The Ideal Quantum Gas

Lecture notes

(February 2005)

Summer School on Open Quantum Systems

Institut Fourier, Grenoble, 2003

Marco Merkli

Institute of Theoretical Physics

ETH Zürich, Switzerland *

Contents

1 Introduction **2**

2 Fock space **4**

2.1 Bosons and Fermions 4

2.2 Creation and annihilation operators 7

2.3 Weyl operators 12

2.4 The C^*-algebras $\mathrm{CAR}_{\mathrm{F}}(\mathfrak{H})$, $\mathrm{CCR}_{\mathrm{F}}(\mathfrak{H})$ 16

2.5 Leaving Fock space 18

3 The CCR and CAR algebras **21**

3.1 The algebra $\mathrm{CAR}(\mathfrak{D})$ 21

3.2 The algebra $\mathrm{CCR}(\mathfrak{D})$ 22

3.3 Schrödinger representation and Stone – von Neumann uniqueness theorem 27

*present address: Department of Mathematics and Statistics, McGill University, Montreal, Canada

3.4 Q–space representation 31
3.5 Equilibrium state and thermodynamic limit 34

4 Araki-Woods representation of the infinite free Boson gas 39
4.1 Generating functionals 39
4.2 Ground state (condensate) 44
4.3 Excited states 49
4.4 Equilibrium states 53
4.5 Dynamical stability of equilibria 57

1 Introduction

The goal of these lecture notes is to give an introduction to the mathematical description of a system of identical non-interacting quantum particles. An important characteristic of the systems considered is their "size", which may refer to *spatial extension* or to the *number of particles*, or to a combination of both. Certain physical phenomena occur only for very large systems, say for systems which occupy an immense region of the universe or for a system the size of a laboratory, if the observed phenomenon takes place on a microscopic level. For the mathematical analysis it is often convenient to make an abstraction and to consider systems which are spatially infinitely extended (and which contain infinitely many particles). From a physical point of view, such a description can only be an approximation which is, however, justified by the fact that the mathematical models lead to correct answers to physical questions. An important part of these lectures is concerned with the description of infinite systems, or the passage of a finite system (a confined one, or one with only finitely many particles) to an infinite one. In some instances, this passage is called the *thermodynamic limit*.

It is natural to consider first a system of finitely many (identical) quantum particles. States of such a system are described by vectors in Fock space, a Hilbert space that has a direct sum decomposition into subspaces, each of which describes a system with a fixed number $n = 0, 1, 2, \ldots$ of particles. The action of operators which are not reduced by this direct sum decomposition is interpreted as creation or annihilation of particles. So Fock space provides us already with a nice toolbox enabling the modelling of many physical processes. However, not all physical situations can be described by Fock space.

Given any state in Fock space the probability of finding at least n particles in it decreases to zero, as $n \to \infty$. Imagine a gas of particles which has a uniform nonzero density (say one particle per unit volume) and which is spatially infinitely extended. Such a state cannot be described by a vector in Fock space!

How can we thus describe the infinitely extended system at positive density? The observable algebra (the one generated by the creation and annihilation operators on Fock space) has a certain structure determined by algebraic relations. Those are called the canonical commutation relations (CCR) or the canonical anticommutation relations (CAR) depending on whether one considers Bosons or Fermions. It can be viewed as an *abstract* algebra, merely determined by its relations, and not a priori represented as an operator algebra on a Hilbert space. Fock space emerges then just as one possible representation Hilbert space of the abstract algebra (called the CCR or the CAR algebra). A fundamental theorem regarding this setting is the von Neumann uniqueness theorem. It says that if we consider only *finitely* many particles then all the representations of the corresponding algebra are (spatially) equivalent. However, in the case of a system with infinitely many particles there are non-equivalent representations of the algebra! This is what happens in the case of the infinitely extended system with nonzero density; it is described by a vector in some Hilbert space which is not compatible with Fock space (the corresponding representations of the algebras are not spatially equivalent).

It is one of the goals of these notes to calculate the representation Hilbert space of the infinitely extended gas for arbitrary densities.

It may have become clear from this short introduction what kind of mathematics is involved in these notes. In the first chapter we will mainly deal with operators on Fock space (bounded and unbounded ones) and, in the second chapter, we move on to some aspects of the theory of C^*algebras in relation with the CCR and CAR algebras. The last chapter is devoted to the Araki-Woods representation, which gives the above mentioned representation of the infinitely extended free Bose gas for arbitrary momentum density distributions.

These notes represent a composition of mostly well known concepts and results relevant to this collection of lecture notes, and they have, in the author's view, an interest on their own. An effort has been made to render the material easy to understand for anybody with basic knowledge in functional analysis.

2 Fock space

Fock space is the Hilbert space suitable to describe a system of arbitrarily many (identical) quantum particles. We start this section by introducing the Bosonic and Fermionic Fock spaces and the corresponding creation and annihilation operators. We will see that in the case of Bosons those operators are unbounded and it is thus convenient to "replace" them by Weyl operators. This leads us to the definition of the C^*algebras CCR_F and CAR_F, for Bosons and Fermions, respectively. We discuss the "shortcomings" of these algebras in the last section, motivating the definition of the abstract CCR and CAR algebras.

2.1 Bosons and Fermions

An ideal quantum gas is a system of identical (meaning indistinguishable), non-interacting quantum particles.

A single particle is described by a complex Hilbert space $\mathfrak{H}$, i.e., a normalized $\psi \in \mathfrak{H}$ is a (pure) state of the particle (ψ is also called the state vector). It is often useful to consider states which are determined by a linear (not necessarily closed) subspace

$$\mathfrak{D} \subseteq \mathfrak{H}. \tag{1}$$

Typically, one may think of $\mathfrak{H} = L^2(\mathbb{R}^3, d^3x)$, then a normalized vector $\psi \in \mathfrak{H}$ is called the *wave function* of the particle and has the following physical interpretation: $|\psi(x)|^2$ is the probability density of finding the particle at location $x \in \mathbb{R}^3$. An example for $\mathfrak{D}$ is the set $\{f \in C_0^\infty(\mathbb{R}^3) \,|\, \mathrm{supp} f \subset V\}$ of all smooth functions with support in some compact set $V \subset \mathbb{R}^3$; $\mathfrak{D}$ is called the *test function space*. We will see that the choice of the test function space often reflects physical properties of the system at hand, e.g., we may want to look only at particles confined to a region V in space.

The Hilbert space of n *distinguishable* particles is given by the n-fold tensor product

$$\mathfrak{H}^n = \mathfrak{H} \otimes \cdots \otimes \mathfrak{H}. \tag{2}$$

If we restrict our attention to one-particle states in $\mathfrak{D}$ then of course only the subspace $\mathfrak{D} \otimes \cdots \otimes \mathfrak{D}$ of $\mathfrak{H}^n$ is relevant. To be able to describe processes involving creation and annihilation of particles, we build the direct sum Hilbert space

$$\mathcal{F}(\mathfrak{H}) = \bigoplus_{n \geq 0} \mathfrak{H}^n, \tag{3}$$

where $\mathfrak{H}^0 = \mathbb{C}$. $\mathcal{F}(\mathfrak{H})$ is called the *Fock space over the Hilbert space* $\mathfrak{H}$. The Hilbert space $\mathfrak{H}^n$ identified as a subspace of Fock space is called the *n-sector* (or the n-th chaos, in quantum probability). The zero-sector is also called the *vacuum sector*. An element ψ of $\mathcal{F}(\mathfrak{H})$ is a sequence $\psi = \{\psi_n\}_{n \geq 0}$ with $\psi_n \in \mathfrak{H}^n$. We write sometimes the n-particle component ψ_n of ψ as $[\psi]_n$. The scalar product on $\mathcal{F}(\mathfrak{H})$ is given by

$$\langle \psi, \phi \rangle = \sum_{n \geq 0} \langle \psi_n, \phi_n \rangle_{\mathfrak{H}^n}, \tag{4}$$

where $\langle \cdot, \cdot \rangle_{\mathfrak{H}^n}$ is the scalar product of $\mathfrak{H}^n$, which we take to be antilinear in the first argument and linear in the second one. The direct sum in (3) is the decomposition of Fock space into spectral subspaces (eigenspaces) of the selfadjoint number operator, N, defined as follows. The domain of N is

$$\mathcal{D}(N) = \left\{ \psi \in \mathcal{F}(\mathfrak{H}) \;\middle|\; \sum_{n \geq 0} n^2 \|\psi_n\|^2_{\mathfrak{H}^n} < \infty \right\}, \tag{5}$$

and the action of N is given, for $\psi \in \mathcal{D}(N)$, by

$$[N\psi]_n = n[\psi]_n. \tag{6}$$

Clearly, the spectrum of N is discrete and consists of all integers $n \in \mathbb{N}$. The vector $\Omega \in \mathcal{F}(\mathfrak{H})$ given by

$$[\Omega]_0 = 1 \in \mathbb{C}, \quad [\Omega]_n = 0 \in \mathfrak{H}^n, \text{if } n > 0, \tag{7}$$

is called the vacuum (vector). It spans the one-dimensional kernel of N. The degree of degeneracy of the eigenvalue n of N equals $\dim(\mathfrak{H}^n) = (\dim \mathfrak{H})^n$.

Let us now consider a system of *indistinguishable* particles. The indistinguishability is reflected in the symmetry of the state vector (wave function) under the exchange of particle labels. We are adopting in these notes the view that all quantum particles fall into two categories: either the state vectors are *symmetric* under permutations of indices, in which case the particles are called *Bosons*, or the state vectors are *anti-symmetric* under permutations of indices, in which case the particles are called *Fermions*. For example, let $\{f_k\}_{k=1}^n \subset \mathfrak{H}$ be n state vectors of a single particle. The vector $f_1 \otimes \cdots \otimes f_n \in \mathfrak{H}^n$ is the state of an n-particle system where the particle labelled by k is in the state f_k. The state describing n Bosons, one of which (but we cannot say which one, because they are indistinguishable) is in the state f_1, one of which is in the state f_2, and so on, is given by the symmetric state vector

$$\frac{1}{n!} \sum_{\pi \in S_n} f_{\pi(1)} \otimes \cdots \otimes f_{\pi(n)} \in \mathfrak{H}^n, \tag{8}$$

where S_n is the group of all permutations π of n objects. The corresponding vector describing n Fermions is given by

$$\frac{1}{n!} \sum_{\pi \in S_n} \epsilon(\pi) f_{\pi(1)} \otimes \cdots \otimes f_{\pi(n)} \in \mathfrak{H}^n, \tag{9}$$

where $\epsilon(\pi)$ is the *signature* of the permutation π. [1]

Let us introduce the symmetrization operator P_+ and the anti-symmetrization operator P_- on $\mathcal{F}(\mathfrak{H})$. Set $P_\pm \Omega = \Omega$ and for $\{f_k\}_{k=1}^n \subset \mathfrak{H}$, $n \geq 1$, set

$$P_+ f_1 \otimes \cdots \otimes f_k = \frac{1}{n!} \sum_{\pi \in S_n} f_{\pi(1)} \otimes \cdots \otimes f_{\pi(n)}, \tag{10}$$

$$P_- f_1 \otimes \cdots \otimes f_k = \frac{1}{n!} \sum_{\pi \in S_n} \epsilon(\pi) f_{\pi(1)} \otimes \cdots \otimes f_{\pi(n)}, \tag{11}$$

[1] Let us recall that every permutation $\pi \in S_n$ is uniquely decomposed into a (commutative) product of cycles and that every cycle is a (non commutative, non unique) product of transpositions (a cycle of length two). The number of transpositions in the decomposition of each cycle is a constant modulo 2. One defines the signature of π to be $\epsilon(\pi) = (-1)^{\#(\text{transp})}$, where $\#(\text{transp})$ is the number of transpositions in any decomposition of π. The permutation π is called even if $\epsilon(\pi) = 1$ and odd if $\epsilon(\pi) = -1$. Each cycle of length $l(\text{cycle}) \geq 2$ is the product of $l(\text{cycle}) - 1$ transpositions, so we have the relations $\epsilon(\pi) = (-1)^{\sum_{c:\text{cycles}} \#(\text{transp in } c)} = (-1)^{\sum_{c:\text{cycles}} (l(\text{cycle})-1)} = (-1)^{n - \#(\text{cycles})} = (-1)^{n + \#(\text{cycles})}$, where we use $\sum_{c:\text{cycles}} l(\text{cycle}) = n$.

and extend the action of $P_\pm$ by linearity to the sets

$$D^n = \left\{ \sum_{k=1}^{K} f_1^{(k)} \otimes \cdots \otimes f_n^{(k)} \;\middle|\; K \in \mathbb{N}, f_l^{(k)} \in \mathfrak{H} \right\} \subset \mathfrak{H}^n, \quad n \geq 1. \tag{12}$$

It is clear that $\|P_\pm f_1 \otimes \cdots \otimes f_n\| \leq \frac{1}{n!} \sum_{n \in S_n} \|f_1\| \cdots \|f_n\| = \|f_1 \otimes \cdots \otimes f_n\|$, so $P_\pm$ is a contraction on D^n, $\|P_\pm \psi\| \leq \|\psi\|$ for $\psi \in D^n$. Consequently the operators $P_\pm$ extend to all of $\mathfrak{H}^n$, for all n, and to $\mathcal{F}(\mathfrak{H})$ by sector-wise action. [2] Of course $P_\pm$ are actually selfadjoint projections; i.e., $P_\pm^2 = P_\pm = {P_\pm}^*$ and they satisfy $\|P_\pm\| = 1$. We define the *Bosonic Fock space*, $\mathcal{F}_+(\mathfrak{H})$, and the *Fermionic Fock space*, $\mathcal{F}_-(\mathfrak{H})$, to be the symmetric and anti-symmetric parts of $\mathcal{F}(\mathfrak{H})$:

$$\mathcal{F}_\pm(\mathfrak{H}) = P_\pm \mathcal{F}(\mathfrak{H}) = \bigoplus_{n \geq 0} P_\pm \mathfrak{H}^n. \tag{13}$$

The number operator (6) leaves $\mathcal{F}_\pm(\mathfrak{H})$ invariant. We will not distinguish in our notation between N and its restriction to those invariant subspaces.

2.2 Creation and annihilation operators

Given $f \in \mathfrak{H}$, we define the annihilation operator $a(f)$ in the following way: $a(f) : \mathfrak{H}^0 \to 0 \in \mathcal{F}(\mathfrak{H})$, $a(f) : \mathfrak{H}^n \to \mathfrak{H}^{n-1}$, $n \geq 1$, and for $\{f_k\}_{k=1}^n \subset \mathfrak{H}$,

$$a(f) f_1 \otimes \cdots \otimes f_n \mapsto \sqrt{n}\, \langle f, f_1 \rangle\, f_2 \otimes \cdots \otimes f_n, \tag{14}$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathfrak{H}$. Similarly, we define the creation operator $a^*(f) : \mathfrak{H}^n \to \mathfrak{H}^{n+1}$ by

$$a^*(f) f_1 \otimes \cdots \otimes f_n \mapsto \sqrt{n+1}\; f \otimes f_1 \otimes \cdots \otimes f_n. \tag{15}$$

The map $f \mapsto a(f)$ is antilinear, while $f \mapsto a^*(f)$ is linear. We extend the action of the creation and annihilation operators by linearity to D^n, see (12), for all n. We have the following relations, for $\psi_n \in D^n$ and $f \in \mathfrak{H}$:

$$\begin{aligned} \|a(f)\psi_n\| &\leq \sqrt{n}\, \|f\|\, \|\psi_n\|, && (16) \\ \|a^*(f)\psi_n\| &= \sqrt{n+1}\, \|f\|\, \|\psi_n\|, && (17) \end{aligned}$$

[2]Formally this means that we consider $\bigoplus_{n \geq 0} P_\pm$ on $\mathcal{F}(\mathfrak{H})$, which we denote simply again by $P_\pm$.

where the symbol $\|\cdot\|$ denotes the norm in the obvious spaces. The bound (16) follows from

$$\begin{aligned}
\|a(f)\psi_n\| &= \sup_{\phi\in\mathfrak{H}^{n-1},\,\|\phi\|=1} |\langle\phi, a(f)\psi_n\rangle| \\
&= \sup_{\phi\in\mathfrak{H}^{n-1},\,\|\phi\|=1} \left|\sqrt{n}\sum_{k=1}^{K}\left\langle f, f_1^{(k)}\right\rangle\left\langle\phi, f_2^{(k)}\otimes\ldots\otimes f_n^{(k)}\right\rangle\right| \\
&= \sup_{\phi\in\mathfrak{H}^{n-1},\,\|\phi\|=1} \left|\sqrt{n}\sum_{k=1}^{K}\left\langle f\otimes\phi, f_1^{(k)}\otimes\cdots\otimes f_n^{(k)}\right\rangle\right| \\
&\leq \sqrt{n}\,\|f\| \sup_{\Phi\in\mathfrak{H}^{n},\,\|\Phi\|=1} \left|\sum_{k=1}^{K}\left\langle\Phi, f_1^{(k)}\otimes\cdots\otimes f_n^{(k)}\right\rangle\right| \\
&= \sqrt{n}\,\|f\|\,\|\psi_n\|.
\end{aligned}$$

Equality (17) is shown as follows

$$\begin{aligned}
\|a^*(f)\psi_n\| &= \sqrt{n+1}\left\|\sum_{k=1}^{K} f\otimes f_1^{(k)}\otimes\cdots\otimes f_n^{(k)}\right\| \\
&= \sqrt{n+1}\left\|f\otimes\left(\sum_{k=1}^{K} f_1^{(k)}\otimes\cdots\otimes f_n^{(k)}\right)\right\| \\
&= \sqrt{n+1}\,\|f\|\,\|\psi_n\|.
\end{aligned}$$

By continuity, the action of $a(f)$ and $a^*(f)$ extends to $\mathfrak{H}^n$, for all n, and hence by component-wise action to the domain $\mathcal{D}(N^{1/2})\subset\mathcal{F}(\mathfrak{H})$. We have

$$\|a^\#(f)\psi\| \leq \|f\|\,\|(N+1)^{1/2}\psi\|, \tag{18}$$

for $\psi\in\mathcal{D}(N^{1/2})$, where $a^\#$ stands for either a or a^*. The bound (18) is easily obtained from $\|a^\#(f)\psi\|^2=\sum_{n\geq 0}\|a^\#(f)\psi_n\|^2$, (16), (17) and the definition of the number operator N, (6). The appearence of the star in $a^*(f)$ is not an arbitrary piece of notation, it signifies that $a^*(f)$ is the adjoint operator $a(f)^*$ of $a(f)$. We show this now. For all $\psi,\phi\in\mathcal{D}(N^{1/2})$, $f\in\mathfrak{H}$, we have

$$\langle\psi, a(f)\phi\rangle = \langle a^*(f)\psi,\phi\rangle\,. \tag{19}$$

Relation (19) follows easily from

$$\langle f_1\otimes\cdots\otimes f_{n-1}, a(f)g_1\otimes\cdots\otimes g_n\rangle = \langle a^*(f)f_1\otimes\cdots\otimes f_{n-1}, g_1\otimes\cdots\otimes g_n\rangle\,,$$

for any n, $f, f_j, g_j \in \mathfrak{H}$, which in turn follows directly from the definitions of $a^{\#}(f)$, see (14), (15). Equality (19) shows that $a^*(f) \subseteq a(f)^*$ (the adjoint of $a(f)$ is an extension of $a^*(f)$), so $a(f)^*$ is densely defined and consequently $a(f)$ is closable (a closed extension of $a(f)$ is $a(f)^{**}$). Similarly, one sees that $a^*(f)$ is a closable operator. We denote from now on by $a^{\#}(f)$ the closed creation and annihilation operators. To show that $a^*(f) = a(f)^*$ it remains to prove that $a^*(f) \supseteq a(f)^*$. Let $\psi \in \mathcal{D}(a(f)^*)$ then

$$\varphi \mapsto \langle \psi, a(f)\varphi \rangle \tag{20}$$

is a bounded linear map on $\mathcal{D}(a(f))$. Given $\varphi \in \mathcal{D}(a(f))$ we choose $\varphi^{(n)}$ to be the vector in Fock space obtained by setting all components φ_k of φ equal to zero, for $k > n$. Due to the boundedness of the map (20) we have

$$\langle \psi, a(f)\varphi \rangle = \lim_{n\to\infty} \left\langle \psi, a(f)\varphi^{(n)} \right\rangle = \lim_{n\to\infty} \sum_{k=0}^{n-1} \langle \psi_k, a(f)\varphi_{k+1} \rangle \,. \tag{21}$$

Equality (19) shows that for each fixed n we can move $a(f)$ to the left factor in the inner product, so

$$\langle \psi, a(f)\varphi \rangle = \lim_{n\to\infty} \sum_{k=0}^{n-1} \langle a^*(f)\psi_k, \varphi_{k+1} \rangle \,. \tag{22}$$

By the density of $\mathcal{D}(a(f))$ the last equality extends to all vectors $\varphi \in \mathcal{F}(\mathfrak{H})$ and choosing $\varphi_{k+1} = a^*(f)\psi_k$ shows that $\sum_{k=0}^{\infty} \|a^*(f)\psi_k\|^2 < \infty$, so that $\psi \in \mathcal{D}(a^*(f))$. We conclude that $\mathcal{D}(a(f)^*) = \mathcal{D}(a^*(f))$. Since $a^*(f)$ is closed we have $a^*(f)\psi = \lim_n a^*(f)\psi^{(n)}$, where $\psi^{(n)}$ is the truncation of ψ as explained above in the case of φ. Using this in (22) gives

$$\langle \psi, a(f)\varphi \rangle = \langle a^*(f)\psi, \varphi \rangle \,, \tag{23}$$

for any φ in the dense set $\mathcal{D}(a(f))$. Consequently, we have $a(f)^*\psi = a^*(f)\psi$ which shows that $a^*(f) \supseteq a(f)^*$. This finishes the proof of the statement $a^*(f) = a(f)^*$.

Notice that $a(f)\Omega = 0$ for all $f \in \mathfrak{H}$ and conversely, if $\psi \in \mathcal{F}(\mathfrak{H})$ is s.t. $a(f)\psi = 0$ for all $f \in \mathfrak{H}$ then $\psi = z\Omega$, for some $z \in \mathbb{C}$.

The annihilation operators $a(f)$ leave the subspaces $\mathcal{F}_\pm(\mathfrak{H})$ invariant.

This can be seen as follows. Let $\tau_{i,j}$ be the bounded linear operator on $\mathcal{F}(\mathfrak{H})$ which interchanges indices i and j in the tensor product, e.g. $\tau_{1,2}$ is determined by $\tau_{1,2} f_1 \otimes f_2 \otimes f_3 \otimes \cdots \otimes f_n = f_2 \otimes f_1 \otimes f_3 \otimes \cdots \otimes f_n$. An element $\psi_n \in \mathfrak{H}^n$ is in the range of $P_\pm$ if and only if $\tau_{i,j}\psi_n = \pm\psi_n$, for all $1 \le i < j \le n$. From the definition (14) of $a(f)$ we have for instance

$$\begin{aligned}
\tau_{1,2} a(f) f_1 \otimes \cdots \otimes f_n &= \sqrt{n}\, \langle f, f_1 \rangle\, f_3 \otimes f_2 \otimes \cdots \otimes f_n \\
&= a(f) f_1 \otimes f_3 \otimes f_2 \otimes \cdots \otimes f_n \\
&= a(f) \tau_{2,3} f_1 \otimes \cdots \otimes f_n,
\end{aligned}$$

and in a similar fashion one sees that $\tau_{i,j} a(f) = a(f)\tau_{i+1,j+1}$. Consequently, if ψ_n is in the range of $P_\pm$, then we have $\tau_{i,j} a(f)\psi_n = a(f)\tau_{i+1,j+1}\psi_n = \pm a(f)\psi_n$, so $a(f)\psi_n$ is in the range of $P_\pm$. We may write this also as $P_\pm a(f) P_\pm = a(f) P_\pm$.

