak{M}_0}' \otimes \mathcal{M} \subset {\mathfrak{M}_2}'$, so taking the commutant gives

$$\mathfrak{M}_0 \otimes \mathcal{M} \supset \mathfrak{M}_2.$$

The reverse inclusion is obtained from $\mathbb{1}_{\mathcal{F}\otimes\mathcal{F}} \otimes \mathcal{M} \subset \mathfrak{M}_2$ and $\mathfrak{M}_0 \otimes \mathbb{1}_{L^2(S^1)} \subset \mathfrak{M}_2$ (see [AW]).

It is well known that $\mathfrak{M}_0$, the von Neumann algebra corresponding to the situation without condensate, is a factor. That means that its center is trivial, $\mathfrak{Z}(\mathfrak{M}_0) = \mathfrak{M}_0 \cap {\mathfrak{M}_0}' \cong \mathbb{C}$. However, we have $\mathfrak{Z}(\mathfrak{M}_2) = (\mathfrak{M}_0 \otimes \mathcal{M}) \cap ({\mathfrak{M}_0}' \otimes \mathcal{M})$, i.e.

$$\mathfrak{Z}(\mathfrak{M}_2) = \mathbb{1}_{\mathcal{F}\otimes\mathcal{F}} \otimes \mathcal{M}, \qquad (48)$$

so the von Neumann algebra $\mathfrak{M}_2$ is *not a factor*. One can decompose $\mathfrak{M}_2$ into a direct integral of factors, or equivalently, one can decompose ω_{ρ,ρ_0} into an integral over factor states. The Hilbert space (37) is the direct integral

$$\mathcal{H}_2 = \int_{[-\pi,\pi]}^{\oplus} \frac{d\theta}{2\pi}\, \mathcal{F} \otimes \mathcal{F}, \qquad (49)$$

and the formula (see (38), (39), (43), (44))

$$\omega_{\rho,\rho_0}(W(f)) = \langle \Omega_2, \pi_2(W(f))\Omega_2\rangle = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{-i\Phi(f,\theta)} \langle \Omega_0, \pi_0(W(f))\Omega_0\rangle \quad (50)$$

shows that π_2 is decomposed as

$$\pi_2 = \int_{[-\pi,\pi]}^{\oplus} \frac{d\theta}{2\pi}\, \pi_\theta, \quad (51)$$

where $\pi_\theta : \mathfrak{W}(\mathcal{D}) \to \mathcal{B}(\mathcal{F} \otimes \mathcal{F})$ is the representation defined by

$$\pi_\theta(W(f)) = e^{-i\Phi(f,\theta)}\pi_0(W(f)). \quad (52)$$

For each fixed θ,

$$\pi_\theta(\mathfrak{W}(\mathcal{D}))'' = \mathfrak{M}_0 \quad (53)$$

is a factor. Accordingly we have

$$\mathfrak{M}_2 = \int_{[-\pi,\pi]}^{\oplus} \frac{d\theta}{2\pi}\, \mathfrak{M}_0. \quad (54)$$

Introducing this decomposition is convenient for us because we will see that it reduces the dynamics of the system, so that one can examine each fiber of the decomposition separately, thus reducing the description of the system with a condensate to one without condensate (but having a dynamics which varies with varying θ).

In what follows we concentrate on the *equilibrium* state of the uncoupled system with a condensate,

$$\omega_{\beta,0}^{\rm con} = \omega_{1,\beta} \otimes \omega_{2,\beta}, \quad (55)$$

where $\omega_{1,\beta}$ is the Gibbs state of the small system (see (33)), and where $\omega_{2,\beta}$ is the equilibrium state of the heat bath at inverse temperature β and above ciritcal density, $\rho_{\rm tot} > \rho_{\rm crit}(\beta)$, determined by the generating functional (35). The index 0 in (55) indicates the absence of an interaction between the two systems. $\omega_{\beta,0}^{\rm con}$ is a state on the C^*-algebra $\mathfrak{A}$, (3). Of course, the GNS representation of $(\mathfrak{A}, \omega_{\beta,0}^{\rm con})$ is just $(\mathcal{H}, \pi, \Omega)$, where

$$\begin{aligned} \mathcal{H} &= \mathcal{H}_1 \otimes \mathcal{H}_2 \\ \pi &= \pi_1 \otimes \pi_2 && (56) \\ \Omega_{\beta,0}^{\rm con} &= \Omega_{1,\beta} \otimes \Omega_2. && (57) \end{aligned}$$

The free dynamics is given by the group of $*$automorphisms α_0^t, (4). Let

$$\mathfrak{M}_\beta^{\rm con} := \pi(\mathfrak{A})'' = \mathfrak{M}_1 \otimes \mathfrak{M}_2 = \int_{[-\pi,\pi]}^{\oplus} \frac{d\theta}{2\pi}\, \mathfrak{M}_1 \otimes \mathfrak{M}_0 \quad \subset \mathcal{B}(\mathcal{H}) \tag{58}$$

be the von Neumann algebra obtained by taking the weak closure of all observables of the combined system, when represented on $\mathcal{H}$. To see how we can implement the uncoupled dynamics in $\mathcal{H}$ we use that (for all $t \in \mathbb{R}$) $\Phi(e^{i\omega t} f, \theta) = \Phi(f, \theta)$, which follows from $\omega(0) = 0$, see (42) and (2). Thus

$$\pi_2(\alpha_2^t(W(f))) = \int_{[-\pi,\pi]}^{\oplus} \frac{d\theta}{2\pi} e^{-i\Phi(f,\theta)} \pi_0(W(e^{i\omega t} f)). \tag{59}$$

It is well known and easy to verify that for $A \in \mathfrak{A}$,

$$(\pi_1 \otimes \pi_0)(\alpha_0^t(A)) = e^{itL_0}(\pi_1 \otimes \pi_0)(A) e^{-itL_0}, \tag{60}$$

where the selfadjoint L_0 on $\mathcal{H}_1 \otimes \mathcal{F} \otimes \mathcal{F}$ is given by

$$L_0 \;=\; L_1 + L_2, \tag{61}$$

$$L_1 \;=\; H_1 \otimes \mathbb{1}_{\mathbb{C}^d} - \mathbb{1}_{\mathbb{C}^d} \otimes H_1, \tag{62}$$

$$L_2 \;=\; \mathrm{d}\Gamma(\omega) \otimes \mathbb{1}_{\mathcal{F}} - \mathbb{1}_{\mathcal{F}} \otimes \mathrm{d}\Gamma(\omega). \tag{63}$$

Here $\mathrm{d}\Gamma(\omega)$ is the second quantization of the operator of multiplication by ω on $L^2(\mathbb{R}^3, d^3k)$. We will omit trivial factors $\mathbb{1}$ or indices ${}_{\mathbb{C}^d}$, ${}_{\mathcal{F}}$ whenever we have the reasonable hope that no confusion can arise (e.g. L_1 really means $L_1 \otimes \mathbb{1}_{\mathcal{F}} \otimes \mathbb{1}_{\mathcal{F}}$). It follows from (58)–(63) that the uncoupled dynamics α_0^t is unitarily implemented in $\mathcal{H}$ by

$$\pi(\alpha_0^t(A)) = e^{it\mathcal{L}_0} \pi(A) e^{-it\mathcal{L}_0}, \tag{64}$$

where the *standard, non-interacting Liouvillian* $\mathcal{L}_0$ is the selfadjoint operator on $\mathcal{H}$ with constant (θ-independent) fiber L_0,

$$\mathcal{L}_0 = \int_{[-\pi,\pi]}^{\oplus} \frac{d\theta}{2\pi}\, L_0. \tag{65}$$

The r.h.s. of (64) extends to a $*$automorphism group σ_0^t of $\mathfrak{M}_\beta^{\rm con}$ which is reduced by the decomposition (58). We write

$$\sigma_0^t = \int_{[-\pi,\pi]}^{\oplus} \frac{d\theta}{2\pi}\, \sigma_{0,\theta}^t, \tag{66}$$

where $\sigma_{0,\theta}^t$ is the $*$automorphism group of $\mathfrak{M}_1 \otimes \mathfrak{M}_0$ generated by L_0. As is well known,

$$\Omega_{\beta,0} = \Omega_{1,\beta} \otimes \Omega_0 \tag{67}$$

is a $(\beta, \sigma_{0,\theta}^t)$-KMS state of $\mathfrak{M}_1 \otimes \mathfrak{M}_0$. The modular conjugation operator J associated to $(\mathfrak{M}_0, \Omega_{1,\beta} \otimes \Omega_0)$ is

$$J = J_1 \otimes J_0, \tag{68}$$

where J_1 is given by (32) and where the action of J_0 on $\mathcal{F} \otimes \mathcal{F}$ is determined by antilinearly extending the relation

$$J_0 \pi_0(W(f))\Omega_0 = W_{\mathcal{F}}(\sqrt{\rho} f) \otimes W_{\mathcal{F}}(\sqrt{1+\rho}\, \overline{f})\Omega_0. \tag{69}$$

J_0 defines an antilinear representation of the Weyl algebra according to $W(f) \mapsto J_0 \pi_0(W(f)) J_0$, which commutes with the representation π_0 given in (43). We view this as a consequence of the Tomita-Takesaki theory which asserts that $\mathfrak{M}_0{}' = J_0 \mathfrak{M}_0 J_0$.

It follows from (57), (58), (66) that

$$\Omega_{\beta,0}^{\rm con} = \int_{[-\pi,\pi]}^{\oplus} \frac{d\theta}{2\pi}\ \Omega_{\beta,0} \tag{70}$$

is a (β, σ_0^t)-KMS state on $\mathfrak{M}_\beta^{\rm con}$, and that the modular conjugation operator $\mathcal{J}$ associated to $(\mathfrak{M}_\beta^{\rm con}, \Omega_{\beta,0}^{\rm con})$ is given by

$$\mathcal{J} = \int_{[-\pi,\pi]}^{\oplus} \frac{d\theta}{2\pi}\ J_1 \otimes J_0. \tag{71}$$

The standard Liouvillian $\mathcal{L}_0$, (65), satisfies the relation

$$\mathcal{J}\mathcal{L}_0 = -\mathcal{L}_0 \mathcal{J}. \tag{72}$$

One can choose different generators to implement the dynamics α_0^t on $\mathcal{H}$ (by adding to the standard $\mathcal{L}_0$ any selfadjoint element affiliated with the commutant $(\mathfrak{M}_\beta^{\rm con})'$). The choice (65) is compatible with the symmetry $\mathfrak{M}_\beta^{\rm con} \cong (\mathfrak{M}_\beta^{\rm con})'$, in that it also implements α_0^t for the antilinear representation $\mathcal{J}\pi(\cdot)\mathcal{J}$. Another way to say this is that the standard Liouvillian (65) is the only generator which implements the non-interacting dynamics α_0^t and satisfies

$$\mathcal{L}_0 \Omega_{\beta,0}^{\rm con} = 0, \tag{73}$$

see e.g. [BR, DJP].

2.1.2 Interacting system

We define the coupled dynamics, i.e. the interaction between the small system and the Bose gas, by specifying a $*$automorphism group σ_λ^t of the von Neumann algebra $\mathfrak{M}_\beta^{\rm con}$ (the "perturbed" or "interacting dynamics"). One may argue that a conceptually more satisfying way is to introduce a representation independent *regularized* dynamics as a $*$automorphism group of $\mathfrak{A}$ and then removing the regularization once the dynamics is represented on a Hilbert space. This procedure can be implemented by following the arguments of [FM1], where it has been carried out for the Bose gas without condensate. The resulting dynamics is of course the same for both approaches. For a technically more detailed exposition of the following construction we refer the reader to [FM1].

