Quantum Measurements of Scattered Particles

Marco Merkli,*  Mark Penney†

November 2, 2012

Abstract

We investigate the history of quantum measurements on scattered probes. Before scattering, the probes are independent, but they become entangled afterwards, due to the interaction with the scatterer. The collection of measurement results (the history) is a stochastic process of dependent random variables. We link the asymptotic properties of this process to spectral characteristics of the dynamics. We show that the process has decaying time correlations and that a zero-one law holds. We deduce that if the incoming probes are not sharply localized with respect to the spectrum of the measurement operator, then the process does not converge. Nevertheless, the scattering modifies the measurement outcome frequencies. We show that those are the fluxes of the von Neumann measurement projections in the initial probe-scatterer state. We illustrate the results on the Jaynes-Cummings model.

1 Introduction and main results

We consider a scattering experiment in which a beam of probes is directed at a scatterer. The probes are sent to interact sequentially, one by one. Before the scattering process, they are identical and independent. The interaction of each probe with the system is governed by a fixed interaction time \( \tau > 0 \) and a fixed interaction operator \( V \). After interacting with the scatterer, a quantum measurement is performed on each “outcoming” probe. The result of the measurement of the \( n \)-th probe is a random variable, denoted

*Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John’s, NL Canada, A1C 5S7; merkli@mun.ca; Supported by an NSERC Discovery Grant
†Current address: Department of Mathematics, University of Toronto, Bahen Centre, 40 St. George St., Toronto, ON Canada, M5S 2E4; mark.penney@mail.utoronto.ca; Partially supported by NSERC USRA
The stochastic process \( \{X_n\}_{n \geq 1} \) is the measurement history. Due to entanglement of the probes with the scatterer, the \( X_n \) are not independent random variables. We analyze asymptotic properties of this process.

A concrete physical setup is given by atoms (being the probes) shot through a cavity containing an electromagnetic field, the modes which interact with the atoms forming the scatterer. We assume that the incoming probe states are stationary with respect to their isolated dynamics.

We study systems with only finitely many degrees of freedom involved in the scattering process. This means that the Hilbert spaces of pure states both of the system and each probe is finite-dimensional. The measurement of a probe is a von Neumann, or projective, measurement associated to a self-adjoint probe measurement observable \( M \). The eigenvalues \( m \) of \( M \) are the possible measurement outcomes. Due to finite dimensionality, the random variables \( X_n \) have finite range.

The present work can be viewed as the continuation of recently developed techniques for the mathematical analysis of repeated interaction quantum systems [12, 13, 14, 15]. In these references, asymptotic properties of the scatterer have been investigated, without considering the fate of the outgoing probes, and without quantum measurements. While the setup of our present work is similar to the one in the given references, our focus here is on the measurement outcomes process. We show that generically, this process does not converge. We describe the fluctuations on the measurement history, provoked by the scattering process, by analyzing the measurement frequencies. A more detailed comparison to related works is given at the end of this section.

As explained in the references above, in absence of quantum measurements on probes, and under a generic ergodicity assumption, one shows that the scatterer approaches a so-called repeated interaction asymptotic state after many interactions. We keep this assumption in the present work.

(A) Assume that if no measurement is performed (\( M = 1 \)), then, under the repeated interaction with the probes, the scatterer approaches a final state. The convergence is exponentially quick in time.

The precise mathematical formulation of this assumption is given in Section 2.4, see before (25). It is a condition on the spectrum of a reduced dynamics operator, and necessitates the introduction of some technicalities.
which we want to avoid in this introduction. Condition (A) is generically satisfied, and is not hard to be verified explicitly, and one even calculates the rate of convergence for concrete models (see the above references).

We now explain our main results. Denote by $\sigma(X_r, \ldots, X_s)$ the sigma-algebra generated by the random variables $X_r, \ldots, X_s$, $1 \leq r \leq s \leq \infty$. We denote by $P$ the probability measure associated with the process $\{X_n\}_{n \geq 1}$.

**Theorem 1 (Decay of correlations)** Suppose that Condition (A) holds. There are constants $c, \gamma' > 0$, such that for $1 \leq k \leq l < m \leq n < \infty$, $A \in \sigma(X_k, \ldots, X_l)$ and $B \in \sigma(X_m, \ldots, X_n)$, we have

$$|P(A \cap B) - P(A)P(B)| \leq cP(A) e^{-\gamma'(m-l)}. \quad (1)$$

We give a proof of the theorem in Section 4.1. Intuitively, the system starts relaxation to its asymptotic state during the time $m - l$ between two consecutive measurements, and hence erases correlations between the two measurements. The rate $\gamma'$ in (1) is linked to the convergence rate in Assumption (A), see Section 2.4.

The tail sigma-algebra is defined by $\mathcal{T} = \cap_{n \geq 1} \sigma(X_n, X_{n+1}, \ldots)$. Decaying correlations imply the following zero-one law.

**Corollary 2 (Zero-one law)** Assume that Condition (A) holds. Any tail event $A \in \mathcal{T}$ satisfies $P(A) = 0$ or $P(A) = 1$.

In textbooks, the Kolmogorov zero-one law is usually presented for independent random variables [8]. However, an adaptation of the proof yields the result for random variables with decaying correlations, see [1] (and also Section 2.4). The tail sigma-algebra captures convergence properties. For instance, given any outcome $m \in \text{spec}(M)$, the set $\{\lim_n X_n = m\}$ is a tail event, hence, according to Corollary 2, it has probability zero or one.

We now explain why fluctuations in the process persist generically, for all times. Let $\omega_{\text{in}}$ be the state of the incoming probes, denote by $E_S$ the spectral projection of the measurement operator $M$ associated to $S \subset \text{spec}(M)$ and denote $E_m = E_{\{m\}}$ for $m \in \text{spec}(M)$. In absence of interaction ($V = 0$ or $\tau = 0$), the $X_j$ are independent random variables. We show in Proposition 12 that the dependence generated by the interaction with the scatterer is small for small interactions, *uniformly in time*. Therefore, since $P(X_n = m) = \ldots$
\[ \omega_{\text{in}}(E_m) + O(\|V\|), \] we have \( P(X_{n+1} = m, X_n = m) = P(X_{n+1} = m)P(X_n = m) + O(\|V\|) \), and consequently,

\[ P(X_{n+1} = X_n) = \sum_{m \in \text{spec}(M)} \omega_{\text{in}}^2(E_m) + O(\|V\|). \]

The numbers \( \omega_{\text{in}}(E_m) \) are probabilities. Thus, \( \sum_m \omega_{\text{in}}^2(E_m) = 1 \) if and only if for a single \( m_0 \) we have \( \omega_{\text{in}}(E_{m_0}) = 1 \) while for all other \( m \), \( \omega_{\text{in}}(E_m) = 0 \). This means that \( P(X_{n+1} = X_n) < 1 \) for small \( V \), whenever there are several \( m \) with \( \omega_{\text{in}}(E_m) > 0 \). Together with the zero-one law, this implies that \( P(X_n \text{ converges}) = 0 \) whenever the incoming state is not localized in a single subspace of \( M \) (and \( V \) is small enough). If \( m \) is a simple eigenvalue of \( M \) with associated eigenvector \( \psi_m \), then \( \omega_{\text{in}}(E_m) = 1 \) is equivalent to \( \omega_{\text{in}}(\cdot) = \langle \psi_m, \cdot \psi_m \rangle \). Statistical fluctuations in the incoming probes (mixture of states localized w.r.t. measurement values) thus get transferred to outgoing probes, even in the limit of large times. The following is a more general statement of this fact.

**Theorem 3** Assume Condition (A) holds. There is a constant \( C \) s.t., for any \( S \subset \text{spec}(M) \) with \( \omega_{\text{in}}(E_S) \neq 1 \), if \( \|V\| \leq C(1 - \omega_{\text{in}}(E_S)) \), then \( P(X_n \in S \text{ eventually}) = 0 \).

The result on non-convergence of \( X_n \) explained before Theorem 3 is a special case of Theorem 3, when \( S = \{m\}, m \in \text{spec}(M) \). We mention that our analysis also gives a condition under which \( P(X_n \in S \text{ eventually}) = 1 \), see Lemma 11.

The process \( X_n \) carries information about the scattering process, encoded in the relative occurrence of a particular measurement outcome. We define the *frequency* of \( m \in \text{spec}(M) \) by

\[ f_m = \lim_{n \to \infty} \frac{1}{n} \{ \text{number of } k \in \{1, \ldots, n\} \text{ s.t. } X_k = m \}. \]

\( f_m \) is a random variable and the limit (whose existence will be shown) is in the almost everywhere sense. The following result analyzes the influence of the scattering process on the frequencies. It shows that the main contribution is expressed as the *flux of the observable*

\[ E_m(\tau) = \frac{1}{\tau} \int_0^\tau e^{isH_P \tau} E_m e^{-isH_P} ds \]

The result on non-convergence of \( X_n \) explained before Theorem 3 is a special case of Theorem 3, when \( S = \{m\}, m \in \text{spec}(M) \). We mention that our analysis also gives a condition under which \( P(X_n \in S \text{ eventually}) = 1 \), see Lemma 11.

The process \( X_n \) carries information about the scattering process, encoded in the relative occurrence of a particular measurement outcome. We define the *frequency* of \( m \in \text{spec}(M) \) by

\[ f_m = \lim_{n \to \infty} \frac{1}{n} \{ \text{number of } k \in \{1, \ldots, n\} \text{ s.t. } X_k = m \}. \]

\( f_m \) is a random variable and the limit (whose existence will be shown) is in the almost everywhere sense. The following result analyzes the influence of the scattering process on the frequencies. It shows that the main contribution is expressed as the *flux of the observable*
in the initial state $\omega_S \otimes \omega_{in}$. $E_m(\tau)$ is the average of the operator $E_m$ over an interaction period $\tau$.

**Theorem 4 (Frequencies)** Assume condition (A) holds, and suppose the interaction operator is of the form $\lambda V$, where $\lambda \in \mathbb{R}$ is a coupling constant. Then $f_m = f_m(\lambda)$ exists as an almost everywhere limit and is deterministic (not random). It is a holomorphic function in $\lambda$ at the origin, and

$$f_m(\lambda) = \omega_{in}(E_m) + \lambda f'_m + O(\lambda^2),$$

where $f'_m = \omega_S \otimes \omega_{in}(i\tau[V,E_m(\tau)])$.