The Bosonic (+) and Fermionic (−) creation and annihilation operators are defined to be the restrictions

$$a_\pm^{\#}(f) = P_\pm a^{\#}(f) P_\pm. \tag{24}$$

One then has $a_\pm(f) = a(f)P_\pm$ and $a_\pm^*(f) = P_\pm a^*(f)$. Using (14) and (15), it is not difficult to verify that

$$a_+(g) a_+^*(f) f_1 \otimes \cdots \otimes f_n = \sum_{k=1}^{n+1} \langle g, f_k \rangle\, P_+ f_1 \otimes \cdots \otimes \widehat{f_k} \otimes \cdots \otimes f_{n+1}, \tag{25}$$

where the hat $\widehat{\ }$ means that the corresponding symbol is omitted, and where we have set $f_{n+1} = f$. Similarly,

$$a_+^*(f) a_+(g) f_1 \otimes \cdots \otimes f_n = \sum_{k=1}^{n} \langle g, f_k \rangle\, P_+ f_1 \otimes \cdots \otimes \widehat{f_k} \otimes \cdots \otimes f_{n+1}. \tag{26}$$

Bosonic creation and annihilation operators satisfy the *canonical commutation relations (CCR)*:

$$[a_+(g), a_+^*(f)] = \langle g, f \rangle\, \mathbb{1}_{\mathcal{F}_+(\mathfrak{H})}, \tag{27}$$

$$[a_+(f), a_+(g)] = [a_+^*(f), a_+^*(g)] = 0, \tag{28}$$

for any $f, g \in \mathfrak{H}$, and where $[x, y] = xy - yx$ is the commutator. Equations (27), (28) are understood in the strong sense on $\mathcal{D}(N)$, on which products of two creation and annihilation operators are defined. Relation (27) follows directly from (25) and (26), and (28) can be established similarly.

Fermionic creation and annihilation operators satisfy the *canonical anti-commutation relations (CAR)*:

$$\{a_-(g), a_-^*(f)\} = \langle g, f\rangle \, \mathbb{1}_{\mathcal{F}_-(\mathfrak{H})}, \tag{29}$$
$$\{a_-(f), a_-(g)\} = \{a_-^*(f), a_-^*(g)\} = 0, \tag{30}$$

for any $f, g \in \mathfrak{H}$, and where $\{x, y\} = xy + yx$ is the anti-commutator (a priori again understood in the strong sense on $\mathcal{D}(N)$. However, it turns out that this relation extends to an equality of bounded operators, as we show now).

Although the CCR and the CAR have a similar structure (just interchange commutators with anti-commutators), they impose very different properties on the respective creation and annihilation operators. For instance, it turns out that the Fermionic creation and annihilation operators extend to bounded operators, while this is not true in the Bosonic case. We see this by using the CAR to obtain

$$\|a_-^*(f)\psi\|^2 = \left\langle \psi, a_-(f)a_-^*(f)\psi \right\rangle = -\|a_-(f)\psi\|^2 + \|f\|^2 \, \|\psi\|^2, \tag{31}$$

for all $\psi \in \mathcal{D}(N)$, from which it follows that $\|a_-^\#(f)\| \leq \|f\|$. On the other hand, $\|a_-^*(f)\Omega\| = \|f\| = \|f\| \, \|\Omega\|$, so $\|a_-(f)a_-^*(f)\Omega\| = \|f\|^2 = \|f\| \, \|a_-^*(f)\Omega\|$, hence

$$\|a_-^\#(f)\| = \|f\|. \tag{32}$$

Notice that this reasoning does not work for Bosons, because the minus sign on the r.h.s. of (31) would have to be replaced by a plus sign.

The fact that $a_+^*(f)$ is an unbounded operator can be seen as follows. Let $\psi_n \in \mathcal{F}_+(\mathfrak{H})$ be the normalized vector whose components are all zero except the n-particle component, which is $f \otimes f \otimes \cdots \otimes f$, for some $f \in \mathfrak{H}$, $\|f\| = 1$. Then we have $a_+^*(f)\psi_n = \sqrt{n+1}\psi_{n+1}$, hence $\|a_+^*(f)\psi_n\| = \sqrt{n+1} \to \infty$, as $n \to \infty$. This reasoning does not work for Fermions, because the vector ψ_n is not in the Fermionic Fock space. More generally, the *Pauli principle* says that it is impossible to have a state of several Fermions in which two among them are in the same one-particle state. This is expressed as

$$a_-^*(f)a_-^*(f) = 0, \tag{33}$$

for all $f \in \mathfrak{H}$, which follows immediately from (30).

2.3 Weyl operators

On a mathematical level, dealing with unbounded operators is a delicate affair so from this point of view Fermionic creation and annihilation operators are more easily handled than the Bosonic ones. It is desirable to replace the set of Bosonic creation and annihilation operators by a set of bounded operators which are in a certain sense equivalent to the set of creation and annihilation operators. These bounded operators are called Weyl operators.

We first form the (normalized) real and imaginary parts of $a_+(f)$

$$\Phi(f) = \frac{a_+(f) + a_+^*(f)}{\sqrt{2}}, \quad \Pi(f) = \frac{a_+(f) - a_+^*(f)}{\sqrt{2}\, i}, \tag{34}$$

defined as operators on $\mathcal{D}(N^{1/2})$. We do not equip Φ and Π with an index $+$ since we are going to use them only for Bosons (although one can do the same procedure for Fermions as well). We have $\Pi(f) = \Phi(if)$, so it suffices to consider the operators $\Phi(f)$. Notice though that $f \mapsto \Phi(f)$ is not a linear nor an antilinear map; it is only a real-linear map. Define the *finite particle subspace* of Fock space by

$$\mathcal{F}_+^0(\mathfrak{H}) = \left\{ \psi = \{\psi_n\}_{n\geq 0} \in \mathcal{F}_+(\mathfrak{H}) \mid \text{all but finitely many } \psi_n \text{ are zero} \right\}. \tag{35}$$

Clearly, $\mathcal{F}_+^0(\mathfrak{H}) \subset \mathcal{D}(N^\nu)$ for any $\nu > 0$. In particular, any polynomial in creation and annihilation operators is well defined as an operator on $\mathcal{F}_+^0(\mathfrak{H})$.

Proposition 2.1

1. *For any $f \in \mathfrak{H}$, $\Phi(f)$ is essentially selfadjoint on $\mathcal{F}_+^0(\mathfrak{H})$. If $\{f_n\}$ is a sequence in $\mathfrak{H}$ converging to $f \in \mathfrak{H}$, i.e. $\|f_n - f\| \to 0$, then $\Phi(f_n) \to \Phi(f)$ in the strong sense on $\mathcal{D}(N^{1/2})$, i.e. $\|(\Phi(f_n) - \Phi(f))\psi\| \to 0$, for all $\psi \in \mathcal{D}(N^{1/2})$.*

2. *On $\mathcal{F}_+^0(\mathfrak{H})$, we have*

$$e^{itN}\Phi(f)e^{-itN} = \Phi(e^{it}f), \tag{36}$$

for any $t \in \mathbb{R}$, $f \in \mathfrak{H}$.

3. *For $f, g \in \mathfrak{H}$, we have the CCR*

$$[\Phi(f), \Phi(g)] = i\mathrm{Im}\,\langle f, g\rangle\,, \tag{37}$$

understood in the strong sense on $\mathcal{D}(N)$.

Proof. An elegant proof of essential selfadjointness can be given using the Glimm-Jaffe-Nelson commutator theorem, c.f. [RS]. We opt for a more pedestrian proof involving analytic vectors, [3] because these are useful for concrete calculations. Nelson's analytic vector theorem says that if the domain of a symmetric operator contains an invariant subspace C which itself contains a dense set (in Hilbert space) of analytic vectors, then the symmetric operator is essentially selfadjoint on C. (See e.g. [RSII, Theorem X.39]).

Let $f \in \mathfrak{H}$ be fixed. The dense set $\mathcal{F}_+^0$ is invariant under $\Phi(f)$. We show that each vector $\psi \in \mathcal{F}_+^0$ is analytic for $\Phi(f)$. Because ψ is a finite sum of vectors $\psi_n \in P_+\mathfrak{H}^n$ (for possibly varying n), it is enough to show that ψ_n is an analytic vector for $\Phi(f)$, for any n. It is clear that $\psi_n \in \mathcal{D}(\Phi(f)^k)$, for all $k \geq 0$ and from

$$\|\Phi(f)^k\psi_n\| \leq \sqrt{2}\|f\| \; \|(N+1)^{1/2}\Phi(f)^{k-1}\Psi_n\| \leq \sqrt{2}\sqrt{n+k} \; \|f\| \, \|\Phi(f)^{k-1}\psi_n\|$$

it follows that

$$\|\Phi(f)^k\psi_n\| \leq 2^{k/2}\sqrt{(n+k)!} \; \|f\|^k \, \|\psi_n\|.$$

This means that the series

$$\sum_{k\geq 0} \frac{t^k}{k!}\|\Phi(f)^k\psi_n\|$$

converges for any $t \in \mathbb{C}$, hence ψ_n is an analytic (even an entire) vector for $\Phi(f)$.

We now show the strong continuity property. Let $\psi \in \mathcal{D}(N^{1/2}) \cap \mathcal{F}_+(\mathfrak{H})$. Then

$$\begin{aligned} \|(\Phi(f_n) - \Phi(f))\psi\| &\leq 2^{-1/2}\|a^*(f_n - f)\psi\| + 2^{-1/2}\|a(f_n - f)\psi\| \\ &\leq \sqrt{2}\|f_n - f\| \; \|(N+1)^{1/2}\psi\|, \end{aligned}$$

and the result follows.

To see 2., simply use the definition of the creation operator to obtain

$$\begin{aligned} e^{itN}a_+^*(f)e^{-itN}P_+f_1 \otimes \cdots f_n &= \sqrt{n+1}e^{it}P_+f \otimes f_1 \otimes \cdots \otimes f_n \\ &= a_+^*(e^{it}f)P_+f_1 \otimes \cdots \otimes f_n, \end{aligned}$$

[3] Let A be a linear operator on a Hilbert space $\mathcal{H}$. A vector $\psi \in \mathcal{H}$ is called *analytic* for A if $\psi \in \cap_{k\geq 0}\mathcal{D}(A^k)$ and the complex power series $\sum_{k\geq 0} t^k\|A^k\psi\|/k!$ has a nonzero radius of convergence. If the radius of convergence is infinite then ψ is said to be *entire* for A.

and similarly for annihilation operators.

The proof of 3. is immediate from (27), (28). ■

From now on we denote by $\Phi(f)$ the selfadjoint closure of (34). It generates a strongly continuous one-parameter group of unitaries on the Hilbert space $\mathcal{F}_+(\mathfrak{H})$,

$$\mathbb{R} \ni t \mapsto e^{it\Phi(f)}. \tag{38}$$

We define the *Weyl operator* $W(f)$, for $f \in \mathfrak{H}$, to be the unitary operator

$$W(f) = e^{i\Phi(f)}. \tag{39}$$

We have encountered the CCR expressed in terms of creation and annihilation operators (see (27), (28)) and in terms of the operators $\Phi(f)$ (see (37)). How are they expressed in terms of the Weyl operators? Taking into account (37), the Baker-Campell-Hausdorff formula gives (formally)

$$W(f)W(g) = e^{-\frac{i}{2}\mathrm{Im}\langle f,g\rangle} W(f+g) = e^{-i\mathrm{Im}\langle f,g\rangle} W(g)W(f).\ ^{4} \tag{40}$$

Relation (40) is called the Weyl form of the CCR. The following result is sometimes useful.

Proposition 2.2 *On the domain $\mathcal{D}(N)$ of the number operator we have*

$$NW(f) = W(f)N + W(f)(\Phi(if) + \|f\|^2/2), \tag{41}$$

for any $f \in \mathfrak{H}$. This means in particular that the Weyl operators leave $\mathcal{D}(N)$ invariant. It follows thus from (40) that any finite sum of products of Weyl operators leave $\mathcal{D}(N)$ invariant.

[4] The BCH formula is the non-commutative analogue of the formula $e^a e^b = e^{a+b}$. Let A, B be bounded operators on a Hilbert space $\mathfrak{H}$. Then $e^A e^B = \exp\{A + B + \frac{1}{2}[A,B] + \frac{1}{12}([A,[A,B]] - [B,[A,B]]) + \cdots\}$ (these are the first explicit terms in the BCH formula). In case the commutator $[A,B]$ is proportional to the identity the BCH formula simply reduces to $e^A e^B = e^{A+B+\frac{1}{2}[A,B]} = e^{A+B} e^{\frac{1}{2}[A,B]}$. Formally (40) follows thus from (37). Recall though that the $\Phi(f)$, $\Phi(g)$ are unbounded operators. It is correct to say that (40) implies (37); this can be seen by noticing that, on $\mathcal{F}_+^0(\mathfrak{H})$, one has $[\Phi(f), \Phi(g)] = \frac{1}{i^2}\partial^2_{st}|_{s=t=0}(W(tf)W(sg) - W(sg)W(tf))$, and then calculating the r.h.s. using (40).

Proof. To show (41) we notice first that $e^{itN}W(f) = W(e^{it}f)e^{itN}$ (which follows from (36)). Using

$$\begin{aligned}
\partial_t|_{t=0}\ \Phi(e^{it}f) &= \Phi(if), \\
\partial_t|_{t=0}\ \Phi(e^{it}f)^n &= n\Phi(f)^{n-1}\Phi(if) - i\|f\|^2\frac{n(n-1)}{2}\Phi(f)^{n-2}, \quad \text{for } n \geq 2,
\end{aligned}$$

we obtain, in the strong sense on $\mathcal{F}_+^0$,

$$\begin{aligned}
\frac{1}{i}\partial_t|_{t=0}\ W(e^{it}f)e^{itN} &= W(f)N + \frac{1}{i}\partial_t|_{t=0} \sum_{n\geq 0} \frac{i^n\Phi(e^{it}f)^n}{n!} \\
&= W(f)N + W(f)\left(\Phi(if) + \|f\|^2/2\right),
\end{aligned}$$

which extends to $\mathcal{D}(N)$, giving (41). ■

We finish this section by examining the continuity properties of the map $f \mapsto W(f)$. Recall that for Fermionic creation and annihilation operators, $f \mapsto a_-^\#(f)$ is a continuous map from $\mathfrak{H}$ into the bounded operators (equipped with the operator-norm topology), see (32). As we show now only a weaker form of continuity holds for the map $f \mapsto W(f)$. This is a source of considerable trouble in many applications.

Theorem 2.1 *If $f_n \to f$ in $\mathfrak{H}$, then $W(f_n) \to W(f)$ in the strong sense on $\mathcal{F}_+(\mathfrak{H})$, i.e., for any $\psi \in \mathcal{F}_+(\mathfrak{H})$, $\|(W(f_n) - W(f))\psi\| \to 0$. However, for any $f \in \mathfrak{H}$, $f \neq 0$, we have $\|W(f) - \mathbb{1}\| = 2$.*

Let e^{ith} be a strongly continuous unitary group on $\mathfrak{H}$ (h being its selfadjoint generator). Due to the theorem we have $\|W(e^{ith}f) - \mathbb{1}\| = 2$ (for $f \neq 0$), which implies that $t \mapsto W(e^{ith}f)$ is not norm continuous (the dynamics defined by e^{ith} is not continuous in the C^*algebra topology).

Proof of Theorem 2.1. The previous proposition tells us that $\Phi(f_n) \to \Phi(f)$, in the strong sense on $\mathcal{F}_+^0(\mathfrak{H})$, which is a joint core for all the operators $\Phi(f_n)$ and $\Phi(f)$. Therefore, $\Phi(f_n)$ converges to $\Phi(f)$ in the strong resolvent sense (see e.g. [RSII, Theorem VII.25]), from which it follows that $e^{it\Phi(f_n)}$ converges to $e^{it\Phi(f)}$ in the strong sense, for all t ([RSII, Theorem VII.21]).

Let us show $\|W(f) - \mathbb{1}\| = 2$, for any $f \neq 0$. The CCR (40) give $W(g)^*W(f)W(g) = e^{-i\mathrm{Im}\langle f,g\rangle}W(f)$, for any $g \in \mathfrak{H}$. Since $W(g)$ is unitary, this tells us that the spectrum of $W(f)$ is invariant under rotations, hence it must be the whole unit circle. The assertion $\|W(f) - \mathbb{1}\| = 2$ follows now from the spectral theorem. ■

2.4 The C^*-algebras $\mathrm{CAR}_{\mathrm{F}}(\mathfrak{H})$, $\mathrm{CCR}_{\mathrm{F}}(\mathfrak{H})$

The set of all Fermionic creation and annihilation operators generates a C^*-algebra of operators on $\mathcal{F}_-(\mathfrak{H})$, which we call $\mathrm{CAR}_{\mathrm{F}}(\mathfrak{H})$. The index $_F$ reminds us that the elements of this C^*-algebra are viewed as operators on Fock space $\mathcal{F}_-(\mathfrak{H})$. Similarly, the set of all Weyl operators generates a C^*-algebra of operators on $\mathcal{F}_+(\mathfrak{H})$, which we shall call $\mathrm{CCR}_{\mathrm{F}}(\mathfrak{H})$. Both algebras are unital C^*-algebras. For $\mathrm{CAR}_{\mathrm{F}}(\mathfrak{H})$ this follows from (29), and for $\mathrm{CCR}_{\mathrm{F}}(\mathfrak{H})$ it follows from $W(0) = \mathbb{1}$.

Theorem 2.2 *Let $a^*(f)$ and $a(f)$ denote the Fermionic creation and annihilation operators, acting on $\mathcal{F}_-(\mathfrak{H})$. The linear span of vectors of the form $a^*(f_1)\cdots a^*(f_n)\Omega$, with $f_k \in \mathfrak{H}$, $n \geq 0$, is dense in $\mathcal{F}_-(\mathfrak{H})$. In particular, Ω is cyclic for $\mathrm{CAR}_{\mathrm{F}}(\mathfrak{H})$ in $\mathcal{F}_-(\mathfrak{H})$.* [5] *Moreover, $\mathrm{CAR}_{\mathrm{F}}(\mathfrak{H})$ acts irreducibly on $\mathcal{F}_-(\mathfrak{H})$.* [6]

Proof. The first statement follows from

$$a^*(f_1)a^*(f_2)\cdots a^*(f_n)\Omega = \sqrt{n!}\; P_- f_1 \otimes f_2 \otimes \cdots \otimes f_n.$$

To see irreducibility, we suppose that T is a bounded operator on $\mathcal{F}_-(\mathfrak{H})$ that commutes with all operators $a^\#(f)$, $f \in \mathfrak{H}$, and show that $T = z\mathbb{1}$, for some $z \in \mathbb{C}$. We have $a(f)T\Omega = Ta(f)\Omega = 0$, for all $f \in \mathfrak{H}$, so $T\Omega = z\Omega$, for some complex number z (see after (19)). It follows that

$$Ta^*(f_1)\cdots a^*(f_n)\Omega = a^*(f_1)\cdots a^*(f_n)T\Omega = za^*(f_1)\cdots a^*(f_n)\Omega,$$

so by cyclicity of Ω, $T\psi = z\psi$, for all $\psi \in \mathcal{F}_-(\mathfrak{H})$. ■

---

[5] Let ψ be a vector in a Hilbert space $\mathcal{H}$ and let $\mathfrak{M}$ be a set of bounded operators on $\mathcal{H}$, $\mathfrak{M} \subseteq \mathcal{B}(\mathcal{H})$. We say that ψ is cyclic for $\mathfrak{M}$ in $\mathcal{H}$ if $\mathfrak{M}\psi = \{M\psi \mid M \in \mathfrak{M}\}$ is dense in $\mathcal{H}$.

[6] Let $\mathfrak{M}$ be a set of bounded operators acting on a Hilbert space $\mathcal{H}$. We say that $\mathfrak{M}$ acts irreducibly if the only closed subspaces of $\mathcal{H}$ which are invariant under the action of $\mathfrak{M}$ are the trivial subspaces $\{0\}$ and $\mathcal{H}$. $\mathfrak{M}$ acts irreducibly on $\mathcal{H}$ if and only if its commutant is trivial, $\mathfrak{M}' = \{T \in \mathcal{B}(\mathcal{H}) \mid TM = MT,\ \forall M \in \mathfrak{M}\} = \mathbb{C}\mathbb{1}$.

Theorem 2.3 *The vacuum vector $\Omega \in \mathcal{F}_+(\mathfrak{H})$ is cyclic for* $\mathrm{CCR}_\mathrm{F}(\mathfrak{H})$ *in* $\mathcal{F}_+(\mathfrak{H})$*, and* $\mathrm{CCR}_\mathrm{F}(\mathfrak{H})$ *acts irreducibly on* $\mathcal{F}_+(\mathfrak{H})$*.*

Proof. As in the case of Fermions, it is clear that the span of

$$\{a^*(f_1)\cdots a^*(f_n)\Omega \mid f_k \in \mathfrak{H}, n \geq 0\}$$

is dense in $\mathcal{F}_+(\mathfrak{H})$. But this is the same as the span of $\{\Phi(f_1)\cdots\Phi(f_n)\Omega \mid f_k \in \mathfrak{H}, n \geq 0\}$, so it is enough to prove that $\mathrm{CCR}_\mathrm{F}(\mathfrak{H})$ is dense in that latter span.

We show first that

$$(N+1)^k W(f)(N+1)^{-k-1} \tag{42}$$

is a bounded operator for all $f \in \mathfrak{H}$ and all $k \geq 0$. We proceed by induction in k. The statement is obvious for $k = 0$. Using (41) of Propositition 2.2 we get

$$\begin{aligned}
&(N+1)^k(N+1)W(f)(N+1)^{-1}(N+1)^{-k-1}\\
&= \;(N+1)^k W(f)(N+1)^{-k-1} \qquad (43)\\
&\quad +(N+1)^k\left\{W(f)(\Phi(if)+\|f\|/2)\right\}(N+1)^{-k-2}, \qquad (44)
\end{aligned}$$

where we commuted $N+1$ through $W(f)$ in the r.h.s. By the induction assumption, (43) is a bounded operator. The term with the field operator in (44) can be written as

$$(N+1)^k W(f)(N+1)^{-k-1}(N+1)^{k+1}\Phi(if)(N+1)^{-k-2},$$

where the product of the first three operators is again bounded. It suffices thus to show that

$$(N+1)^k\Phi(f)(N+1)^{-k-1} \tag{45}$$

is bounded, for all $f \in \mathfrak{H}$ and $k \geq 0$. Clearly we have $a(f)N = (N+1)a(f)$, $a^*(f)N = (N-1)a^*(f)$, and (45) follows easily. This finishes the proof of (42).