The interaction between the two subsystems is given formally by (17), which we understand as an operator in a regular representation of the Weyl algebra, so that the creation and annihilation operators are well defined. We could treat interactions which are sums over finitely many terms of the form (17), simply at the expense of more complicated notation.

The field operator $\varphi(f) = \frac{1}{i}\partial_t|_{t=0}\pi(W(tf))$ in the representation π, (56), is easily calculated to be

$$\begin{aligned} \varphi(f) &= \int_{[-\pi,\pi]}^{\oplus} \frac{d\theta}{2\pi}\, \varphi_\theta(f), && (74)\\ \varphi_\theta(f) &= \varphi_{\mathcal{F}}(\sqrt{1+\rho}f)\otimes \mathbb{1} + \mathbb{1}\otimes \varphi_{\mathcal{F}}(\sqrt{\rho}\overline{f}) - \Phi(f,\theta), && (75)\end{aligned}$$

where $\Phi(f,\theta)$ is given in (42), and where $\varphi_{\mathcal{F}}(f)$ is given in (40). Define the interaction operator by

$$V = G\otimes \mathbb{1}_{\mathbb{C}^d}\otimes\varphi(g), \tag{76}$$

which corresponds formally to $\pi\big(G\otimes\frac{1}{\sqrt{2}}(a^*(g)+a(g))\big)$. V is an unbounded selfadjoint operator on $\mathcal{H}$ which is affiliated with $\mathfrak{M}_\beta^{\rm con}$. For $t\in\mathbb{R}$, $A\in\mathfrak{M}_\beta^{\rm con}$ we set

$$\begin{aligned}\sigma_\lambda^t(A) &= \sum_{n\geq 0}(i\lambda)^n\int_0^t dt_1\dots\int_0^{t_{n-1}}dt_n\big[e^{it_n\mathcal{L}_0}Ve^{-it_n\mathcal{L}_0},\big[\cdots\\ &\qquad\cdots\big[e^{it_1\mathcal{L}_0}Ve^{-it_1\mathcal{L}_0},A\big]\cdots\big]\big]. && (77)\end{aligned}$$

The series is understood in the strong sense on a dense set of vectors (e.g. vectors which are analytic with respect to the the total number operator $N = \int^{\oplus}\{\mathrm{d}\Gamma(\mathbb{1}) \otimes \mathbb{1}_{\mathcal{F}} + \mathbb{1}_{\mathcal{F}} \otimes \mathrm{d}\Gamma(\mathbb{1})\}$, [FM1]), on which it converges for any $A \in \mathfrak{M}_\beta^{\mathrm{con}}$, $\lambda, t \in \mathbb{R}$. Since V is affiliated with $\mathfrak{M}_\beta^{\mathrm{con}}$ and $e^{it\mathcal{L}_0} \cdot e^{-it\mathcal{L}_0}$ leaves $\mathfrak{M}_\beta^{\mathrm{con}}$ invariant, one sees that the integrand in (77) does not change when one adds to each $e^{it_j\mathcal{L}_0}Ve^{-it_j\mathcal{L}_0}$ a term $-\mathcal{J}e^{it_j\mathcal{L}_0}Ve^{-it_j\mathcal{L}_0}\mathcal{J} = -e^{it_j\mathcal{L}_0}\mathcal{J}V\mathcal{J}e^{-it_j\mathcal{L}_0}$ (which is affiliated with the commutant $(\mathfrak{M}_\beta^{\mathrm{con}})'$). In other words, V in (77) can be replaced by $V - \mathcal{J}V\mathcal{J}$. The r.h.s. of (77) is then identified as the Dyson series expansion of

$$e^{it\mathcal{L}_\lambda}Ae^{-it\mathcal{L}_\lambda}, \tag{78}$$

where the standard, interacting Liouvillian $\mathcal{L}_\lambda$ is the selfadjoint operator

$$\mathcal{L}_\lambda = \mathcal{L}_0 + \lambda(V - \mathcal{J}V\mathcal{J}) \equiv \mathcal{L}_0 + \lambda\mathcal{I}. \tag{79}$$

Subtracting the term $\mathcal{J}V\mathcal{J}$ serves to preserve the symmetry (72) under the perturbation, i.e., we have $\mathcal{J}\mathcal{L}_\lambda = -\mathcal{L}_\lambda\mathcal{J}$. It is not hard to verify that (78) defines a $*$automorphism group

$$\sigma_\lambda^t(A) = e^{it\mathcal{L}_\lambda}Ae^{-it\mathcal{L}_\lambda} \tag{80}$$

of $\mathfrak{M}_\beta^{\mathrm{con}}$, [FM1]. This defines the interacting dynamics. The Liouvillian $\mathcal{L}_\lambda$ is reduced by the direct integral decomposition,

$$\mathcal{L}_\lambda = \int_{[-\pi,\pi]}^{\oplus} \frac{d\theta}{2\pi}\ L_{\lambda,\theta}, \tag{81}$$

where the selfadjoint operator $L_{\lambda,\theta}$ is

$$L_{\lambda,\theta} = L_0 + \lambda I_\theta. \tag{82}$$

Here L_0 is given in (61) and we define

$$I_\theta = I + K_\theta, \tag{83}$$

$$I = G \otimes \mathbb{1}_{\mathbb{C}^d} \otimes \left\{ a_{\mathcal{F}}{}^*(\sqrt{1+\rho}\, g) \otimes \mathbb{1}_{\mathcal{F}} + \mathbb{1}_{\mathcal{F}} \otimes a_{\mathcal{F}}(\sqrt{\rho}\,\overline{g}) \right\}$$
$$-\mathbb{1}_{\mathbb{C}^d} \otimes \mathcal{C}_1 G \mathcal{C}_1 \otimes \left\{ a_{\mathcal{F}}(\sqrt{\rho} g) \otimes \mathbb{1}_{\mathcal{F}} + \mathbb{1}_{\mathcal{F}} \otimes a_{\mathcal{F}}{}^*(\sqrt{1+\rho}\,\overline{g}) \right\} \tag{84}$$

$$K_\theta = -\Phi(g,\theta)\left\{ G \otimes \mathbb{1}_{\mathbb{C}^d} - \mathbb{1}_{\mathbb{C}^d} \otimes \mathcal{C}_1 G \mathcal{C}_1 \right\}, \tag{85}$$

with $\mathcal{C}_1$, Φ defined in (32), (42) and where the creation and annihilation operators $a_{\mathcal{F}}{}^*$, $a_{\mathcal{F}}$ are defined by (41). It is convenient to write (compare with (66))

$$\sigma_\lambda^t = \int_{[-\pi,\pi]}^{\oplus} \frac{d\theta}{2\pi}\, \sigma_{\lambda,\theta}^t, \tag{86}$$

where $\sigma_{\lambda,\theta}^t$ is the $*$automorphism group on $\mathfrak{M}_1 \otimes \mathfrak{M}_0$ generated by $L_{\lambda,\theta}$, (82).

To the interacting dynamics (80) corresponds a β-KMS state on $\mathfrak{M}_\beta^{\rm con}$, the equilibrium state of the interacting system. It is given by the vector

$$\Omega_{\beta,\lambda}^{\rm con} = (Z_{\beta,\lambda}^{\rm con})^{-1} \int_{[-\pi,\pi]}^{\oplus} \frac{d\theta}{2\pi}\, \Omega_{\beta,\lambda}^{\theta}, \tag{87}$$

where $Z_{\beta,\lambda}^{\rm con}$ is a normalization factor ensuring that $\|\Omega_{\beta,\lambda}^{\rm con}\| = 1$, and where

$$\Omega_{\beta,\lambda}^{\theta} = (Z_{\beta,\lambda}^{\theta})^{-1} e^{-\beta(L_0+\lambda I_{\theta,\ell})/2}\Omega_{\beta,0} \in \mathcal{H}_1 \otimes \mathcal{F} \otimes \mathcal{F}. \tag{88}$$

$Z_{\beta,\lambda}^{\theta}$ is again a normalization factor, and $I_{\theta,\ell}$ is obtained by dropping the second term both in (84) and in (85). The fact that $\Omega_{\beta,0}$, (67), is in the domain of the unbounded operator $e^{-\beta(L_0+\lambda I_{\theta,\ell})/2}$, provided

$$\|g/\sqrt{\omega}\|_{L^2(\mathbb{R}^3)} < \infty, \tag{89}$$

can be seen by expanding the exponential in a Dyson series and verifying that the series applied to $\Omega_{\beta,0}$ converges, see e.g. [BFS]. It then follows from the generalization of Araki's perturbation theory of KMS states, given in [DJP], that $\Omega_{\beta,\lambda}^{\theta}$ is a $(\beta, \sigma_{\lambda,\theta}^t)$-KMS state on $\mathfrak{M}_1 \otimes \mathfrak{M}_0$, and that

$$L_{\lambda,\theta}\, \Omega_{\beta,\lambda}^{\theta} = 0. \tag{90}$$

We conclude that $\Omega_{\beta,\lambda}^{\rm con}$ is a $(\beta, \sigma_\lambda^t)$-KMS state on $\mathfrak{M}_\beta^{\rm con}$, and that $\mathcal{L}_\lambda \Omega_{\beta,\lambda}^{\rm con} = 0$.

2.2 Main results

We make two assumptions on the the coupling operator G and the form factor g determining the interaction (see (17), (76)).

(A1) *Regularity.* The form factor g is a function in $C^4(\mathbb{R}^3)$ and satisfies

$$\|(1 + 1/\sqrt{\omega})(k \cdot \nabla_k)^j g\|_{L^2(\mathbb{R}^3, d^3k)} < \infty,$$

for $j = 0, \ldots, 4$, and $\|\, (1 + \omega)^2 g\|_{L^2(\mathbb{R}^3, d^3k)} < \infty$.

(A2) *Effective coupling.* Let φ_n be the eigenvector of H_1 with eigenvalue E_n, see (26). We assume that for all $m \neq n$, $\langle \varphi_m, G\varphi_n \rangle \neq 0$ and $\int_{S^2} d\sigma\, g(|E_m - E_n|, \sigma) \neq 0$. Here, g is represented in polar coordinates.

Remarks. 1) Condition (A1) is used in the application of the virial theorem – we choose the generator of dilations $\frac{1}{2}(k \cdot \nabla_k + \nabla_k \cdot k)$ to be the conjugate operator in the theory.

2) Condition (A2) is often called the *Fermi Golden Rule Condition.* It guarantees that the processes of absorption and emission of field quanta by the small system, which are the origin of the stability of the equilibrium, are effective, see the discussion in the introduction.

Theorem 2.1 (Stability of equilibrium with condensate). *Assume conditions (A1) and (A2). Let ω be a normal state on $\mathfrak{M}_\beta^{\rm con}$ and let $A \in \mathfrak{M}_\beta^{\rm con}$. We have*

$$\lim_{\lambda\to 0} \lim_{T\to\infty} \frac{1}{T} \int_0^T dt\ \omega(\sigma_\lambda^t(A)) = \omega_{\beta,0}^{\rm con}(A), \tag{91}$$

where $\omega_{\beta,0}^{\rm con}$ is the equilibrium state of non-interacting system, see (55).

Remark. As mentioned in the introduction, we expect that the stronger result $\lim_{T\to\infty} \int_0^T dt\ \omega(\sigma_\lambda^t(A)) = \omega_{\beta,\lambda}^{\rm con}(A)$ is true, where $\omega_{\beta,\lambda}^{\rm con}$ is the interacting KMS state given by (87). This relation, called *Return to Equilibrium*, has been proven for systems without a condensate (with varying conditions on the interaction and varying modes of convergence) in several papers, see [JP1, BFS, M1, DJ, FM2]. The obstruction to applying the strategies of these papers is that they all need the condition that either $g(0) = 0$, or $g(k) \sim |k|^{-1/2}$, as $|k| \to 0$. The first case is uninteresting in the presence of a condensate (no coupling to the modes of the condensate!), and the second type of form factor does not enter into the description of a system with a condensate (see (42)). See Subsection 2.2.1 for a more detailed discussion of this point.