Note that by invariance of $\omega_S \otimes \omega_{in}$ under the free dynamics generated by $H_0 = H_S + H_P$, we have $\omega_S \otimes \omega_{in}(i\lambda \tau[V,E_m(\tau)]) = \tau \omega_S \otimes \omega_{in}(i[H,E_m(\tau)])$, where $H = H_0 + \lambda V$. Thus

$$f'_m = \tau \frac{d}{dt} \bigg|_{t=0} \omega_S \otimes \omega_{in}(e^{itH}E_m(t)e^{-itH}),$$

which is ($\tau$ times) the change in the quantity $E_m(\tau)$ per unit time, i.e., the flux.

**Theorem 5 (Mean)** Let $\overline{X}_n = \frac{1}{n}(X_1 + \cdots + X_n)$ be the empirical average of the process $\{X_n\}$. We have a law of large numbers,

$$\lim_{n \to \infty} \overline{X}_n = \mu_\infty,$$

almost everywhere, for some deterministic (non random) $\mu_\infty \in \mathbb{R}$. If the interaction is of the form $\lambda V$, where $\lambda \in \mathbb{R}$ is a coupling constant, then $\mu_\infty = \mu_\infty(\lambda)$ is a holomorphic function in $\lambda$ at the origin, and

$$\mu_\infty(\lambda) = \omega_{in}(M) + \lambda \mu'_\infty + O(\lambda^2),$$

where $\mu'_\infty = \omega_S \otimes \omega_{in}(i\tau[V,M(\tau)])$ is the flux of the averaged measurement operator $M(\tau) = \frac{1}{\tau} \int_0^\tau e^{isH_P}Me^{-isH_P}ds$.

**Relation to previous work.** The literature on repeated interaction systems we are aware of can be classified into two categories. In a first one, effective evolution equations are derived by taking continuous interaction limits
and in a second category, the dynamics is left discrete, but the time-asymptotics is investigated. To our knowledge, repeated interaction systems have been first proposed in [6, 7], as approximations for system-environment type models. It is proven that the discrete evolution converges to that of a quantum Langevin equation, in the limit of ever shorter system-probe interaction times. The study of the time-asymptotics was initiated in [12] (and further developed in [13, 14, 15]), where the goal was not to reach a continuous-time limit, but rather to determine the time-asymptotics of the system state. It is shown that the (reduced) system state converges to a final state. The latter is described explicitly (by rigorous perturbation theory in the system-environment coupling) and its physical (thermodynamic) properties are found. The present work falls into the second category. It would be interesting to examine how our mathematical approach leads to effective evolution equations, but we have not studied this yet.

The mathematical formalism we have developed in [12, 13, 14, 15] is based on a spectral approach to the time-asymptotics of open quantum systems (algebraic formalism of quantum mechanics, Liouville generator of dynamics, spectral analysis of the latter – via spectral deformation in the infinite-dimensional situation). The present work extends this formalism to the setting of repeated probe quantum measurements. The references [9, 10] also deal with such measurements. Their analysis is based on the theory of classical stochastic processes (law of large numbers, martingale convergence, large deviation principle). The two mathematical approaches are completely different. The models are different too, and so are the questions addressed, as we now explain.

The main difference in the model we and [9, 10] consider is in our assumption (A) in contrast to their non-demolition assumption on the evolution. The latter assumes that there is a preferred basis of system states, called pointer states, which is preserved by the interacting system-probe dynamics. The pointer states remain unchanged under the successive measurements and evolution. As a consequence of this assumption, the system-probe interaction operator commutes with the system Hamiltonian, so there is no energy transfer between the system and the probes. Assuming this manifold of invariant states, it is shown in [9] that any initial state of the system converges, under the repeated measurement evolution, to one of the pointer states. Moreover, the measurement outcome determines which pointer state is chosen. The
results are derived for a homogeneous model (same setting for each interaction cycle) and under a ‘non-degeneracy condition’. They are generalized in [10] to non-homogeneous settings and without the non-degeneracy condition (under which the final dynamics becomes more complicated). Our assumption (A) is in some sense exactly the ‘opposite’ of a non-demolition assumption. Namely, it forces the dynamics (without measurement) to have a single stationary state (as opposed to an entire basis of stationary states). Both assumptions are reasonable, but they describe different physical situations. As pointed out in [9, 10], the non-demolition setting is realized in some experiments in quantum optics. Our setting describes scattering processes where energy is exchanged. A typical example is that of a ‘one atom maser’, where atoms (probes) interact with modes of the electromagnetic field in a cavity (system) by exciting the field modes, leading to subsequent photon emission [27]. These processes necessitate energy exchange. A famous model describing this situation is the Jaynes-Cummings model, in which energy is not conserved, and which we discuss in some detail in Section 3.

According to the different physical setups, the questions addressed here and in [9, 10] are different. Their goal is to gain knowledge on the system by measuring the outgoing probes. This can be done since both the system and the measurements converge for large times. (The convergence of the measurement outcomes is not mentioned as one of the main results in [9, 10], but it is established in the text of the article, see e.g. the $\pi_\infty(i)$ in [9].) As our results show, in the energy-exchanging situation, the measurement process typically does not converge. We then focus on the question how, on average, the input (incoming probes) is related to the output (outgoing probes) of the scattering process. Related to this is that incoming probes are taken to be in pure states in [9, 10] (they may be selected randomly from a set of pure states), while in our work, they may be mixed. As we explain in the introduction above, the statistical uncertainty in the mixed incoming states contributes to the fluctuation of the measurement process. The inverse scattering problem (namely, the state of the scattering centre, the system) is addressed in Lemmas 13 and 14. The first lemma says that the evolution of the average of the system state (averaged over measurement outcomes) equals the evolution in absence of measurement. In particular, the average system state always converges – the final state being the same one reached without measurements (see also condition (A) above). The second lemma shows that if the measurement process converges, then so does the system state (without
taking its average).

On top of generalizing the results of [9] to a more general non-demolition setting (with random characteristics of the system, and e.g. possible feedback in the measurement process, and also considering the continuous-time limit), the authors in [10] prove the (almost sure) convergence of measurement outcome frequencies (sections 4.1 and 4.3 in [10]). This is analogous to the convergence statement in our Theorem 4. Their argument is based on the law of large numbers for Markov chain states. Our approach gives a concrete construction of the final outcome frequencies which is amenable to explicit calculations (analytic perturbation theory) and physical interpretation (Theorem 4).

There is an interesting similarity between repeated interaction and quantum random walk models. The literature on the latter is huge and it is not our goal to present it here, with the exception of [2], which has been pointed out to us by the referee of the present paper. In that reference, the authors consider the dynamics of an open system quantum random walker. The pair \((\rho_n, X_n)\) defines a quantum trajectory process, where \(\rho_n\) is the state of the walker and \(X_n\) is its position (in \(\mathbb{Z}^d\)). Under a suitable ergodicity assumption on the dynamics of \(\rho_n\) (resembling our condition (A)), a central limit theorem is proven for the random process \(X_n\). In a sense, the \(X_n\) is analogous to our measurement outcome process, and it would be interesting to establish a central limit theorem in our case as well, giving a refinement of the law or large numbers established in Theorem 5. This question, together with an elucidation of the relation between repeated interaction systems and quantum random walks, presents a circle of interesting problems for future analysis.

2 Quantum dynamical system setup

2.1 Formalism

The general formalism of quantum dynamical systems is presented in [5, 16] (see also [12]). Both \(\mathcal{S}\) and \(\mathcal{P}\) are described as quantum \((W^*)\) dynamical systems in standard form. The states of such a system are given by unit vectors in a Hilbert space \(\mathcal{H}\), observables form a von Neumann algebra \(\mathcal{M} \subset \mathcal{B}(\mathcal{H})\) and the dynamics is given by a group of *automorphisms \(\alpha^t\) on \(\mathcal{M}\),
There is a distinguished (reference) vector $\psi \in \mathcal{H}$ which is cyclic and separating for $\mathcal{M}$, and such that the dynamics is represented as $\alpha^t(A) = e^{itL}Ae^{-itL}$, for $A \in \mathcal{M}$, and where $L$ is a selfadjoint operator on $\mathcal{H}$ satisfying $L\psi = 0$. This operator is called the standard Liouville operator.

Accordingly, the system $\mathcal{S}$ is determined by a Hilbert space $\mathcal{H}_S$, a von Neumann algebra $\mathcal{M}_S$, a cyclic and separating vector $\psi_S$ and a dynamics $\alpha^t_S = e^{itL_S}e^{-itL_S}$, with $L_S\psi_S = 0$. A single probe is described by the same ingredients (with index $\mathcal{S}$ replaced by $\mathcal{P}$). We assume throughout the paper that $\dim \mathcal{H}_S < \infty$, $\dim \mathcal{H}_P < \infty$. The Hilbert space of the chain of all probes is the tensor product $\mathcal{H}_C = \otimes_{n \geq 1} \mathcal{H}_P$, stabilized on the reference vector $\psi_P \in \mathcal{H}_P$. The von Neumann algebra of observables of $C$ is $\mathcal{M}_C = \otimes_{n \geq 1} \mathcal{M}_P$ and its dynamics is $\alpha^t_C = \otimes_{n \geq 1} \alpha^t_P$.

The full system is described by the Hilbert space

$$\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_C$$

on which acts the von Neumann algebra of observables

$$\mathcal{M} = \mathcal{M}_S \otimes \mathcal{M}_C.$$  \hspace{1cm} (3)

The non-interacting Liouville operator is given by

$$\tilde{L}_0 = L_S + \sum_{n \geq 1} L_{n,P},$$

where $L_{n,P}$ is the operator acting trivially on all factors of $\mathcal{H}$ except on the $n$-th factor of $\mathcal{H}_C$, on which it acts as $L_P$. The interaction between $\mathcal{S}$ and the $n$-th probe lasts for a duration of $\tau > 0$ and is determined by an operator $V_n$, acting trivially on all factors of $\mathcal{H}$ except $\mathcal{H}_S$ and the $n$-th one in $\mathcal{H}_C$, where it acts as a fixed selfadjoint operator

$$V = V^* \in \mathcal{M}_S \otimes \mathcal{M}_P.$$ \hspace{1cm} (5)

Let $\Psi_0 \in \mathcal{H}$ be an initial state of the full system. Then the state of the system at time-step $n$ is given by the vector

$$\Psi_n = U_n \cdots U_2 U_1 \Psi_0,$$

where

$$U_n = e^{i\tau (L_0 + V_n)}.$$ \hspace{1cm} (7)
is the unitary generating the one-step time evolution at instant \( n \).