Since the product of Weyl operators is again a Weyl operator (modulo a phase) we get a bounded operator also if we replace $W(f)$ in (42) by any sum of products of Weyl operators. Given any $\epsilon > 0$ there exists a $T_n(\epsilon) < \infty$ such that

$$\Phi(f_1)\cdots\Phi(f_n)\Omega = \Phi(f_1)\cdots\Phi(f_{n-1})\frac{W(t_nf_n)-\mathbb{1}}{it_n}\Omega + O(\epsilon),$$

provided $t_n \leq T_n(\epsilon)$, and where $O(\epsilon)$ denotes a vector with norm less than ϵ. There exists a $T_{n-1}(\epsilon, t_n)$ such that

$$\begin{aligned}
&\Phi(f_1)\cdots\Phi(f_n)\Omega \\
&\quad = \quad \Phi(f_1)\cdots\Phi(f_{n-2})\frac{W(t_{n-1}f_{n-1})-\mathbb{1}}{it_{n-1}}\ \frac{W(t_nf_n)-\mathbb{1}}{it_n}\Omega + O(\epsilon),
\end{aligned}$$

provided $t_{n-1} \leq T_{n-1}(\epsilon, t_n)$. Continuing this process we see that there are numbers $T_n(\epsilon)$, $T_k(\epsilon, t_n, \ldots, t_{k+1})$, $1 \leq k \leq n-1$, such that if $t_k \leq T_k(\epsilon, t_n, \ldots, t_{k+1})$ and $t_n \leq T_n(\epsilon)$ then

$$\begin{aligned}
&\Phi(f_1)\cdots\Phi(f_n)\Omega \\
&\quad = \quad \frac{W(t_1f_1)-\mathbb{1}}{it_1}\cdots\frac{W(t_{n-1}f_{n-1})-\mathbb{1}}{it_{n-1}}\ \frac{W(t_nf_n)-\mathbb{1}}{it_n}\Omega + O(\epsilon).
\end{aligned}$$

Since the operator acting on Ω in the above r.h.s. is an element of $\mathrm{CCR}_F(\mathfrak{H})$ cyclicity of Ω is shown.

We finish the proof by showing irreducibility. Suppose T is a bounded operator on $\mathcal{F}_+(\mathfrak{H})$ that commutes with all $W(f)$, $f \in \mathfrak{H}$. It follows that for any $\psi \in \mathcal{D}(\Phi(f))$,

$$\frac{e^{it\Phi(f)}-\mathbb{1}}{it}T\psi = T\frac{e^{it\Phi(f)}-\mathbb{1}}{it}\psi \longrightarrow T\Phi(f)\psi,$$

as $t \to 0$. This shows that $T\psi \in \mathcal{D}(\Phi(f))$ and that $\Phi(f)T\psi = T\Phi(f)\psi$, i.e. T leaves the domain of every $\Phi(f)$ invariant and T commutes strongly with every $\Phi(f)$. Since $a(f) = 2^{-1/2}(\Phi(f) + i\Phi(if))$, this means that T commutes with $a(f)$, in the strong sense, for all $f \in \mathfrak{H}$. Irreducibility is now shown exactly as in Theorem 2.2. ■

2.5 Leaving Fock space

We explain in this section why Fock space is not always the right Hilbert space to describe a physical system.

As we have pointed out in Section 1.1, the very definition of Fock space gives the existence of a number operator, N, which is the operator of multiplication by n on the n-sector. Let $\psi \in \mathcal{F}(\mathfrak{H})$ be a (pure) state of the quantum gas (the following reasoning applies equally well to mixed states given by density matrices, i.e., convex combinations of pure states). The probability of finding more than a fixed number n of particles in the state ψ is given by

$$\langle \psi, P(N \geq n)\psi \rangle = \sum_{k \geq n} \| [\psi]_k \|^2, \tag{46}$$

where $P(N \geq n)$ is the spectral projection of N onto the set $\{n, n+1, \ldots\}$. The probability (46) vanishes in the limit $n \to \infty$, simply because ψ is in Fock space (the series converges). This shows that, a priori, any state described by a vector (or a density matrix) in Fock space has only finitely many particles in the sense that the probability of finding n particles approaches zero as n increases to infinity.

We will be interested in describing an ideal quantum gas which is extended in all of physical space $\mathbb{R}^3$, and which has a *nonzero density*, say one particle per unit volume. Such a state cannot be described by a vector (or density matrix) in Fock space! We may describe such a state as a limit of states "living" in Fock space (i.e., given by a density matrix on Fock space), e.g. by saying that the system should first be confined to a finite box $\Lambda_0 \subset \mathbb{R}^3$, in which case it is described by a vector $\psi_{\Lambda_0} \in \mathcal{F}(L^2(\Lambda_0))$ (of course, since the box is finite, and we specify a fixed density, there are only finitely many particles and Fock space can describe such a state). One then takes a sequence of nested boxes, $\Lambda_0 \subset \Lambda_1 \subset \cdots$ which increase to all of $\mathbb{R}^3$, $\cup_{k\geq 0}\Lambda_k = \mathbb{R}^3$, hence obtaining a sequence of states $\psi_{\Lambda_k} \in \mathcal{F}(L^2(\Lambda_k))$. If one can show that ψ_{Λ_k} has a limit ψ_∞, in a suitable sense, and where the density or particles is fixed, as $k \to \infty$, then ψ_∞ can be regarded as being the infinitely extended state with nonzero density. This limit is called the thermodynamic limit.

The limit state ψ_∞ is naturally not a vector in Fock space any more. What kind of object is it? To answer this, we have to say in what sense we take the thermodynamic limit. To be specific, we carry out the following discussion for Bosons. It can be repeated for Fermions. For any finite box Λ, the vector $\psi_\Lambda \in \mathcal{F}_+(L^2(\Lambda))$ gives rise to a positive, linear, normalized map on the von Neumann algebra of all bounded operators on $\mathcal{F}_+(L^2(\Lambda))$ by the assignment

$$\mathcal{B}\big(\mathcal{F}_+(L^2(\Lambda))\big) \ni A \mapsto \omega_\Lambda(A) = \langle \psi_\Lambda, A\psi_\Lambda \rangle \tag{47}$$

(for a mixed state determined by the density matrix ρ_Λ, we set $\omega_\Lambda(A) = \mathrm{tr}(\rho_\Lambda A)$). Since $\mathrm{CCR}_{\mathrm{F}}(L^2(\Lambda))$ is irreducible (see Theorem 2.3), its weak closure is the set of all bounded operators (indeed, irreducibility implies that $\mathrm{CCR_F}(\mathfrak{H})' = \mathbb{C}\mathbb{1}$, so $\mathrm{CCR_F}(\mathfrak{H})'' = \mathcal{B}(\mathfrak{H})$). Without loss of generality, we may therefore consider (47) only for $A \in \mathrm{CCR_F}(L^2(\Lambda))$, i.e. we view ω_Λ as a state on $\mathrm{CCR_F}(L^2(\Lambda))$, in the sense of the theory of C^*-algebras. [7]

Consider the (so-called *quasi-local*) C^*-algebra

$$\mathfrak{A}_0 = \overline{\bigcup_{n\geq 0} \mathrm{CCR_F}(L^2(\Lambda_n))}^{\mathrm{norm}} \subset \mathcal{B}\left(\mathcal{F}_+(L^2(\mathbb{R}^3))\right)$$

where $\overline{}^{\mathrm{norm}}$ means that we take the norm closure (in the operator norm of $\mathcal{B}\big(\mathcal{F}_+(L^2(\mathbb{R}^3))\big)$). Assume that the limit

$$\omega_\infty(A) = \lim_{k\to\infty} \omega_{\Lambda_k}(A) \tag{48}$$

exists, for any $A \in \mathrm{CCR_F}(L^2(\Lambda_n))$, any n. Then ω_∞ defines a state on $\mathfrak{A}_0$. We point out once more that in general, ω_∞ cannot be represented by a density matrix on Fock space $\mathcal{F}_+(L^2(\mathbb{R}^3, d^3x))$. One says that ω_∞ is not *normal* with respect to the states ω_{Λ_k}. [8] In the GNS representation $(\mathcal{H}_\infty, \pi_\infty, \psi_\infty)$ of $(\mathfrak{A}_0, \omega_\infty)$, the state ω_∞ is represented as

$$\omega_\infty(A) = \langle \psi_\infty, \pi_\infty(A)\psi_\infty\rangle\,.$$

In Section 4 we will discuss in detail the construction of the infinite-volume limit of a state describing a Bose gas with a given momentum density distribution and we will explicitly construct the corresponding GNS representation (the Araki-Woods representation).

One may wonder about the dependence of the C^*-algebra $\mathrm{CCR_F}(\mathfrak{H})$ on its underlying Hilbert space, $\mathcal{F}_+(\mathfrak{H})$. After all, we have just seen that density matrices on $\mathcal{F}_+(\mathfrak{H})$ cannot describe certain states of physical interest. Therefore Fock space should not play a central role in the definition of a physical system. In an attempt to detach ourselves from Fock space we may define the CCR and CAR algebras as *abstract C^*-algebras*, without referring to a Hilbert space. Fock space is then just the GNS representation space of a certain state on the abstract algebras, represented by the Fock vacuum vector

[7] Let $\mathfrak{A}$ be a (unital) C^*-algebra. A state ω on $\mathfrak{A}$ is a positive linear functional $\omega : \mathfrak{A} \to \mathbb{C}$ which is normalized as $\omega(\mathbb{1}) = 1$.

[8] Let ω_1 and ω_2 be two states on a C^*-algebra $\mathfrak{A}$. Then ω_1 is called normal with respect to ω_2 iff $\omega_1(A) = \mathrm{tr}(\rho\pi_2(A))$, where ρ is a trace class operator (density matrix) on $\mathcal{H}_2$, and where $(\mathcal{H}_2, \pi_2, \Omega_2)$ is the GNS representation of $(\mathfrak{A}, \omega_2)$.

(recall that the Fock vacuum vector is cyclic for $\mathrm{CCR}_{\mathrm{F}}(\mathfrak{H})$ and $\mathrm{CAR}_{\mathrm{F}}(\mathfrak{H})$, as we have shown in Theorems 2.2 and 2.3 above).

3 The CCR and CAR algebras

In this section we introduce abstract CAR and CCR algebras and review some of their properties. Useful references are [BRI,II] and [T].

We remind the reader of the notion of the "test function space" $\mathfrak{D} \subseteq \mathfrak{H}$, introduced at the beginning of Section 2.1, see (1).

3.1 The algebra $\mathrm{CAR}(\mathfrak{D})$

An (abstract) CAR algebra $\mathrm{CAR}(\mathfrak{D})$ over $\mathfrak{D} \subseteq \mathfrak{H}$ (where $\mathfrak{H}$ is a Hilbert space) is defined to be a unital C^*-algebra generated by elements written as $a(f)$, $f \in \mathfrak{D}$, where the assignment $f \mapsto a(f)$ is an antilinear map, and where the following relations hold

$$\{a(f), a(g)\} = 0, \quad \{a(f), a^*(g)\} = \langle f, g\rangle \, \mathbb{1}. \tag{49}$$

Here $a^*(f)$ is the element in the C^*algebra obtained by applying the $*$operation to $a(f)$, and $\{a, b\} = ab + ba$ is the anticommutator. We have already seen in the previous section that a C^*-algebra with these properties exists. Let us mention that the CAR (49) imply that

$$\|a(f)\| = \|f\|, \tag{50}$$

where $\|\cdot\|$ on the left hand side is the C^* norm and on the right hand side it is the norm of $\mathfrak{D}$ induced by $\mathfrak{H}$. This follow since $(a(f)a(f) = 0$ by the Pauli principle, see (33))

$$\big(a^*(f)a(f)\big)^2 = a^*(f)\{a(f), a^*(f)\}a(f) = \|f\|^2 a^*(f)a(f),$$

so that by the C^*norm property ($\|A^*A\| = \|A\|^2$), we have $\|a(f)\|^4 = \|f\|^2\,\|a(f)\|^2$; alternatively, boundedness of the Fermionic creation and annihilation operators follows from the fact that

$$\|\pi(a(f))\| = \|f\| \tag{51}$$

in any representation π of the CAR, which is shown as in (32). Let f_α be a net in $\mathfrak{D}$ converging to $f \in \overline{\mathfrak{D}}$ (the closure of $\mathfrak{D} \subseteq \mathfrak{H}$). Then $\|a(f) - a(f_\alpha)\| = \|f - f_\alpha\| \to 0$, so $a(f) \in \mathrm{CAR}(\overline{\mathfrak{D}})$ because $\mathrm{CAR}(\mathfrak{D})$, being a C^*-algebra, is uniformly closed. This shows that

$$\mathrm{CAR}(\mathfrak{D}) = \mathrm{CAR}(\overline{\mathfrak{D}}). \tag{52}$$

The next result tells us that given $\mathfrak{D}$, the corresponding CAR algebra $\mathrm{CAR}(\mathfrak{D})$ is unique.

Theorem 3.1 (Uniqueness of the CAR algebra). *Let $\mathfrak{D} \subseteq \mathfrak{H}$ be a given test function space (see (1)), and let $\mathfrak{A}_1$, $\mathfrak{A}_2$ be two CAR algebras over $\mathfrak{D}$ (generated by $a_1(f)$ and $a_2(f)$, respectively, with $f \in \mathfrak{D}$). There is a unique $*$isomorphism $\alpha : \mathfrak{A}_1 \to \mathfrak{A}_2$ such that $\alpha(a_1(f)) = a_2(f)$, for all $f \in \mathfrak{D}$.*

A proof can be found for instance in [BRII]. Once uniqueness is known in the sense above, one can easily prove the following result.

Theorem 3.2 *The C^*algebra $\mathrm{CAR}(\mathfrak{D})$ is simple.* [9]

Proof. Let $\mathfrak{I} \neq \mathfrak{A}_1 = \mathrm{CAR}(\mathfrak{D})$ be a closed two-sided ideal of $\mathrm{CAR}(\mathfrak{D})$. Define $\mathfrak{A}_2 = \mathrm{CAR}(\mathfrak{D})/\mathfrak{I}$ to be the C^*algebra generated by the equivalence classes $a_2(f) = [a(f)]$. Theorem 3.1 tells us that the projection P : $\mathrm{CAR}(\mathfrak{D}) \mapsto \mathrm{CAR}(\mathfrak{D})/\mathfrak{I}$ is an isomorphism. Therefore the kernel of P, which is the span of $\mathfrak{I}$, must be zero: $\mathfrak{I} = \{0\}$. ■

An interesting consequence of the simplicity is that every representation of $\mathrm{CAR}(\mathfrak{D})$ is faithful (has trivial kernel). Indeed, let π be a (nonzero) representation of $\mathrm{CAR}(\mathfrak{D})$. It is readily verified that $\ker \pi$ is a two-sided, closed ideal of $\mathrm{CAR}(\mathfrak{D})$. Hence by Theorem 3.2, $\ker \pi = \{0\}$.

3.2 The algebra $\mathrm{CCR}(\mathfrak{D})$

An (abstract) Weyl algebra, or CCR algebra $\mathrm{CCR}(\mathfrak{D})$ over a test function space $\mathfrak{D} \subseteq \mathfrak{H}$ is defined to be the unital C^*algebra generated by elements $W(f)$, $f \in \mathfrak{D}$, satisfying the relations

$$W(-f) = W(f)^*, \quad W(f)W(g) = e^{-\frac{i}{2}\mathrm{Im}\langle f,g\rangle} W(f+g). \tag{53}$$

[9] A C^*algebra $\mathfrak{A}$ is called simple if it has no nontrivial closed two-sided ideals, i.e., if the only closed two-sided ideals are $\{0\}$ and $\mathfrak{A}$. A subspace $\mathfrak{I} \subseteq \mathfrak{A}$ is a two-sided ideal if $A \in \mathfrak{A}$ and $I \in \mathfrak{I}$ implies that IA and AI are in $\mathfrak{I}$.

We have seen in the previous section that an algebra with these properties exists. The CCR (53) imply that $f \mapsto W(f)$ is not continuous (in the C^* norm topology). Indeed, the proof of Theorem 2.1 shows that we have $\|W(f) - \mathbb{1}\| = 2$, for any $f \neq 0$. Similarly to the CAR case, the Weyl algebra is unique.

Theorem 3.3 (Uniqueness of the Weyl algebra). *Let $\mathfrak{D} \subseteq \mathfrak{H}$ be given and let $\mathfrak{W}_1$ and $\mathfrak{W}_2$ be two Weyl algebras over $\mathfrak{H}$ (generated by $W_1(f)$ and $W_2(f)$, $f \in \mathfrak{D}$). There is a unique $*$isomorphism $\alpha : \mathfrak{W}_1 \to \mathfrak{W}_2$ such that $\alpha(W_1(f)) = W_2(f)$, for all $f \in \mathfrak{D}$.*

A proof can be found in [BRII, P]. As for the CAR algebra, simplicity of the CCR algebra follows from uniqueness.

Theorem 3.4 *The C^* algebra* $\mathrm{CCR}(\mathfrak{D})$ *is simple.*

Due to the lack of continuity of the map $f \mapsto W(f)$ it is not true that the Weyl algebra over $\mathfrak{D}$ is the same as the one over $\overline{\mathfrak{D}}$ if $\mathfrak{D} \neq \overline{\mathfrak{D}}$. One can show that if $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are two linear (not necessarily closed) subspaces of $\mathfrak{H}$ then

$$\mathrm{CCR}(\mathfrak{D}_1) = \mathrm{CCR}(\mathfrak{D}_1) \Longleftrightarrow \mathfrak{D}_1 = \mathfrak{D}_2,$$

see e.g. [BRII, Proposition 5.2.9]. In particular, $\mathrm{CCR}(\overline{\mathfrak{D}}) = \mathrm{CCR}(\mathfrak{D})$ if and only if $\mathfrak{D}$ is closed. Another difficulty is generated by the lack of continuity of the map

$$t \mapsto W(e^{ith} f), \tag{54}$$

where $t \in \mathbb{R}$ and h is some selfadjoint operator on $\mathfrak{H}$ (leaving $\mathfrak{D}$ invariant). The assignment (54) is called a *Bogoliubov transformation.* It represents a *dynamics* of the system, where h is interpreted as the one-particle Hamiltonian. The lack of continuity prevents us from treating the dynamics with ease on an algebraic level; for instance, one cannot take the derivative (nor the integral) of the r.h.s. of (54) w.r.t. t – and these operations are important e.g. to define a perturbed dynamics. There are representations of the CCR for which weaker continuity properties hold; we look at them now.

By a *regular representation* π of $\mathrm{CCR}(\mathfrak{D})$ we understand one with the property that $t \mapsto \pi(W(tf))$ is continuous in the strong operator topology on the representation Hilbert space $\mathcal{H}$, for all $f \in \mathfrak{D}$. A state ω on $\mathrm{CCR}(\mathfrak{D})$ is called a *regular state* if its GNS representation is regular (see also Theorem

4.1). For a regular representation the map $t \mapsto \pi(W(tf))$ is a strongly continuous one-parameter group of unitaries on $\mathcal{H}$. [10] The Stone-von Neumann theorem tells us that this group has a selfadjoint generator on $\mathcal{H}$, which we denote by $\Phi_\pi(f)$,

$$\pi(W(tf)) = e^{it\Phi_\pi(f)}.$$

It is convenient to introduce annihilation and creation operators in the regular representation π by setting

$$a_\pi(f) = \frac{\Phi_\pi(f) + i\Phi_\pi(if)}{\sqrt{2}}, \quad a_\pi^*(f) = \frac{\Phi_\pi(f) - i\Phi_\pi(if)}{\sqrt{2}}. \tag{55}$$

Compare this with (34)! Definition (55) needs some explanation because $\Phi_\pi(f)$ and $\Phi_\pi(if)$ are both unbounded operators on $\mathcal{H}$.

Proposition 3.1 *Let $F = \{f_1, \ldots, f_n\}$ be a finite collection of elements in $\mathfrak{D}$. The operators $\{\Phi_\pi(f_j), \Phi_\pi(if_j)\}_{j=1}^N$ have a common set of analytic vectors which is dense in the representation Hilbert space $\mathcal{H}$. This means that, for $f \in \mathfrak{D}$ fixed, the domain*

$$\mathcal{D}_{\pi,f} := \mathcal{D}(a_\pi(f)) := \mathcal{D}(a_\pi^*(f)) := \mathcal{D}(\Phi_\pi(f)) \cap \mathcal{D}(\Phi_\pi(if)) \tag{56}$$

is dense in $\mathcal{H}$. We understand the equalities (55) in the sense of operators on $\mathcal{D}_{\pi,f}$. Both $a_\pi(f)$ and $a_\pi^*(f)$ are closed operators on $\mathcal{D}_{\pi,f}$.

We have proved after equation (19) above that, for $a^\#(f)$ defined as in Section 2.2, the adjoint operator of $a(f)$ is $a^*(f)$. This can be shown for any regular representation, i.e., we have

$$a_\pi^*(f) = a_\pi(f)^*. \tag{57}$$

A proof of (57) can be found in [BRII].

Proof of Proposition 3.1. The following "smoothing" is useful: let $f \in \mathfrak{D}$ and consider the integral (understood in the strong sense on $\mathcal{H}$)

$$\sqrt{\frac{n}{\pi}} \int_{\mathbb{R}} ds\ e^{-ns^2} W_\pi(sf), \tag{58}$$

[10] The group properties follow from $\pi(W(tf))\pi(W(sf)) = \pi(W(tf)W(sf)) = e^{\frac{i}{2}st\mathrm{Im}\langle f,f\rangle}\pi(W((s+t)f)) = \pi(W((s+t)f))$ and $\pi(W(f))^* = \pi(W(f)^*) = \pi(W(-f)) = \pi(W(f)^{-1}) = \pi(W(f))^{-1}$.

where $n > 0$ and where we set $\pi(W(f)) = W_\pi(f)$. The strong limit of (58), as $n \to \infty$, is just the identity operator on $\mathcal{H}$. We apply the operator $W_\pi(tf)$ to the integral in (58) and obtain, after a change of variable,

$$W_\pi(tf) \int_{\mathbb{R}} ds\ e^{-ns^2} W_\pi(sf) = \int_{\mathbb{R}} ds\ e^{-n(s-t)^2} W_\pi(sf). \qquad (59)$$

The r.h.s. of (59) has an analytic extension in t to the whole complex plane. Similarly, if f_k is any element in F then the map

$$t \mapsto W_\pi(tf_k) \left(\frac{n}{\pi}\right)^{N/2} \int_{\mathbb{R}} ds_1 \cdots \int_{\mathbb{R}} ds_N\ e^{-n(s_1^2+\cdots+s_N^2)} W_\pi\left(\sum_{j=1}^N s_j f_j\right) \qquad (60)$$

is easily seen to have an analytic extension in t to all of $\mathbb{C}$, and the r.h.s. of (60) converges in the strong sense to $W_\pi(tf_k)$, as $n \to \infty$. This means that any vector of the form

$$\left(\frac{n}{\pi}\right)^{N/2} \int_{\mathbb{R}} ds_1 \cdots \int_{\mathbb{R}} ds_N\ e^{-n(s_1^2+\cdots+s_N^2)} W_\pi\left(\sum_{j=1}^N s_j f_j\right) \psi, \qquad (61)$$

where $\psi \in \mathcal{H}$ is arbitrary, is an analytic (entire) vector for all operators in the set $\{\Phi_\pi(f_j), \Phi_\pi(if_j)\}_{j=1}^N$. The set (61), where ψ varies over all of $\mathcal{H}$, is dense in $\mathcal{H}$, because (61) converges to ψ, as $n \to \infty$. This shows the first part of the proposition.