In order to state the virial theorem and to measure the regularity of eigenvectors of $L_{\lambda,\theta}$, (82), we introduce the non-negative selfadjoint operator

$$\Lambda = \mathrm{d}\Gamma(\omega) \otimes \mathbb{1}_{\mathcal{F}} + \mathbb{1}_{\mathcal{F}} \otimes \mathrm{d}\Gamma(\omega), \tag{92}$$

where $\mathrm{d}\Gamma(\omega)$ is the second quantization of the operator of multiplication by $\omega(k)$ on $L^2(\mathbb{R}^3, d^3k)$, c.f. (2). The kernel of Λ is spanned by the vector

$\Omega_0 = \Omega_{\mathcal{F}} \otimes \Omega_{\mathcal{F}}$ (c.f. (44)) and Λ has no nonzero eigenvalues. The operator Λ represents the quadratic form $i[L_0, A]$, the commutator of L_0 with the *conjugate operator*

$$A = \mathrm{d}\Gamma(a_{\mathrm{d}}) \otimes \mathbb{1}_{\mathcal{F}} - \mathbb{1}_{\mathcal{F}} \otimes \mathrm{d}\Gamma(a_{\mathrm{d}}), \tag{93}$$

where a_{d} is the selfadjoint generator of dilations on $L^2(\mathbb{R}^3, d^3k)$,

$$a_{\mathrm{d}} = i\left(k \cdot \nabla_k + \frac{3}{2}\right). \tag{94}$$

The formal relation $\Lambda = i[L_0, A]$ follows from $i[\omega, a_{\mathrm{d}}] = \omega$ (for relativistic Bosons, see (2); in the non-relativistic case, $i[\omega, a_{\mathrm{d}}/2] = \omega$, so we could redefine a_{d} by dividing the r.h.s. of (94) by a factor two, in order to have $\Lambda = i[L_0, A]$). The selfadjoint operator representing the quadratic form $i[L_{\lambda,\theta}, A]$ is easily calculated to be (see (84))

$$C_1 = \Lambda + \lambda I_1 \tag{95}$$

$$\begin{aligned} I_1 = {} & G \otimes \mathbb{1}_{\mathbb{C}^d} \otimes \left\{ a_{\mathcal{F}}{}^*(a_{\mathrm{d}}\sqrt{1+\rho}\, g) \otimes \mathbb{1}_{\mathcal{F}} - \mathbb{1}_{\mathcal{F}} \otimes a_{\mathcal{F}}(a_{\mathrm{d}}\sqrt{\rho}\, \overline{g}) \right\} \\ & - \mathbb{1}_{\mathbb{C}^d} \otimes \mathcal{C}_1 G \mathcal{C}_1 \otimes \left\{ a_{\mathcal{F}}(a_{\mathrm{d}}\sqrt{\rho} g) \otimes \mathbb{1}_{\mathcal{F}} - \mathbb{1}_{\mathcal{F}} \otimes a_{\mathcal{F}}{}^*(a_{\mathrm{d}}\sqrt{1+\rho}\, \overline{g}) \right\}. \end{aligned} \tag{96}$$

Similar expressions are obtained for the higher commutators of $L_{\lambda,\theta}$ with A, see Section 3. Assumption (A1) guarantees that $(1 + 1/\sqrt{\omega})(a_{\mathrm{d}})^j \sqrt{\rho}\, g$ and $(1 + 1/\sqrt{\omega})(a_{\mathrm{d}})^j \sqrt{1+\rho}\, g$ are in $L^2(\mathbb{R}^3, d^3k)$, for $j = 0, \ldots, 4$, so the commutators of $L_{\lambda,\theta}$ with A, up to order four, are represented by selfadjoint operators (satisfying the technical requirements needed in the proof of the virial theorem).

Theorem 2.2 (Virial Theorem, regularity of eigenvectors of $L_{\lambda,\theta}$). *Assume condition (A1) and let $\theta \in [-\pi, \pi]$ be fixed. If ψ is an eigenfunction of $L_{\lambda,\theta}$ then ψ is in the form domain of C_1, (95), and*

$$\langle \psi, C_1 \psi \rangle = 0. \tag{97}$$

There is a constant c which does not depend on $\theta \in [-\pi, \pi]$ nor on $\beta \geq \beta_0$, for any $\beta_0 > 0$ fixed, such that

$$\|\Lambda^{1/2}\psi\| \leq c|\lambda|\, \|\psi\|. \tag{98}$$

Remarks. 1) Relation (97) seems "obvious" from a formal point of view, writing $C_1 = i[L_{\lambda,\theta}, A] = i[L_{\lambda,\theta} - e, A]$, and using that $(L_{\lambda,\theta} - e)^* = L_{\lambda,\theta} - e$, where $L_{\lambda,\theta}\psi = e\psi$. A *proof* of (97) is certainly not trivial, though, and considerable effort has been spent by many authors to establish "Virial Theorems" (see e.g. [ABG] and [GG] for an overview, and also [M1], [FM1] for approaches similar to ours).

2) The regularity bound (98) follows easily from (97) and (95) and from the standard fact that I_1 is infinitesimally small relative to $\Lambda^{1/2}$ (Kato), so that $0 = \langle\psi, C_1\psi\rangle \geq (1-\epsilon)\langle\psi, \Lambda\psi\rangle - \frac{\lambda^2}{\epsilon} c\|\psi\|^2$, for any $\epsilon > 0$, for some constant c independent of θ and β, as mentioned in the theorem. We refer for a more complete exposition of this to [FM1].

We prove Theorem 2.2 in Section 3.2 by showing that the hypotheses leading to Theorem 3.2, a more general result, are satisfied in the present situation. Our next result describes the structure of $\ker L_{\lambda,\theta}$. Let $P(\Lambda \leq x)$ stand for the spectral projection of Λ onto the interval $[0, x]$.

Theorem 2.3 (Structure of the kernel of $L_{\lambda,\theta}$). *Assume Conditions (A1), (A2) and let $\theta \in [-\pi, \pi]$ be fixed. There is a number $\lambda_0 > 0$ s.t. if $0 < |\lambda| < \lambda_0$ then any normalized $\psi_\lambda \in \ker(L_{\lambda,\theta})$ satisfies*

$$\psi_\lambda = \Omega_{1,\beta} \otimes \big(P(\Lambda \leq |\lambda|)\chi_{\lambda,\theta}\big) + o(\lambda), \tag{99}$$

for some $\chi_{\lambda,\theta} \in \mathcal{F} \otimes \mathcal{F}$ satisfying $\|\chi_{\lambda,\theta}\| \geq 1 - o(\lambda)$. In (99) $o(\lambda)$ denotes a vector in $\mathcal{H}_1 \otimes \mathcal{F} \otimes \mathcal{F}$ whose norm vanishes in the limit $\lambda \to 0$ (uniformly in $\theta \in [-\pi, \pi]$ and in $\beta \geq \beta_0$, for any $\beta_0 > 0$ fixed), and $\Omega_{1,\beta}$ is the Gibbs vector (33). The constant λ_0 does not depend on $\theta \in [-\pi, \pi]$, and it is uniform in $\beta \geq \beta_0$, for any fixed $\beta_0 > 0$.

Our proof of this theorem, given in Section 5, relies on a positive commutator estimate and Theorem 2.2. Expansion (99) implies that the only vector in the kernel of $L_{\lambda,\theta}$ which does not converge weakly to zero, as $\lambda \to 0$, is the interacting KMS state $\Omega^\theta_{\beta,\lambda}$, (88). This information on the kernel of $L_{\lambda,\theta}$ *alone* enters our proof of Theorem 2.1.

Corollary 2.4 *Assume Conditions (A1) and (A2) and let $P^\theta_{\beta,\lambda}$ the projection onto the subspace spanned by the interacting KMS state $\Omega^\theta_{\beta,\lambda}$, (88). Let $\theta \in [-\pi, \pi]$ be fixed. Any normalized element $\psi_\lambda \in \ker(L_{\lambda,\theta}) \cap \big(\mathrm{Ran}\, P^\theta_{\beta,\lambda}\big)^\perp$ converges weakly to zero, as $\lambda \to 0$. The convergence is uniform in $\theta \in [-\pi, \pi]$*

and in $\beta \geq \beta_0$, for any $\beta_0 > 0$ fixed.

We prove the corollary in Section 5. The virial theorem we present in Section 3, Theorem 3.2, is applicable to systems without a condensate, in which case one is interested in form factors g which have a singularity at the origin. Theorem 3.2 can handle a wide range of such singularities (see the remark after Theorem 2.4) and is therefore relevant in the study of *return to equilibrium* and *thermal ionization* for systems without condensate, as will be explained in [FM3].

Theorem 2.4 (Improved Virial Theorem for systems without condensate). *Let L_λ be the Liouvillian of a system without condensate, $L_\lambda = L_0 + \lambda I$ (i.e., $K_\theta = 0$), see (83), (84), (85) and suppose that the form factor g is in $C^4(\mathbb{R}^3 \backslash \{0\})$ and satisfies the condition*

$$(1+1/\sqrt{\omega})(a_{\rm d})^j\sqrt{1+\rho}\, g,\ (1+1/\sqrt{\omega})(a_{\rm d})^j\sqrt{\rho}\, g\ \in L^2(\mathbb{R}^3, d^3k), \quad (100)$$

$$(1+\omega)^2(a_{\rm d})^j\sqrt{1+\rho}\, g,\ (1+\omega)^2(a_{\rm d})^j\sqrt{\rho}\, g\ \in L^2(\mathbb{R}^3, d^3k), \quad (101)$$

for $j = 0, \ldots, 4$. Then the conclusions (97), (98) of Theorem 2.2 hold.

Remark. An admissible infrared behaviour of g satisfying (100), (101) is $g(k) \sim |k|^p$, as $|k| \sim 0$, with $p > -1/2$ for relativistic Bosons (c.f. (2)). The range of treatable values of p obtained in previous works, [M1,DJ,FM1,FM2], is $p = -1/2, 1/2, 3/2,\ p > 2$. Theorem 2.4 fills in the gaps between the discrete values of these admissible p. This means that one has now a virial theorem for Liouville operators for the continuous range $p \geq -1/2$.

The proof of Theorem 2.4 is the same as the one of Theorem 2.2, see Section 3.2.

Theorem 2.5 *Assume the setting of Theorem 2.4, that (A2) holds and that $|g(k)| \leq c|k|^p$, for $|k| < c'$, for some constants c, c', and where $p > -1/2$ (for relativistic Bosons, and $p > 0$ for nonrelativistic ones). There is a number $\lambda_0 > 0$ s.t. if $0 < |\lambda| < \lambda_0$ then any normalized $\psi_\lambda \in \ker(L_\lambda)$ satisfies*

$$\psi_\lambda = \Omega_{1,\beta} \otimes \big(P(\Lambda \leq |\lambda|)\chi_\lambda\big) + o(\lambda), \quad (102)$$

for some $\chi_\lambda \in \mathcal{F} \otimes \mathcal{F}$ satisfying $\|\chi_\lambda\| \geq 1 - o(\lambda)$. In (102) $o(\lambda)$ denotes a vector in $\mathcal{H}_1 \otimes \mathcal{F} \otimes \mathcal{F}$ whose norm vanishes in the limit $\lambda \to 0$ (uniformly in $\beta \geq \beta_0$, for any $\beta_0 > 0$ fixed), and $\Omega_{1,\beta}$ is the Gibbs vector (33). The constant λ_0 does not depend on $\beta \geq \beta_0$, for any fixed $\beta_0 > 0$.

We give the proof Theorem 2.5 together with the proof of Theorem 2.3 in Section 5.