We consider initial states of the form

\[ \Psi_0 = \Psi_S \otimes_{n \geq 1} B \psi_P, \quad (8) \]

where \( B \) is an operator in the commutant von Neumann algebra

\[ \mathcal{M}_C' = \{ A \in B(\mathcal{H}_P) : AX =XA \forall X \in \mathcal{M}_C \}, \]

such that \( \|B\psi_P\| = 1 \) and \( L_P B = BL_P \) (normalized, invariant state). Since \( \psi_P \) is cyclic for \( \mathcal{M}_P' \) (\( \Leftrightarrow \) separating for \( \mathcal{M}_P \), where \( \mathcal{M}_P' \) the commutant of \( \mathcal{M}_P \)) and since \( \dim \mathcal{H}_P \) is finite, every \( \psi \in \mathcal{H}_P \) is exactly represented as \( \psi = B\psi_P \) for a unique \( B \in \mathcal{M}_P' \).

Let \( J \) and \( \Delta \) denote the modular conjugation and the modular operator associated to the cyclic and separating vector \( \psi_S \otimes \psi_P \) for the von Neumann algebra \( \mathcal{M}_S \otimes \mathcal{M}_P \). By the Tomita-Takesaki theorem [8], we know that \( \Delta^{-it}(\mathcal{M}_S \otimes \mathcal{M}_P)\Delta^{it} = \mathcal{M}_S \otimes \mathcal{M}_P \) for all \( t \in \mathbb{R} \) (\( \Delta^{-it} \) is a group of unitaries), and that \( J(\mathcal{M}_S \otimes \mathcal{M}_P)J = \mathcal{M}_S' \otimes \mathcal{M}_P' \). Consequently, \( J\Delta^{it}V\Delta^{-it}J \in \mathcal{M}_S' \otimes \mathcal{M}_P' \), where \( V \) is the interaction operator defined in (5). In the finite-dimensional case as considered here, an easy analyticity argument shows that the last relation stays valid for any \( t \in \mathbb{C} \). In particular,

\[ J\Delta^{1/2}V\Delta^{-1/2}J \in \mathcal{M}_S' \otimes \mathcal{M}_P'. \quad (9) \]

It will be convenient to represent the joint dynamics of the system \( S \) and the probe \( P \) interacting at the given moment with \( S \) by the following operator acting on \( \mathcal{H}_S \otimes \mathcal{H}_P \) (see Subsection 2.3),

\[ K = L_S + L_P + \lambda V - \lambda J\Delta^{1/2}V\Delta^{-1/2}J. \quad (10) \]

Here, \( \lambda \in \mathbb{R} \) is a coupling constant. Due to property (9) we have

\[ e^{itK}Ae^{-itK} = e^{it[L_S+L_P+\lambda V]}Ae^{-it[L_S+L_P+\lambda V]} \]

for all \( A \in \mathcal{M}_S \otimes \mathcal{M}_P \) and all \( t \in \mathbb{R} \), as is not hard to see for instance by using the Trotter product formula. The term \( -\lambda J\Delta^{1/2}V\Delta^{-1/2}J \) in (10) is introduced in order to have the property

\[ \dot{K}\psi_S \otimes \psi_P = 0. \quad (12) \]
The latter relation follows from \((L_S + L_P)\psi_S \otimes \psi_P = 0\) and the fact that 
\(J\Delta^{1/2}V\Delta^{-1/2}J\psi_S \otimes \psi_P = V\psi_S \otimes \psi_P\) (which in turn is implied by \(\Delta^{-1/2}J = J\Delta^{1/2}\) and \(J\Delta^{1/2}A\psi_S \otimes \psi_P = A^*\psi_S \otimes \psi_P\) for all \(A \in \mathcal{M}_S \otimes \mathcal{M}_P\)). The operator \(K\) has been used in [25] for the study of non-equilibrium open quantum systems and in the setting of repeated interaction open systems in [12, 13, 14, 15].

### 2.2 Multitime measurement process

In this subsection, we describe the process of multitime measurement of the outcoming probes. We refer to [29, 17, 24, 19] for a detailed introduction to quantum measurement theory. Let \(M \in \mathcal{M}_P\) be a selfadjoint “measurement” operator on \(\mathcal{H}_P\) with spectrum \(\text{spec}(M) = \{m_1, \ldots, m_\mu\}\), where \(1 \leq \mu \leq \dim \mathcal{H}_P\) (distinct eigenvalues). Let \(S\) be any subset of \(\text{spec}(M)\) and denote by \(E_S\) the spectral projection of \(M\) associated to \(S\).

Suppose that the entire system is in a state \(\Psi_0 \in \mathcal{H}\) initially, and that the following experiment is performed: the system evolves according to \(U_1\) and then a measurement of the observable \(M\) is made on the outcoming probe, yielding a value in \(S_1 \subset \text{spec}(M)\), and then the system evolves according to \(U_2\) and after this evolution a measurement of \(M\) is made on the outcoming probe and yields a result lying in \(S_2 \subset \text{spec}(M)\), and this procedure is repeated \(n\) times. According to the principles of quantum mechanics, the probability for obtaining the multitime measurement result lying in \(S_1, \ldots, S_n\) is given by

\[
P(S_1, \ldots, S_n) = \|E_{S_n}U_{n-1} \cdots E_{S_2}U_2 E_{S_1}U_1 \Psi_0\|^2.
\]

Furthermore, if this probability is nonzero (so that the outcome of the specific experiment is actually realizable), then the state of the system immediately after the \(n\)-th measurement is given by the normalized vector

\[
\Psi_n = \frac{E_{S_n}U_{n-1} \cdots E_{S_2}U_2 E_{S_1}U_1 \Psi_0}{\sqrt{P(S_1, \ldots, S_n)}}.
\]

We have \(E_{\text{spec}(M)} = \mathbb{1}\), which corresponds to the situation where at the given time step no measurement is performed.

The stochastic process associated to the measurements is constructed as follows. Let

\[
\Omega = \Sigma^\mathbb{N} = \{\omega = (\omega_1, \omega_2, \ldots) : \omega_j \in \text{spec}(M)\}\]
and let $\mathcal{F}$ be the $\sigma$-algebra of subsets of $\Omega$ generated by all cylinder sets of the form

$$\{\omega \in \Omega : \omega_1 \in S_1, \ldots, \omega_n \in S_n, n \in \mathbb{N}, S_j \subset \text{spec}(M)\}.$$ 

On $(\Omega, \mathcal{F})$ we define the random variables $X_n : \Omega \to \text{spec}(M)$ by

$$X_n(\omega) = \omega_n, \quad \text{for } n = 1, 2, \ldots$$

The random variable $X_n$ represents the outcome of the measurement at time-step $n$. The finite-dimensional distribution of the process $\{X_n\}_{n \geq 1}$ is given by

$$P(X_1 \in S_1, \ldots, X_n \in S_n) = P(S_1, \ldots, S_n), \quad (15)$$

for any $n \in \mathbb{N}$, any subsets $S_1, \ldots, S_n$ of $\text{spec}(M)$ and where the right hand side is defined in $(13)$. $P$ extends uniquely to a probability measure on $(\Omega, \mathcal{F})$ by the Kolmogorov extension theorem.

### 2.3 Representation of joint probabilities

The main result of this subsection is formula $(22)$. The expectation of an observable $A \in \mathfrak{M}_S$ of $S$ at time step $n$ is given by $\langle A \rangle_n = \langle \Psi_n, A \Psi_n \rangle$, where $\Psi_n$, defined in $(14)$, is the normalization of the vector $\Psi_n = E_S U_n \cdots E_S U_1 \Psi_0$. Note that only the $n$ first factors in the product of the probe Hilbert space $\mathcal{H}_C$ (see before $(2)$) play a nontrivial role in the definition of $\Psi_n$. We have

$$\langle \Psi_n, A \tilde{\Psi}_n \rangle = \langle \Psi_0, e^{itL_1} E_S \cdots e^{itL_n} E_S A \ E_S e^{-itL_n} \cdots E_S e^{-itL_1} \Psi_0 \rangle$$

$$= \langle \Psi_0, [U_n^{-}]^* e^{itL_1} E_S \cdots e^{itL_n} E_S [U_n^{-}]^* A U_n^{-} E_S e^{-itL_n} \cdots E_S e^{-itL_1} U_n^+ \Psi_0 \rangle$$

where

$$L_j = L_S + L_{j,p} + V_j \quad (17)$$

acts non-trivially only on the Hilbert space of $S$ and the $j$-th probe Hilbert space, and where we introduced the unitaries

$$U_n^- = \exp \left[ -it \sum_{j=1}^n (n-j) L_{j,p} \right] \quad \text{and} \quad U_n^+ = \exp \left[ -it \sum_{j=2}^n (j-1) L_{j,p} \right]. \quad (18)$$
Since $A$ commutes with $U_n^\pm$ we have $[U_n^n]^*AU_n^- = A$. The operator $C_j = e^{-i\tau(j-1)L,P}Be^{i\tau(j-1)L,P}$ belongs to the commutant $\mathcal{M}_P^\prime$, since $B$ does, and since the dynamics generated by $L,P$ leaves $\mathcal{M}_P^\prime$ invariant. Recalling that $\Psi_0 = (\mathbf{1}_S \otimes B \otimes B \cdots)\Psi_{\text{ref}}$, with $\Psi_{\text{ref}} = \psi_S \otimes \psi_P \otimes \psi_P \cdots$, we obtain from (16)

$$\left\langle \tilde{\Psi}_n, A\tilde{\Psi}_n \right\rangle = \left\langle C_1 \cdots C_n \Psi_{\text{ref}}, e^{i\tau L_1}E_{S_1} \cdots e^{i\tau L_n}E_{S_n}Ae^{i\tau L_n} \cdots E_{S_1}e^{-i\tau L_1}C_1 \cdots C_n \Psi_{\text{ref}} \right\rangle$$

$$= \left\langle \Psi_{\text{ref}}, C_1^*C_1 \cdots C_n^*C_n e^{i\tau K_1}E_{S_1} \cdots e^{i\tau K_n}E_{S_n}A\Psi_{\text{ref}} \right\rangle$$

$$= \left\langle \Psi_{\text{ref}}, [P_1C_1^*C_1 e^{i\tau K_1}E_{S_1}P_1] \cdots [P_nC_n^*C_n e^{i\tau K_n}E_{S_n}P_n]A\Psi_{\text{ref}} \right\rangle,$$

where $P_j$ is the projection acting trivially on all factors of $\mathcal{H}$ except on the $j$-th $\mathcal{H}_P$, where it acts as the rank-one orthogonal projection onto $\psi_P$. We define the operator $T_j = P_jC_j^*C_j e^{i\tau K_j}E_{S_j}P_j$, which, under the hypothesis that $e^{isL_P}Be^{-isL_P} = B$, becomes $T_j = P_jB^*Be^{i\tau K_j}E_{S_j}P_j$. We identify $T_j$ as an operator on the Hilbert space $\mathcal{H}_S$, and as such, write

$$T_S = PB^*Be^{i\tau K}E_{S_j}P,$$  

(20)

where $P$ the orthogonal projection onto $\psi_P \otimes \mathcal{H}_C$, $S \subset \text{spec}(M)$ determines the measurement performed at the given time-step, and where $K$ is given in (10). We write simply $T$ for $T_{\text{spec}(M)}$. Remark that since $E_{S_j} \in \mathcal{M}_P$ and $\psi_P$ is separating for $\mathcal{M}_P$, we have $E_{S_j}P_j \neq 0$ for all $j$. With this definition, we arrive at

$$\left\langle \tilde{\Psi}_n, A\tilde{\Psi}_n \right\rangle = \left\langle \psi_S, T_{S_1}T_{S_2} \cdots T_{S_n}A\psi_S \right\rangle.$$  