Let us now prove that the $a_\pi^\#(f)$ are closed operators on $\mathcal{D}_{\pi,f}$, where $f \in \mathfrak{D}$ is fixed. For any $\psi \in \mathcal{D}_{\pi,f}$ we have, by (55),

$$\|\Phi_\pi(f)\psi\|^2 + \|\Phi_\pi(if)\psi\|^2 = \|a_\pi(f)\psi\|^2 + \|a_\pi^*(f)\psi\|^2. \qquad (62)$$

We use $W_\pi(sf)W_\pi(itf) = e^{-ist\|f\|^2} W_\pi(itf)W_\pi(sf)$ to get

$$\begin{aligned} \frac{1}{i^2}\partial_{st}^2|_{s=t=0} \langle \psi, W_\pi(sf)W_\pi(itf)\psi\rangle &= \langle \Phi_\pi(f)\psi, \Phi_\pi(if)\psi\rangle \\ &= \langle \Phi_\pi(if)\psi, \Phi_\pi(f)\psi\rangle + i\|f\|^2\,\|\psi\|^2, \end{aligned}$$

which implies that

$$\|a_\pi^*(f)\psi\|^2 = \|a_\pi(f)\psi\|^2 + \|f\|^2\,\|\psi\|^2. \qquad (63)$$

Combining (62) and (63) yields the identity

$$\|\Phi_\pi(f)\psi\|^2 + \|\Phi_\pi(if)\psi\|^2 = 2\|a_\pi(f)\psi\|^2 + \|f\|^2\|\psi\|^2. \tag{64}$$

To show that $a_\pi(f)$ is a closed operator on $\mathcal{D}_{\pi,f}$ assume that $\psi_n \in \mathcal{D}_{\pi,f}$ is a sequence of vectors converging to some $\psi \in \mathcal{H}$, such that $a_\pi(f)\psi_n$ converges as $n \to \infty$, i.e., $\|a_\pi(f)(\psi_n - \psi_m)\| \to 0$, as $n, m \to \infty$. It follows from (64) that both $\|\Phi_\pi(f)(\psi_n - \psi_m)\|$ and $\|\Phi_\pi(if)(\psi_n - \psi_m)\|$ converge to zero as $n \to \infty$. Since $\Phi_\pi(f)$ and $\Phi_\pi(if)$ are closed operators (they are selfadjoint) we conclude that $\psi \in \mathcal{D}(\Phi_\pi(f))$ and $\psi \in \mathcal{D}(\Phi_\pi(if))$, i.e., $\psi \in \mathcal{D}_{\pi,f}$, and that $\Phi_\pi(f)\psi_n \to \Phi_\pi(f)\psi$, $\Phi_\pi(if)\psi_n \to \Phi_\pi(if)\psi$. Another application of (64) (with ψ replaced by $\psi_n - \psi$) shows that $\|a_\pi(f)(\psi_n - \psi)\|^2 \to 0$ as $n \to \infty$. Consequently $a_\pi(f)$ is a closed operator. In the same way one sees that $a_\pi^*(f)$ is a closed operator. ■

The *Fock representation* of $\mathrm{CCR}(\mathfrak{D})$ is the regular representation defined by $\pi_{\mathrm{F}} : \mathrm{CCR}(\mathfrak{D}) \to \mathcal{B}(\mathcal{F}_+(\mathfrak{H}))$,

$$\pi_{\mathrm{F}}(W(f)) = W_{\mathrm{F}}(f), \tag{65}$$

where the operator on the r.h.s. is given by (39), and where the Bosonic Fock space $\mathcal{F}_+(\mathfrak{H})$ was defined in (13).

We mention another structural property of the Weyl algebra. Let $\mathfrak{D}_1 \subseteq \mathfrak{H}_1$ and $\mathfrak{D}_2 \subseteq \mathfrak{H}_2$ be two linear subspaces and let $\mathfrak{D}_1 \oplus \mathfrak{D}_2 \subseteq \mathfrak{H}_1 \oplus \mathfrak{H}_2$ be their direct sum (i.e., the not necessarily closed set of all $f \oplus g$, $f \in \mathfrak{D}_1, g \in \mathfrak{D}_2$ equipped with the usual direct sum operations). We have the relation

$$\mathrm{CCR}(\mathfrak{D}_1 \oplus \mathfrak{D}_2) = \mathrm{CCR}(\mathfrak{D}_1) \otimes \mathrm{CCR}(\mathfrak{D}_2). \tag{66}$$

This follows simply from the CCR (53),

$$W(f_1 \oplus f_2) = W(f_1 \oplus 0 + 0 \oplus f_2) = e^{\frac{i}{2}\mathrm{Im}\langle f_1 \oplus 0, 0 \oplus f_2\rangle_{\mathfrak{H}\oplus\mathfrak{H}}} W(f_1 \oplus 0) W(0 \oplus f_2),$$

$\langle f_1 \oplus 0, 0 \oplus f_2\rangle_{\mathfrak{H}\oplus\mathfrak{H}} = \langle f_1, 0\rangle + \langle 0, f_2\rangle = 0$ and the identifications

$$W(f_1 \oplus 0) \mapsto W(f_1) \otimes W(0), \quad W(0 \oplus f_2) \mapsto W(0) \otimes W(f_2).$$

3.3 Schrödinger representation and Stone – von Neumann uniqueness theorem

Let us consider the easiest Weyl algebra CCR($\mathbb{C}$), where $\mathfrak{H} = \mathbb{C}$ is a one-dimensional Hilbert space. The Weyl operators are given by $W(z)$, with $z = s + it \in \mathbb{C}$, $s, t \in \mathbb{R}$. They satisfy

$$W(z) = W(s+it) = e^{\frac{i}{2}\mathrm{Im}\langle s,it\rangle} W(s)W(it) = e^{\frac{i}{2}st} W(s)W(it).$$

Let us assume that we are in a regular representation of the CCR, i.e., $\tau \mapsto W(\tau z)$ is a strongly continuous one parameter group ($\tau \in \mathbb{R}$) of unitaries on a (representation) Hilbert space. In particular, there are selfadjoint operators Φ, Π such that

$$W(\tau) = e^{i\tau\Phi}, \quad W(i\tau) = e^{i\tau\Pi}.$$

It is suggestive to write $\Phi = \Phi(1)$ and $\Pi = \Phi(i)$, compare with (39). The generators satisfy the commutation relations $[\Phi, \Pi] = [\Phi(1), \Phi(i)] = i\mathrm{Im}\,\langle 1, i\rangle = i\mathbb{1}$ which can be seen by noticing that

$$W(s)W(it)W(-s) = e^{-i\mathrm{Im}\langle s,it\rangle} W(it) = e^{-ist} W(it), \tag{67}$$

which yields (by applying $-i\partial_s|_{s=0}$) $[\Phi, W(it)] = -tW(it)$, and hence (by applying $-i\partial_t|_{t=0}$) $[\Phi, \Pi] = i\mathbb{1}$.

These commutation relations remind us of $[x, -i\partial_x] = i$, where x and $-i\partial_x$ are selfadjoint operators on $L^2(\mathbb{R}, dx)$. We can define a regular representation π_{S} of CCR($\mathbb{C}$) on $L^2(\mathbb{R}, dx)$ by

$$\pi_{\mathrm{S}}(W(z)) = e^{\frac{i}{2}st} U(s)V(t), \tag{68}$$

where $z = s + it \in \mathbb{C}$, and $U(s)$ and $V(t)$ are the one-parameter ($s, t \in \mathbb{R}$) unitary groups on $L^2(\mathbb{R}, dx)$ given by

$$\begin{aligned} (U(s)\psi)(x) &= e^{isx}\psi(x), \\ (V(t)\psi)(x) &= \psi(x+t), \end{aligned}$$

with selfadjoint generators $\Phi_{\mathrm{S}} = x$ and $\Pi_{\mathrm{S}} = -i\partial_x$. The representation (68) is called the *Schrödinger representation* of the CCR.

Since this representation is regular we can introduce creation and annihilation operators (c.f. (34)) by

$$a_{\mathrm{S}} = \frac{\Phi_{\mathrm{S}} + i\Pi_{\mathrm{S}}}{\sqrt{2}} = \frac{1}{\sqrt{2}}\left(x + \partial_x\right) \tag{69}$$

$$a_{\mathrm{S}}^* = \frac{\Phi_{\mathrm{S}} - i\Pi_{\mathrm{S}}}{\sqrt{2}} = \frac{1}{\sqrt{2}}\left(x - \partial_x\right). \tag{70}$$

Since both Φ_{S} and Π_{S} are unbounded operators one has to take care in the exact definition of the unbounded (non-selfadjoint) operators $a_{\mathrm{S}}^{\#}$ in (69), (70). This can be done by proceeding as in Proposition 3.1.

These considerations show that $L^2(\mathbb{R}, dx)$ carries a Fock space structure, i.e., there are two (densely defined, closed, unbounded, non symmetric) operators a_{S} and a_{S}^* acting on $L^2(\mathbb{R}, dx)$ and satisfying the commutation relation $[a_{\mathrm{S}}, a_{\mathrm{S}}^*] = \mathrm{id}$. The commutator is understood in the strong sense on some dense set of vectors (e.g. the functions in C_0^∞).

The vacuum vector $\Omega_{\mathrm{S}} \in L^2(\mathbb{R}, dx)$ is given by the normalized solution of $(x + \partial_x)\Omega_{\mathrm{S}}(x) = 0$ (i.e. $a_{\mathrm{S}}\Omega_{\mathrm{S}} = 0$),

$$\Omega_{\mathrm{S}}(x) = \pi^{-1/4} e^{-x^2/2}.$$

We introduce a sequence of one-dimensional subspaces $\mathcal{H}_n \subset L^2(\mathbb{R}, dx)$ spanned by $(a_{\mathrm{S}}^*)^n\Omega_{\mathrm{S}}$. Using the commutation relations for the creation and annihilation operator, one easily sees that the operator

$$N_{\mathrm{S}} = a_{\mathrm{S}}^* a_{\mathrm{S}} = \frac{1}{2}\left(-\partial_x^2 + x^2 - 1\right) \tag{71}$$

leaves each $\mathcal{H}_n$ invariant, and that $N_{\mathrm{S}} \restriction \mathcal{H}_n = n \,\mathrm{id} \restriction \mathcal{H}_n$. Notice that N_{S} is just the Schrödinger operator (Hamiltonian) corresponding to a one-dimensional quantum harmonic oscillator (modulo the constant term $-1/2$). There are various ways to see that we have

$$L^2(\mathbb{R}, dx) = \bigoplus_{n \geq 0} \mathcal{H}_n. \tag{72}$$

For instance, one knows that the eigenvalues of N_{S} are $0, 1, 2, 3, \ldots$ and they are simple (harmonic oscillator!), so (72) is a consequence of the fact that the eigenvectors of N_{S} span the entire space. The eigenvector ψ_n of N_{S} with eigenvalue $n \in \mathbb{N}$ satisfies the equation $N_{\mathrm{S}}\psi_n = n\psi_n$, which is equivalent to (c.f. (71))

$$(-\partial_x^2 + x^2)\psi_n = (2n+1)\psi_n, \tag{73}$$

i.e. ψ_n is a harmonic oscillator eigenvector. The ψ_n are the Hermite functions, they have the form

$$\psi_n(x) = \frac{1}{\sqrt{n!}}(a_{\mathrm{S}}^*)^n\Omega_{\mathrm{S}}(x) = \frac{1}{\sqrt{2^n n!}}(-1)^n\pi^{-1/4}e^{\frac{1}{2}x^2}(\partial_x)^n e^{-x^2}, \tag{74}$$

where $\frac{1}{\sqrt{n!}}$ is a normalization factor.

The Schrödinger representation of $\mathrm{CCR}(\mathbb{C}^n)$ is defined as the n-fold tensor product representation of $\mathrm{CCR}(\mathbb{C})$,

$$\pi_{\mathrm{S}}(W(z_1,\ldots,z_n)) = \prod_{j=1}^n e^{\frac{i}{2}s_j t_j}U_j(s_j)V_j(t_j), \tag{75}$$

acting on $L^2(\mathbb{R}^n, d^n x)$ and where $U_j(s)$, $V_j(t)$ act on the variable x_j in the obvious way. We may view $\mathbb{C}^n$ as $\mathbb{C}\oplus\cdots\oplus\mathbb{C}$ and compare (75) with (66).

The above discussion shows that the Schrödinger and the Fock representations of $\mathrm{CCR}(\mathbb{C})$ are unitarily equivalent, the correspondence being

$$(a_{\mathrm{S}}^*)^n\Omega_{\mathrm{S}} \mapsto (a_{\mathrm{F}}^*)^n\Omega_{\mathrm{F}}, \tag{76}$$

where a_{F}^* is the creation operator in the Fock representation, (65). This is not a coincidence, it can be viewed as a consequence of the following result.

Theorem 3.5 (Stone – von Neumann uniqueness theorem). *Let $\mathfrak{H}$ be a finite dimensional Hilbert space. Any irreducible regular representation of* $\mathrm{CCR}(\mathfrak{H})$ *is unitarily equivalent to the Fock representation of* $\mathrm{CCR}(\mathfrak{H})$.

It is instructive to have a look at the mechanism behind the proof of the Stone – von Neumann uniqueness theorem.

Outline of the proof of Theorem 3.5. Let $\{f_1,\ldots,f_n\}$ be an orthonormal basis of $\mathfrak{H}$ and define the non-negative operator (the number operator)

$$N_\pi = \sum_{j=1}^n a_\pi^*(f_j)a_\pi(f_j), \tag{77}$$

where the creation and annihilation operators are defined as in Proposition 3.1. One can show that N_π is a non-negative selfadjoint operator on the representation Hilbert space which we shall call $\mathcal{H}$. Using the CCR we find that, for any $f \in \mathfrak{H}$,

$$N_\pi a_\pi(f) = a_\pi(f)(N_\pi - 1). \tag{78}$$

Let $n_0 > -\infty$ be the infimum of the spectrum of N_π and let $P(N_\pi \leq n_0+1/2)$ denote the spectral projection of N_π associated with the interval $[n_0, n_0+1/2]$. Take any normalized $\Omega_\pi \in \operatorname{Ran} P(N_\pi \leq n_0+1/2)$. Relation (78) tells us that $P(N_\pi \leq x)a_\pi(f) = a_\pi(f)P(N_\pi \leq x+1)$ for any x, so we have $a_\pi(f)\Omega_\pi = 0$, for any $f \in \mathfrak{H}$.

Since π is irreducible the set $\mathcal{H}_\pi = \{W_\pi(f)\Omega_\pi \mid f \in \mathfrak{H}\}$, where $W_\pi(f) = \pi(W(f))$, is dense in $\mathcal{H}$ (the closure of $\mathcal{H}_\pi$ is a closed subspace of $\mathcal{H}$ which is invariant under $\pi(\mathrm{CCR}(\mathfrak{H}))$, and $\mathcal{H}_\pi \neq \{0\}$ since $\mathbb{1} \in \mathcal{H}_\pi$). Proceeding as in the proof of Theorem 2.3 one shows that the closure of $\mathcal{H}_\pi$ is the same as the closure of the set of vectors of the form $a_\pi^*(f_1)\cdots a_\pi^*(f_n)\Omega_\pi$,

$$\mathcal{H} = \mathrm{closure}\{a_\pi^*(f_1)\cdots a_\pi^*(f_n)\Omega_\pi \mid n \in \mathbb{N},\ f_1,\ldots,f_n \in \mathfrak{H}\}. \tag{79}$$

Now we define the linear map $U : \mathcal{H} \to \mathcal{F}_+(\mathfrak{H})$ by

$$U a_\pi^*(f_1)\cdots a_\pi^*(f_n)\Omega_\pi = a_{\mathrm{F}}^*(f_1)\cdots a_{\mathrm{F}}^*(f_n)\Omega_{\mathrm{F}}. \tag{80}$$

It is easy to verify that U extends to a unitary map because the norms of $a_\#^*(f_1)\cdots a_\#^*(f_n)\Omega_\#$, $\# = \pi, \mathrm{F}$, can be calculated purely by using the fact that $a_\#(f_j)\Omega_\# = 0$ and the canonical commutation relations. This finishes the outline of the proof of the Stone – von Neumann uniqueness theorem. ∎

Since every representation can be decomposed into a direct sum of irreducible representations, Theorem 3.5 says that every regular representation of $\mathrm{CCR}(\mathfrak{H})$, $\dim \mathfrak{H} < \infty$, is a direct sum of Fock representations (in which case we say that the representation is *quasi-equivalent* to the Fock representation). If $\dim \mathfrak{H} = \infty$ this is no longer true. In particular, the GNS representation corresponding to states of the infinitely extended free Bose gas with nonzero density which we will construct in Section 4 are not quasi-equivalent to the Fock representation.

There is however a characterization of representations of $\mathrm{CCR}(\mathfrak{H})$, where $\dim \mathfrak{H} = \infty$, which are quasi-equivalent to the Fock representation. In view of the outline of the proof of the Stone – von Neumann uniqueness theorem this characterization is very natural, although its exact formulation is somewhat technical. The central object in the above proof of Theorem 3.5 is the number operator (77). It can be generalized by putting

$$N_\pi = \sup_F \sum_j a_\pi^*(f_j) a_\pi(f_j), \tag{81}$$

where the supremum is over all finite-dimensional subspaces F of $\mathfrak{H}$, and the sum extends over an orthonormal basis $\{f_j\}$ of F. It is clear that a rigorous definition of (81) is not trivial. It can be given using quadratic forms rather than operators, see e.g. [BRII] Section 5.2.3. By proceeding as in the above outline of the proof of the Stone – von Neumann theorem one can show that *a representation* π *of* $\mathrm{CCR}(\mathfrak{H})$ *is a direct sum of Fock representations of* $\mathrm{CCR}(\mathfrak{H})$ *if and only if the number operator (81) can be defined as a densely defined selfadjoint operator.* This may be phrased as "π is quasi-equivalent to the Fock representation if and only if there is a number operator in the representation space of π". A precise statement of this result can be found in [BRII], Theorem 5.2.14.

3.4 Q–space representation

Our goal is to examine the unitary equivalence obtained from (76) when $\mathbb{C}$ is first replaced by $\mathbb{C}^n$, and then n is taken to infinity. This will provide us with another representation of $\mathrm{CCR}(\mathfrak{H})$, where $\mathfrak{H}$ is a separable Hilbert space. The representation Hilbert space we construct is $L^2(Q, d\mu)$, where μ is a probability measure on Q, $\mu(Q) = 1$. We give an explicit unitary equivalence between $L^2(Q, d\mu)$ and the bosonic Fock space $\mathcal{F}(\mathfrak{H})$ (we write $\mathcal{F}$ instead of $\mathcal{F}_+$). The Q-space representation is particularly useful in the analysis of interacting fields, see e.g. [RSII].

The assignment

$$\begin{aligned} &a_1^*(f_1)\cdots a_1^*(f_m)\Omega_1 \otimes a_2^*(g_1)\cdots a_2^*(g_n)\Omega_2 \\ &\quad \mapsto a^*(f_1 \oplus 0)\cdots a^*(f_m \oplus 0) a^*(0 \oplus g_1)\cdots a^*(0 \oplus g_n)\Omega, \end{aligned}$$

where $f_j \in \mathfrak{H}_1$, $g_j \in \mathfrak{H}_2$, establishes a unitary map between the Fock spaces $\mathcal{F}(\mathfrak{H}_1) \otimes \mathcal{F}(\mathfrak{H}_2)$ and $\mathcal{F}(\mathfrak{H}_1 \oplus \mathfrak{H}_2)$ (compare with (66)). This means that

$$\mathcal{F}(\mathbb{C}^n) = \mathcal{F}(\mathbb{C} \oplus \cdots \oplus \mathbb{C}) \cong \mathcal{F}(\mathbb{C}) \otimes \cdots \otimes \mathcal{F}(\mathbb{C}), \tag{82}$$

and taking into account the identification (76) we obtain

$$\mathcal{F}(\mathbb{C}^n) \cong L^2(\mathbb{R}, dx) \otimes \cdots \otimes L^2(\mathbb{R}, dx) \cong L^2(\mathbb{R}^n, d^n x). \tag{83}$$

Let C be a conjugation on $\mathfrak{H}$, i.e., C is an antilinear isometry satisfying $C^2 = 1$. One may think of C as the operation of taking the complex conjugate of coordinates in a given basis of $\mathfrak{H}$. [11] Let $\{e_j\}_{j=1}^{\infty}$ be an orthonormal basis of $\mathfrak{H}$ such that each e_j is invariant under C, $Ce_j = e_j$. A consequence of introducing a basis of C invariant vectors is that if $f = Cf$ and $g = Cg$ then $\langle f, g\rangle = \langle Cf, Cg\rangle = \overline{\langle f, g\rangle}$, so the corresponding Weyl operators commute, $W(f)W(g) = W(g)W(f)$; c.f. (53) (similarly the field operators in a regular representation commute in the strong sense on a dense set of vectors).

Let $\{f_1, \dots f_n\}$ be a finite collection of elements in $\{e_j\}$ and define

$$\mathcal{F}_n = \mathrm{closure}\{P(a^*(f_1), \dots, a^*(f_n))\Omega \mid P \text{ a polynomial}\} \subset \mathcal{F}(\mathfrak{H}), \tag{84}$$

where Ω is the Fock vacuum and the a^* are the creation operators in Fock representation, defined by (34) (we write a^* instead of a^*_+). Clearly, the map

$$a^*(f_1)^{k_1} \cdots a^*(f_n)^{k_n}\Omega \mapsto a^*(\zeta_1)^{k_1} \cdots a^*(\zeta_n)^{k_n}\Omega_{\mathcal{F}(\mathbb{C}^n)}, \tag{85}$$

where $\Omega_{\mathcal{F}(\mathbb{C}^n)}$ is the vacuum vector in $\mathcal{F}(\mathbb{C}^n)$ and $\zeta_j \in \mathbb{C}^n$ has zero components except for the j-th which equals one, extends to a unitary map between $\mathcal{F}_n$ and $\mathcal{F}(\mathbb{C}^n)$. The r.h.s. of (85) can be identified, via (83), (70), with the vector

$$\zeta_1^{k_1} \cdots \zeta_n^{k_n} \left(\frac{x_1 - \partial_{x_1}}{\sqrt{2}}\right)^{k_1} \cdots \left(\frac{x_n - \partial_{x_n}}{\sqrt{2}}\right)^{k_n} \Omega_n \in L^2(\mathbb{R}^n, d^n x), \tag{86}$$

where

$$\Omega_n = \pi^{-n/4} \exp\left(-\frac{1}{2}\sum_{j=1}^{n} x_j^2\right). \tag{87}$$

We normalize Ω_n to be the constant function by introducing the unitary map

$$(Tf)(x) = \pi^{n/4} \exp\left(\frac{1}{2}\sum_{j=1}^{n} x_j^2\right) f(x)$$

[11] If $\{f_j\}$ is any basis of $\mathfrak{H}$ then define $f'_j = f_j + Cf_j$. The f'_j are invariant under C, $Cf'_j = f'_j$, and they span $\mathfrak{H}$. A Gram-Schmidt procedure yields an orthonormal basis $\{e_j\}$ of vectors satisfying $Ce_j = e_j$. The action of C on coordinates w.r.t. the basis $\{e_j\}$ is complex conjugation.

between $L^2(\mathbb{R}^n, d^n x)$ and $L^2(\mathbb{R}^n, d\mu_1 \times \cdots \times d\mu_n)$, where

$$d\mu_j = \pi^{-1/2} e^{-x_j^2} dx_j.$$

Thus (85), (86) give a unitary map U_n between $\mathcal{F}_n$ and $L^2(\mathbb{R}^n, d\mu_1 \times \cdots \times d\mu_n)$ such that $U_n \Omega = 1$ (the constant function) and $U_n a^*(f_j) U_n^{-1} = \frac{2x_j - \partial_{x_j}}{\sqrt{2}}$, $U_n a(f_j) U_n^{-1} = \frac{1}{\sqrt{2}} \partial_{x_j}$ [12] so that

$$U_n \Phi(f_j) U_n^{-1} = U_n \frac{a^*(f_j) + a(f_j)}{\sqrt{2}} U_n^{-1} = x_j.$$

Let P_j, $j = 1, \ldots, n$ be n polynomials in one variable. The unitarity of U_n gives

$$\begin{aligned} \langle \Omega, P_1(\Phi(f_1)) \cdots P_n(\Phi(f_n)) \Omega \rangle &= \int_{\mathbb{R}^n} P_1(x_1) \cdots P_n(x_n) d\mu_1 \cdots d\mu_n \\ &= \prod_{j=1}^{n} \langle \Omega, P_j(\Phi(f_j)) \Omega \rangle . \end{aligned} \tag{88}$$

Let $Q = \times_{j=1}^{\infty} \mathbb{R}$ be the set of sequences $q = (q_1, q_2, \ldots)$ equipped with the σ–algebra generated by countable products of measurable sets in $\mathbb{R}$, and let $\mu = \otimes_{j=1}^{\infty} \mu_j$. The pair (Q, μ) is a measure space (see e.g. Chapter VI of [J]), and the set of all polynomials $P(q_1, \ldots, q_n)$, $n \in \mathbb{N}$, is dense in $L^2(Q, d\mu)$.