2.2.1 Discussion of "stability of $\omega_{\beta,0}^{\rm con}$" v.s. "return to equilibrium", and relation with infrared regularity of the coupling

A central tool in our analysis of the time-asymptotic behaviour of the system is the virial theorem, whose use imposes regularity conditions on the interaction. In particular, we must be able to control the commutators of $L_{\lambda,\theta}$ with the conjugate operator A of degree up to four (see Section 3.1). Depending on the choice of A this will impose more or less restrictive requirements on the interaction. A very convenient choice for A is obtained by representing $\mathcal{F}\otimes\mathcal{F}$ as $\mathcal{F}(L^2(\mathbb{R}\times S^2, du\times d\sigma))$ and choosing $A = i\mathrm{d}\Gamma(\partial_u)$ (*translation generator*). This choice, introduced in [JP1], has proven to be very useful and was adopted in [M1, DJ, FM1, FMS, FM2]. The commutator of the non-interacting Liouvillian $L_0 = \mathrm{d}\Gamma(u)$ with A (multiplied by i) is just $N = \mathrm{d}\Gamma(\mathbb{1})$, the number operator in $\mathcal{F}(L^2(\mathbb{R}\times S^2, du\times d\sigma))$, which has a one-dimensional kernel and a *spectral gap* at zero. We may explain the usefulness of the gap as follows. If one carries out the proof of Theorem 2.3 with the translation generator as the conjugate operator then the role of Λ, (92), is taken by N, and relation (99) is replaced by $\|P_{1,\beta}P(N\leq|\lambda|)\psi_\lambda\| = 1-o(\lambda)$, where $P_{1,\beta} = |\Omega_{1,\beta}\rangle\langle\Omega_{1,\beta}|$. But for $|\lambda|<1$, $P_{\beta,1}P(N\leq|\lambda|)$ is just the projection $|\Omega_{\beta,0}\rangle\langle\Omega_{\beta,0}|$ onto the span of the non-interacting KMS state, so one has $|\langle\Omega_{\beta,0},\psi_\lambda\rangle| = 1-o(\lambda)$. Since $\Omega_{\beta,0}$ is close to $\Omega_{\beta,\lambda}^\theta$ for small values of λ, this means that there are no elements in the kernel of $L_{\lambda,\theta}$ which are orthogonal to $\Omega_{\beta,\lambda}^\theta$, provided $|\lambda|$ is small enough, i.e., the kernel of $L_{\lambda,\theta}$ has dimension one. A consequence of the simplicity of $\ker L_{\lambda,\theta}$ is that return to equilibrium holds.

The disadvantage of the translation generator is that its use requires (too) restricitve infrared regularity on the form factor. Indeed, the j-th commutator of the interaction with the translation generator involves the j-th derivative of the fuction $\frac{g}{\sqrt{e^{\beta\omega}-1}}$, so an infrared singular behaviour of this function is worsened by each application of the commutator (and we require those derivatives to be square integrable!). As a result, the case $g(0)\neq 0$ cannot be treated.

The remedy is to develop the theory with a conjugate operator A which does not affect the infrared behaviour of $\frac{g}{\sqrt{e^{\beta\omega}-1}}$ in a negative way. The choice (93) (*dilation generator*) is a good candidate (one could as well take operators interpolating between the translation and the dilation generator). The disadvantage of using the dilation generator is that its commutator with the non-interacting Liouvillian gives the operator Λ, which still has a one-dimensional kernel, but does *not* have a spectral gap at zero. This means that we cannot show that the kernel of $L_{\lambda,\theta}$ is simple, but we only have expansion (99), which, in turn, allows us only to show stability of $\omega_{\beta,0}^{\rm con}$, in the sense of Theorem 2.1, but not return to equilibrium.

We remark that the dilation generator has been used in [BFSS] to show instability of excited eigenvalues in zero-temperature models. We expect that it is a relatively easy exercise to modify the techniques of [M1] and show absence of *nonzero* eigenvalues of $L_{\lambda,\theta}$ (which we see as the "excited eigenvalues" in the positive temperature case) by using the dilation instead of the translation generator. Notice though that if one succeeds to show that the kernel of $L_{\lambda,\theta}$ is simple, then one knows *automatically* that $L_{\lambda,\theta}$ cannot have any non-zero eigenvalues, see e.g. [JP2].

2.2.2 Proof of Theorem 2.1

The normal state ω is a convex combination of vector states on $\mathfrak{M}_\beta^{\rm con}$, so it is enough to show (91) for $\omega(A)=\langle\psi,A\psi\rangle$, for an arbitrary normalized vector $\psi=\int_{[-\pi,\pi]}^{\oplus}\frac{d\theta}{2\pi}\,\psi_\theta\in\mathcal{H}$, and an arbitrary observable $A=\int_{[-\pi,\pi]}^{\oplus}\frac{d\theta}{2\pi}\,A_\theta\in\mathfrak{M}_\beta^{\rm con}$. Since

$$\omega(\sigma_\lambda^t(A))=\int_{-\pi}^{\pi}\frac{d\theta}{2\pi}\left\langle\psi_\theta,e^{itL_{\lambda,\theta}}A_\theta e^{-itL_{\lambda,\theta}}\psi_\theta\right\rangle \tag{103}$$

it suffices to prove that, for each θ,

$$\lim_{\lambda\to0}\lim_{T\to\infty}\frac{1}{T}\int_0^T dt\left\langle\psi_\theta,e^{itL_{\lambda,\theta}}A_\theta e^{-itL_{\lambda,\theta}}\psi_\theta\right\rangle=\langle\Omega_{\beta,0},A_\theta\Omega_{\beta,0}\rangle\,. \tag{104}$$

Let $\epsilon>0$ be fixed. Since $\Omega_{\beta,0}$ is cyclic for $(\mathfrak{M}_1\otimes\mathfrak{M}_0)'$ there exists a $B_{\theta,\epsilon}\in(\mathfrak{M}_1\otimes\mathfrak{M}_0)'$ such that

$$\psi_\theta=B_{\theta,\epsilon}\Omega_{\beta,0}+O(\epsilon)\,. \tag{105}$$

It follows that

$$\begin{aligned}&\left\langle\psi_\theta,e^{itL_{\lambda,\theta}}A_\theta e^{-itL_{\lambda,\theta}}\psi_\theta\right\rangle \qquad (106)\\&\quad=\left\langle(B_{\theta,\epsilon})^*\psi_\theta,e^{itL_{\lambda,\theta}}A_\theta\Omega_{\beta,\lambda}^\theta\right\rangle+O\left(\epsilon\|A\|+\|B_{\theta,\epsilon}\|\,\|A\|\,\|\Omega_{\beta,\lambda}^\theta-\Omega_{\beta,0}\|\right),\end{aligned}$$

where we use here that $e^{itL_{\lambda,\theta}} A_\theta e^{-itL_{\lambda,\theta}}$ commutes with $B_{\theta,\epsilon}$, and relation (90). Taking the limit $T \to \infty$ of the ergodic average $\frac{1}{T}\int_0^T dt$ on both sides and invoking von Neumann's ergodic theorem shows that

$$\begin{aligned} & \lim_{T\to\infty} \frac{1}{T} \int_0^T dt \left\langle \psi_\theta, e^{itL_{\lambda,\theta}} A_\theta e^{-itL_{\lambda,\theta}} \psi_\theta \right\rangle \\ & = \left\langle (B_{\theta,\epsilon})^* \psi_\theta, \Pi_{\lambda,\theta} A_\theta \Omega^\theta_{\beta,\lambda} \right\rangle + O\left(\epsilon \|A\| + \|B_{\theta,\epsilon}\| \, \|A\| \, \|\Omega^\theta_{\beta,\lambda} - \Omega_{\beta,0}\| \right), \end{aligned} \tag{107}$$

where $\Pi_{\lambda,\theta}$ is the projection onto $\ker(L_{\lambda,\theta})$,

$$\Pi_{\lambda,\theta} = |\Omega^\theta_{\beta,\lambda}\rangle\langle\Omega^\theta_{\beta,\lambda}| + \sum_{j=1}^\infty |\psi^\theta_{j,\lambda}\rangle\langle\psi^\theta_{j,\lambda}|, \tag{108}$$

where $\{\Omega^\theta_{\beta,\lambda}, \psi^\theta_{j,\lambda}\}$ is an orthonormal basis of $\ker(L_{\lambda,\theta})$. From Corollary 2.4 we know that the $\psi^\theta_{j,\lambda}$ converge weakly to zero, as $\lambda \to 0$, so

$$\lim_{\lambda\to 0} \left\langle (B_{\theta,\epsilon})^* \psi_\theta, \psi^\theta_{j,\lambda} \right\rangle \left\langle \psi^\theta_{j,\lambda}, A_\theta \Omega^\theta_{\beta,\lambda} \right\rangle = 0. \tag{109}$$

Using this in (107), together with $\lim_{\lambda\to 0} \|\Omega^\theta_{\beta,\lambda} - \Omega_{\beta,0}\| = 0$ (this limit is uniform in θ and in β, see [FM2]), shows that

$$\lim_{\lambda\to 0} \lim_{T\to\infty} \frac{1}{T} \int_0^T dt \left\langle \psi_\theta, e^{itL_{\lambda,\theta}} A_\theta e^{-itL_{\lambda,\theta}} \psi_\theta \right\rangle = \langle \Omega_{\beta,0}, A_\theta \Omega_{\beta,0} \rangle + O(\epsilon). \tag{110}$$

Since ϵ is arbitrary we are done. ■

3 Another abstract Virial Theorem with concrete applications

In this section we introduce a virial theorem in an abstract setting covering the cases of interest in the present paper (but which is general enough to allow for future generalizations). The virial theorem developed in [FM1], where the dominant part of $[L, A]$ commutes with A, does not apply to the present situation; here the leading term of $[[L, A], A]$ is L.

3.1 The abstract Virial Theorem

Let $\mathcal{H}$ be a Hilbert space, $\mathcal{D} \subset \mathcal{H}$ a core for a selfadjoint operator $Y \geq \mathbb{1}$, and X a symmetric operator on $\mathcal{D}$. We say the triple $(X, Y, \mathcal{D})$ satisfies the *GJN (Glimm-Jaffe-Nelson) Condition*, or that $(X, Y, \mathcal{D})$ is a *GJN-triple*, if there is a constant $k < \infty$, s.t. for all $\psi \in \mathcal{D}$:

$$\|X\psi\| \leq k\|Y\psi\| \tag{111}$$

$$\pm i \{\langle X\psi, Y\psi\rangle - \langle Y\psi, X\psi\rangle\} \leq k \langle \psi, Y\psi\rangle . \tag{112}$$

Notice that if $(X_1, Y, \mathcal{D})$ and $(X_2, Y, \mathcal{D})$ are GJN triples, then so is $(X_1 + X_2, Y, \mathcal{D})$. Since $Y \geq \mathbb{1}$, inequality (111) is equivalent to

$$\|X\psi\| \leq k_1\|Y\psi\| + k_2\|\psi\|,$$

for some $k_1, k_2 < \infty$. Condition (111) is phrased equivalently as "$X \leq kY$, in the sense of Kato on $\mathcal{D}$".