(21)

In particular, the probability (14) can be expressed as

$$P(S_1, \ldots, S_n) = \left\langle \psi_S, T_{S_1} \cdots T_{S_n}\psi_S \right\rangle.$$  

(22)

### 2.4 Analysis of joint probabilities

**Lemma 6** The spectrum of $T_S$, (20), lies in the closed unit disk centered at the origin of the complex plane. For $S = \text{spec}(M)$, i.e., $E_S = \mathbf{1}$, we have in addition $T\psi_S = \psi_S$.

We consider the probability $P(X_n \in S$ eventually), for $S \subset \text{spec}(M)$. This quantity can be expressed using the Riesz spectral projections $I$ and
Π_S of the operators T and T_S associated to the eigenvalue 1. They are defined by

\[ \Pi_S = \frac{1}{2\pi i} \oint (z - T_S)^{-1} \, dz, \quad \Pi = \Pi_{\text{spec}(M)}, \quad (23) \]

where the integral is over a simple closed contour in the complex plane encircling no spectrum of T_S except the point 1. If 1 is not an eigenvalue then \( \Pi_S = 0 \). For the next result, we recall the following definition,

\[ \{ X_n \in S \text{ eventually} \} = \{ \omega \mid \text{there exists a } k \text{ s.t. } X_n(\omega) \in S \text{ for all } n \geq k \}. \]

**Lemma 7** We have \( P(X_n \in S \text{ eventually}) = \langle \psi_S, \Pi S \psi_S \rangle \).

**Proof.** The set \( \{ X_n \in S \text{ eventually} \} \) is the increasing union of \( \{ X_n \in S \forall n \geq k \} \), so \( P(X_n \in S \text{ eventually}) = \lim_{k \to \infty} P(X_n \in S \forall n \geq k) \). Next, \( \{ X_n \in S \forall n \geq k \} \) is the intersection of the decreasing sequence \( \{ X_n \in S, n = k, \ldots, k + l \} \), so

\[ P(X_n \in S \text{ eventually}) = \lim_{k \to \infty} \lim_{l \to \infty} P(X_n \in S, n = k, \ldots, k + l). \quad (24) \]

We have \( P(X_n \in S, n = k, \ldots, k + l) = \langle \psi_S, T^{k-1}T_S^{l+1} \psi_S \rangle \). Since for each \( k \) fixed, the limit of \( \langle \psi_S, T^{k-1}T_S^{l+1} \psi_S \rangle \) as \( l \to \infty \) exists (it is the probability \( P(X_n \in S \forall n \geq k) \)), we have

\[ \lim_{l \to \infty} \langle \psi_S, T^{k-1}T_S^{l+1} \psi_S \rangle = \lim_{l \to \infty} \frac{1}{L} \sum_{i=1}^{L} \langle \psi_S, T^{k-1}T_S^{l+1} \psi_S \rangle = \langle \psi_S, T^{k-1} \tilde{\Pi}_S \psi_S \rangle, \]

where \( \tilde{\Pi}_S \) is the ergodic projection of \( T_S \) associated to the eigenvalue 1. (\( \tilde{\Pi}_S = 0 \) if 1 is not an eigenvalue of \( T_S \).) Arguing in the same way for the limit \( k \to \infty \), we obtain \( P(X_n \in S \text{ eventually}) = \langle \psi_S, \Pi \tilde{\Pi}_S \psi_S \rangle \). Invoking Lemma 21 we replace the ergodic projections with the Riesz projections. \( \blacksquare \)

The probability is given entirely by information on the spectrum of \( T \) and \( T_S \) at the point 1 (Riesz projections). The result holds even if \( T \) or \( T_S \) have spectrum on the unit circle other than possibly at 1.

Finer information about the asymptotic dynamics depends on all the spectrum of \( T \) on the unit circle. We make the following ergodicity assumption (compare with Lemma 6).
**Condition A.** The point \(z = 1\) is the unique eigenvalue of \(T\) with \(|z| = 1\), and this eigenvalue is simple (with eigenvector \(\psi_S\)). We define the gap by

\[
\gamma = 1 - \sup \{ |z| : z \in \text{spec}(T), z \neq 1 \}.
\]  

(25)

For \(\lambda = 0\) the operator \(T\) is \(e^{i\tau L_S}\) and has spectrum on the unit circle, with degenerate eigenvalue 1. Assumption A is verified in practice typically by perturbation theory (\(\lambda\) small, nonzero). It is sometimes called a “Fermi golden rule condition”. In this setting, condition A implies \(\gamma > 0\) for small nonzero \(\lambda\). Condition A implies the dynamical behaviour of the assumption (A) stated before Theorem 1.

The random variables \(X_n\) are not independent but their correlations decay.

**Theorem 8 (Decay of correlations)** Suppose Condition A holds. For any \(\epsilon > 0\) there is a constant \(C_\epsilon\) such that for \(1 \leq k \leq l < m \leq n < \infty\), \(A \in \sigma(X_k, \ldots X_l)\) and \(B \in \sigma(X_m, \ldots, X_n)\), we have

\[
|P(A \cap B) - P(A)P(B)| \leq C_\epsilon P(A) \ e^{-(m-l)[\ln(\frac{1}{1-\gamma}) - \epsilon]}.
\]  

(26)

We give a proof of Theorem 8 in Section 4.1. For fixed \(\epsilon < \ln(\frac{1}{1-\gamma})\), define the function \(C : \mathbb{N} \to \mathbb{R}_+\) by

\[
C(d) = C_\epsilon e^{-d[\ln(\frac{1}{1-\gamma}) - \epsilon]}.
\]  

(27)

Theorem 8 implies that the random variables \(X_n\) have decaying correlations in the following sense: for all \(1 \leq k \leq l < m \leq n\), all \(A \in \sigma(X_k, \ldots X_l)\) and all \(B \in \sigma(X_m, \ldots, X_n)\), we have

\[
|P(A \cap B) - P(A)P(B)| \leq C(m - l),
\]  

(28)

where the function \(C\) is independent of \(A, B, k, l, m, n\) and satisfies \(C(d) \to 0\) as \(d \to \infty\).

Let \(X_k, k = 1, 2, \ldots\) be a sequence of random variables. We denote by \(\sigma(X_n, X_{n+1}, \ldots)\) the sigma algebra generated by \(\{X_k\}_{k \geq n}\). The tail sigma algebra of the process \(\{X_n\}_{n \geq 1}\) is defined by \(\mathcal{T} = \cap_{n \geq 1} \sigma(X_n, X_{n+1}, \ldots)\).

The following result is a generalization of the Kolmogorov zero-one law, valid for a process \(\{X_n\}\) where the random variables are not independent, but have decaying correlations.
Theorem 9 (Extended Kolmogorov zero-one law) Let \( X_n \) be a sequence of random variables with decaying correlations, as in (28). Then we have \( P(A) = 0 \) or \( P(A) = 1 \) for any tail event \( A \in \mathcal{T} \).

A proof of this result can be obtained by extending proofs of the Kolmogorov zero-one law for independent variables, see [1]. Under Condition A, we write the rank-one Riesz projection of \( T \) associated to \( z = 1 \) as

\[ \Pi = |\psi_S\rangle\langle \psi_S^*|, \tag{29} \]

where \( T\psi = \psi, \ T^*\psi^* = \psi \) and \( \langle \psi^*_S, \psi_S \rangle = 1, \|\psi_S\| = 1. \) (\( T^* \) is the adjoint of \( T \).)

Corollary 10 Assume that Condition A holds and let \( S \subset \text{spec}(M) \). Then

\[ P(X_n \in S \text{ eventually}) = \langle \psi^*_S, \Pi_S \psi_S \rangle \in \{0, 1\}. \]

Remark. Both \( \psi^*_S \) and \( \Pi_S \) depend on \( \lambda \) (\( \psi_S \) does not). If condition A holds for \( \lambda \in I\backslash\{0\} \) for some neighbourhood \( I \subset \mathbb{R} \) of zero, then for \( \lambda \) sufficiently small we have \( P(X_n \in S \text{ eventually}) = \lim_{\lambda \to 0} \langle \psi^*_S(\lambda), \Pi_S(\lambda)\psi_S \rangle \).

This follows from the facts that the mapping \( \lambda \mapsto \langle \psi^*_S, \Pi_S \psi_S \rangle \) is continuous in a deleted neighbourhood of \( \lambda = 0 \) and that the image is discrete. We point out that the map \( \lambda \mapsto \langle \psi^*_S, \Pi_S \psi_S \rangle \) is actually holomorphic in a punctured neighbourhood of the origin, and stays bounded there (the only possible image points being 0 or 1, even for complex \( \lambda \), by the identity principle). Hence zero is a removable singularity of this map. As objects on their own, \( \psi^*_S \) and \( \Pi_S \) are not holomorphic at the origin in general (eigenvalue splitting), but their combination as in the inner product is.

Theorem 3 gives a criterion for \( P(X_n \in S \text{ eventually}) = 0 \). The next result is a characterization of when this probability is one.

Lemma 11 Assume that Condition A holds. If \( \langle \psi_S, \Pi_S \psi_S \rangle \neq 0 \) for some small enough \( \lambda \) then \( P(X_n \in S \text{ eventually}) = 1 \) for sufficiently small values of \( \lambda \).