The space $\mathcal{F}_n$, (84), equals the closure of $\{P(\Phi(f_1), \ldots, \Phi(f_n))\Omega\}$, where P ranges over all polynomials in n variables (see also the proof of Theorem 2.3). For any $n \in \mathbb{N}$ and any polynomial P in n variables,

$$P(x_{j_1}, \ldots, x_{j_n}) = \sum_{p_1, \ldots, p_n} c(p_1, \ldots, p_n) x_{j_1}^{p_1} \cdots x_{j_n}^{p_n},$$

set

$$U P(\Phi(f_{j_1}), \ldots, \Phi(f_{j_n}))\Omega = P(q_{j_1}, \ldots, q_{j_n}) \in L^2(Q, d\mu). \tag{89}$$

[12]We have $T \partial_{x_j} T^{-1} = \partial_{x_j} - x_j$

Let us verify that U is norm preserving:

$$\begin{aligned}
&\|P(\Phi(f_{j_1}),\ldots,\Phi(f_{j_n}))\Omega\|^2 \\
&= \sum_{p_1,\ldots,p_n}\sum_{p'_1,\ldots,p'_n}\overline{c(p_1,\ldots,p_n)}c(p'_1,\ldots,p'_n)\left\langle\Omega,\Phi(f_{j_1})^{p_1+p'_1}\cdots\Phi(f_{j_n})^{p_n+p'_n}\Omega\right\rangle \\
&= \sum_{p_1,\ldots,p_n}\sum_{p'_1,\ldots,p'_n}\overline{c(p_1,\ldots,p_n)}c(p'_1,\ldots,p'_n)\int_{\mathbb{R}^n} q_{j_1}^{p_1+p'_1}\cdots q_{j_n}^{p_n+p'_n}d\mu_{j_1}\cdots d\mu_{j_n} \\
&= \int_Q |P(q_{j_1},\ldots,q_{j_n})|^2 d\mu. \qquad (90)
\end{aligned}$$

We use in the first step that the Φ's commute, which is due to the fact that $Cf_j = f_j$ and in the second step we make use of (88). Since the set of vectors $P(\Phi(f_{j_1}),\ldots,\Phi(f_{j_n}))\Omega$ is dense in $\mathcal{F}(\mathfrak{H})$ formula (90) shows that U extends to a unitary map from $\mathcal{F}(\mathfrak{H})$ to $L^2(Q,d\mu)$, s.t. $U\Omega = 1$ and $U\Phi(f_j)U^{-1} = q_j$.

3.5 Equilibrium state and thermodynamic limit

We focus in this subsection on Bosons and refer for more detail, as well as for the Fermionic case, to [BRII], Section 5.2.5.

Let H be the one-particle Hamiltonian, acting on the one-particle Hilbert space $\mathfrak{H}$, and denote by $\mathrm{d}\Gamma(H)$ its second quantization acting on Bosonic Fock space $\mathcal{F}_+(\mathfrak{H})$. $\mathrm{d}\Gamma(H)$ acts on the n sector as

$$H\otimes\cdots\otimes\mathbb{1}+\mathbb{1}\otimes H\otimes\cdots\otimes\mathbb{1}+\cdots+\mathbb{1}\otimes\cdots\otimes H.$$

We set $N = \mathrm{d}\Gamma(\mathbb{1})$, put

$$K_\mu = \mathrm{d}\Gamma(H) - \mu N = \mathrm{d}\Gamma(H-\mu\mathbb{1}), \qquad (91)$$

where $\mu\in\mathbb{R}$ is called the chemical potential, and assume that

$$Z_{\beta,\mu} = \mathrm{tr} e^{-\beta K_\mu} \qquad (92)$$

exists, for some inverse temperature $\beta > 0$. Here, tr denotes the trace on the Hilbert space $\mathcal{F}_+(\mathfrak{H})$. It is not hard to show that (92) is finite if and only if

$$\mathrm{tr} e^{-\beta H} < \infty \quad \text{and} \quad H-\mu\mathbb{1} > 0, \qquad (93)$$

see [BRII], Proposition 5.2.27; the trace here is of course over $\mathfrak{H}$. From the latter inequality it follows ($H - \mu\mathbb{1}$ has purely discrete spectrum) that there is a number $\eta > 0$ s.t.

$$\mathrm{d}\Gamma(H - \mu\mathbb{1}) = K_\mu \geq \eta\mathrm{d}\Gamma(\mathbb{1}) = \eta N. \tag{94}$$

The *Gibbs (equilibrium) state* on CCR($\mathfrak{H}$) is defined by

$$\omega_{\beta,\mu}(A) = Z_{\beta,\mu}^{-1} \operatorname{tr}\left(e^{-\beta K_\mu} A\right). \tag{95}$$

It depends on the inverse temperature β and the chemical potential μ. The Gibbs state satisfies the KMS relation

$$\omega_{\beta,\mu}(A\alpha_t(B)) = \omega_{\beta,\mu}\left(\alpha_{t-i\beta}\left(e^{-\beta\mu N} B e^{\beta\mu N}\right) A\right), \tag{96}$$

where $\alpha_t(A) = e^{it\mathrm{d}\Gamma(H)} A e^{-it\mathrm{d}\Gamma(H)}$ is the Heisenberg dynamics generated by the Hamiltonian H. Identity (96) makes sense for operators B s.t.

$$e^{\beta(\mathrm{d}\Gamma(H) - \mu N)} B e^{-\beta(\mathrm{d}\Gamma(H) - \mu N)}$$

exists. If $\mu = 0$, (96) reduces to the usual KMS relation $\omega_\beta(A\alpha_t(B)) = \omega_\beta(\alpha_{t-i\beta}(B)A)$. In order to calculate (95) explicitly it is useful to extend the domain of definition of $\omega_{\beta,\mu}$ to arbitrary (finite) products of creation and annihilation operators, i.e., to the polynomial $*$algebra $\mathfrak{P}$ of unbounded operators on $\mathcal{F}_+(\mathfrak{H})$, generated by $\{a^\#(f) \mid f \in \mathfrak{H}\}$. This can be done in the following way.

From (94) we see that

$$\|N^k e^{-tK_\mu}\| < \infty, \tag{97}$$

for any $t > 0$ and for any $k \geq 0$ [13] . The operator $e^{-\beta K_\mu/2}$ leaves the finite particle subspace $\mathcal{F}_+^0$ invariant (see (35)). If $Q \in \mathfrak{P}$ is any polynomial in creation and annihilation operators then Qe^{-tK_μ} is well defined on $\mathcal{F}_+^0$ and, by (97), extends to a bounded operator on $\mathcal{F}_+$, satisfying

$$\|a^\#(f_1) \cdots a^\#(f_k) e^{-tK_\mu}\| \leq C\|f_1\| \cdots \|f_k\|. \tag{98}$$

[13] This follows from $\|N^k e^{-tK_\mu}\psi\| \leq \left\langle N^k\psi, e^{-2t\eta N} N^k\psi\right\rangle^{1/2} = \|N^k e^{-t\eta N}\psi\|$.

Let $\mu > 0$. For $\psi \in \mathcal{F}_+(\mathfrak{H})$ we have

$$\begin{aligned}
&\left|\left\langle \psi, e^{-\beta K_\mu/2} Q e^{-\beta K_\mu/2} \psi \right\rangle\right| \\
&= \left|\left\langle \psi, e^{-\beta K_\mu/2} e^{\beta\mu N/4} e^{-\beta\mu N/4} Q e^{-\beta\mu N/4} e^{\beta\mu N/4} e^{-\beta K_\mu/2} \psi \right\rangle\right| \\
&\leq \|Q e^{-\beta\mu N/4}\| \; \left|\left\langle \psi, e^{-\beta K_\mu} e^{\beta\mu N/2} \psi \right\rangle\right| \\
&= \|Q e^{-\beta\mu N/4}\| \; \left|\left\langle \psi, e^{-\beta K_{\mu/2}} \psi \right\rangle\right| . \qquad (99)
\end{aligned}$$

Since $e^{-\beta K_\mu}$ is trace class $e^{-\beta K_{\mu/2}}$ is too (see (93)), so (99) shows that for any $Q \in \mathfrak{P}$,

$$\operatorname{tr}\left(e^{-\beta K_\mu/2} Q e^{-\beta K_\mu/2}\right) \leq C, \qquad (100)$$

where $C < \infty$ depends on Q, β and $\mu > 0$. Therefore $\omega_{\beta,\mu}$ can be extended to $\mathfrak{P}$, and we have

$$\omega_{\beta,\mu}\left(a^{\#}(f_1)\cdots a^{\#}(f_k)\right) \leq C\|f_1\|\cdots\|f_k\|. \qquad (101)$$

Note that since $e^{-\beta K_\mu}$ commutes with the number operator, the l.h.s. of (101) is actually zero unless k is even and $k/2$ of the operators $a^{\#}$ are creation operators.

We have in the strong sense on $\mathcal{F}_+^0$

$$e^{-\beta K_\mu/2} a^*(f) = a^*\left(e^{-\beta(H-\mu)/2} f\right) e^{-\beta K_\mu/2}, \qquad (102)$$

and hence, using the cyclicity of the trace and the CCR, we obtain

$$\begin{aligned}
\omega_{\beta,\mu}(a^*(f)a(g)) &= Z_{\beta,\mu}^{-1} \operatorname{tr}\left(a^*(e^{-\beta(H-\mu)/2} f)\; e^{-\beta K_\mu}\; a(e^{-\beta(H-\mu)/2} g)\right) \\
&= \omega_{\beta,\mu}\left(a(e^{-\beta(H-\mu)/2} g) a^*(e^{-\beta(H-\mu)/2} f)\right) \\
&= \left\langle g, e^{-\beta(H-\mu)} f \right\rangle + \omega_{\beta,\mu}\left(a^*(e^{-\beta(H-\mu)/2} f) a(e^{-\beta(H-\mu)/2} g)\right) .
\end{aligned}$$

Iterating this m times gives

$$\begin{aligned}
\omega_{\beta,\mu}(a^*(f)a(g)) &= \sum_{j=1}^{m} \left\langle g, e^{-j\beta(H-\mu)} f \right\rangle \\
&\quad + \omega_{\beta,\mu}\left(a^*(e^{-m\beta(H-\mu)/2} f) a(e^{-m\beta(H-\mu)/2} g)\right) . \qquad (103)
\end{aligned}$$

In the limit $m \to \infty$, the last term on the r.h.s. of (103) tends to zero, which follows from $\lim_m \|e^{-m\beta(H-\mu)/2} f\| = 0$ $(H - \mu > 0\,!)$ and the continuity of

$\omega_{\beta,\mu}$, (101). The first term on the r.h.s. of (103) can be summed explicitly and we obtain

$$\omega_{\beta,\mu}(a^*(f)a(g)) = \left\langle g, \frac{1}{e^{\beta(H-\mu)}-1} f \right\rangle . \tag{104}$$

Viewed as a function on $\mathfrak{H} \times \mathfrak{H}$, (104) is called the two-point function of the state $\omega_{\beta,\mu}$. Similarly, one defines n-point functions for all $n \geq 1$ by

$$\omega_{\beta,\mu}(a^*(f_1) \cdots a^*(f_n)a(g_1) \cdots a(g_n)). \tag{105}$$

Notice that the average of a product of m creation operators and n annihilation operators in the state $\omega_{\beta,\mu}$ vanishes unless $m = n$. A state with this property is called *gauge invariant*. The average of an arbitrary polynomial $Q \in \mathfrak{P}$ is expressed in terms of the n-point functions by first normal ordering Q. This means that the CCR are used repeatedly to write Q as a sum of polynomials in $a^\#$, where in each polynomial all creation operators stand to the left of all annihilation operators.

Proceeding in the same way as above, one can show that the n-point function (105) can be expressed as a sum of products of two-point functions. Consequently, (104) determines the state uniquely. Any state which is determined uniquely by its one- and two-point functions is called *quasi-free*.

Using the quasi-free structure one can show that

$$\begin{aligned}\omega_{\beta,\mu}(W(f)) &= \exp\left\{-\frac{1}{4}\left\langle f, \frac{e^{\beta(H-\mu)}+1}{e^{\beta(H-\mu)}-1} f \right\rangle\right\} \\ &= \exp\left\{-\frac{1}{4}\left\langle f, \coth\left(\frac{\beta(H-\mu)}{2}\right) f \right\rangle\right\}.\end{aligned}$$

So far, we have treated a general Hilbert space $\mathfrak{H}$ and a Hamiltonian H with the property that $H - \mu\mathbb{1} > 0$ is trace class. We consider now the case of the free Bose gas. The following discussion of the thermodynamic limit of the free Bose gas is summarized in [BRII, Proposition 5.2.29], see also [P].

Let $\{\Lambda_k\}_{k\geq 0} \subset \mathbb{R}^3$ be an increasing sequence of bounded regions in $\mathbb{R}^3$, s.t. $\bigcup_k \Lambda_k = \mathbb{R}^3$. Denote by $-H_k$ the selfadjoint Laplace operator on $L^2(\Lambda_k, d^3x)$ corresponding to a classical boundary condition. We choose μ s.t. there is a $C > 0$ satisfying

$$H_k - \mu\mathbb{1} \geq C\mathbb{1}, \tag{106}$$

uniformly in k. Let $\omega_{\beta,\mu}^{\Lambda_k}$ denote the Gibbs state on $\mathrm{CCR}(L^2(\Lambda_k, d^3x))$, see (95). The following results hold.

1. For any k and any $A \in \mathrm{CCR}(L^2(\Lambda_k, d^3x))$, the limit
$$\lim_{k'\to\infty} \omega_{\beta,\mu}^{\Lambda_{k'}}(A) = \omega_{\beta,\mu}(A) \tag{107}$$
exists and defines a state $\omega_{\beta,\mu}$ on $\mathrm{CCR}(\mathfrak{D})$, where $\mathfrak{D}$ is the dense subspace of $L^2(\mathbb{R}^3, d^3x)$ given by
$$\mathfrak{D} = \bigcup_{k\geq 0} L^2(\Lambda_k, d^3x). \tag{108}$$
The generating functional of $\omega_{\beta,\mu}$ is given by
$$\omega_{\beta,\mu}(W(f)) = \exp\left\{-\frac{1}{4}\left\langle f, \coth\left(\frac{\beta(H-\mu)}{2}\right) f\right\rangle\right\}, \tag{109}$$
for $f \in \mathfrak{D}$ and where $-H$ is the selfadjoint Laplace operator on $L^2(\mathbb{R}^3, d^3x)$. Note that due to (106) we can extend (109) to all $f \in L^2(\mathbb{R}^3, d^3x)$.

2. The GNS representation $(\mathcal{H}_{\beta,\mu}, \pi_{\beta,\mu}, \Omega_{\beta,\mu})$ of $(\mathrm{CCR}(\mathfrak{D}), \omega_{\beta,\mu})$ is regular. Let $a_{\beta,\mu}^{\#}(f)$, $f \in \mathfrak{D}$, denote the creation and annihilation operators in this representation. The state $\omega_{\beta,\mu}$ can be extended to the polynomial algebra $\mathfrak{P}_{\beta,\mu}$ generated by $\{a_{\beta,\mu}^{\#}(f) \mid f \in \mathfrak{D}\}$. The extension is the gauge-invariant quasi-free state with two-point function
$$\omega_{\beta,\mu}\left(a_{\beta,\mu}^*(f) a_{\beta,\mu}(g)\right) = \left\langle g, \frac{1}{e^{\beta(H-\mu)}-1} f\right\rangle. \tag{110}$$

3. Let $f \in L^2(\mathbb{R}^3, d^3x)$ and let $\{f_n\} \subset \mathfrak{D}$ be a sequence approximating f, i.e., $\|f - f_n\| \to 0$. The strong limit
$$\lim_n \pi_{\beta,\mu}(W(f_n)) = W_{\beta,\mu}(f) \tag{111}$$
exists and defines a unitary operator $W_{\beta,\mu}(f)$ in the von Neumann algebra $\pi_{\beta,\mu}(\mathfrak{A})'' \subset \mathcal{B}(\mathcal{H}_{\beta,\mu})$. The operators $W_{\beta,\mu}(f)$, for $f \in L^2(\mathbb{R}^3, d^3x)$, satisfy the Weyl CCR, (40). In other words, they define a representation of $\mathrm{CCR}(L^2(\mathbb{R}^3, d^3x))$.

4. The state $\omega_{\beta,\mu}$, viewed as a state on the von Neumann algebra $\pi_{\beta,\mu}(\mathfrak{A})''$ determined by the vector $\Omega_{\beta,\mu}$, is a (β, α_t)-KMS state, where α_t is the $*$automorphism group given by $\alpha_t(W_{\beta,\mu}(f)) = W_{\beta,\mu}(e^{-iHt} f)$, for $f \in L^2(\mathbb{R}^3, d^3x)$.

We point out that condition (106) gives a restriction on the possible values of μ. We must require $\mu < \mu_0$, where μ_0 depends on the choice of the boundary condition. On the other hand, μ is related to the particle density of the system. It turns out that under condition (106), one cannot describe high particle densities – e.g. the situation where most particles are in the state of lowest energy. In order to describe this phenomenon, called Bose-Einstein condensation, one needs a more careful analysis of the thermodynamic limit. We refer for more detail to [BRII, Section 5.2.5].

4 Araki-Woods representation of the infinite free Boson gas

The goal of this section is to find the GNS representation of states ω on $\mathrm{CCR}(\mathfrak{D})$ which represent the infinitely extended ideal Bose gas in which the momentum density distribution of the particles is prescribed. Our approach is based on the original paper [AW]. In a first step we show that the states of $\mathrm{CCR}(\mathfrak{D})$ are in one-to-one correspondence with so-called *generating* functionals on $\mathfrak{D}$. Then we calculate explicitly the generating functional corresponding to the Bose gas in a box and with a prescribed momentum density distribution. We take the thermodynamic limit of the finite-volume generating functionals, where the box size tends to infinity and the momentum density distribution approaches a given limit. The infinite-limit generating functional corresponds to a unique state on $\mathrm{CCR}(\mathfrak{D})$. We construct explicitly its GNS representation, which is commonly called the Araki-Woods representation.

4.1 Generating functionals

We consider in the remaining part of the notes the C^*algebra $\mathrm{CCR}(\mathfrak{D})$, where $\mathfrak{D} \subseteq \mathfrak{H}$. Given a state ω on $\mathrm{CCR}(\mathfrak{D})$, we may consider the (nonlinear) generating functional defined by

$$\begin{aligned} E : \mathfrak{D} &\rightarrow \mathbb{C} \\ f &\mapsto E(f) = \omega(W(f)). \end{aligned} \tag{112}$$

The generating functional satisfies the following properties:

1. (normalization) $E(0) = 1$
2. (unitarity) $\overline{E(f)} = E(-f)$, $f \in \mathfrak{D}$
3. (positivity) for any $K \geq 1$, $z_k \in \mathbb{C}$, $f_k \in \mathfrak{D}$, $k = 1 \ldots K$,

$$\sum_{k,k'=1}^{K} z_k \ \overline{z_{k'}} \ e^{-\frac{i}{2}\mathrm{Im}\langle f_k, f_{k'}\rangle} E(f_k - f_{k'}) \geq 0. \tag{113}$$

Properties 1. and 2. are obvious, and (113) is a consequence of the positivity of the state ω. Any positive element in a C^*algebra can be written as A^*A, so in $\mathrm{CCR}(\mathfrak{D})$, any positive element is approximated in C^*algebra norm by elements of the form

$$\left(\sum_{k=1}^{K} z_k W(f_k)\right)^* \left(\sum_{k=1}^{K} z_k W(f_k)\right) = \sum_{k,k'=1}^{K} z_k \ \overline{z_{k'}} \ e^{-\frac{i}{2}\mathrm{Im}\langle -f_{k'}, f_k\rangle} W(f_k - f_{k'}),$$

for some $z_k \in \mathbb{C}$ and $f_k \in \mathfrak{D}$. Hence (113) is equivalent to $\omega(A^*A) \geq 0$, for any $A \in \mathrm{CCR}(\mathfrak{D})$.

We now show that conversely, if a functional E with properties 1.-3. is given, then it determines uniquely a state on $\mathrm{CCR}(\mathfrak{D})$, with respect to which it is the generating functional.

Theorem 4.1 *Suppose a map $E : \mathfrak{D} \to \mathbb{C}$ satisfies 1.-3. above. For $f \in \mathfrak{D}$, set $\omega(W(f)) = E(f)$ and extend ω by linearity to the linear span of the Weyl operators,*

$$\omega\left(\sum_{k=1}^{K} z_k W(f_k)\right) = \sum_{k=1}^{K} z_k \ E(f_k).$$

Then ω extends uniquely to a state on $\mathrm{CCR}(\mathfrak{D})$. Moreover, E is continuous in the topology of $\mathfrak{D} \subseteq \mathfrak{H}$ if and only if $f \mapsto \pi_\omega(W(f))$ is a strongly continuous map from $\mathfrak{D}$ into the bounded operators on the GNS Hilbert space associated to $(\mathrm{CCR}(\mathfrak{D}), \omega)$.

Remark. The statement "$f \mapsto E(f)$ is continuous in the topology of $\mathfrak{D}$" is equivalent to the statement "$f \mapsto E(f + f_0)$ is continuous in the topology of $\mathfrak{D}$, for any fixed $f_0 \in \mathfrak{D}$". We incorporate the proof of this remark in the proof of Theorem 4.1 given below.