Theorem 3.1 (GJN commutator theorem) *If $(X, Y, \mathcal{D})$ satisfies the GJN Condition, then X determines a selfadjoint operator (again denoted by X), s.t. $\mathcal{D}(X) \supset \mathcal{D}(Y)$. Moreover, X is essentially selfadjoint on any core for Y, and (111) is valid for all $\psi \in \mathcal{D}(Y)$.*

Based on the GJN commutator theorem we next describe the setting for our general *virial theorem*. Suppose one is given a selfadjoint operator $Y \geq \mathbb{1}$ with core $\mathcal{D} \subset \mathcal{H}$, and operators $L, A, \Lambda \geq 0, D, C_n$, $n = 0, \ldots, 4$, all symmetric on $\mathcal{D}$, and being interrelated as

$$\langle \varphi, D\psi\rangle = i \{\langle L\varphi, \Lambda\psi\rangle - \langle \Lambda\varphi, L\psi\rangle\} \tag{113}$$

$$C_0 = L$$

$$\langle \varphi, C_n\psi\rangle = i \{\langle C_{n-1}\varphi, A\psi\rangle - \langle A\varphi, C_{n-1}\psi\rangle\}, \quad n = 1, \ldots, 4, \tag{114}$$

where $\varphi, \psi \in \mathcal{D}$. We assume that

(VT1) $(X, Y, \mathcal{D})$ satisfies the GJN Condition, for $X = L, \Lambda, D, C_n$. Consequently, all these operators determine selfadjoint operators (which we denote by the same letters).

(VT2) A is selfadjoint, $\mathcal{D} \subset \mathcal{D}(A)$, e^{itA} leaves $\mathcal{D}(Y)$ invariant, and

$$e^{itA} Y e^{-itA} \leq k e^{k'|t|} Y, \quad t \in \mathbb{R}, \tag{115}$$

in the sense of Kato on $\mathcal{D}$, for some constants k, k'.

(VT3) The operator D satisfies $D \leq k\Lambda^{1/2}$ in the sense of Kato on $\mathcal{D}$, for some constant k.

(VT4) Let the operators V_n be defined as follows: for $n = 1, 3$ set $C_n = \Lambda + V_n$, and set $C_2 = L_2 + V_2$, $C_4 = L_4 + V_4$. We assume the following relative bounds, all understood in the sense of Kato on $\mathcal{D}$:

$$\begin{aligned} V_n &\leq k\Lambda^{1/2}, \quad \text{for } n = 1, \ldots, 4, & (116)\\ L_4 &\leq k\Lambda, & (117)\\ L_2 &\leq k\Lambda^r, \quad \text{for some } r > 0. & (118) \end{aligned}$$

Remark. The invariance condition $e^{itA}\mathcal{D}(Y) \subset \mathcal{D}(Y)$ implies that the bound (115) holds in the sense of Kato on $\mathcal{D}(Y)$, see [ABG], Propositions 3.2.2 and 3.2.5.

Theorem 3.2 (Virial Theorem) *We assume the setting and assumptions introduced in this section so far. If $\psi \in \mathcal{H}$ is an eigenvector of L then ψ is in the form domain of C_1 and*

$$\langle C_1 \rangle_\psi = 0. \tag{119}$$

We prove this theorem in Section 4.

3.2 The concrete applications

The proofs of Theorems 2.2 and 2.4 reduce to an identification of the involved operators and domains and a subsequent verification of the assumptions of Section 3.1. Let us define

$$\mathcal{D} = \mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathcal{F}_0(C_0^\infty(\mathbb{R}^3, d^3k)) \otimes \mathcal{F}_0(C_0^\infty(\mathbb{R}^3, d^3k)), \tag{120}$$

where $\mathcal{F}_0$ is the finite-particle subspace of Fock space. Take

$$Y = \mathrm{d}\Gamma(\omega + 1) \otimes \mathbb{1}_{\mathcal{F}} + \mathbb{1}_{\mathcal{F}} \otimes \mathrm{d}\Gamma(\omega + 1) + \mathbb{1}, \tag{121}$$

and let the operators L, Λ, A of Section 3.1 be given, repectively, by the operators $L_{\lambda,\theta}$ (see (82), or L_λ in the case of Theorem 2.4), (92), and (93).

It is an easy task to calculate the operators C_j; C_1 is given in (95), $C_2 = L_2 + \lambda I_2$, $C_3 = \Lambda + \lambda I_3$, $C_4 = L_2 + \lambda I_4$, where L_2 is given in (63), and where the I_j are obtained similarly to I_1 (see (96)). The operator D, (113), is just $i\lambda[I, \Lambda]$. It is a routine job to verify that Conditions (VT1)–(VT4) hold, with $V_n = I_n$ and $L_4 = L_2$, $r = 1$. To check Condition (VT2) one can use the explicit action of e^{itA}, see also [FM1], Section 8.

4 Proof of Theorem 3.2

Before immersing ourselves into the details of the proof we present some facts we shall use repeatedly.

- If a unitary group e^{itX} leaves the domain $\mathcal{D}(Y)$ invariant then there exist constants k, k' s.t. $\|Ye^{itX}\psi\| \leq ke^{k'|t|}\|Y\psi\|$, for all $\psi \in \mathcal{D}(Y)$. Moreover, if $(X, Y, \mathcal{D})$ is a GJN triple then the unitary group e^{itX} leaves $\mathcal{D}(Y)$ invariant.

- Let $(X, Y, \mathcal{D})$ and $(Z, Y, \mathcal{D})$ be GNS triples, and suppose that the quadratic form of the commutator of X with Z, multiplied by i, is represented by a symmetric operator on $\mathcal{D}$, denoted by $i[X, Z]$, and that $(i[X, Z], Y, \mathcal{D})$ is a GNJ triple. Then we have

$$e^{itX}Ze^{-itX} - Z = \int_0^t dt_1\, e^{it_1X} i[X, Z]e^{-it_1X}. \tag{122}$$

  This equality is understood in the sense of operators on $\mathcal{D}(Y)$. Of course, if the higher commutators of X with Z also form GJN triples with $Y, \mathcal{D}$ then one can iterate formula (122).

We refer to [FM1] and the references therein for more detail and further results of this sort. Let us introduce the cutoff functions

$$f_1(x) = \int_{-\infty}^x dy\, e^{-y^2}, \quad f(x) = e^{-y^2/2}, \tag{123}$$

$$g = g_1^2, \tag{124}$$

where $g_1 \in C_0^\infty((-1,1))$ satisfies $g_1(0) = 1$. The derivative $(f_1)'$ equals f^2 which is strictly positive and the ratio $(f')^2/f$ decays faster than eponentially at infinity. The Gaussian f is the fixed point of the Fourier transform

$$\widehat{f}(s) = (2\pi)^{-1/2} \int_{\mathbb{R}} dx\, e^{-isx} f(x), \tag{125}$$

i.e., $\widehat{f}(s) = e^{-s^2/2}$, and we have $\widehat{(f_1)'} = is\widehat{f_1} = \widehat{f^2}$ which is a Gaussian itself. This means that $\widehat{f_1}$ decays like a Gaussian for large $|s|$ and has a singularity of type s^{-1} at the origin. We define cutoff operators, for $\nu, \alpha > 0$, by

$$\begin{aligned} g_{1,\nu} &= g_1(\nu\Lambda) = (2\pi)^{-1/2} \int_{\mathbb{R}} ds\, \widehat{g_1}(s) e^{is\nu\Lambda} && (126) \\ g_\nu &= g_{1,\nu}^2 && (127) \\ f_\alpha &= f(\alpha A) = (2\pi)^{-1/2} \int_{\mathbb{R}} ds \widehat{f}(s) e^{is\alpha A}. && (128) \end{aligned}$$

Since $\widehat{f_1}$ has a singularity at the origin, we cut a small interval $(-\eta, \eta)$ out of the real axis, where $\eta > 0$, and define

$$f_{1,\alpha}^\eta = \alpha^{-1}(2\pi)^{-1/2} \int_{\mathbb{R}_\eta} ds\, \widehat{f_1}(s) e^{is\alpha A}, \tag{129}$$

where we set $\mathbb{R}_\eta = \mathbb{R}\backslash(-\eta, \eta)$. Standard results about invariance of domains show that the cutoff operators $g_\nu, f_\alpha, f_{1,\alpha}^\eta$ are bounded selfadjoint operators leaving the domain $\mathcal{D}(Y)$ invariant, and it is not hard to see that $\|f_{1,\alpha}^\eta\| \leq k/\alpha$, uniformly in η (see [FM1]).

Suppose that ψ is a normalized eigenvector of L with eigenvalue e, $L\psi = e\psi$, $\|\psi\| = 1$. Let $\varphi \in \mathcal{H}$ be s.t. $\psi = (L+i)^{-1}\varphi$ and let $\{\varphi_n\} \subset \mathcal{D}$ be a sequence approximating φ, $\varphi_n \to \varphi$. Then we have

$$\psi_n = (L+i)^{-1}\varphi_n \longrightarrow \psi, \quad n \to \infty, \tag{130}$$

and $\psi_n \in \mathcal{D}(Y)$. The latter statement holds since the resolvent of L leaves $\mathcal{D}(Y)$ invariant (which in turn is true since $(L, Y, \mathcal{D})$ is a GJN triple). It follows that the *regularized eigenfunction*

$$\psi_{\alpha,\nu,n} = f_\alpha g_\nu \psi_n \tag{131}$$

is in $\mathcal{D}(Y)$, and that $\psi_{\alpha,\nu,n} \to \psi$, as $\alpha, \nu \to 0$ and $n \to \infty$. It is not hard to see that $(L-e)\psi_n \to 0$ as $n \to \infty$, a fact we write as

$$(L-e)\psi_n = o(n). \tag{132}$$

Since $f_{1,\alpha}^{\eta}$ leaves $\mathcal{D}(Y)$ invariant, and since $\mathcal{D}(Y) \subset \mathcal{D}(L)$, the commutator $-i[f_{1,\alpha}^{\eta}, L]$ is defined in the usual (strong) way on $\mathcal{D}(Y)$. We consider its expectation value in the state $g_\nu \psi_n \in \mathcal{D}(Y)$,

$$-i \left\langle [f_{1,\alpha}^{\eta}, L] \right\rangle_{g_\nu \psi_n} = -i \left\langle [f_{1,\alpha}^{\eta}, L-e] \right\rangle_{g_\nu \psi_n}. \tag{133}$$

The idea is to write (133) on the one hand as $\langle C_1 \rangle_{\psi_{\alpha,\nu,n}}$ modulo some small term for appropriate α, ν, n ("positive commutator"), and on the other hand to see that (133) itself is small, using the fact that $(L-e)\psi = 0$.