Proof of Lemma 11. The inequality \( P(X_n \in S \text{ eventually}) \geq P(X_n \in S \ \forall n \geq 1) \) is the same as \( \langle \psi_S, \Pi \Pi_S \psi_S \rangle \geq \langle \psi_S, \Pi_S \psi_S \rangle \) (see also Appendix A). Result (b) follows from the fact that the left side can only take the values zero or one, independently of \( \lambda \) for \( \lambda \) sufficiently small (guaranteeing that \( \langle \psi_S, \Pi \Pi_S \psi_S \rangle \) is continuous).
Proposition 12 Let $A_j \in \sigma(X_j)$, $j \geq 1$. We have

$$\sup_{n \geq 1} |P(A_n, \ldots, A_{n+k}) - P(A_n) \cdots P(A_{n+k})| \leq C_k \|V\|$$

for any $k \geq 1$, and for some constant $C_k$.

Proof of Proposition 12. It suffices to show that

$$P(X_n \in S_n, \ldots, X_{n+k} \in S_{n+k}) - P(X_n \in S_n) \cdots P(X_{n+k} \in S_{n+k}) = O(\|V\|),$$

uniformly in $n$, and where $S_j \subseteq \text{spec}(M)$. Since $T_S = PB^*Be^{i\tau K}E_S P = e^{i\tau L_S}\omega_{in}(E_S) + O(\|V\|)$, the joint probability on the left side is

$$\langle \psi_S, T_n^{-1}T_{S_n} \cdots T_{S_{n+k}}\psi_S \rangle = \omega_{in}(E_{S_n}) \cdots \omega_{in}(E_{S_{n+k}}) + O(\|V\|).$$

Similarly, $P(X_j \in S_j) = \omega_{in}(E_{S_j}) + O(\|V\|)$, and so the result follows. \[\square\]

Let $\omega_n$ be the state of $S$ at time step $n$ (obtained by reducing the state of the entire system to $S$).

Lemma 13 (Evolution of averaged system state) The system state at time step $n$, $\omega_n$, is a random variable (determined by the random measurement history). Its expectation, $\mathbb{E}[\omega_n]$, equals the state obtained by evolving the initial condition according to the dynamics without measurement.

Proof of Lemma 13. For a given measurement path $X_1 = m_1, \ldots, X_n = m_n$, the system state is

$$\omega_n(A) = \frac{\langle \psi_S, T_1 \cdots T_n A \psi_S \rangle}{\langle \psi_S, T_1 \cdots T_n \psi_S \rangle},$$

where $A$ is any system observable and $T_j = T_{\{m_j\}}$. Since $P(X_1 = m_1, \ldots, X_n = m_n) = \langle \psi_S, T_1 \cdots T_n \psi_S \rangle$, this yields

$$\mathbb{E}[\omega_n(A)] = \sum_{m_1, \ldots, m_n} P(X_1 = m_1, \ldots, X_n = m_n) \frac{\langle \psi_S, T_1 \cdots T_n A \psi_S \rangle}{\langle \psi_S, T_1 \cdots T_n \psi_S \rangle}$$

$$= \sum_{m_1, \ldots, m_n} \langle \psi_S, T_1 \cdots T_n A \psi_S \rangle$$

$$= \langle \psi_S, T^n A \psi_S \rangle.$$
In the last step, we have used that
\[
\sum_{m} T_{\{m\}} = \sum_{m} PB^* Be^{i\tau K} E_{\{m\}} P = PB^* Be^{i\tau K} P = T.
\]

The right hand side is the single-step dynamics operator of the system without probe measurements.

Lemma 13 shows in particular that the expectation of the system state converges to the repeated interaction state,
\[
\lim_{n \to \infty} E[\omega_n(A)] = \langle \psi_S^*, (A \otimes \mathbb{1}_S) \psi_S \rangle,
\]
see also (29). This, of course, does not mean that \(\omega_n\) itself converges, in general. However, if the measurement process converges, then we do have the following result.

**Lemma 14 (Asymptotic state of \(S\))** Suppose that the measurement outcomes
\[
X_1 \in S_1, \ldots, X_{n-1} \in S_{n-1}, \quad X_k \in S, \quad k \geq n
\]
are observed for some \(n \geq 1\), and that 1 is a simple eigenvalue of \(T_S\) with Riesz projection \(\Pi_S = |\psi\rangle \langle \psi^*|\). We have for any observable \(A\) of \(S\)
\[
\lim_{n \to \infty} \omega_n(A) = \omega_\infty(A) = \frac{\langle \psi^*, A \psi_S \rangle}{\langle \psi^*, \psi_S \rangle}.
\]

This is an inverse scattering result: knowing that the scattered particles are measured to lie in \(S\) we can deduce the state of the scattering object \(S\). The final state does not depend on the initial outcomes \(X_n\) for \(n < k\), any \(k\). However we show in the proof of Lemma 14 that if the eigenvalue 1 of \(T_S\) is not simple, then the system converges to a final state which depends on the whole measurement path \(X_1, X_2, \ldots\).

**Proof of Lemma 14.** The asymptotic state of the system is
\[
\omega_\infty(A) = \lim_{l \to \infty} \frac{\langle \psi_S, T_1 \cdots T_{n-1} T_S^l A \psi_S \rangle}{\langle \psi_S, T_1 \cdots T_{n-1} T_S^l \psi_S \rangle},
\]
see (21). Now \( \langle \psi_S, T_1 \cdots T_{n-1} T_S \psi_S \rangle \) converges to the nonzero probability of observing (30). Therefore, by (the proof of) Lemma 21
\[
\lim_{l \to \infty} \langle \psi_S, T_1 \cdots T_{n-1} T_l \psi_S \rangle = \langle \psi_S, T_1 \cdots T_{n-1} \Pi_S \psi_S \rangle.
\]
By simplicity of the eigenvalue 1 of \( T_S \),
\[
\omega_\infty(A) = \frac{\langle \psi_S, T_1 \cdots T_{n-1} \psi \rangle \langle \psi^*, A \psi_S \rangle}{\langle \psi_S, T_1 \cdots T_{n-1} \psi \rangle \langle \psi^*, \psi_S \rangle}.
\]
If 1 is not a simple eigenvalue of \( T_S \), so that \( \Pi_S = \sum_{j=1}^{r} |\psi_j \rangle \langle \psi_j^*| \), the final state is
\[
\omega_\infty(A) = \frac{\sum_{j=1}^{r} \langle \psi_S, T_1 \cdots T_{n-1} \psi_j \rangle \langle \psi_j^*, A \psi_S \rangle}{\sum_{i=1}^{r} \langle \psi_S, T_1 \cdots T_{n-1} \psi_i \rangle \langle \psi_i^*, \psi_S \rangle}.
\]
The final state then depends on the whole measurement history. ■

3 The Jaynes-Cummings model

We consider a simple system where both the scatterer and the probes have only two degrees of freedom participating in the scattering process. The pure state space of \( S \) and \( P \) is \( \mathbb{C}^2 \), and the Hamiltonians are given by the Pauli \( \sigma_z \) operator,
\[
H_S = H_P = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.
\]
(31)
The interaction between \( S \) and \( P \) is determined by the operator
\[
\lambda V = \lambda \left( a_S^* \otimes a_P + a_S \otimes a_P^* \right),
\]
(32)
with coupling constant \( \lambda \in \mathbb{R} \), and where
\[
a = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad a^* = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}
\]
(33)
are the annihilation and creation operators. In the usual Jaynes-Cummings model (used e.g. in quantum optics), the system \( S \) has many levels (harmonic oscillator), see e.g. [29] and also [18]. Our model is a truncation, but it still
describes energy exchange between $S$ and $P$. In what follows, we can treat all values of $\lambda$, not necessarily small ones only. This is so since the model is essentially exactly solvable. The total Hamiltonian $H = H_S + H_P + \lambda V$ describes exchange of energy between $S$ and $P$, while the total number of excitations, $N = a_S^*a_S + a_P^*a_P$, is conserved (commutes with $H$). This allows for a treatment of the system separately in the invariant sectors $N = 0, 1, 2$.

For an arbitrary probe observable $X \in \mathcal{B}(\mathbb{C}^2)$ we write $X_{ij} = \langle \varphi_i, X \varphi_j \rangle$, where $\varphi_1$, $\varphi_2$ are the orthonormal eigenvectors of $H_P$ (with $H_P\varphi_1 = \varphi_1$).

Incoming states are determined by $p \in [0, 1]$ via

$$\omega_{\text{in}}(X) = pX_{11} + (1 - p)X_{22}, \quad (34)$$

where $X \in \mathcal{B}(\mathbb{C}^2)$ is an arbitrary probe observable.

We will use the notation and definitions of Section 2 in what follows. In particular, the single step operator $T_S$ is defined in (20). For the following explicit formula, we take the reference state $\Psi_S$ to be the trace state.

**Theorem 15 (Explicit reduced dynamics operator)** Set $\varphi_{ij} = \varphi_i \otimes \varphi_j$ and let $X$ be any operator of $\mathcal{P}$. In the basis $\{\varphi_{11}, \varphi_{12}, \varphi_{21}, \varphi_{22}\}$ we have

$$PB^*B e^{i r K} X P = \omega_{\text{in}}(X) e^{i r L_S} +$$

$$\begin{bmatrix}
(1 - p)X_{22}a & (1 - p)X_{21}b \\
-pX_{12}e^{2i\tau_1 \sin(\lambda \tau)} & e^{2i\tau_1 \sin(\lambda \tau)}(\cos(\lambda \tau) - 1)\omega_{\text{in}}(X) \\
pX_{21}e^{-2i\tau_1 \sin(\lambda \tau)} & 0 \\
-pX_{22}a & -pX_{21}b
\end{bmatrix}$$

$$\begin{bmatrix}
-(1 - p)X_{12}b & -(1 - p)X_{11}a \\
0 & -(1 - p)X_{12}e^{2i\tau_1 \sin(\lambda \tau)} \\
e^{-2i\tau_1 \sin(\lambda \tau)}(\cos(\lambda \tau) - 1)\omega_{\text{in}}(X) & -(1 - p)X_{21}e^{-2i\tau_1 \sin(\lambda \tau)} \\
pX_{12}b & pX_{11}a
\end{bmatrix},$$

where $a = -\sin^2(\lambda \tau)$, $b = -i\sin(\lambda \tau)\cos(\lambda \tau)$.

We point out that the vector $[p, 0, 0, 1 - p]^t$ is an eigenvector of the adjoint of (35) with eigenvalue $\omega_{\text{in}}(X)$.