This theorem can be viewed as a *non-commutative analog of the Bochner-Minlos theorem* (see e.g. [H], Section 3.2 or [GJ], Sections 3.4 and A.6). Let $\mathcal{S}$ be the Schwartz space on $\mathbb{R}^n$, [14] $\mathcal{S}'$ its dual (the set of continuous linear functionals on $\mathcal{S}$), and let $\mathcal{S}'_{\mathbb{R}}$ be the set of real Schwartz distributions, i.e. the set of $\chi \in \mathcal{S}'$ satisfying $\overline{\langle\chi; f\rangle} = \langle\chi; \overline{f}\rangle$, for all $f \in \mathcal{S}$, where $\langle\cdot\,;\cdot\rangle$ is the dual pairing. Let ν be a positive regular Borel measure on $\mathcal{S}'_{\mathbb{R}}$, [15] s.t. $\int_{\mathcal{S}'_{\mathbb{R}}} d\nu(\chi) = 1$. We define the Fourier transform of ν by

$$E(f) = \int_{\mathcal{S}'_{\mathbb{R}}} e^{-i\langle\chi;f\rangle} d\nu(\chi),$$

$f \in \mathcal{S}$. Then E satisfies

1'. $E(0) = 1$,

2'. $f \mapsto E(f)$ is continuous,

3'. for any $K \geq 1$, $z_k \in \mathbb{C}$, $f_k \in \mathcal{S}$, $k = 1 \ldots K$, we have

$$\sum_{k,k'=1}^{K} z_k\ \overline{z_{k'}} E(f_k - \overline{f_{k'}}) \geq 0. \tag{114}$$

Inequality (114) holds because the l.h.s. is just $\int_{\mathcal{S}'_{\mathbb{R}}} \left|\sum_{k=1}^{K} z_k e^{-i\langle\chi;f_k\rangle}\right|^2 d\nu(\chi)$. Here is the Bochner–Minlos theorem:

Theorem 4.2 *Suppose a map $E : \mathcal{S} \to \mathbb{C}$ satisfies 1'.-3'. above. Then there exists a unique normalized positive regular Borel measure ν on the real Schwartz distribution space $\mathcal{S}'_{\mathbb{R}}$ such that E is the Fourier transform of ν.*

---

[14] the set of $f \in C^\infty(\mathbb{R}^n)$ s.t. all seminorms $\|f\|_{\mathbf{k},\mathbf{l}} = \sup_{x\in\mathbb{R}^n} \|x^{\mathbf{k}}\partial^{\mathbf{l}} f(x)\|$ are finite, for any multi-indices $\mathbf{k}, \mathbf{l} \in \mathbb{N}^n$. The topology of $\mathcal{S}$ is the one induced by these seminorms.

[15] A Borel measure μ on Q is called regular if for any Borel subset E of Q we have $\mu(E) = \inf\{\mu(U) \mid U \supset E, U \text{ open }\}$ and $\mu(E) = \sup\{\mu(K) \mid K \subset E, K \text{ compact }\}$.

The Borel σ-algebra of $\mathcal{S}'$ is generated by the open sets of the weak$*$ topology on $\mathcal{S}'$. A base for this topology is given by the collection of all *cylinder sets*. Cylinder sets are of the form $\{\chi \in \mathcal{S}' \mid (\langle\chi; f_1\rangle, \ldots, \langle\chi; f_n\rangle) \in B \subset \mathbb{C}^n\}$, where $f_1, \ldots, f_n \in \mathcal{S}$ and B is open. An open set in the weak$*$ topology is a union of cylinder sets.

Proof of Theorem 4.1. Parts of our proof are inspired by [A]. Let $\mathcal{S}$ be the $*$algebra generated by the Weyl operators $W(f)$, i.e., $\mathcal{S}$ is the $*$algebra of finite linear combinations of products of elements $W(f) \in \mathrm{CCR}(\mathfrak{D})$. $\mathcal{S}$ is a subalgebra of $\mathrm{CCR}(\mathfrak{D})$ and inherits the notion of positivity induced by $\mathrm{CCR}(\mathfrak{D})$. Inequality (113) implies that ω is a positive linear map on $\mathcal{S}$. Positivity implies boundedness as follows: let first $A \in \mathcal{S}$ be a *selfadjoint* element satisfying $\|A\| < 1$. Then we have $\mathcal{S} \ni \mathbb{1} - A > 0$, so $\omega(\mathbb{1}) - \omega(A) = \omega(\mathbb{1} - A) \geq 0$, and consequently $\omega(A) \leq \omega(\mathbb{1}) = E(0) = 1$. Next consider any $A \in \mathcal{S}$ s.t. $\|A\| < 1$. From $A^*A \leq \|A^*A\|\mathbb{1} \leq \|A\|^2\mathbb{1} < \mathbb{1}$ and the Cauchy-Schwarz inequality, $|\omega(A^*B)|^2 \leq \omega(A^*A)\omega(B^*B)$, which is valid for $A, B \in \mathcal{S}$ (note that we only need $\mathcal{S}$ to be a $*$algebra here, it does not have to be a Banach algebra), we obtain the estimate

$$|\omega(A)|^2 = |\omega(\mathbb{1}A)|^2 \leq \omega(\mathbb{1})\omega(A^*A) \leq \|A\|^2\omega(\mathbb{1})^2 = \|A\|^2.$$

This shows that $|\omega(A)| \leq \|A\|$, for $A \in \mathcal{S}$. Thus ω extends to a state on $\mathrm{CCR}(\mathfrak{D})$.

Next we show that if $f \mapsto E(f)$ is continuous then ω is a regular state. Let $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ be the GNS representation of $(\mathrm{CCR}(\mathfrak{D}), \omega)$. Suppose $f_n \to f$ is a convergent sequence in $\mathfrak{D}$. Define a family of unitary operators $U_n = \pi_\omega(W(f_n))$ and $U = \pi_\omega(W(f))$ on the Hilbert space $\mathcal{H}_\omega$. We show that

$$\lim_{n\to\infty} \|(U_n - U)\psi\| = 0, \tag{115}$$

for any $\psi \in \mathcal{H}_\omega$. Due to unitarity it suffices to show weak convergence, i.e. (115) is equivalent to

$$\lim_{n\to\infty} \langle \phi, (U_n - U)\psi \rangle = 0, \tag{116}$$

for all $\phi, \psi \in \mathcal{H}_\omega$. Because Ω_ω is cyclic, we have that for any $\epsilon > 0$, there are vectors $\phi_\epsilon = \sum_{k=1}^K z_k \pi_\omega(W(g_k))\Omega_\omega$ and $\psi_\epsilon = \sum_{l=1}^L \zeta_l \pi_\omega(W(h_l))\Omega_\omega$, s.t. $\|\phi - \phi_\epsilon\| < \epsilon$ and $\|\psi - \psi_\epsilon\| < \epsilon$. Now

$$\begin{aligned} |\langle \phi, (U_n - U)\psi \rangle - \langle \phi_\epsilon, (U_n - U)\psi_\epsilon \rangle| &\leq \|\phi - \phi_\epsilon\| \, \|(U_n - U)\| \, \|\psi\| \\ &\quad + 2\|\phi\| \, \|(U_n - U)\| \, \|\psi - \psi_\epsilon\| \\ &\leq 4\epsilon(\|\phi\| + \|\psi\|), \end{aligned}$$

uniformly in n (we use here that $\|\phi_\epsilon\| < 2\|\phi\|$, for small ϵ). Thus it is enough to prove that

$$\lim_{n\to\infty} \langle \pi_\omega(W(g))\Omega_\omega, (U_n - U)\pi_\omega(W(h))\Omega_\omega \rangle = 0, \tag{117}$$

for any $g, h \in \mathfrak{D}$. This scalar product is just

$$\begin{aligned}
&\langle \Omega_\omega, \pi_\omega(W(-g)(W(f_n) - W(f))W(h))\Omega_\omega \rangle \\
&\quad = \; e^{-\frac{i}{2}\mathrm{Im}(\langle -g, f_n\rangle + \langle -g+f_n, h\rangle)} E(-g + f_n + h) \\
&\quad\quad - e^{-\frac{i}{2}\mathrm{Im}(\langle -g, f\rangle + \langle -g+f, h\rangle)} E(-g + f + h),
\end{aligned}$$

which converges to zero as $n \to \infty$.

Next we show that if ω is a regular state then $f \mapsto E(f)$ is continuous. Let f_n be a sequence in $\mathfrak{D}$ converging to $f \in \mathfrak{D}$. We have

$$\begin{aligned}
E(f_n) - E(f) &= \omega(W(f_n) - W(f)) = \omega\big((W(f_n)W(-f) - \mathbb{1})W(f)\big) \\
&= \omega\left(\left(e^{\frac{i}{2}\mathrm{Im}\langle f_n, f\rangle} W(f_n - f) - \mathbb{1}\right) W(f)\right) \\
&= \left\langle \Omega_\omega, \left(e^{\frac{i}{2}\mathrm{Im}\langle f_n, f\rangle} \pi_\omega(W(f_n - f)) - \mathbb{1}\right) \pi_\omega(W(f))\Omega_\omega \right\rangle . \quad (118)
\end{aligned}$$

Since $\pi_\omega(W(f_n - f))$ converges strongly to zero as $n \to \infty$ we have the desired continuity of E.

Finally we prove the assertion of the remark after Theorem 4.1. Suppose that $f_n \to 0$ and that $E(f_n) \to 1$. Our goal is to show that $E(f_n + f_0) \to E(f_0)$, where $f_0 \in \mathfrak{H}$ is fixed. The above considerations leading to (115) show that $\pi_\omega(W(f_n)) \to \mathbb{1}$ in the strong sense on $\mathcal{H}_\omega$. Then we write, as in (118),

$$\begin{aligned}
&E(f_n + f_0) - E(f_0) \\
&\quad = \left\langle \Omega_\omega, \left(e^{\frac{i}{2}\mathrm{Im}\langle f_n, f_0\rangle} \pi_\omega(W(f_n)) - \mathbb{1}\right) \pi_\omega(W(f_0))\Omega_\omega \right\rangle .
\end{aligned}$$

The r.h.s. converges to zero as $n \to \infty$. ■

Suppose that E_k, $k = 1, 2, \ldots$ is a sequence of generating functionals, each satisfying conditions 1.-3. above. If E_k has a limit in the sense that there is a map $E : \mathfrak{D} \to \mathbb{C}$ s.t. $E(f) = \lim_{k\to\infty} E_k(f)$, for any $f \in \mathfrak{D}$, then it is clear that E satisfies conditions 1.-3. as well. In the next section we use this fact to construct the generating functional and the GNS representation of the free Bose gas extended to all of physical space in a state determined by a given momentum density distribution.

We close this section with the calculation of $E_{\mathrm{F}}(f) = \langle \Omega, W(f)\Omega \rangle$, the Fock generating functional, corresponding to the vacuum state on $\mathrm{CCR}_{\mathrm{F}}(\mathfrak{H})$.

Using the series expansion of the Weyl operator in Fock space, we can write

$$E_{\mathrm{F}}(f) = \sum_{n\geq 0} \frac{i^{2n}}{(2n)!} \left\langle \Omega, \Phi(f)^{2n}\Omega \right\rangle, \tag{119}$$

where we have used that all odd powers of $\Phi(f)$ have a vanishing vacuum expectation value. We use the commutation relations (27), (28), and the fact that $a(f)\Omega = 0$ to get

$$\left\langle \Omega, \Phi(f)^{2n}\Omega \right\rangle = \frac{1}{\sqrt{2}} \left\langle \Omega, a(f)\Phi(f)^{2n-1}\Omega \right\rangle = \frac{2n-1}{2} \|f\|^2 \left\langle \Omega, \Phi(f)^{2n-2}\Omega \right\rangle.$$

By induction, we arrive at $\langle \Omega, \Phi(f)^{2n}\Omega \rangle = \left(\frac{\|f\|^2}{2}\right)^n \frac{(2n)!}{2^n n!}$, which we use in (119) to obtain

$$E_{\mathrm{F}}(f) = e^{-\|f\|^2/4}. \tag{120}$$

4.2 Ground state (condensate)

We construct in this section the representation of the CCR describing the infinitely extended Bose gas in its ground state where all particles are in the same state. The ground state is an example of a *condensate* (macroscopic occupation of a particular one-particle state of an infinitely extended system), it is parametrized by the particle density $\rho \geq 0$.

Consider first the free non-relativistic Bose gas confined to a finite box $V = \frac{1}{8}[-|V|^{1/3}, |V|^{1/3}]^3 \subset \mathbb{R}^3$ of volume $|V|$. We will let the volume and the number of particles, n, tend to infinity while keeping the density $\rho = n/|V|$ fixed. For any finite n, the Bose gas is described using Fock space $\mathcal{F}(L^2(V, d^3x))$, it is just a system of n non-interacting particles whose symmetric wave function $\psi \in L^2(V^n, d^{3n}x)$ evolves according to the Schrödinger equation

$$i\partial_t \psi(x_1, \ldots, x_n) = (H\psi)(x_1, \ldots, x_n), \tag{121}$$

where

$$H = \sum_{j=1}^{n} -\Delta_j \tag{122}$$

is the selfadjoint Hamiltonian operator on $L^2(V^n, d^{3n}x)$ with periodic boundary conditions (of course, Δ_j is the Laplacian with respect to the variable

x_j). The system is in its ground state Ψ_V (the one having the lowest energy) if each of the n particles is in the state f_V of minimal energy (relative to $-\Delta_j$), given by

$$f_V(x) = |V|^{-1/2}, \quad x \in V, \tag{123}$$

which we have normalized as $\|f_V\|_{L^2(V,d^3x)} = 1$. Consequently,

$$\Psi_V = \frac{1}{\sqrt{n!}} a_{\mathrm{F}}^*(f_V)^n \Omega_{\mathrm{F}}, \tag{124}$$

where $\frac{1}{\sqrt{n!}}$ is a normalization factor, a_{F}^* is the Bosonic creation operator on Fock space, and Ω_{F} is the Fock vacuum. The generating functional correspondig to Ψ_V is

$$E_V(f) = \langle \Psi_V, W(f)\Psi_V \rangle = \frac{1}{n!} \langle \Omega_{\mathrm{F}}, a_{\mathrm{F}}(f_V)^n W_{\mathrm{F}}(f) a_{\mathrm{F}}^*(f_V)^n \Omega_{\mathrm{F}} \rangle , \tag{125}$$

where $W_{\mathrm{F}} = e^{i\Phi_{\mathrm{F}}(f)}$ is a Weyl operator in the Fock representation, $\Phi_{\mathrm{F}} = \frac{1}{\sqrt{2}}(a_{\mathrm{F}}^*(f) + a_{\mathrm{F}}(f))$. Our plan is to calculate the right hand side of (125) explicitly and take the limit $n \to \infty$, keeping ρ fixed. This provides us with a generating functional E_{GS} (depending on the number ρ) which we interpret as the generating functional of the ground state of the infinite system. Knowing E_{GS}, we explicitly construct the GNS representation of the ground state of the infinite system (it will not be the Fock representation – i.e., there is no vector (or density matrix) on Fock space representing the ground state of the infinite system – we have already discussed this in Section 1.5.).

In order to calculate the r.h.s. of (125) we "pull" (or commute) the annihilation operators to the right, through $W_{\mathrm{F}}(f)$ and through the creation operators, by using the canonical commutation relations. Whenever an annihilation operator is completely pulled through, it hits the vacuum Ω_{F} yielding zero. The value of the r.h.s. of (125) is given by all extra terms (contractions) one generates, using the CCR, in this procedure. Let us first show how to pull the annihilation operators through the Weyl operator. Using the series expansion of $W_{\mathrm{F}}(f) = e^{i\Phi_{\mathrm{F}}(f)}$ and that

$$[a_{\mathrm{F}}(f), \Phi_{\mathrm{F}}(g)^k] = 2^{-1/2} k \langle f, g \rangle \Phi_{\mathrm{F}}(g)^{k-1}, \tag{126}$$

which follows easily from the CCR (27), (28), one verifies without difficulty that

$$[a_{\mathrm{F}}(f), W_{\mathrm{F}}(g)] = 2^{-1/2} i \langle f, g \rangle W_{\mathrm{F}}(g). \tag{127}$$

All these relations can be understood in the strong sense on the finite particle subspace. We view the pulling through procedure as follows. Consider $a_{\mathrm{F}}(f_V)^n W_{\mathrm{F}}(f)$. Among the n annihilation operators, k $(= 0, 1, \ldots, n)$ are commuted through $W_{\mathrm{F}}(f)$ to the right while $n-k$ have undergone a contraction of the form (127). For each fixed value of k, there are $\binom{n}{k}$ ways of choosing which annihilation operators are safely pulled through the Weyl operator. We obtain

$$a_{\mathrm{F}}(f_V)^n W_{\mathrm{F}}(f) = \sum_{k=0}^{n} \binom{n}{k} \left(2^{-1/2} i \langle f_V, f\rangle\right)^{n-k} W_{\mathrm{F}}(f) a_{\mathrm{F}}(f_V)^k. \tag{128}$$

One can of course prove (128) as well by induction, which is an easy task. The generating functional can thus be written as

$$E_V(f) = \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} \left(2^{-1/2} i \langle f_V, f\rangle\right)^{n-k} \left\langle \Omega_{\mathrm{F}}, W_{\mathrm{F}}(f) a_{\mathrm{F}}(f_V)^k a_{\mathrm{F}}^*(f_V)^n \Omega_{\mathrm{F}} \right\rangle . \tag{129}$$

A similar pull through argument as above, plus the facts that $a_{\mathrm{F}}(f_V)\Omega_{\mathrm{F}} = 0$ and $\|f_V\| = 1$ yields

$$\begin{aligned} a_{\mathrm{F}}(f_V)^k a_{\mathrm{F}}^*(f_V)^n \Omega_{\mathrm{F}} &= n(n-1)\cdots(n-k+1) \langle f_V, f_V\rangle^k a_{\mathrm{F}}^*(f_V)^{n-k} \Omega_{\mathrm{F}} \\ &= \frac{n!}{(n-k)!}\, a_{\mathrm{F}}^*(f_V)^{n-k} \Omega_{\mathrm{F}}, \end{aligned} \tag{130}$$

from which we get

$$E_V(f) = \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} \frac{n!}{(n-k)!} \left(2^{-1/2} i \langle f_V, f\rangle\right)^{n-k} \left\langle \Omega_{\mathrm{F}}, W_{\mathrm{F}}(f) a_{\mathrm{F}}^*(f_V)^{n-k} \Omega_{\mathrm{F}} \right\rangle . \tag{131}$$

We pull the $n-k$ creation operators to the left through the Weyl operator by using the adjoint relation to (128) (recall also that $W_{\mathrm{F}}(f)^* = W_{\mathrm{F}}(-f)$)

$$W_{\mathrm{F}}(f) a_{\mathrm{F}}^*(f_V)^{n-k} = \sum_{l=0}^{n-k} \binom{n-k}{l} \left(-2^{-1/2} i\, \overline{\langle f_V, -f\rangle}\right)^{n-k-l} a_{\mathrm{F}}^*(f_V)^l W_{\mathrm{F}}(f). \tag{132}$$

Clearly, only the term $l = 0$, where no creation annihilator arrives safely to the left of $W_{\mathrm{F}}(f)$ will give a non-zero contribution to expression (131) (because $a_{\mathrm{F}}(f_V)\Omega_{\mathrm{F}} = 0$, once again), so

$$E_V(f) = \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}\frac{n!}{(n-k)!}\left(-\frac{1}{2}|\langle f_V, f\rangle|^2\right)^{n-k}\langle\Omega_{\mathrm{F}}, W_{\mathrm{F}}(f)\Omega_{\mathrm{F}}\rangle .$$

We denote the Fock vacuum generating functional by $E_{\mathrm{F}}(f) = \langle\Omega, W_{\mathrm{F}}(f)\Omega\rangle$ (see (119)) and observe that for f with compact support and large enough $|V|$, we have

$$\langle f_V, f\rangle = |V|^{-1/2}\int_V f(x)d^3x = \left((2\pi)^3\frac{\rho}{n}\right)^{1/2}\widehat{f}(0),$$

where

$$\widehat{f}(k) = \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3} d^3k\; e^{-ikx}f(x) \tag{133}$$

is the Fourier transform. Consequently we have

$$E_V(f) = E_{\mathrm{F}}(f)\;\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}\frac{n!}{(n-k)!}\left(-(2\pi)^3\frac{\rho}{2n}|\widehat{f}(0)|^2\right)^{n-k}. \tag{134}$$

We recall that the Laguerre polynomials are defined by

$$L_n(z) = \frac{1}{n!}e^z\frac{d^n}{dz^n}(e^{-z}z^n) = \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}\frac{n!}{(n-k)!}(-z)^{n-k} \tag{135}$$

for $n = 0, 1, \dots$ and $z \in \mathbb{C}$. Next, it is known that (see e.g. [AS], formula 22.15.2)

$$\lim_{n\to\infty} L_n(z/n) = J_0(2\sqrt{z}), \tag{136}$$

where the Bessel function J_0 satisfies

$$\int_{-\pi}^{\pi}\frac{d\theta}{2\pi}e^{-i(\alpha\cos\theta+\beta\sin\theta)} = J_0\left(\sqrt{\alpha^2+\beta^2}\right), \tag{137}$$

for any $\alpha, \beta \in \mathbb{R}$ (see e.g. [MF], formula (5.3.66)). In conclusion, we have calculated the infinite volume generating functional to be

$$E_{\mathrm{GS}}(f) = E_{\mathrm{F}}(f)J_0\left((2\pi)^{3/2}\sqrt{2\rho}\;|\widehat{f}(0)|\right). \tag{138}$$

This generating functional defines a state on CCR($\mathfrak{D}$) which we view as being the ground state of the infinite Bose gas. The test function space is given by

$$\mathfrak{D} = \left\{ f \in L^2(\mathbb{R}^3, d^3x) \mid \widehat{f}(0) \text{ exists } \right\}. \tag{139}$$

Any function in Schwarz space is contained in $\mathfrak{D}$, so $\mathfrak{D}$ is dense in $L^2(\mathbb{R}^3, d^3x)$.