The latter is easily seen by first writing

$$(L-e) g_\nu \psi_n = g_\nu (L-e)\psi_n + g_{1,\nu}[L, g_{1,\nu}]\psi_n + [L, g_{1,\nu}] g_{1,\nu} \psi_n \tag{134}$$

and then realizing that, due to condition (VT3),

$$g_{1,\nu}[L, g_{1,\nu}] = \frac{\nu}{(2\pi)^{1/2}} \int_{\mathbb{R}} ds\, \widehat{g}_1(s) e^{is\nu\Lambda} \int_0^s ds_1\, e^{-is_1\nu\Lambda} g_{1,\nu} D e^{is\nu\Lambda} = O\left(\sqrt{\nu}\right),$$

and similarly, $[L, g_{1,\nu}] g_{1,\nu} = O\left(\sqrt{\nu}\right)$, so that

$$-i \left\langle [f_{1,\alpha}^{\eta}, L] \right\rangle_{g_\nu \psi_n} = O\left(\frac{o(n) + \sqrt{\nu}}{\alpha} \right). \tag{135}$$

Next we figure out a lower bound on (133). A repeated application of formula (122) gives, in the strong sense on $\mathcal{D}(Y)$,

$$\begin{aligned} -i[f_{1,\alpha}^{\eta}, L] &= f_{1,\alpha}' C_1 - i\frac{\alpha}{2!} f_{1,\alpha}'' C_2 - \frac{\alpha^2}{3!} f_{1,\alpha}''' C_3 \\ &\quad + \frac{i\alpha^3}{(2\pi)^{1/2}} \int_{\mathbb{R}_\eta} ds\, \widehat{f}_1(s) e^{is\alpha A} \int_0^s ds_1 \int_0^{s_1} ds_2 \int_0^{s_2} ds_3 \int_0^{s_4} ds_4\, e^{-is_4\alpha A} C_4 e^{is_4\alpha A} \\ &\quad + R_{\eta,1} C_1 + \frac{\alpha}{2!} R_{\eta,2} C_2 + \frac{\alpha^2}{3!} C_3, \end{aligned} \tag{136}$$

where we use that

$$(2\pi)^{-1/2} \int_{\mathbb{R}} ds\, (is)^n \widehat{f}(s) e^{isx} = f^{(n)}(x),$$

and where we set $f'_{1,\alpha} = (f_1)'(\alpha A)$, $f''_{1,\alpha} = (f_1)''(\alpha A)$, e.t.c., and

$$R_{\eta,n} = -i(2\pi)^{-1/2} \int_{-\eta}^{\eta} ds\, s^n \widehat{f}_1(s) e^{is\alpha A}. \tag{137}$$

Using that $f'_{1,\alpha} = f^2(\alpha A) = f_\alpha^2$ and applying again expansion (122) yields

$$\begin{aligned} f'_{1,\alpha} C_1 &= f_\alpha C_1 f_\alpha + i\alpha f_\alpha f'_\alpha C_2 + \frac{\alpha^2}{2!} f_\alpha f''_\alpha C_3 \\ &\quad - \frac{\alpha^3}{(2\pi)^{1/2}} f_\alpha \int_{\mathbb{R}} ds\, \widehat{f}(s) e^{is\alpha A} \int_0^s ds_1 \int_0^{s_1} ds_2 \int_0^{s_2} ds_3\, e^{-is_3\alpha A} C_3 e^{is_3\alpha A}. \end{aligned} \tag{138}$$

Plugging this into the r.h.s. of (136) and using that $f''_{1,\alpha} = 2 f_\alpha f'_\alpha$, we obtain

$$\begin{aligned} &-i \left\langle [f^\eta_{1,\alpha}, L] \right\rangle_{g_\nu \psi_n} \\ &= \langle C_1 \rangle_{\psi_{\alpha,\nu,n}} + \alpha^2 \mathrm{Re} \left\langle \frac{1}{2} f_\alpha f''_\alpha C_3 - \frac{1}{3!} f'''_{1,\alpha} C_3 \right\rangle_{g_\nu \psi_n} + O\left(\frac{\eta}{\nu^r} + \frac{\eta}{\sqrt{\nu}} + \frac{\alpha^3}{\nu} \right). \end{aligned} \tag{139}$$

We take the real part on the r.h.s. for free since the l.h.s. is real. The error term in (139) is obtained as follows. Clearly we have $R_{\eta,n} = O(\eta)$ and condition (VT4) gives $C_n g_\nu = O\left(\nu^{-r} + \nu^{-1/2}\right)$, which accounts for the term $O\left(\eta/\nu^r + \eta/\sqrt{\nu}\right)$. The term $O\left(\alpha^3/\nu\right)$ is an upper bound for the expectation of the terms in (136) and (138) involving the multiple integrals, in the state $g_\nu \psi_n$. For instance, the contribution coming from (136) is bounded above as follows. Due to condition (VT4) we have

$$\| e^{-is_4\alpha A} C_4 e^{is_4\alpha A} g_\nu \psi_n \| \le k \| \Lambda e^{is_4\alpha A} g_\nu \psi_n \| = e^{k'\alpha|s_4|} O\left(\frac{1}{\nu}\right),$$

which gives the following upper bound on the relevant term:

$$\alpha^3 \int_{\mathbb{R}_\eta} ds \left| \widehat{f}_1(s) \right| s^4 e^{k'|s|} \cdot O\left(\frac{1}{\nu}\right).$$

The integral is finite because $\widehat{f}_1$ has Gaussian decay.

Our next task is to esimtate the real part in (139). It suffices to consider the terms

$$\alpha^2 \mathrm{Re} \left\langle f''_\alpha f_\alpha C_3 \right\rangle_{g_\nu \psi_n} \quad \text{and} \quad \alpha^2 \mathrm{Re} \left\langle (f'_\alpha)^2 C_3 \right\rangle_{g_\nu \psi_n}, \tag{140}$$

because $f'''_{1,\alpha} = 2(f'_\alpha)^2 + 2f''_\alpha f_\alpha$. Let us start with the first term in (140). Using the decompostion $C_3 = \Lambda + V_3$ and the relative bound of V_3 given in (VT4) we estimate

$$\begin{aligned}\alpha^2 \mathrm{Re}\,\langle f''_\alpha f_\alpha C_3\rangle_{g_\nu\psi_n} &= \alpha^2 \mathrm{Re}\,\langle f''_\alpha f_\alpha \Lambda\rangle_{g_\nu\psi_n} + O\left(\frac{\alpha^2}{\sqrt{\nu}}\right) \\ &= \alpha^2 \mathrm{Re}\,\langle f''_\alpha \Lambda f_\alpha\rangle_{g_\nu\psi_n} + O\left(\frac{\alpha^2}{\sqrt{\nu}} + \frac{\alpha^3}{\nu}\right). \quad (141)\end{aligned}$$

We bound the first term on the r.h.s. from above as

$$\alpha^2 \left|\mathrm{Re}\,\langle f''_\alpha \Lambda f_\alpha\rangle_{g_\nu\psi_n}\right| \le \alpha^2 \|\Lambda^{1/2} f''_\alpha g_\nu \psi_n\|\, \|\Lambda^{1/2}\psi_{\alpha,\nu,n}\| \quad (142)$$

and use that

$$\langle f''_\alpha \Lambda f''_\alpha\rangle_{g_\nu\psi_n} \le \int_{\mathbb{R}} ds\, |\widehat{f''}(s)| \left|\left\langle f''_\alpha \Lambda e^{is\alpha\Lambda}\right\rangle_{g_\nu\psi_n}\right| = O\left(\frac{1}{\nu}\right)$$

to see that for any $c > 0$,

$$\alpha^2 \left|\mathrm{Re}\,\langle f''_\alpha \Lambda f_\alpha\rangle_{g_\nu\psi_n}\right| \le \frac{\alpha^4}{c\nu} + c\,\langle\Lambda\rangle_{\psi_{\alpha,\nu,n}}. \quad (143)$$

Choose $c = \alpha^{1+\xi}$, for some $\xi > 0$ to be determined later. Then, inserting again a term V_1 into the last expectation value (by adding a correction of size $O\left(\alpha^{1+\xi}/\sqrt{\nu}\right)$), we get

$$|(141)| \le \alpha^{1+\xi} \left|\langle C_1\rangle_{\psi_{\alpha,\nu,n}}\right| + O\left(\frac{\alpha^2}{\sqrt{\nu}} + \frac{\alpha^3}{\nu} + \frac{\alpha^{1+\xi}}{\sqrt{\nu}} + \frac{\alpha^{3-\xi}}{\nu}\right). \quad (144)$$

Next we tackle the second term in (140). The Gaussian f is strictly positive, so we can write

$$\begin{aligned}\alpha^2 \mathrm{Re}\,\left\langle (f'_\alpha)^2 C_3\right\rangle_{g_\nu\psi_n} &= \alpha^2 \mathrm{Re}\left\langle \frac{(f'_\alpha)^2}{f_\alpha} f_\alpha C_3\right\rangle_{g_\nu\psi_n} \\ &= \alpha^2 \mathrm{Re}\left\langle \frac{(f'_\alpha)^2}{f_\alpha} \Lambda f_\alpha\right\rangle_{g_\nu\psi_n} + O\left(\frac{\alpha^3}{\nu}\right), \quad (145)\end{aligned}$$

where we have taken into account condition (VT4) in the same way as above. It follows that

$$|(145)| \le \alpha^2 \left\|\Lambda^{1/2} \frac{(f'_\alpha)^2}{f_\alpha} g_\nu \psi_n\right\| \, \|\Lambda^{1/2}\psi_{\alpha,\nu,n}\| + O\left(\frac{\alpha^3}{\nu}\right),$$

and proceeding as in (142)–(143) we see that

$$\alpha^2 \left|\left\langle (f'_\alpha)^2 C_3 \right\rangle_{g_\nu \psi_n}\right| \leq \alpha^{1+\xi} \left|\langle C_1\rangle_{\psi_{\alpha,\nu,n}}\right| + O\left(\frac{\alpha^2}{\sqrt{\nu}} + \frac{\alpha^3}{\nu} + \frac{\alpha^{1+\xi}}{\sqrt{\nu}} + \frac{\alpha^{3-\xi}}{\nu}\right). \tag{146}$$

Estimates (144) and (146) together with (139) give the bound

$$\begin{aligned} \left|-i \left\langle [f^\eta_{1,\alpha}, L]\right\rangle_{g_\nu \psi_n}\right| &\geq \left(1 - O\left(\alpha^{1+\xi}\right)\right) \left|\langle C_1\rangle_{\psi_{\alpha,\nu,n}}\right| \\ &+ O\left(\frac{\alpha^2}{\sqrt{\nu}} + \frac{\alpha^3}{\nu} + \frac{\alpha^{1+\xi}}{\sqrt{\nu}} + \frac{\alpha^{3-\xi}}{\nu} + \frac{\eta}{\nu^r} + \frac{\eta}{\sqrt{\nu}} + \frac{\sqrt{\nu}}{\alpha} + \frac{o(n)}{\alpha}\right). \end{aligned} \tag{147}$$

We combine this upper bound with the lower bound obtained in (135) to arrive at

$$\begin{aligned} &\left(1 - O\left(\alpha^{1+\xi}\right)\right) \left|\langle C_1\rangle_{\psi_{\alpha,\nu,n}}\right| \\ &\quad = O\left(\frac{\sqrt{\nu} + o(n)}{\alpha} + \frac{\alpha^2}{\sqrt{\nu}} + \frac{\alpha^3}{\nu} + \frac{\alpha^{1+\xi}}{\sqrt{\nu}} + \frac{\alpha^{3-\xi}}{\nu} + \frac{\eta}{\nu^r} + \frac{\eta}{\nu} + \frac{\sqrt{\nu}}{\alpha} + \frac{o(n)}{\alpha}\right). \end{aligned} \tag{148}$$

Choose α so small that $1 - O\left(\alpha^{1+\xi}\right) > 1/2$ and take the limits $\eta \to 0$, $n \to \infty$ to get

$$\left|\langle C_1\rangle_{f_\alpha g_\nu \psi}\right| = O\left(\frac{\sqrt{\nu}}{\alpha} + \frac{\alpha^2}{\sqrt{\nu}} + \frac{\alpha^3}{\nu} + \frac{\alpha^{1+\xi}}{\sqrt{\nu}} + \frac{\alpha^{3-\xi}}{\nu}\right). \tag{149}$$

Take for example $\xi = 1/2$, $\nu = \nu(\alpha) = \alpha^{9/4}$. Then the r.h.s. of (149) is $O\left(\alpha^{1/4}\right)$, so

$$\lim_{\alpha \to 0} \langle C_1\rangle_{f_\alpha g_{\nu(\alpha)} \psi} = 0.$$

Since the operator C_1 is semibounded its quadratic form is closed, hence it follows from $f_\alpha g_{\nu(\alpha)} \psi \to \psi$, $\alpha \to 0$, that ψ is in the form domain of C_1 and that $\langle C_1\rangle_\psi = 0$. ■

5 Proofs of Theorem 2.3 and of Corollary 2.4

In order to alleviate the notation we drop in this section the variable θ labelling the fiber in the decomposition (49) (imagining $\theta \in [-\pi, \pi]$ to be fixed). The operator $L_{\lambda,\theta}$, (82), is thus denoted

$$L_\lambda = L_0 + \lambda(I + K), \tag{150}$$

where I and K are given in (84), (85). In parallel we can imagine that $K = 0$ and that Condition (A1) is replaced by (101).