**Proof.** The proof is a rather lengthy but straightforward, explicit calculation. Since $H_S$ and $H_P$ commute with $I = V - V'$, where $V' = J\Delta^{1/2}V\Delta^{-1/2}J$
(see (10)), it suffices to calculate

\[ PB^* B e^{i\tau} XP = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left( \frac{(i\tau)^n}{n!} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} PB^* BV^k (V')^{n-k} XP. \]

Here it is understood that all operators are considered in the “doubled” (GNS) Hilbert space, e.g.,

\[ V = a_S^* \otimes 1_S \otimes a_p \otimes 1_P + a_S \otimes 1_S \otimes a_p^* \otimes 1_P. \]

Powers of \( V \) and \( V' \) can be calculated explicitly. For instance, for \( k \geq 2 \) even, we have

\[ V^k = \hat{n}^{k/2} \otimes 1 \otimes (1 - \hat{n})^{k/2} \otimes 1 + (1 - \hat{n})^{k/2} \otimes 1 \otimes \hat{n}^{k/2} \otimes 1, \]

where \( \hat{n} = a^* a \). One obtains similar expressions for \( k \) odd, and for \( (V')^k \). Using these expression in the above series, one gets the result of Theorem 15. ■

**Resonant and non-resonant system.** If \( \lambda \tau \) is a multiple of \( \pi \) then (35) reduces to

\[ PB^* B e^{i\tau K} XP = \omega_{in}(X) \ \text{diag}(1, \pm 1, \pm 1, 1) \]

with plus and minus signs if the multiple is even and odd, respectively. Then, by using the expression \( P(X_1 \in S_1, \ldots, X_n \in S_n) \) given in (22), it is readily seen that the random variables \( X_j \) are independent, and \( P(X_j \in S) = \omega_{in}(E_S) \). When \( \lambda \tau \in \pi \mathbb{Z} \) we call the system **resonant** [18], otherwise we call it **non-resonant**. One can understand the resonant regime as follows: consider the dynamics generated on \( S \) and a single probe \( P \) by the Hamiltonian \( H = H_S + H_P + \lambda V \). The probability of transition from the initial state \( \varphi_S^0 \otimes \varphi_P^0 \), where the \( S \) is in the ground state and \( P \) in the excited state, to the opposite state \( \varphi_S^1 \otimes \varphi_P^1 \), at time \( t \), is given by \( P_t = \left| \langle \varphi_S^1 \otimes \varphi_P^1, e^{itH} \varphi_S^0 \otimes \varphi_P^0 \rangle \right|^2 = \sin^2(\lambda t) \). For \( \lambda t \in \pi \mathbb{Z} \) this probability vanishes. If the interaction time \( \tau \) in the repeated interaction system is a multiple of \( \pi / \lambda \), then interaction effects are suppressed. It is not hard to see that in this case, the system does not feel the interaction with the probes in the sense that \( \omega_n(A) = \omega_0(\alpha^S_n(A)) \) for all \( n \geq 1 \), where \( \alpha^S_n(A) \) is the reduced dynamics of \( S \) alone. We focus now on the non-resonant situation.

**Asymptotics.** We show that

For \( \omega_{in} = |\uparrow\rangle \langle \uparrow| \), the process \( X_n \) converges to \( |\uparrow\rangle \langle \uparrow| \) (meaning that \( P(X_n = |\uparrow\rangle \langle \uparrow| \) eventually) = 1) and the incoming state of the probes is copied onto the scatterer, in the limit of large times.

This result is non-perturbative and holds for all \( \lambda \in \mathbb{R} \). We point out that this copying mechanism has been described before (“homogenization”), see
Let \( \omega_{\text{in}} = |\uparrow\rangle\langle\uparrow| \), corresponding to \( p = 1 \). The matrix in (35) becomes lower triangular, and

\[
\text{spec}(PB^*Be^{i\tau K}XP) = X_{11} \{1, e^{2i\tau \cos(\lambda \tau)} , e^{-2i\tau \cos(\lambda \tau)} , \cos^2(\lambda \tau)\}.
\]

Condition A is always satisfied in the non-resonant case. \( \psi_S \) and \( \varphi_{11} \) are the eigenvectors associated to the eigenvalue 1 of \( PB^*Be^{i\tau K}P \) and its adjoint, respectively. So the Riesz projection associated with the eigenvalue 1 is \( \Pi = \sqrt{2} \langle \psi_S | \varphi_{11} \rangle \langle \psi_S | \) (see (23)). We can now readily obtain the following results, for an arbitrary measurement operator \( M \).

The asymptotic mean is given by \( \mu_\infty = \omega_{\text{in}}(M) \) (expression (61)). The frequencies are \( f_m = \omega_{\text{in}}(E_m) \) (expression (52)). This suggests that the cavity becomes ‘transparent’ for large times (no effect on incoming probes). And indeed, the Riesz projection \( \Pi_S \) associated to the value 1 of \( T_S \) (with \( S \subset \text{spec}(M) \)) vanishes unless \( (E_S)_{11} = 1 \). Thus we have \( P(X_n \in S \text{ eventually}) = 0 \) if \( (E_S)_{11} \neq 1 \) (see Lemma 7). If \( (E_S)_{11} = 1 \) then \( E_S \) is the projection \( |\varphi_1\rangle\langle\varphi_1| \), and one finds easily that \( \Pi_S = |\varphi_{11}\rangle\langle\varphi_{11}| \) and (using Lemma 7), that \( P(X_n = |\uparrow\rangle \text{ eventually}) = 1 \). In the situation where the measurement outcomes stabilize, \( X_n \to |\uparrow\rangle \), we can determine the asymptotic state of the scatterer \( S \) from Lemma 14. It is given by \( \omega_\infty(A) = \omega_{\text{in}}(A) \), \( A \in \mathcal{B}(\mathbb{C}^2) \). The above ‘asymptotic transparency’ statement follows.

**Large deviations for the mean.** The logarithmic moment generating function [23] is defined by

\[
\Lambda(\alpha) = \lim_{n \to \infty} \frac{1}{n} \log \mathbb{E}[e^{\alpha X_n}],
\]

for \( \alpha \in \mathbb{R} \) s.t. the limit exists as an extended real number. Using expression (66) and Theorem 15 (with \( p = 1 \)), we find that \( \Lambda(\alpha) = \log \omega_{\text{in}}(e^{\alpha M}) \), for \( \alpha \in \mathbb{R} \). The Legendre transformation of \( \Lambda(\alpha) \),

\[
\Lambda^*(x) = \sup_{\alpha \in \mathbb{R}} \alpha x - \Lambda(\alpha),
\]

\( x \in \mathbb{R} \), is called the *rate function*. Its usefulness in the present context is due to the Gärtner-Ellis theorem [23], which asserts that for any closed set
$F \subset \mathbb{R}$ and any open set $G \subset \mathbb{R}$, we have
\[
\limsup_{n\to\infty} \frac{1}{n} \log P(\overline{X}_n \in F) \leq - \inf_{x \in F} \Lambda^*(x)
\]
\[
\liminf_{n\to\infty} \frac{1}{n} \log P(\overline{X}_n \in G) \geq - \inf_{x \in G \cap F} \Lambda^*(x).
\] (38)

Here, $\mathcal{F}$ denotes the set of ‘exposed points of $\Lambda^*$’. A stronger “full large deviation principle” holds if an additional condition, called steepness of $\Lambda$, is verified. (See [23], Theorem 2.3.6 (c).) However, this condition does not hold in the present example. (Indeed, $|\Lambda'(\alpha)| \not\to \infty$ as $\alpha \to \pm \infty$, as is apparent from the explicit form $\Lambda(\alpha) = \log \omega_{\text{in}}(e^{\alpha M})$.)

**Proposition 16** Suppose that $\text{Var}(M) = \omega_{\text{in}}(M^2) - \omega_{\text{in}}(M)^2$, the variance of $M$ in the state $\omega_{\text{in}}$, does not vanish. Then $\Lambda^*$ is holomorphic at $x = \omega_{\text{in}}(M)$, and
\[
\Lambda^*(x) = \frac{(x - \omega_{\text{in}}(M))^2}{2 \text{Var}(M)} + O((x - \omega_{\text{in}}(M))^4).
\]

**Proof.** Note that $\Lambda$ is twice differentiable, and the second derivative w.r.t. $\alpha$ of the argument of the supermum in (37) is less than or equal to zero. Therefore, for fixed $x$, the supremum is taken at $\alpha \in \mathbb{R}$ satisfying
\[
x = \Lambda'(\alpha) = \frac{\omega_{\text{in}}(Me^{\alpha M})}{\omega_{\text{in}}(e^{\alpha M})}. \tag{39}
\]

For $\alpha = 0$ we have $x = \omega_{\text{in}}(M)$. If $\Lambda''(0) = \text{Var}(M) \neq 0$, then equation (39) has an implicit solution $\alpha = \alpha(x)$, locally around $x = \omega_{\text{in}}(M)$. Since $\Lambda'(\alpha)$ is holomorphic at $\alpha = 0$, the implicit solution is holomorphic at $x = \omega_{\text{in}}(M)$ (see e.g. [32], p.163, equation (12.4)). The Taylor expansion of (39) is
\[
x = \omega_{\text{in}}(M) + \alpha \text{Var}(M) + c\alpha^2 + O(\alpha^3), \tag{40}
\]
where $c = \frac{1}{2}\{\omega_{\text{in}}(M^3) - 3\omega_{\text{in}}(M^2)\omega_{\text{in}}(M) + 2\omega_{\text{in}}(M^3)\}$. We solve equation (40) implicitly for $\alpha = \alpha(x)$, which is the point where the supremum in (37) is taken. The explicit formula for the supremum given in Proposition 16 follows. $\blacksquare$

**Example: Measuring the outgoing spin angle.** Since $\omega_{\text{in}}$ is the state ‘spin up’, we have $\omega_{\text{in}}(M) = M_{11}$ and $\text{Var}(M) = |M_{12}|^2$. Imagine an experiment
where we measure the angle of the spins as they exit the scattering process. Let \( \theta \in [0, \pi/2] \) and \( \phi \in [0, 2\pi) \) be the angles measuring the altitude (\( \theta = 0 \) is spin up) and azimuth (\( \phi = 0 \) is the plane orthogonal to the axis of the cavity). The measurement operator “spin in direction (\( \theta, \phi \))” is given by

\[
M = \begin{bmatrix}
\cos \theta & e^{-i\phi} \sin \theta \\
e^{i\phi} \sin \theta & -\cos \theta
\end{bmatrix},
\]

see e.g. [20], Chapitre IV, (A-19). The eigenvectors of \( M \) associated to the eigenvalues \( \pm 1 \) of \( M \) are

\[
\chi_+ = e^{-i\phi/2} \cos(\theta/2) \varphi_1 + e^{i\phi/2} \sin(\theta/2) \varphi_2 \\
\chi_- = -e^{-i\phi/2} \sin(\theta/2) \varphi_1 + e^{i\phi/2} \cos(\theta/2) \varphi_2.
\]

The eigenprojection \( E_+ \) measures the spin in the positive direction (\( \theta, \phi \)). By using Lemma 7 is easy to see that

\[
P(X_n \text{ is in direction (\( \theta, \phi \)) eventually}) = \begin{cases}
1 & \text{if } \theta = 0 \\
0 & \text{if } \theta \neq 0.
\end{cases}
\]

This is another manifestation of the asymptotic transparency of the cavity.