Let us now construct the GNS representation of the infinite Bose gas. Consider the Hilbert space

$$\mathcal{H}_{\mathrm{GS}} = \mathcal{F}(L^2(\mathbb{R}^3, d^3x)) \otimes L^2\left(S^1, d\sigma\right), \tag{140}$$

where the left factor is the Bosonic Fock space over $L^2(\mathbb{R}^3, d^3x)$ and the right one is the Hilbert space of all square integrable functions on the unit circle with uniform measure. It is convenient to parametrize the circle by the angle $\theta \in [-\pi, \pi]$. Set

$$\Omega_{\mathrm{GS}} = \Omega_{\mathrm{F}} \otimes 1, \tag{141}$$

where $1 \in L^2\left(S^1, d\sigma\right)$ is the constant function. We define the representation map $\pi_{\mathrm{GS}} : \mathrm{CCR}(\mathfrak{D}) \to \mathcal{B}(\mathcal{H}_{\mathrm{GS}})$ as

$$\pi_{\mathrm{GS}} : W(f) \mapsto W_{\mathrm{F}}(f) \otimes e^{-i(2\pi)^{3/2}\sqrt{2\rho}\ \left(\mathrm{Re}\widehat{f}(0)\cos\theta + \mathrm{Im}\widehat{f}(0)\sin\theta\right)}. \tag{142}$$

Using relation (137) it is easily seen that for any $f \in \mathfrak{D}$ we have

$$\langle \Omega_{\mathrm{GS}}, \pi_{\mathrm{GS}}(W(f))\Omega_{\mathrm{GS}} \rangle = E_{\mathrm{GS}}(f) \tag{143}$$

so the representation gives the correct generating functional. To show that the GNS Hilbert space of $(\mathrm{CCR}(\mathfrak{D}), \omega_{\mathrm{GS}})$, where the state ω_{GS} is represented by Ω_{GS} in $\mathcal{H}_{\mathrm{GS}}$, is actually the entire $\mathcal{H}_{\mathrm{GS}}$, we need to verify that Ω_{GS} is cyclic for π_{GS}. To show this, define the family of functions

$$f_{z,s}(x) = \sqrt{\frac{\pi}{2}}\ sz \frac{e^{-s|x|}}{x^2+1},$$

where $x \in \mathbb{R}^3$, $z \in \mathbb{C}$ and $s > 0$. Clearly, $\|f_{z,s}\|_{L^2(\mathbb{R}^3,d^3x)} \to 0$ as $s \to 0_+$, while $\widehat{f}_{z,s}(0) \to z$ for $s \to 0_+$. Due to the strong continuity of the Weyl operators in Fock space, we have for any $z \in \mathbb{C}$

$$\pi_{\mathrm{GS}}(W(f_{z,s})) \to \mathbb{1} \otimes e^{-i(2\pi)^{3/2}\sqrt{2\rho}\ (\mathrm{Re}z\cos\theta + \mathrm{Im}z\sin\theta)},$$

in the strong sense on $\mathcal{H}_{\mathrm{GS}}$, as $s \to 0_+$. Since the constant function 1 is cyclic in $L^2\left(S^1, d\sigma\right)$ for the set of multiplication operators $\{e^{i(a\cos\theta + b\sin\theta)} \mid a, b \in \mathbb{R}\}$ [16] , we see that

$$\Omega_{\mathrm{F}} \otimes L^2\left(S^1, d\sigma\right)$$

is contained in the closure of

$$\pi_{\mathrm{GS}}\left(\mathrm{CCR}(\mathfrak{D})\right)\Omega_{\mathrm{GS}}.$$

Because Ω_{F} is cyclic for $\{W_{\mathrm{F}}(f) \mid f \in \mathfrak{D}\}$ in $\mathcal{F}(L^2(\mathbb{R}^3, d^3x))$ (we use here that $\mathfrak{D}$ is dense in $L^2(\mathbb{R}^3, d^3x)$) we can approximate arbitrarily well any given $\psi \in \mathcal{F}(L^2(\mathbb{R}^3, d^3x))$ by some $W_{\mathrm{F}}(f)\Omega_{\mathrm{F}}$, $f \in \mathfrak{D}$. It follows that for an appropriate choice of $z \in \mathbb{C}$ and for s small enough the vector $\psi \otimes 1 \in \mathcal{H}_{\mathrm{GS}}$ is approximated arbitrarily well by $\pi_{\mathrm{GS}}(W(f)W(f_{z,s}))\Omega_{\mathrm{GS}}$. This shows that Ω_{GS} is cyclic for π_{GS} in $\mathcal{H}_{\mathrm{GS}}$.

The representation $(\mathcal{H}_{\mathrm{GS}}, \pi_{\mathrm{GS}}, \Omega_{\mathrm{GS}})$ is a regular. The creation and annihilation operators are given by

$$\begin{aligned} a^*_{\mathrm{GS}}(f) &= a^*_{\mathrm{F}}(f) \otimes 1 - (2\pi)^{3/2}\sqrt{\rho}\ \widehat{f}(0)\ \mathbb{1} \otimes e^{-i\theta}, \\ a_{\mathrm{GS}}(f) &= a_{\mathrm{F}}(f) \otimes 1 - (2\pi)^{3/2}\sqrt{\rho}\ \overline{\widehat{f}(0)}\ \mathbb{1} \otimes e^{i\theta}. \end{aligned}$$

At zero density, $\rho = 0$, the ground state representation of the infinite Bose gas coincides with (is isomorphic to) the Fock representation.

4.3 Excited states

Our goal for this section is to extend the above method to construct the generating functional and the GNS representation corresponding to an (infinite volume) state of the CCR with a continuous momentum distribution $\rho(k)$.

Consider first the situation where, in our box V (as in the last section), we have n_j particles with momentum k_j, i.e., with wave function $f^j_V(x) = |V|^{-1/2}e^{ik_jx}$, where $j = 1, \ldots, p$, and where $|k_j|^2$ are (discrete) eigenvalues of the Laplacian (in the box $V \subset \mathbb{R}^3$ with periodic boundary conditions). The f^j_V are eigenfunctions of the Laplacian and satisfy the orthonormality condition $\left\langle f^j_V, f^l_V\right\rangle = \delta_{jl}$ (Kronecker symbol). We will let the box tend to all

[16] Any function $f \in L^2(S^1, d\sigma)$ has a Fourier series expansion.

of $\mathbb{R}^3$ with the result that in the limit, the values of k can range continuously throughout $\mathbb{R}^3$ (this reflects the fact that $-\Delta$ on $L^2(\mathbb{R}^3, d^3x)$ has purely absolutely continuous spectrum).

The state of the gas in the box with densities $\rho_j = n_j/|V|$ of particles with momenta k_j, $j = 1, \ldots, p$, is given by

$$\Psi_V = \frac{1}{\sqrt{n_1! \cdots n_p!}} a_{\mathrm{F}}^*(f_V^1)^{n_1} \cdots a_{\mathrm{F}}^*(f_V^p)^{n_p} \Omega_{\mathrm{F}},$$

and the corresponding generating functional

$$E_V(f) = \langle \Psi_V, W_{\mathrm{F}}(f) \Psi_V \rangle$$

can be calculated just as in the previous section. It is an easy exercise to obtain the expression

$$E_V(f) = E_{\mathrm{F}}(f) L_{n_1}\left((2\pi)^3 \frac{\rho_1}{2n_1} |\widehat{f}(k_1)|^2 \right) \cdots L_{n_p}\left((2\pi)^3 \frac{\rho_p}{2n_p} |\widehat{f}(k_p)|^2 \right),$$

where $L_n(z)$ are the Laguerre polynomials defined in (135). We have used that for any f with compact support,

$$\langle f_j, f \rangle = |V|^{-1/2} \int_V e^{-ik_j x} f(x) d^3x = \left(\frac{\rho_j}{n_j} \right)^{1/2} (2\pi)^{3/2} \widehat{f}(k_j),$$

for $|V|$ big enough and where $\widehat{f}$ is the Fourier transform (133). Using (136), we take the limits $n_j \to \infty$, $j = 1, \ldots, p$, while leaving ρ_j fixed. The infinite volume generating functional is

$$E(f) = E_{\mathrm{F}}(f) J_0 \left((2\pi)^{3/2} \sqrt{2\rho_1}\, |\widehat{f}(k_1)| \right) \cdots J_0 \left((2\pi)^{3/2} \sqrt{2\rho_p}\, |\widehat{f}(k_p)| \right). \tag{144}$$

Our next task is to let the discrete distribution ρ_j, $j = 1, \ldots, p$, tend to a continuous distribution $\rho(k)$. We do this for simplicity first in the one-dimensional case, $k \in \mathbb{R}$, and we will deduce the general formula afterwards. Let thus $\mathbb{R} \ni k \mapsto \rho(k) \in \mathbb{R}_+$ be a given momentum density and consider an interval $[-L, L]$. We partition $[-L, L]$ into small intervals with endpoints $k_j = -L + 2Lj/p$, $j = 0, \ldots, p$, each of length $2L/p$, and we will let $p \to \infty$.

The density ρ_j of particles having momenta in the interval with left endpoint k_j is given by $\rho_j = 2L\rho(k_j)/p$. Our goal is to calculate

$$\lim_{p\to\infty} \log\left(\frac{E(f)}{E_{\mathrm{F}}(f)}\right) = \lim_{p\to\infty} \sum_{j=0}^{p} \log J_0\left((2\pi)^{1/2}\sqrt{\frac{4L\rho(k_j)}{p}}\,|\widehat{f}(k_j)|\right). \tag{145}$$

Notice that the power of 2π is now $1/2$, in one dimension. A simple Taylor expansion shows that the leading term of $\log J_0(\epsilon)$ for small ϵ is $-\frac{\epsilon^2}{4}$, so that we have

$$\begin{aligned}&\lim_{p\to\infty} \log\left(\frac{E(f)}{E_{\mathrm{F}}(f)}\right)\\ &\qquad = -\frac{1}{2}2\pi \lim_{p\to\infty} \sum_{j=0}^{p} \frac{2L\rho(k_j)}{p}\,|\widehat{f}(k_j)|^2 = -\frac{1}{2}2\pi \int_{-L}^{L} dk\ \rho(k)|\widehat{f}(k)|^2.\end{aligned}$$

If we take f such that $\sqrt{\rho}\widehat{f}$ is square integrable then we can take $L \to \infty$ and obtain for the generating functional of the infinite Bose gas *in three dimensions* and with momentum density ρ

$$E_\rho(f) = E_{\mathrm{F}}(f) \exp\left\{-\frac{(2\pi)^3}{2}\int_{\mathbb{R}^3} d^3k\ \rho(k)|\widehat{f}(k)|^2\right\}. \tag{146}$$

The test function space consists of functions s.t. the integral in (146) exists,

$$\mathfrak{D}' = \left\{f \in L^2(\mathbb{R}^3, d^3x) \mid \sqrt{\rho}\widehat{f} \in L^2(\mathbb{R}^3, d^3k)\ \right\}, \tag{147}$$

it depends on the function ρ. It is convenient to pass to a representation of the one-particle Hilbert space where the energy operator is diagonal; in the case of the Laplacian this means that we consider $L^2(\mathbb{R}^3, d^3k)$, the Fourier-transformed position space $L^2(\mathbb{R}^3, d^3x)$. The Fourier transform is an isometric isomorphism between $L^2(\mathbb{R}^3, d^3x)$ and $L^2(\mathbb{R}^3, d^3k)$ which induces a C^*algebra isomorphism between $\mathrm{CCR}(\mathfrak{D}')$ and $\mathrm{CCR}(\mathfrak{D})$, where

$$\mathfrak{D} = \left\{f \in L^2(\mathbb{R}^3, d^3k) \mid \sqrt{\rho} f \in L^2(\mathbb{R}^3, d^3k)\ \right\}, \tag{148}$$

and the corresponding generating functional is given by

$$E_\rho(f) = E_{\mathrm{F}}(f) \exp\left\{-\frac{(2\pi)^3}{2}\int_{\mathbb{R}^3} d^3k\ \rho(k)|f(k)|^2\right\},\quad f \in \mathfrak{D}, \tag{149}$$

where $E_{\mathrm{F}}(f)=e^{-\|f\|^2/4}$, for $f\in\mathfrak{D}$ (see (120)). Hence

$$E_\rho(f)=\exp\left\{-\frac{1}{4}\left\langle f,(1+16\pi^3\rho)f\right\rangle\right\},\quad f\in\mathfrak{D}. \tag{150}$$

One can carry out the construction for a general selfadjoint Hamiltonian H (not necessarily of the form (122)) and one arrives at (146) where $\widehat{f}$ stands for the eigenfunction expansion of f corresponding to H.

Formula (150) gives a generating functional which defines a state ω_ρ on $\mathrm{CCR}(\mathfrak{D})$, according to Theorem 4.1. We give now the GNS representation of $(\mathrm{CCR}(\mathfrak{D}),\omega_\rho)$ for densities $\rho(k)$ such that

$$k\mapsto\rho(k)\text{ is continuous},\quad \rho(k)>0\text{ a.e.},\quad \int_{\mathbb{R}^3}d^3k\ \rho(k)<\infty. \tag{151}$$

The representation Hilbert space $\mathcal{H}_\rho$ and the cyclic vector Ω_ρ are

$$\begin{aligned}\mathcal{H}_\rho &= \mathcal{F}(L^2(\mathbb{R}^3,d^3k))\otimes\mathcal{F}(L^2(\mathbb{R}^3,d^3k)) &(152)\\ \Omega_\rho &= \Omega_{\mathrm{F}}\otimes\Omega_{\mathrm{F}}, &(153)\end{aligned}$$

where $\mathcal{F}(L^2(\mathbb{R}^3,d^3k))$ is the Bosonic Fock space over the one-particle space $L^2(\mathbb{R}^3,d^3k)$ and Ω_{F} is the vacuum therein. The representation map π_ρ : $\mathrm{CCR}(\mathfrak{D})\to\mathcal{B}(\mathcal{H}_\rho)$ is given by

$$\begin{aligned}\pi_\rho(W(f)) &= W_{\mathrm{F}}\left(\sqrt{1+\mu}\,f\right)\otimes W_{\mathrm{F}}\left(\sqrt{\mu}\,\overline{f}\,\right), &(154)\\ \mu(k) &= 8\pi^3\rho(k). &(155)\end{aligned}$$

Notice that in the Weyl operator on the right factor there appears the complex conjugate of f. Using expression (120) it is an easy matter to verify that

$$\langle\Omega_\rho,\pi_\rho(W(f))\Omega_\rho\rangle=E_\rho(f), \tag{156}$$

where $E_\rho(f)$ is given by (150). π_ρ is a regular representation and the creation and annihilation operators are given by

$$\begin{aligned}a_\rho^*(f) &= a_{\mathrm{F}}^*\left(\sqrt{1+\mu}\,f\right)\otimes\mathbb{1}+\mathbb{1}\otimes a_{\mathrm{F}}\left(\sqrt{\mu}\,\overline{f}\,\right), &(157)\\ a_\rho(f) &= a_{\mathrm{F}}\left(\sqrt{1+\mu}f\right)\otimes\mathbb{1}+\mathbb{1}\otimes a_{\mathrm{F}}^*\left(\sqrt{\mu}\,\overline{f}\,\right). &(158)\end{aligned}$$

Since the $a_\rho^\#(f)$ are obtained from the represented Weyl operators by strong differentiation it follows that Ω_ρ is cyclic for π_ρ if Ω_ρ is cyclic for the polynomial algebra $\mathfrak{P}$ generated by all creation and annihilation operators $a_\rho^\#(f)$, $f \in \mathfrak{D}$. The set $\{\sqrt{\mu} f \mid f \in \mathfrak{D}\}$ is dense in $L^2(\mathbb{R}^3, d^3k)$ due to condition (151). Since Ω_{F} is cyclic for the Fock creation operators it follows from (158) that $\Omega_{\mathrm{F}} \otimes \mathcal{F}(L^2(\mathbb{R}^3, d^3k))$ lies in the closure of $\mathfrak{P}\Omega_\rho$. Similarly (157) shows that $\mathcal{F}(L^2(\mathbb{R}^3, d^3k)) \otimes \Omega_{\mathrm{F}}$ is in that closure. Hence Ω_ρ is cyclic for π_ρ.

If $\rho(k) = 0$ then $\mu(k) = 0$ and the representation (154) reduces to the Fock representation.

4.4 Equilibrium states

The results of the previous two sections can be combined to describe the infinitely extended free Bose gas with a momentum density distribution which has some condensate part characterized by the density $\rho_0 \in \mathbb{R}_+$ and some continuous part given by $\rho(k)$. The corresponding generating functional is

$$\begin{aligned} &E_{\rho_0,\rho}(f) \\ &\quad = E_{\mathrm{F}}(f) \exp\left\{-\frac{(2\pi)^3}{2}\int_{\mathbb{R}^3} d^3k\ \rho(k)|f(k)|^2\right\} J_0\left((2\pi)^{3/2}\sqrt{2\rho_0}\ |f(0)|\right), \end{aligned} \tag{159}$$

compare with (138) and (149), for functions f in the test function space

$$\mathfrak{D} = \left\{f \in L^2(\mathbb{R}^3, d^3k) \mid \sqrt{\rho} f \in L^2(\mathbb{R}^3, d^3k),\ |f(0)| < \infty\right\}. \tag{160}$$

The GNS representation Hilbert space $\mathcal{H}_{\rho_0,\rho}$ and the cyclic vector $\Omega_{\rho_0,\rho}$ associated to $(\mathrm{CCR}(\mathfrak{D}), \omega_{\rho_0,\rho})$, where $\omega_{\rho_0,\rho}$ is the state defined by (159), are given by

$$\begin{aligned} \mathcal{H}_{\rho_0,\rho} &= \mathcal{F}(L^2(\mathbb{R}^3, d^3k)) \otimes \mathcal{F}(L^2(\mathbb{R}^3, d^3k)) \otimes L^2\left(S^1, d\sigma\right), \\ \Omega_{\rho_0,\rho} &= \Omega_{\mathrm{F}} \otimes \Omega_{\mathrm{F}} \otimes 1, \end{aligned} \tag{161}$$

and the representation map is

$$\pi_{\rho_0,\rho}(W(f)) = W_{\mathrm{F}}(\sqrt{1+\mu}\, f) \otimes W_{\mathrm{F}}(\sqrt{\mu}\, \overline{f}\,) \otimes e^{-i\Phi(f,\theta)}, \tag{162}$$

with $\mu(k) = 8\pi^3\rho(k)$, and where we introduce the phase

$$\Phi(f,\theta) = (2\pi)^{3/2}\sqrt{2\rho_0}\ \left(\mathrm{Re}f(0)\cos\theta + \mathrm{Im}f(0)\sin\theta\right). \tag{163}$$

Note that Φ is real linear in the first argument. The creation and annihilation operators in this regular representation are not difficult to calculate:

$$\begin{aligned} a^*_{\rho_0,\rho}(f) &= a^*_{\mathrm{F}}\left(\sqrt{1+\mu}\, f\right) \otimes \mathbb{1} \otimes 1 + \mathbb{1} \otimes a_{\mathrm{F}}\left(\sqrt{\mu}\,\overline{f}\,\right) \otimes 1 \\ &\quad -(2\pi)^{3/2}\sqrt{\rho_0}\ f(0)\ \mathbb{1} \otimes \mathbb{1} \otimes e^{-i\theta}, \qquad (164) \\ a_{\rho_0,\rho}(f) &= a_{\mathrm{F}}\left(\sqrt{1+\mu} f\right) \otimes \mathbb{1} \otimes 1 + \mathbb{1} \otimes a^*_{\mathrm{F}}\left(\sqrt{\mu}\,\overline{f}\,\right) \otimes 1 \\ &\quad -(2\pi)^{3/2}\sqrt{\rho_0}\ \overline{f(0)}\ \mathbb{1} \otimes \mathbb{1} \otimes e^{i\theta}. \qquad (165) \end{aligned}$$

The dynamics on $\mathrm{CCR}(\mathfrak{D})$ generated by the Hamiltonian (122) is given by

$$W(f) \mapsto \alpha_t(W(f)) = W(e^{it\omega} f), \tag{166}$$

where $\omega(k) = |k|^2$. It is clear from (159) that $E_{\rho_0,\rho}(e^{it\omega} f) = E_{\rho_0,\rho}(f)$, for all $t \in \mathbb{R}$. Consequently $\omega_{\rho_0,\rho}$ is a time translation invariant i.e. stationary state, for any choice of $\rho_0, \rho(k)$. We wish to examine which particular momentum density distributions correspond to *equilibrium states* of the system.

We have

$$W_{\mathrm{F}}(e^{it\omega} f) = e^{itH} W_{\mathrm{F}}(f) e^{-itH}, \tag{167}$$

where H is the free field Hamiltonian (in Fock space) given by the second quantization of the multiplication by $\omega(k)$. It is easy to see from (162) and (167) that the dynamics (166) is unitarily implemented as

$$\pi_{\rho_0,\rho}(\alpha_t(W(f)) = e^{itL} \pi_{\rho_0,\rho}(W(f)) e^{-itL}, \tag{168}$$

where L is the so-called Liouvillian, given by

$$L = H \otimes \mathbb{1} \otimes 1 - \mathbb{1} \otimes H \otimes 1. \tag{169}$$

An equilibrium state ω is a state that satisfies the KMS condition

$$\omega(A\alpha_t(B)) = \omega(\alpha_{t-i\beta}(B)A), \tag{170}$$

see also (96). We assume here that B is such that $\alpha_z(B)$ exists for values of z in a strip around the real axis. Since an equilibrium state is necessarily α_t–invariant, (170) is equivalent to $\omega(AB) = \omega(\alpha_{-i\beta}(B)A)$. It is evident from the explicit form of $\Omega_{\rho_0,\rho}$ that $\omega_{\rho_0,\rho}$ can be extended to the polynomial algebra generated by the creation and annihilation operators (164), (165), giving a gauge-invariant quasifree state (see after (105)). To see which densities give an equilibrium state it is thus necessary and sufficient to solve the equation

$$\left\langle \Omega_{\rho_0,\rho}, a^*_{\rho_0,\rho}(f) a_{\rho_0,\rho}(g) \Omega_{\rho_0,\rho} \right\rangle = \left\langle \Omega_{\rho_0,\rho}, e^{\beta L} a_{\rho_0,\rho}(g) e^{-\beta L} a^*_{\rho_0,\rho}(f) \Omega_{\rho_0,\rho} \right\rangle, \quad (171)$$

which should hold for all f, g, for ρ_0 and $\rho(k)$. We calculate

$$\left\langle \Omega_{\rho_0,\rho}, a^*_{\rho_0,\rho}(f) a_{\rho_0,\rho}(g) \Omega_{\rho_0,\rho} \right\rangle = \langle g, \mu f \rangle + (2\pi)^3 \rho_0 \, f(0) \overline{g(0)}$$

and

$$\left\langle \Omega_{\rho_0,\rho}, e^{\beta L} a_{\rho_0,\rho}(g) e^{-\beta L} a^*_{\rho_0,\rho}(f) \Omega_{\rho_0,\rho} \right\rangle = \left\langle g, e^{-\beta\omega}(1+\mu) f \right\rangle + (2\pi)^3 \rho_0 \, f(0) \overline{g(0)}.$$

Consequently $\rho_0 \geq 0$ can be arbitrary and μ must satisfy $\mu = e^{-\beta\omega}(1+\mu)$, i.e., $\mu(k) = \frac{1}{e^{\beta\omega(k)}-1}$. The density is thus given by (see (155))

$$\rho(k) = (2\pi)^{-3} \frac{1}{e^{\beta\omega(k)} - 1}, \quad (172)$$

which is the Planck distribution of black body radiation.

Let us focus on massless relativistic Bosons, where $\omega(k) = |k|$. Other dispersion relations are discussed in an analogous way. The total density of particles in the state of equilibrium is

$$\rho_{\text{tot}} = \rho_0 + \int_{\mathbb{R}^3} d^3k \rho(k) = \rho_0 + \frac{1}{8\pi^3} \int_{\mathbb{R}^3} \frac{d^3k}{e^{\beta|k|} - 1} = \rho_0 + \frac{c}{\beta^3}, \quad (173)$$

where $c = \frac{1}{2\pi^2} \int_0^\infty \frac{s^2}{e^s - 1} ds$ is a fixed constant. We can deduce from (173) the following qualitative behaviour of the system. Suppose ρ_{tot} is fixed and suppose we decrease the temperature of the system ($\beta \to \infty$). Then ρ_0 tends to ρ_{tot} which means that for low temperatures the system likes to form a condensate. If we fix an inverse temperature β and increase the total density ρ_{tot} of the system then ρ_0 increases as well. These considerations show that we are likely to observe a condensate if either the temperature is low or the density is high (this, of course, is also an experimental fact).