Let $\epsilon, \rho, \theta > 0$ be parameters (θ reappears here as a different variable in order for the notation in this section to be compatible with [FM1]!). Set

$$\begin{aligned} P_\rho &= P_0 P(\Lambda \leq \rho) && (151) \\ P_0 &= P(L_1 = 0) \\ A_0 &= i\theta\lambda(P_\rho I \overline{R}_\epsilon^2 - \overline{R}_\epsilon^2 I P_\rho) && (152) \\ \overline{R}_\epsilon &= \overline{P}_\rho R_\epsilon \\ R_\epsilon &= (L_0^2 + \epsilon^2)^{-1/2} && (153) \end{aligned}$$

where $\overline{P}_\rho = \mathbb{1} - P_\rho$. We also set $\overline{P}_0 = \mathbb{1} - P_0$. The product in (151) is understood in the sprit of leaving out trivial factors ($P_\rho = P_0 \otimes P(\Lambda \leq \rho)$). We also define the selfadjoint operator (c.f. (95), (96))

$$B = C_1 + i[L_\lambda, A_0] = \Lambda + I_1 + i[L_\lambda, A_0], \tag{154}$$

where the last commutator is a bounded operator. Let us decompose

$$B = P_\rho B P_\rho + \overline{P}_\rho B \overline{P}_\rho + 2\mathrm{Re} P_\rho B \overline{P}_\rho. \tag{155}$$

Our goal is to obtain a lower bound on $\langle B \rangle_{\psi_\lambda}$, the expectation value of B in the state given by the normalized eigenvector ψ_λ of L_λ. We look at each term in (155) separately. In what follows we use the standard form bound

$$\lambda I_1 \geq -\frac{1}{2}\Lambda - O\left(\lambda^2\right), \tag{156}$$

and the estimates $\|\Lambda^{1/2}\psi_\lambda\| = O(\lambda)$, $\|\overline{P}_0 P(\Lambda \leq \rho)\psi_\lambda\| = O(\lambda)$. The former estimate follows from Theorem 2.2 (or Theorem 2.4 for the system without condensate) and the latter is easily obtained like this: let $\chi \in C_0^\infty(\mathbb{R})$ be such that $0 \leq \chi \leq 1$, $\chi(0) = 1$ and such that χ has support in a neighborhood of the origin containing no other eigenvalue of L_1 than zero. Then, for ρ sufficiently small, we have $\overline{P}_0 P(\Lambda \leq \rho)\chi(L_0) = 0$, so $\overline{P}_0 P(\Lambda \leq \rho)\psi_\lambda = \overline{P}_0 P(\Lambda \leq \rho)(\chi(L_\lambda) - \chi(L_0))\psi_\lambda = O(\lambda)$, by standard functional calculus.

Taking into account (156) we estimate

$$
\begin{aligned}
&\langle P_\rho B P_\rho \rangle_{\psi_\lambda} \\
&\geq \; -\theta\lambda^2 \left\langle P_\rho [I+K, P_\rho I \overline{R}_\epsilon^2 - \overline{R}_\epsilon^2 I P_\rho] P_\rho \right\rangle_{\psi_\lambda} - O\left(\lambda^2\right) \\
&= \; 2\theta\lambda^2 \left\langle P_\rho I \overline{R}_\epsilon^2 I P_\rho \right\rangle_{\psi_\lambda} + \theta\lambda^2 \left\langle P_\rho I \overline{R}_\epsilon^2 K P_\rho + P_\rho K \overline{R}_\epsilon^2 I P_\rho \right\rangle_{\psi_\lambda} - O\left(\lambda^2\right) \\
&\geq \; 2\theta\lambda^2 \left\langle P_\rho I \overline{R}_\epsilon^2 I P_\rho \right\rangle_{\psi_\lambda} - \frac{\theta\lambda^2}{\epsilon} O\left(\frac{\epsilon}{\theta} + \epsilon\right), \qquad (157)
\end{aligned}
$$

where we use in the last step that $\overline{P}_\rho = \overline{P}_0 P(\Lambda \leq \rho) + P(\Lambda > \rho)$ to arrive at

$$
\|P_\rho I \overline{R}_\epsilon^2 K P_\rho\| = \|P_\rho I R_\epsilon^2 \overline{P}_0 P(\Lambda \leq \rho) K P_\rho\| \leq c.
$$

The last estimate is due to $\|R_\epsilon^2 \overline{P}_0 P(\Lambda \leq \rho)\| < c$ and $\|P_\rho I P(\Lambda < \rho)\| < c$ (this follows in a standard way assuming condition (89)).

Next we estimate

$$
\left\langle \overline{P}_\rho B \overline{P}_\rho \right\rangle_{\psi_\lambda} \geq \frac{1}{2} \left\langle \overline{P}_\rho \Lambda \right\rangle_{\psi_\lambda} - 2\theta\lambda^2 \mathrm{Re} \left\langle \overline{P}_\rho (I+K) P_\rho I \overline{R}_\epsilon^2 \right\rangle_{\psi_\lambda} - O\left(\lambda^2\right) \qquad (158)
$$

and

$$
\left\langle \overline{P}_\rho (I+K) P_\rho I \overline{R}_\epsilon^2 \right\rangle_{\psi_\lambda} = \|\overline{P}_\rho \psi_\lambda\|^2 \; O\left(\frac{1}{\epsilon} \|P_\rho I R_\epsilon\|\right) = O\left(\frac{\lambda^2}{\rho \epsilon^{3/2}}\right), \qquad (159)
$$

where we use $\|\overline{P}_\rho \psi_\lambda\| \leq \|\overline{P}_0 P(\Lambda \leq \rho)\psi_\lambda\| + \|P(\Lambda > \rho)\psi_\lambda\| = O\left(\lambda/\sqrt{\rho}\right)$, and $\|P_\rho I R_\epsilon\| = O\left(1/\sqrt{\epsilon}\right)$. The former estimate follows from the observations after (156) and from $\|P(\Lambda > \rho)\psi_\lambda\| \leq 1/\sqrt{\rho} \|P(\Lambda > \rho)\Lambda^{1/2}\psi_\lambda\| = O(\lambda/\sqrt{\rho})$. The estimate $\|P_\rho I R_\epsilon\| = O\left(1/\sqrt{\epsilon}\right)$ is standard in this business, it follows from $P_\rho I R_\epsilon^2 I P_\rho = O\left(1/\epsilon\right)$ (see e.g. [BFSS] and also the explanations before (165) here below). Combining (158) and (159), and taking into account that $\left\langle \overline{P}_\rho \Lambda \right\rangle_{\psi_\lambda} \geq \langle P(\Lambda > \rho)\Lambda \rangle_{\psi_\lambda} \geq \rho \left\langle P(\Lambda > \rho) \right\rangle_{\psi_\lambda} \geq \rho(\left\langle \overline{P}_\rho \right\rangle_{\psi_\lambda} - O\left(\lambda^2\right))$ gives

$$
\left\langle \overline{P}_\rho B \overline{P}_\rho \right\rangle_{\psi_\lambda} \geq \frac{\rho}{2} \left\langle \overline{P}_\rho \right\rangle_{\psi_\lambda} - \frac{\theta\lambda^2}{\epsilon} O\left(\frac{\epsilon}{\theta} + \frac{\lambda^2}{\rho\sqrt{\epsilon}}\right). \qquad (160)
$$

Our next task it to estimate

$$
\left\langle P_\rho B \overline{P}_\rho \right\rangle_{\psi_\lambda} = \lambda \left\langle P_\rho I_1 \overline{P}_\rho \right\rangle_{\psi_\lambda} - \theta\lambda \left\langle P_\rho (L_\lambda P_\rho I R_\epsilon^2 - I \overline{R}_\epsilon^2 L_\lambda) \overline{P}_\rho \right\rangle_{\psi_\lambda}. \qquad (161)
$$

It is not difficult to see that

$$\begin{aligned}\left\langle P_\rho I_1 \overline{P}_\rho \right\rangle_{\psi_\lambda} &= \left\langle P_\rho I_1 \overline{P}_0 P(\Lambda \le \rho) \right\rangle_{\psi_\lambda} + \langle P_\rho I_1 P(\Lambda > \rho) \rangle_{\psi_\lambda} \\ &= O(\lambda) + O\left(\|(I_1)_{\mathrm{a}} \Lambda^{-1/2}\| \, \|\Lambda^{1/2}\psi_\lambda\| \right) \\ &= O(\lambda),\end{aligned}$$

where $(I_1)_{\mathrm{a}}$ means that we take in I_1 only the terms containing annihilation operators (see (84)) and where we use $\|(I_1)_{\mathrm{a}}\Lambda^{-1/2}\| < c$. The second term on the r.h.s. of (161) is somewhat more difficult to estimate. We have

$$\begin{aligned}&\theta\lambda \left\langle P_\rho (L_\lambda P_\rho I R_\epsilon^2 - I \overline{R}_\epsilon^2 L_\lambda) \overline{P}_\rho \right\rangle_{\psi_\lambda} \\ &= -\theta\lambda^2 \left\langle (I+K) P_\rho I \overline{R}_\epsilon^2 \right\rangle_{\psi_\lambda} - \theta\lambda \left\langle P_\rho I R_\epsilon^2 L_0 \overline{P}_\rho \right\rangle_{\psi_\lambda} \\ &\quad + \theta\lambda^2 \left\langle P_\rho ((I+K) P_\rho I \overline{R}_\epsilon^2 - I \overline{R}_\epsilon^2 \overline{P}_\rho (I+K)) \overline{P}_\rho \right\rangle_{\psi_\lambda}, \qquad (162)\end{aligned}$$

where the first term on the r.h.s. comes from the contribution $\left\langle P_\rho L_0 I \overline{R}_\epsilon^2 \right\rangle_{\psi_\lambda}$ in the l.h.s. by using that $P_\rho L_0 = L_0 P_\rho = L_\lambda P_\rho - \lambda(I+K)P_\rho$ and that $L_\lambda \psi_\lambda = 0$. We treat the first term on the r.h.s. of (162) as

$$\begin{aligned}&\left\langle (I+K) P_\rho I \overline{R}_\epsilon^2 \right\rangle_{\psi_\lambda} \\ &= \left\langle (I+K) P_\rho I R_\epsilon^2 \overline{P}_0 P(\Lambda \le \rho) \right\rangle_{\psi_\lambda} + \left\langle (I+K) P_\rho I R_\epsilon^2 P(\Lambda > \rho) \right\rangle_{\psi_\lambda} \\ &= O\left(\lambda \|\overline{P}_0 P(\Lambda \le \rho) \psi_\lambda\| \right) + O\left(\epsilon^{-2} \|(I_1)_{\mathrm{a}} \Lambda^{-1/2}\| \, \|\Lambda^{1/2}\psi_\lambda\| \right) \\ &= O\left(\lambda + \frac{\lambda}{\epsilon^2} \right). \qquad (163)\end{aligned}$$

The second term on the r.h.s. of (162) has the bound

$$\begin{aligned}\left\langle P_\rho I R_\epsilon^2 L_0 \overline{P}_\rho \right\rangle_{\psi_\lambda} &= \left\langle P_\rho I R_\epsilon^2 L_0 \overline{P}_0 P(\Lambda \le \rho) \right\rangle_{\psi_\lambda} + \left\langle P_\rho I R_\epsilon^2 L_0 P(\Lambda > \rho) \right\rangle_{\psi_\lambda} \\ &= O(\lambda) + O\left(\|P_\rho (I)_{\mathrm{a}} R_\epsilon \Lambda^{-1/2} P(\Lambda > \rho)\| \, \|\Lambda^{1/2}\psi_\lambda\| \right) \\ &= O\left(\frac{\lambda}{\sqrt{\epsilon}} \right), \qquad (164)\end{aligned}$$

where we use that (with $(I)_{\mathrm{c}} = ((I)_{\mathrm{a}})^*$)

$$\|P_\rho (I)_{\mathrm{a}} R_\epsilon \Lambda^{-1/2} P(\Lambda > \rho)\|^2 = \|P(\Lambda > \rho) \Lambda^{-1/2} R_\epsilon (I)_{\mathrm{c}} P_\rho\|^2 = O\left(\frac{1}{\epsilon} \right).$$