We obtain from theorem 5 (with \( \mu_\infty = \cos \theta \)) that for any \( \epsilon > 0 \),

\[
\lim_{n \to \infty} P(|X_n - \cos \theta| \geq \epsilon) = 0.
\]

The speed of convergence can be estimated using (38) and Proposition 16. It is easy to see that the logarithmic generating function and the rate function associated to the shifted random variable \( X_n - \cos \theta \) are given by \( \Lambda_{\text{shift}}(\alpha) = \Lambda(\alpha) - \alpha \cos \theta \) and \( \Lambda^*_{\text{shift}}(x) = \Lambda^*(x + \cos \theta) \), respectively. Next, we note that all points in the vicinity of zero belong to the set \( \mathcal{F}_{\text{shift}} \), the set of exposed points of \( \Lambda^*_{\text{shift}} \). Indeed, if \( x = \Lambda^*_{\text{shift}}(\alpha) \) for some \( \alpha \in \mathbb{R} \), then \( x \in \mathcal{F}_{\text{shift}} \) ([23], Lemma 2.3.9). But \( x = 0 = \Lambda^*_{\text{shift}}(0) \), and \( \Lambda^*_{\text{shift}} \) is invertible around zero (as \( \Lambda''_{\text{shift}}(0) \neq 0 \)). This shows that \( \mathcal{F}_{\text{shift}} \) contains a neighbourhood of the origin.

Take \( 0 < \epsilon < \epsilon' \ll 1 \), set \( G = (-\epsilon', -\epsilon) \cup (\epsilon, \epsilon') \), and let \( F \) be the closure of \( G \). Then \( \inf_{x \in F} \Lambda^*_{\text{shift}}(x) = \inf_{x \in G \cap \mathcal{F}_{\text{shift}}} \Lambda^*_{\text{shift}}(x) = \frac{\epsilon^2}{2 \text{Var}(M)} + O((\epsilon')^4) \). (We use Proposition 16.) Combining this with the two bounds (38) (for the shifted random variable), we obtain

\[
P(\epsilon \leq |X_n - \cos \theta| \leq \epsilon') \sim \exp \left[ -n \left\{ \frac{\epsilon^2}{2 \sin^2 \theta} + O((\epsilon')^4) \right\} \right], \quad n \to \infty,
\]

which is a large deviation statement for the average \( X_n \).
4 Proofs

4.1 Proof of Theorems 1 and 8

Theorem 8 is a stronger version of Theorem 1, so it suffices to prove the former.

Let \( A \in \sigma(X_k, \ldots, X_l) \) and \( B \in \sigma(X_m, \ldots, X_n) \). The range \( \Sigma \) of the \( X_n \) is finite, so \( \sigma(X_k, \ldots, X_l) \) consists of the collection of all sets of the form \( \{ \omega : (X_k(\omega), \ldots, X_l(\omega)) \in H \} \), where \( H \subseteq \Sigma^{l-k+1} \) ([8], Thm. 5.1). Therefore we have

\[
A = \bigcup_{j=1}^{J} X^{-1}_k(\{ s^{(j)}_k \}) \cap \cdots \cap X^{-1}_l(\{ s^{(j)}_l \}) =: \bigcup_{j=1}^{J} A_j
\]

\[
B = \bigcup_{i=1}^{I} X^{-1}_m(\{ s^{(i)}_m \}) \cap \cdots \cap X^{-1}_n(\{ s^{(i)}_n \}) =: \bigcup_{i=1}^{I} B_i,
\]

where \( s^{(j)}_r, s^{(i)}_r \in \Sigma, \) and \( A_i \cap A_j = \emptyset, B_i \cap B_j = \emptyset \) for \( i \neq j \). Thus,

\[
P(A) = \sum_j P(A_j), \quad P(B) = \sum_i P(B_i), \quad P(A \cap B) = \sum_{i,j} P(A_j \cap B_i).
\]

Setting \( T_{m}^{(n)} := T_{s^{(n)}_m} \), we have

\[
P(A \cap B) = \sum_{i,j} \langle \Psi_S, T^{k-1}T^{(j)}_k \cdots T^{(j)}_l T^{m-1}T^{(i)}_m \cdots T^{(i)}_n \Psi_S \rangle.
\] (41)

We now approximate \( T^{m-l-1} \) by its value for large \( m - l \). To do so, let \( P_S \) denote the Riesz spectral rank-one projection onto the eigenvalue one of \( T \), and let \( \overline{P}_S = 1_S - P_S \). We have \( P_S = |\psi_S\rangle\langle \psi_S^*| \), where \( \psi_S^* \in \mathcal{H}_S \) satisfies \( T^* \psi_S^* = \psi_S^* \) and \( \langle \psi_S^*, \psi_S \rangle = 1 \). The operator \( T \) has the spectral representation [26], 185

\[
T = |\psi_S\rangle\langle \psi_S^*| + \sum_{r=1}^{d} \{ z_r P_r + D_r \},
\] (42)

where \( P_r \) is the Riesz projection associated to the eigenvalue \( z_j \) and \( D_r \) is the associated eigen-nilpotent. We have \( D_r^{\nu_r} = 0 \) (\( \nu_r \) is the index of \( z_r \)).
Consequently,

\[ T^k = |\psi_S\rangle\langle \psi_S^*| + \sum_{r=1}^{d} \sum_{q=0}^{\nu_r-1} \binom{k}{q} z_r^{k-q} P_r D_q^r =: |\psi_S\rangle\langle \psi_S^*| + R_k. \] (43)

Note that it suffices to consider the nonzero \( z_r \) in (43). Using the bound \( \sum_{q=0}^{\nu_r-1} \binom{k}{q} \leq \nu k^{\nu-1} \leq k^\nu \), and the fact that for any \( \epsilon > 0 \) there is a constant \( C_\epsilon \) s.t. \( k^\nu \leq C_\epsilon e^{\epsilon k} \) for all \( k \geq 1 \), we obtain

\[ ||R_k|| \leq C_\epsilon e^{\epsilon k} \max_{1 \leq r \leq d} |z_r|^k \leq C_\epsilon e^{\epsilon k} (1 - \gamma)^k, \] (44)

where we invoke Assumption A, \( |z_r| \leq 1 - \gamma \). We now replace \( T_m - l - 1 \) in (41) using (43),

\[ P(A \cap B) = P(A) \sum_i \langle \psi_S, T^{(i)}_m \ldots T^{(i)}_n \psi_S \rangle + \sum_{i,j} \langle \psi_S, T^{k-1}_k T^{(j)}_i \ldots T^{(i)}_l R_{m-l-1} T^{(i)}_m \ldots T^{(i)}_n \psi_S \rangle. \] (46)

The sum in (45) equals

\[ \sum_i \langle \psi_S, T^{(i)}_m \ldots T^{(i)}_n \psi_S \rangle = \sum_i \langle \psi_S, (|\psi_S\rangle\langle \psi_S^*|) T^{(i)}_m \ldots T^{(i)}_n \psi_S \rangle \\
= \sum_i \langle \psi_S, (T^{m-1} - R_{m-1}) T^{(i)}_m \ldots T^{(i)}_n \psi_S \rangle \\
= P(B) - \sum_i \langle \psi_S, R_{m-1} T^{(i)}_m \ldots T^{(i)}_n \psi_S \rangle. \] (47)

**Lemma 17** There is a constant \( C \) s.t. \( ||\sum_i T^{(i)}_m \ldots T^{(i)}_n|| \leq C \), independently of the range of values of \( i, m, n \) and of the \( s_l^{(i)} \) defining the \( T^{(i)}_l \).

We give a proof below.Lemma 17 together with (45), (46) and (47) gives

\[ |P(A \cap B) - P(A)P(B)| \leq CP(A)\|R_{m-1}\| \\
+ \sum_{i,j} \langle \psi_S, T^{k-1}_k T^{(j)}_i \ldots T^{(i)}_l R_{m-l-1} T^{(i)}_m \ldots T^{(i)}_n \psi_S \rangle. \] (48)
The remainder $R_{m-l-1}$ is given by the sum in (43) (with $k = m - l - 1$). We expand
\[ P_r D_r^q = \sum_{s,s'} \omega(s,s') |\chi_s\rangle \langle \chi_{s'}| \]
in an orthonormal basis $\{\chi_s\}$, where $\omega(s,s') \in \mathbb{C}$ are matrix elements (also depending on $r, q$). The modulus of the sum in (48) is bounded above by
\[ C \sum_{r=1}^d \sum_{q=0}^{m-l-1} \binom{m-l-1}{q} |z_r|^{m-l-1-q} \sum_{s,s'} |\omega(s,s')| \]
\[ \times \sum_j \left| \langle \psi_S, T^{k-1}_r T^{(j)}_{r-1} \cdots T^{(j)}_{r-l} \chi_s \rangle \right|. \] (50)

**Lemma 18** Let $A_s \in \mathfrak{M}_S$ be the unique operator s.t. $\chi_s = A_s \psi_S$. We have
\[ \sum_j \left| \langle \psi_S, T^{k-1}_r T^{(j)}_{r-1} \cdots T^{(j)}_{r-l} \chi_s \rangle \right| \leq \|A_s\| P(A). \]

We give a proof of Lemma 18 below. Using the result of the Lemma in (50) we obtain, for any $\epsilon > 0$, the upper bound $P(A) C_e^{(m-l)(1-\gamma)^m}$ for the sum in (48). Thus for all $\epsilon > 0$ there is a $C_\epsilon$ s.t.
\[ |P(A \cap B) - P(A) P(B)| \leq C_\epsilon P(A) \left\{ (1-\gamma)^m e^{\gamma \epsilon m} + (1-\gamma)^{m-l} e^{(m-l)} \right\}. \]
Since $m \mapsto (1-\gamma)^m e^{\gamma m}$ is decreasing (for $\epsilon$ small enough), we get (26). This concludes the proof of Theorem 8 modulo proofs of Lemmata 17 and 18.