We close this section with a result about the thermodynamic limit of Gibbs states which is due to Cannon, [C]. Fix an inverse temperature $0 < \beta < \infty$ and define the *critical density* by

$$\rho_{\text{crit}}(\beta) = (2\pi)^{-3} \int_{\mathbb{R}^3} \frac{d^3k}{e^{\beta\omega} - 1}, \quad (174)$$

which coincides with the total density (173) in the equilibrium state for $\rho_0 = 0$. Let V be the box defined by $-L/2 \le x_j \le L/2$ ($j = 1, 2, 3$) and define the canonical state at inverse temperature β and density ρ_{tot} by

$$\langle A\rangle^{\mathrm{c}}_{\beta,\rho_{\mathrm{tot}},V} = \frac{\mathrm{tr} A P_{\rho_{\mathrm{tot}}V} e^{-\beta H_V}}{\mathrm{tr} P_{\rho_{\mathrm{tot}}V} e^{-\beta H_V}}, \tag{175}$$

where the trace is over Fock space over $L^2(V, d^3x)$, $P_{\rho_{\mathrm{tot}}V}$ is the projection onto the subspace of Fock space with $\rho_{\mathrm{tot}}V$ particles (if $\rho_{\mathrm{tot}}V$ is not an integer take a convex combination of canonical states with integer values $\rho_1 V$ and $\rho_2 V$ extrapolating $\rho_{\mathrm{tot}}V$). The Hamiltonian H_V is negative the Laplacian with periodic boundary conditions. The observable A in (175) belongs to the Weyl algebra over the test function space C_0^∞, realized as a C^*algebra acting on Fock space. Cannon shows that for any $\beta, \rho_{\mathrm{tot}} > 0$ and $f \in C_0^\infty$,

$$\langle W(f)\rangle^{\mathrm{c}}_{\beta,\rho_{\mathrm{tot}},V} \longrightarrow \left\{ \begin{array}{ll} e^{-\frac{1}{4}\|f\|^2} e^{-\frac{1}{2}\left\langle f, \frac{z_\infty}{e^{\beta\omega}-z_\infty} f\right\rangle}, & \rho_{\mathrm{tot}} \le \rho_{\mathrm{crit}}(\beta) \\ E_{\rho_0,\rho}(f), & \rho_{\mathrm{tot}} \ge \rho_{\mathrm{crit}}(\beta) \end{array} \right. \tag{176}$$

for any sequence $L \to \infty$. Here, $z_\infty \in [0, 1]$ is such that for subcritical density, the momentum density distribution of the gas is given by

$$\rho(k) = (2\pi)^{-3} \frac{z_\infty}{e^{\beta\omega} - z_\infty}, \tag{177}$$

so that z_∞ is the solution of

$$\rho_{\mathrm{tot}} = (2\pi)^{-3} \int \frac{z}{e^{\beta\omega} - z} d^3k. \tag{178}$$

The generating functional $E_{\rho_0,\rho}$ in (176) is the one obtained by Araki and Woods, (159), where ρ is the continuous momentum density distribution prescribed by Planck's law of black body radiation (172), and where

$$\rho_0 = \rho_{\mathrm{tot}} - \rho_{\mathrm{crit}}. \tag{179}$$

This gives the following picture for equilibrium states: if the system has density $\rho_{\mathrm{tot}} \le \rho_{\mathrm{crit}}$ then the particle momentum distribution of the equilibrium state is purely continuous, meaning that below critical density there is no condensate. As ρ_{tot} increases and surpasses the critical value, $\rho_{\mathrm{tot}} > \rho_{\mathrm{crit}}$, the "excess" particles form a condensate which is immersed in a gas of particles radiating according to Planck's law.

Finally we mention the work [LP] which treats the thermodynamic limit for the grand-canonical ensemble.

4.5 Dynamical stability of equilibria

Take the infinitely extended Bose gas initially in a state which differs from the equilibrium state at a given temperature only inside a bounded region of space. As time goes on we expect the local perturbation to spread out and propagate off to spatial infinity. This property, sometimes called the property of return to equilibrium, is a priori not built into the definition of equilibrium states, i.e., the KMS condition, but it has to be verified "by hand". In this section we investigate the large time limit of initial states which are local perturbations of an equilibrium state.

Let us first describe sates which are *local perturbations* of a given state ω of the infinitely extended Bose gas. Let $f \in \mathfrak{D} \subset L^2(\mathbb{R}^3, d^3x)$ be a test function which is supported in a compact region $\Lambda_0 \subset \mathbb{R}^3$. If g is supported in the complement $\mathbb{R}^3 \backslash \Lambda_0$ then we have

$$W(f)W(g) = e^{-i\mathrm{Im}\langle f,g\rangle} W(g)W(f) = W(g)W(f). \tag{180}$$

Consequently the state $A \mapsto \omega'(A) := \omega(W(f)^* A W(f))$ does not differ from the state ω on observables supported away from Λ_0 (i.e., on observables $A = \sum_{j=1}^{n} z_j W(f_j)$, where the f_j are supported away from Λ_0). The state ω' is a local perturbation of ω. More generally, if B is an observable (an element of the Weyl algebra) we say the state

$$\omega_B(\cdot) := \frac{\omega(B^* \cdot B)}{\omega(B^* B)} \tag{181}$$

is a local perturbation of ω. The set of all local perturbations of ω is defined to be the set of all convex combinations of states of the form (181). The dynamical stability of an equilibrium state ω_β (w.r.t. the dynamics α_t) is expressed as

$$\lim_{t \to \infty} \omega_B(\alpha_t(A)) = \omega_\beta(A), \tag{182}$$

for all observables A, B.

We start our investigation of return to equilibrium by some purely algebraic considerations. Let A be an element in the Weyl algebra $\mathrm{CCR}(\mathfrak{D})$. Given any ϵ there are complex numbers z_j and test functions $f_j \in \mathfrak{D}$ s.t.

$$\left\| A - \sum_{j=1}^{n} z_j W(f_j) \right\| = \left\| \alpha_t(A) - \sum_{j=1}^{n} z_j W(e^{i\omega t} f_j) \right\| < \epsilon, \tag{183}$$

where we use the fact that α_t is an isometry. Let g be fixed. We have

$$\left\| W(g)^* \left(\alpha_t(A) - \sum_{j=1}^{n} z_j W(e^{i\omega t} f_j) \right) W(g) \right\| \\ = \left\| \alpha_t(A) - \sum_{j=1}^{n} z_j W(e^{i\omega t} f_j) \right\| < \epsilon,$$

and the l.h.s. equals

$$\left\| W(g)^* \alpha_t(A) W(g) - \sum_{j=1}^{n} z_j e^{-\frac{i}{2}\mathrm{Im}[\langle -g, e^{i\omega t} f_j\rangle + \langle e^{i\omega t} f_j, g\rangle]} W(e^{i\omega t} f_j) \right\| < \epsilon. \quad (184)$$

Since $\lim_{t\to\infty} \langle -g, e^{i\omega t} f_j\rangle + \langle e^{i\omega t} f_j, g\rangle = 0$ (this follows from the Riemann-Lebesgue Lemma) there is a number $T_0(\epsilon) < \infty$ s.t. if $t > T_0$ then

$$\left\| \sum_{j=1}^{n} z_j e^{-\frac{i}{2}\mathrm{Im}[\langle -g, e^{i\omega t} f_j\rangle + \langle e^{i\omega t} f_j, g\rangle]} W(e^{i\omega t} f_j) - \sum_{j=1}^{n} z_j \alpha_t(W(f_j)) \right\| < \epsilon. \quad (185)$$

It follows from (183), (184) and (185) that $\|W(g)^* \alpha_t(A) W(g) - \alpha_t(A)\| < 2\epsilon$, for $t > T_0(\epsilon)$, and consequently

$$\lim_{t\to\infty} \|W(g)^* \alpha_t(A) W(g) - \alpha_t(A)\| = 0, \quad (186)$$

for all observables A and all test functions g. Relation (186), which merely involves observables and the dynamics (and no reference to any state is made), is a form of *asymptotic abelianness* w.r.t. α_t. In fact, it follows easily from (186) and (183) that

$$\lim_{t\to\infty} \|B\alpha_t(A) - \alpha_t(A)B\| = 0, \quad (187)$$

for any observables $A, B \in \mathrm{CCR}(\mathfrak{D})$.

If ω is any α_t–invariant state then (186) shows that

$$\lim_{t\to\infty} \omega(W(g)^* \alpha_t(A) W(g)) = \omega(A). \quad (188)$$

To prove that (188) holds if ω is an equilibrium state, and for $W(g)$ replaced by *any* observable B, i.e. to show return to equilibrium as in (182), we need to use special properties of equilibrium states. The property of asymptotic abelianness, (187), does not suffice.

Let ω_β denote an equilibrium state of the free Bose gas with a continuous density (172) and with a fixed condensate density $\rho_0 \geq 0$. The expectation functional is given by (159). We have

$$\omega_\beta(W(g)W(e^{i\omega t}f)W(h)) = e^{-\frac{i}{2}\mathrm{Im}[\langle g, e^{i\omega t}f\rangle + \langle g+e^{i\omega t}f, h\rangle]}\omega_\beta(W(g+e^{i\omega t}f+h))$$

and (using again the Riemann-Lebesgue Lemma)

$$\begin{aligned}
&\lim_{t\to\infty}\omega_\beta(W(g)W(e^{i\omega t}f)W(h)) \\
&= e^{-\frac{i}{2}\mathrm{Im}\langle g,h\rangle}E_{\mathrm{F}}(g+h)\exp\left\{-\frac{(2\pi)^3}{2}\|\sqrt{\rho}(g+h)\|^2\right\} \\
&\quad \times E_{\mathrm{F}}(f)\exp\left\{-\frac{(2\pi)^3}{2}\|\sqrt{\rho}f\|^2\right\} \\
&\quad \times J_0\left((2\pi)^{3/2}\sqrt{2\rho_0}\,|g(0)+f(0)+h(0)|\right). \qquad (189)
\end{aligned}$$

In the absence of a condensate, $\rho_0 = 0$, $J_0(0) = 1$, equation (189) is just

$$\lim_{t\to\infty}\omega_\beta(W(g)W(e^{i\omega t}f)W(h)) = \omega_\beta(W(g)W(h))\ \omega_\beta(W(f)). \qquad (190)$$

Using an easy approximation argument (as in (183)), this yields the property of return to equilibrium ($\rho_0 = 0$),

$$\lim_{t\to\infty}\omega_\beta(B\alpha_t(A)C) = \omega_\beta(BC)\omega_\beta(A), \qquad (191)$$

for any $A, B, C \in \mathrm{CCR}(\mathfrak{D})$.

What happens in presence of a condensate, $\rho_0 > 0$? Formula (191) is *not* valid in this case, because the Bessel function in (189) does not split into a product. However, the integrand in the representation (137) of J_0 does split into a product according to

$$\begin{aligned}
&J_0\left((2\pi)^{3/2}\sqrt{2\rho_0}\,|g(0)+f(0)+h(0)|\right) \\
&= \int_{-\pi}^{\pi}\frac{d\theta}{2\pi}\exp -i(2\pi)^{3/2}\sqrt{2\rho_0}[\mathrm{Re}(g(0)+h(0))\cos\theta + \mathrm{Im}(g(0)+h(0))\sin\theta] \\
&\qquad \times\exp -i(2\pi)^{3/2}\sqrt{2\rho_0}[\mathrm{Re}(f(0))\cos\theta + \mathrm{Im}(f(0))\sin\theta]. \qquad (192)
\end{aligned}$$

This suggests that for an equilibrium state with a condensate and a fixed value of θ, the property of return to equilibrium holds. To cast this into a precise form we write

$$\omega_\beta(W(f)) = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi}\, \omega_\beta^\theta(W(f)), \tag{193}$$

where

$$\omega_\beta^\theta(W(f)) = e^{-i\Phi(f,\theta)} \left\langle \Omega_\rho, \pi_\rho(W(f))\Omega_\rho \right\rangle, \tag{194}$$

with Ω_ρ given in (153) and π_ρ defined in (154), (155). This decomposition is in accordance with the decomposition of the Hilbert space into a direct integral,

$$\mathcal{H}_{\rho_0,\rho} = \int_{S^1}^{\oplus} d\sigma\ \mathcal{H}_\rho, \tag{195}$$

see (152), (161). The GNS representation of $(\mathrm{CCR}(\mathfrak{D}), \omega_\beta^\theta)$ is given by $(\mathcal{H}_\rho, \pi_\beta^\theta, \Omega_\rho)$, where

$$\pi_\beta^\theta(W(f)) = e^{-i\Phi(f,\theta)} \pi_\rho(W(f)). \tag{196}$$

The representation map π_β associated to the state ω_β is the direct integral of the fibers π_β^θ, and the representation vector Ω_β of ω_β is the direct integral with constant fiber Ω_ρ:

$$\pi_\beta = \int_{[-\pi,\pi]}^{\oplus} \frac{d\theta}{2\pi}\ \pi_\beta^\theta, \quad \Omega_\beta = \int_{[-\pi,\pi]}^{\oplus} \frac{d\theta}{2\pi}\ \Omega_\rho. \tag{197}$$

It is clear from (194) that for each θ fixed, ω_β^θ is a (α_t, β)-KMS state. The (α_t, β)-KMS state ω_β is a uniform superposition of the ω_β^θ, $\theta \in S^1$. We can form other equilibrium states by taking different superpositions of the ω_β^θ: Given any probability measure $d\mu$ on $[-\pi, \pi]$,

$$\omega_\mu(W(f)) := \int_{-\pi}^{\pi} d\mu(\theta)\ \omega_\beta^\theta(W(f)) \tag{198}$$

is an (α_t, β)-KMS state. As follows directly from (191) and (194), for each fixed θ we have $\lim_{t\to\infty} \omega_\beta^\theta(B\alpha_t(A)C) = \omega_\beta^\theta(BC)\omega_\beta^\theta(A)$, so

$$\lim_{t\to\infty} \omega_\mu(B\alpha_t(A)C) = \int_{-\pi}^{\pi} d\mu(\theta)\ \omega_\beta^\theta(BC)\, \omega_\beta^\theta(A). \tag{199}$$

In general, the r.h.s. of (199) does *not* equal ω_μ, so the time-asymptotic state depends on the initial state. If the perturbation of the state ω_μ is given by B, C s.t. $\omega_\beta^\theta(BC) = 1$ for all θ then we have return to ω_μ in the usual sense.

What is special about the equilibrium states ω_β^θ? They are *factorial*[17] equilibrium states. The fact that each ω_β^θ is factorial follows from $\mathfrak{M} := \pi_{\omega_\beta^\theta}(\mathcal{A})'' = \mathcal{B}(\mathcal{F}(L^2(\mathbb{R}^3, d^3k))) \otimes \mathbb{1}$, $\mathfrak{M}' = \mathbb{1} \otimes \mathcal{B}(\mathcal{F}(L^2(\mathbb{R}^3, d^3k)))$, hence $\mathfrak{M} \cap \mathfrak{M}' = \mathbb{C}\mathbb{1} \otimes \mathbb{1}$. On the other hand, it is clear that ω_β, (193), is not factorial since $\mathbb{1} \otimes \mathbb{1} \otimes \mathcal{M}$ (denoting the multiplication operators on $L^2(S^1, d\sigma)$) belongs to the center of its von Neumann algebra, see (162). The decomposition (195)–(197) is called a *factor decomposition* of the state ω_β, or a decomposition into extremal states.

Let us now see how the emergence of a multitude of equilibrium states for a fixed inverse temperature β can be viewed as a consequence of *spontaneous symmetry breaking* – here, the gauge group symmetry is broken. The general scheme is this: suppose a dynamics α_t on a C^*-algebra $\mathfrak{A}$ has a symmetry group σ_s, i.e. σ_s is a group of automorphisms of $\mathfrak{A}$ satisfying $\sigma_s \circ \alpha_t = \alpha_t \circ \sigma_s$, for all s, t (s may belong to a discrete or continuous set, $t \in \mathbb{R}$). If ω is any (β, α_t)-KMS state then $\omega_s := \omega \circ \sigma_s$ is a (β, α_t)-KMS state as well:

$$\begin{aligned}\omega_s(A\alpha_t(B)) &= \omega(\sigma_s(A)\alpha_t(\sigma_s(B))) \\ &= \omega(\alpha_{t-i\beta}(\sigma_s(B))\sigma_s(A)) = \omega_s(\alpha_{t-i\beta}(B)A).\end{aligned} \tag{200}$$

(We implicitly assume that $\alpha_{t-i\beta}(\sigma_s(B))$ is well defined.) If there is a (β, α_t)-KMS state which is not invariant under σ_s for some s, i.e., $\omega \circ \sigma_s \neq \omega$, then we say the symmetry σ_s is spontaneously broken, because there are equilibrium states which “have less symmetry” than the dynamics. This gives rise to the existence of several equilibrium states at the same temperature.

Consider the continuous symmetry group σ_s on $\mathrm{CCR}(\mathfrak{D})$ given by

$$\sigma_s(W(f)) = W(e^{is}f), \quad s \in \mathbb{R}, f \in \mathfrak{D}, \tag{201}$$

[17] A state ω on a C^*–algebra $\mathfrak{A}$ is called factorial iff the von Neumann algebra $\pi_\omega(\mathfrak{A})''$ is a factor. (Here, π_ω is the GNS representation map associated to $(\mathfrak{A}, \omega)$.) A von Neumann algebra $\mathfrak{M} \subset \mathcal{B}(\mathcal{H})$ is a factor iff its center is trivial, $\mathfrak{Z} := \mathfrak{M} \cap \mathfrak{M}' = \mathbb{C}\mathbb{1}$.

We point out that it follows from general considerations that an equilibrium state is factorial iff it is *extremal* (see [BRII, Theorem 5.3.30]). A state ω is called extremal iff the relation $\{\omega = \lambda\omega_1 + (1-\lambda)\omega_2$, for some $0 < \lambda < 1$ and some states $\omega_1, \omega_2\}$ implies that $\omega_1 = \omega_2 = \omega$.

called the gauge group. Clearly we have $\alpha_t \circ \sigma_s = \sigma_s \circ \alpha_t$ (where α_t is given in (166)). Using (194) we obtain

$$\omega_\beta^\theta(\sigma_s(W(f))) = e^{-i\Phi(e^{is}f,\theta)} \left\langle \Omega_\rho, \pi_\rho(W(e^{is}f))\Omega_\rho \right\rangle, \tag{202}$$

and (150), (156) show that $\langle \Omega_\rho, \pi_\rho(W(e^{is}f))\Omega_\rho\rangle = \langle \Omega_\rho, \pi_\rho(W(f))\Omega_\rho\rangle$, while (163) gives

$$\begin{aligned} \Phi(e^{is}f,\theta) &= (2\pi)^{3/2}\sqrt{2\rho_0}\ \big(\mathrm{Re}(e^{is}f(0))\cos\theta + \mathrm{Im}(e^{is}f(0))\sin\theta\big) \\ &= (2\pi)^{3/2}\sqrt{2\rho_0}\ (\mathrm{Re}(f(0))\cos(\theta - s) + \mathrm{Im}(f(0))\sin(\theta - s)) \\ &= \Phi(f,\theta - s). \end{aligned} \tag{203}$$

This shows that the equilibrium states ω_β^θ break the gauge group symmetry, hence giving rise to an S^1-multitute of equilibrium states ((203) shows that we get the whole family ω_β^θ by varying s in any interval of length 2π).

Let us finally examine the *mixing properties* of the equilibrium states with respect to space translations. Given a vector $a \in \mathbb{R}^3$ we define

$$\tau_a(W(f)) := W(f_a), \tag{204}$$

where $f_a(x) := f(x - a)$ is the translate of f by a. τ_a defines a (three parameter) group of automorphisms on $\mathrm{CCR}(\mathfrak{D})$. We say that a state ω on $\mathrm{CCR}(\mathfrak{D})$ has the property of *strong mixing* w.r.t. space translations if

$$\lim_{|a|\to\infty} \omega(W(f)\tau_a(W(g))) = \omega(W(f))\omega(W(g)), \tag{205}$$

for any $f, g \in \mathfrak{D}$. This means that if two observables ($W(f)$ and $W(g)$) are spatially separated far from each other then the expectation of the product of the observables is close to the product of the expectations (independence of random variables). Intuitively, this means that the state ω has a certain property of locality in space: what happens far out in space does not influence events taking place, say, around the origin. Condition (205) is also called a *cluster property*. It is easy to calculate explicitly the l.h.s. of (205) for $\omega = \omega_\beta$, the equilibrium state of the free Bose gas with a continuous density (172) and with a fixed condensate density $\rho_0 \geq 0$ (whose expectation functional is given by (159)):

$$\begin{aligned} &\lim_{|a|\to\infty} \omega_\beta\big(W(f)\tau_a(W(g))\big) \\ &\qquad = \omega_\beta(W(f))\omega_\beta(W(g)) \exp\left[-8\pi^3\rho_0\, \mathrm{Re}(\overline{f(0)}g(0))\right]. \end{aligned} \tag{206}$$

Consequently, the equilibrium state is strongly mixing w.r.t. space translations if and only if $\rho_0 = 0$, i.e., if and only if there is no condensation. In presence of a condensate, the system exhibits *long range correlations* (what happens far out does influence what happens say at the origin). On the other hand, it is easily verified that each state ω_β^θ is strongly mixing. We may understand that limit states ($\lim_{t\to\infty}$ of states of the form (198)) depend on the initial state as a consequence of the long-range correlations in presence of a condensate. Even as time reaches its asymptotic value the system "remembers" the initial state.

References

[A] Araki, H.: *Hamiltonian Formalism and the Canonical Commutation Relations in Quantum Field Theory.* J. Math. Phys., **1**, No. 6, 492-504 (1960)

[AS] Abramowitz, A., Stegun, I. A.: *Handbook of mathematical functions.* Dover Publications, Inc., New York, Ninth Printing, 1970

[AW] Araki, H., Woods, E.: *Representations of the canonical commutation relations describing a non-relativistic infinite free bose gas.* J. Math. Phys. **4**, 637-662 (1963)

[BRI, II] Bratteli, O., Robinson, D.: *Operator Algebras and Quantum Statistical Mechanics I, II.* Texts and Monographs in Physics, Berlin: Springer-Verlag, 2nd edition, 1987

[C] Cannon, J.T.: *Infinite Volume Limits of the Canonical Free Bose Gas States on the Weyl Algebra.* Comm. Math. Phys., **29**, 89-104 (1973)

[GJ] Glimm, J., Jaffe, A.: *Quantum Physics. A functional integral point of view.* Springer-Verlag, New York, second edition, 1981

[H] Hida, T.: *Brownian Motion.* Applications of Mathematics 11, Springer-Verlag 1980

[J] Jacobs, K.: *Measure and Integral.* Probability and mathematical statistics. A Series of Monographs and Textbooks, Academic Press, 1978

[LP] Lewis, J.T., Pulè, J.V.: *The Equilibrium States of the Free Boson Gas.* Comm. Math. Phys. **36**, 1-18 (1974)

[MF] Morse, P. M., Feshbach, H.: *Methods of Theoretical Physics, Part I.* McGraw-Hill Book Company, Inc., New York, Toronto, London, 1953

[P] Petz, D.: *An invitation to the Algebra of Canonical Commutation Relations.* Leuven Notes in Mathematical and Theoretical Physics, Volume 2, Series A: Mathematical Physics, Leuven University Press, 1990

[RS] Reed, M., Simon, B.: *Methods of Mathematical Physics, Volumes I-IV.* Academic Press, 1980

[T] Takesaki, M. :*Theory of Operator Algebra I.* Encyclopaedia of mathematical sciences, Volume 124, Springer Verlag