The latter bound can be shown by using the explicit form of the interaction I, given in (84), and by using standard pull-through formulae to see that a typical contraction term in $P_\rho(I)_{\rm a}R_\epsilon^2\Lambda^{-1}P(\Lambda>\rho)(I)_{\rm c}P_\rho$ has the form

$$\int d^3k\ \frac{|g(k)|^2}{e^{\beta\omega}-1}P_\rho(G\otimes \mathbb{1}_{\mathbb{C}^d})\frac{P(\Lambda+|k|>\rho)}{(\Lambda+|k|)\left((L_0\pm|k|)^2+\epsilon^2\right)}(G\otimes \mathbb{1}_{\mathbb{C}^d})P_\rho$$

and is thus bounded from above, in norm, by a constant times $1/\epsilon$, provided $p>-1/2$ (recall that p characterizes the infrared behaviour of the form factor, see Theorem 2.5; in the case of the system with condensate we have $p=0$). To see this use $(\Lambda+|k|)^{-1}\leq|k|^{-1}$, and then standard estimates which show that the resulting operator is of order ϵ^{-1}; the mechanism is that the main part comes from the restriction of the operator to $\mathrm{Ran}\,P_0P_{\Omega_0}$ ($\rho=0$) and there the resolvent, when multiplied by ϵ, converges to the Dirac delta distribution $\delta(L_1\pm|k|)$, so the integral is $1/\epsilon$ times a bounded operator. See also [BFSS].

Next we estimate the third term in the r.h.s. of (162) as

$$\begin{aligned}&\left\langle P_\rho((I+K)P_\rho IR_\epsilon^2-I\overline{R}_\epsilon^2(I+K))\overline{P}_\rho\right\rangle_{\psi_\lambda}\\ &\quad= O\left(\epsilon^{-3/2}\|\overline{P}_\rho\psi_\lambda\|\right)+O\left(\|P_\rho I\overline{R}_\epsilon^2(I+K)\overline{P}_\rho\psi_\lambda\|\right)\\ &\quad= O\left(\frac{\lambda}{\sqrt{\rho}\epsilon^2}\right),\end{aligned}\tag{165}$$

where we use again that $\|P_\rho IR_\epsilon\|=O(1/\sqrt{\epsilon})$, $\|\overline{P}_\rho\psi_\lambda\|=O(\lambda/\sqrt{\rho})$, and that $\|P_\rho I\overline{R}_\epsilon^2I\|=O(1/\epsilon^2)$. Collecting the effort we put into estimates (163), (164) and (165) rewards us with the bound

$$\left\langle P_\rho B\overline{P}_\rho\right\rangle_{\psi_\lambda}=\frac{\theta\lambda^2}{\epsilon}\ O\left(\frac{\epsilon}{\theta}+\sqrt{\epsilon}+\frac{\lambda}{\epsilon\sqrt{\rho}}\right),\tag{166}$$

which we combine with (157) and (160) to obtain

$$\begin{aligned}&\langle B\rangle_{\psi_\lambda}\\ &\quad\geq 2\theta\lambda^2\left\langle P_\rho I\overline{R}_\epsilon^2IP_\rho\right\rangle_{\psi_\lambda}+\frac{\rho}{2}\left\langle\overline{P}_\rho\right\rangle_{\psi_\lambda}-\frac{\theta\lambda^2}{\epsilon}O\left(\frac{\epsilon}{\theta}+\frac{\lambda^2}{\rho\sqrt{\epsilon}}+\sqrt{\epsilon}+\frac{\lambda}{\epsilon\sqrt{\rho}}\right).\end{aligned}\tag{167}$$

The non-negative operator $P_\rho I\overline{R}_\epsilon^2 IP_\rho$ has appeared in various guises in many previous papers on the subject ("level shift operator"). The following result follows from a rather straightforward calculation, using the explicit form of the interaction I, (84). We do not write down the analysis, one can follow closely e.g. [BFSS], [M1], [BFS].

Lemma 5.1 *We have the expansion*

$$P_\rho I\overline{R}_\epsilon^2 IP_\rho = \frac{1}{\epsilon} P_0\big(\Gamma + o(\epsilon)\big)P_0 \otimes P(\Lambda \leq \rho) + O\left(\frac{\rho^{2+2p}}{\epsilon^2} + \frac{\rho}{\epsilon^3}\right), \tag{168}$$

where p is the parameter characterizing the infrared behaviour of the form factor (see Theorem 2.5; in the situation of Theorem 2.3 we set $p = 0$), $o(\epsilon)$ is an operator whose norm vanishes in the limit $\epsilon \to 0$, and where Γ is the non-negative operator on $\operatorname{Ran} P_0$ *given by*

$$\Gamma = \sum_{m,n} P_0 \int d^3k\ |g(k)|^2 \delta(E_{mn} - |k|) \times \\ \times \big\{(X_{mn})^* X_{mn} + J_1 (X_{mn})^* X_{mn} J_1\big\} P_0, \tag{169}$$

where J_1 is the modular conjugation operator given in (32), $E_{mn} = E_m - E_n$, and where the rank-one operators X_{mn} are

$$X_{mn} = \sqrt{1+\rho(k)}\ (P_n G) \otimes P_m - \sqrt{\rho(k)}\ P_n \otimes (P_m \mathcal{C}_1 G \mathcal{C}_1). \tag{170}$$

Here, $\rho(k) = (e^{\beta\omega(k)} - 1)^{-1}$, P_n is the rank-one projection onto the span of the eigenvector φ_n of L_1, and $\mathcal{C}_1$ is defined in (32).

Moreover, if Condition (A2) holds, then the kernel of Γ is spanned by the Gibbs state (33), $\ker(\Gamma) = \mathbb{C}\Omega_{1,\beta}$, *and the spectrum of Γ has a gap at zero which is of size at least*

$$\gamma = \min_{E_{mn}>0} \frac{|G_{mn}|^2 e^{\beta E_{mn}}}{e^{\beta E_{mn}} - 1} \int_{S^2} d\Sigma\ |g(E_{mn}, \Sigma)|^2 > 0. \tag{171}$$

It follows from the lemma that

$$2\theta\lambda^2 \left\langle P_\rho I\overline{R}_\epsilon^2 IP_\rho \right\rangle_{\psi_\lambda} \\ \geq\ 2\frac{\theta\lambda^2}{\epsilon}\gamma \left\langle \overline{P}_{1,\beta} P_\rho \right\rangle_{\psi_\lambda} - \frac{\theta\lambda^2}{\epsilon}\ \left(o(\epsilon) + O\left(\frac{\rho^{2+2p}}{\epsilon} + \frac{\rho}{\epsilon^2}\right)\right), \tag{172}$$

where $\overline{P}_{1,\beta} = \mathbb{1} - P_{1,\beta}$, and $P_{1,\beta} = |\Omega_{1,\beta}\rangle\langle\Omega_{1,\beta}|$ is the projection onto the span of the Gibbs state (33). Using this estimate in (167) gives

$$\begin{aligned} \langle B\rangle_{\psi_\lambda} \quad &\geq \quad \min\left\{\frac{2\theta\lambda^2}{\epsilon}\gamma, \frac{\rho}{2}\right\} \|\psi_\lambda\|^2 - \frac{2\theta\lambda^2}{\epsilon}\gamma \left\langle P_{1,\beta} P(\Lambda \leq \rho)\right\rangle_{\psi_\lambda} \\ &\quad -\frac{\theta\lambda^2}{\epsilon}\, O\left(\frac{\epsilon}{\theta} + \frac{\lambda^2}{\rho\sqrt{\epsilon}} + \frac{\lambda}{\epsilon\sqrt{\rho}} + o(\epsilon) + \frac{\rho^{2+2p}}{\epsilon} + \frac{\rho}{\epsilon^2}\right). \qquad (173) \end{aligned}$$

Let us choose the parameters like this: $\epsilon = \lambda^{49/100}$, $\theta = \lambda^{1/100}$, $\rho = \lambda$, $p > -1/2$. Then the minimum in (173) is given by $\frac{2\theta\lambda^2}{\epsilon}\gamma$ (provided $\lambda \leq (4\gamma)^{-25/13}$) and the error term in (173) is $O\left(\lambda^{1/100} + o(\lambda)\right) = o(\lambda)$. The virial theorem tells us that $\langle B\rangle_{\psi_\lambda} = 0$, so

$$\left\langle P_{1,\beta} P(\Lambda \leq \lambda)\right\rangle_{\psi_\lambda} \geq 1 - o(\lambda). \qquad (174)$$

Consequently,

$$\psi_\lambda = P_{1,\beta} P(\Lambda \leq \lambda)\psi_\lambda + o(\lambda) = \Omega_{1,\beta} \otimes \left(P(\Lambda \leq \lambda)\chi_\lambda\right) + o(\lambda), \qquad (175)$$

for some vector $\chi_\lambda \in \mathcal{F} \otimes \mathcal{F}$ with norm $\|\chi_\lambda\| \geq 1 - o(\lambda)$. We point out that all estimates are uniform in the parameter $\theta \in [-\pi, \pi]$ (which we actually suppressed in the notation); indeed, this parameter only appears through the interaction term $K = K_\theta$, see (85), which can always be bounded uniformly in $\theta \in [-\pi, \pi]$. This finishes the proof of Theorem 2.3.

Proof of Corollary 2.4. We denote by $P_{1,\beta}$, $P_{\beta,0}$ and $P^\theta_{\beta,\lambda}$ the projections onto the spans of $\Omega_{1,\beta}$, $\Omega_{\beta,0}$ and $\Omega^\theta_{\beta,\lambda}$, see (33), (67) and (88). Since $\|P_{\beta,0} - P^\theta_{\beta,\lambda}\| \to 0$ as $\lambda \to 0$ (uniformly in $\theta \in [-\pi, \pi]$ and in β, [FM2]) it follows that

$$\begin{aligned} \psi_\lambda = (P^\theta_{\beta,\lambda})^\perp \psi_\lambda \quad &= \quad \overline{P}_{\beta,0}\psi_\lambda + o(\lambda) \\ &= \quad \left(\overline{P}_{1,\beta} \otimes P_{\Omega_0}\right)\psi_\lambda + \overline{P}_{\Omega_0}\psi_\lambda + o(\lambda) \\ &= \quad \Omega_{1,\beta} \otimes \left(\overline{P}_{\Omega_0} P(\Lambda \leq \lambda)\chi_\lambda\right) + o(\lambda) \end{aligned}$$

where we used (99) in the last step. It suffices now to observe that $\overline{P}_{\Omega_0} P(\Lambda \leq \lambda)$ converges strongly to zero, as $\lambda \to 0$. This follows from $\overline{P}_{\Omega_0} = \overline{P}_{\Omega_\mathcal{F}} \otimes P_{\Omega_\mathcal{F}} + \mathbb{1}_\mathcal{F} \otimes \overline{P}_{\Omega_\mathcal{F}}$,

$$P(\Lambda \leq \lambda) = \Big(P(\mathrm{d}\Gamma(\omega) \leq \lambda) \otimes P(\mathrm{d}\Gamma(\omega) \leq \lambda)\Big) P(\Lambda \leq \lambda)$$

and the fact that $d\Gamma(\omega)$ has absolutely continuous spectrum covering $\mathbb{R}_+$ and a simple eigenvalue at zero, $\Omega_{\mathcal{F}}$ being the eigenvector. ■

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