**Proof of Lemma 17.** Any vector $\psi \in \mathcal{H}_S$ is of the form $\psi = \tilde{A} \Psi_S$, for some $\tilde{A} \in \mathfrak{M}_S$. We have
\[ \left\| \sum_{i} T^{(i)}_m \cdots T^{(i)}_n \tilde{A} \Psi_S \right\| = \sup_{\|\tilde{B} \Psi_S\|=1} \left| \langle \tilde{B} \Psi_S, \sum_{i} T^{(i)}_m \cdots T^{(i)}_n \tilde{A} \Psi_S \rangle \right| \]
\[ = \sup_{\|\tilde{B} \Psi_S\|=1} \left| \langle \tilde{B} \Psi_S \otimes B \Psi_{P_m} \otimes \cdots \otimes B \Psi_{P_n}, e^{i\tau L_m} \cdots e^{i\tau L_n} Q \tilde{A} \times \right| \]
\[ \times e^{-i\tau L_m} \cdots e^{-i\tau L_n} \tilde{B} \Psi_S \otimes B \Psi_{P_m} \otimes \cdots \otimes B \Psi_{P_n} \rangle, \]
where $Q := \sum_{i} E_m(\{s^{(i)}_m\}) \cdots E_n(\{s^{(i)}_n\})$ is a selfadjoint projection, $\|Q\| = 1$. Thus $\| \sum_{i} T^{(i)}_m \cdots T^{(i)}_n \tilde{A} \Psi_S \| \leq \|\tilde{A}\|$. The result now follows from the uniform boundedness principle. This proves Lemma 17.
Proof of Lemma 18. By using the definition \( T_k^{(j)} = P_k B^* B \exp(iK E_j^{(j)}) P_k \), where \( E_k^{(j)} = E_k(\{s_k^{(j)}\}) \), see (20), we see that
\[
\langle \psi_S, T_k^{k-1} T_k^{(j)} \cdots T_l^{(j)} A_s \psi_S \rangle = \langle \Psi^{(j)}, A_s \Psi^{(j)} \rangle,
\]
where \( \Psi^{(j)} = E^{(j)}_k \cdots E^{(j)}_l \exp(iL_k \cdots e^{-iL_k} \psi_S \otimes B \psi_p \cdots \otimes B \psi_p \) \( l \) probes). To arrive at the form of \( \Psi^{(j)} \), we replace the action of \( \exp(iK) \) in the operators \( T_k^{(j)} \) (see just above) by the action of the Liouville operators \( \exp(iL_k \cdot e^{-iL_k} \psi_S \otimes B \psi_p \cdots \otimes B \psi_p \), see (11), (19). Since a positive linear functional on \( M_S \) is bounded, with norm equal to its value for the observable \( \mathbb{1} \) ([16] Prop. 2.3.11), we have
\[
\left| \langle \psi_S, T_k^{k-1} T_k^{(j)} \cdots T_l^{(j)} A_s \psi_S \rangle \right| \leq \|A_s\| \langle \psi_S, T_k^{k-1} T_k^{(j)} \cdots T_l^{(j)} \psi_S \rangle.
\]
Note that the scalar product on the r.h.s. is non-negative, as it is a probability. It now follows that
\[
\sum_j \left| \langle \psi_S, T_k^{k-1} T_k^{(j)} \cdots T_l^{(j)} \chi_s \rangle \right| \leq \|A_s\| \sum_j \langle \psi_S, T_k^{k-1} T_k^{(j)} \cdots T_l^{(j)} \psi_S \rangle,
\]
the latter sum being \( P(A) \).

4.2 Proof of Theorem 3
We have \( P(X_n \in S \text{ eventually}) = \langle \psi_S, \Pi S \psi_S \rangle \), where \( \Pi \) and \( \Pi S \) are the Riesz projections of \( T \) and \( T_S \) respectively, associated to the point 1, see Lemma 7. We know that \( T_S = \omega_{\text{in}}(E_S) + O(\|V\|) \). Since \( E_S \) is an orthogonal projection and \( \omega_{\text{in}} \) is a state, \( \omega_{\text{in}}(E_S) \neq 1 \) means \( \omega_{\text{in}}(E_S) < 1 \). Thus, for small enough \( V \), \( T_S \) does not have an eigenvalue at the point 1, and consequently \( \Pi S = 0 \).

4.3 Proof of Theorem 4
The proof is in two parts. First we show that the expectation of \( f_m \) has the indicated limit. Then we upgrade this convergence to almost everywhere convergence, using the decay of correlations, Theorem 1.

Step 1. We show convergence of the expectation of the frequency,
\[
F_m := \mathbb{E}[f_m] = \lim_{n \to \infty} \frac{1}{n} \mathbb{E} \left[ \text{number of } k \in \{1, \ldots, n\} \text{ s.t. } X_k = m \right]. \tag{51}
\]
Let \( m \in \text{spec}(M) \) be fixed and define \( \chi \) by \( \chi(x) = 1 \) if \( x = m \) and \( \chi(x) = 0 \) otherwise. Then the expectation in (51) equals

\[
\sum_{m_1, \ldots, m_n} \left[ \sum_{j=1}^{n} \chi(m_j) \right] P(X_1 = m_1, \ldots, X_n = m_n)
\]

\[
= \sum_{j=1}^{n} \sum_{m_k, k \neq j} P(X_1 = m_1, \ldots, X_j = m, \ldots, X_n = m_n)
\]

\[
= \sum_{j=1}^{n} \langle \psi_S, T_j^{-1}T_m\psi_S \rangle.
\]

Therefore, since \( \frac{1}{n} \sum_{j=0}^{n-1} T^j \to \Pi \) as \( n \to \infty \) (the ergodic projection which equals the Riesz projection of \( T \) associated to one, see Appendix A), we obtain

\[
F_m = \langle \psi_S, \Pi T_m \psi_S \rangle.
\]

Recall that \( T_m = PB^*Be^{i\tau K}E_mP \), where \( K = L_0 + \lambda I \). To expand (52) in a power series in \( \lambda \), we use

\[
e^{i\tau K} = e^{i\tau L_0} \sum_{n \geq 0} (i\lambda)^n \int_0^\tau d\tau_1 \cdots \int_0^{\tau_n} d\tau_n I(\tau_1) \cdots I(\tau_n),
\]

where \( I(s) = e^{-isL_0}Ie^{isL_0} \). The series converges in operator norm, for all \( \lambda \in \mathbb{R} \). This gives \( T = e^{i\tau L_S} + T' \), where

\[
T' = e^{i\tau L_S} \sum_{n \geq 1} (i\lambda)^n \int_0^\tau d\tau_1 \cdots \int_0^{\tau_{n-1}} d\tau_n PB^*BI(\tau_1) \cdots I(\tau_n)P.
\]

For \( z \in \mathbb{C} \) in the resolvent sets of \( e^{i\tau L_S} \) and \( T \), we have

\[
(T - z)^{-1} = (e^{i\tau L_S} - z)^{-1} \sum_{n \geq 0} (-T'(e^{i\tau L_S} - z)^{-1})^n.
\]

The last series converges for \( \lambda \) small enough (\( T' \) is of order \( \lambda \)). Using (55) and (54) in the expression \( \Pi = \frac{1}{2\pi i} \oint (T - z)^{-1}dz \), where the integral is on a small circle encircling the point one of the complex plane, but no other
spectrum of $T$, we obtain
\[
\Pi T_m
\]
\[
= \frac{-1}{2\pi i} \oint dz (e^{irL_S} - z)^{-1} \left[ I - i\lambda e^{iL_S} \int_0^\tau d\tau_1 PB^* BI(\tau_1)P(e^{iL_S} - z)^{-1}
+ \lambda^2 e^{iL_S} \int_0^\tau d\tau_1 \int_0^{\tau_1} d\tau_2 PB^* BI(\tau_1)I(\tau_2)P(e^{iL_S} - z)^{-1}
- \lambda^2 \left( e^{iL_S} \int_0^\tau d\tau_1 PB^* BI(\tau_1)P(e^{iL_S} - z)^{-1} \right)^2
+ O(\lambda^3) \right] e^{iL_S}
\]
\[
\times PB^* B \left[ I + i\lambda \int_0^\tau d\tau_1 I(\tau_1) - \lambda^2 \int_0^\tau d\tau_1 \int_0^{\tau_1} d\tau_2 I(\tau_1)I(\tau_2) + O(\lambda^3) \right] E_mP.
\]
By taking averages in the state $\psi_S$, we obtain $F_m$. The term of order $\lambda^0$ of $F_m$ is thus (use that $L_S\psi_S = 0$)
\[
\frac{-1}{2\pi i} \oint dz \langle \psi_S, (e^{iL_S} - z)^{-1} e^{iL_S} PB^* BE_mP\psi_S \rangle = \omega_{in}(E_m).
\]

There are, at first sight, two terms of order $\lambda^1$. However, the one containing two resolvents $(e^{iL_S} - z)^{-1}$ vanishes, as by taking the average in the state $\psi_S$, each resolvent reduces to $(1 - z)^{-1}$, and $\frac{-1}{2\pi i} \oint dz (1 - z)^{-2} = 0$. The non-vanishing term of order $\lambda^1$ in $F_m$ is $i\lambda \int_0^\tau \langle \psi_S, PB^* Be^{-i\tau L_0} e^{i\tau L_0} E_mP\psi_S \rangle d\tau_1$.

Writing $I = V - J\Delta^{1/2}VJ\Delta^{1/2}$, for $V \in \mathfrak{M}_S \otimes \mathfrak{M}_P$, the integrand of the last integral has the form
\[
\langle \psi_S, PB^* B\{V(\tau_1) - J\Delta^{1/2}V(\tau_1)J\Delta^{1/2}\} E_mP\psi_S \rangle
= \langle \psi_S \otimes \psi_p, B^* B\{V(\tau_1) - J\Delta^{1/2}V(\tau_1)J\Delta^{1/2}\} E_m\psi_S \otimes \psi_p \rangle
= \langle \psi_S \otimes \psi_p, B^* B\{V(\tau_1)E_m - E_mV(\tau_1)\}\psi_S \otimes \psi_p \rangle
= \omega_S \otimes \omega_{in}(\{V(\tau_1), E_m\}).
\]

The form of $f'_m$ given in Theorem 4 follows. We point out that in the above four formulas, we use the same notation for the operator $V$ and its representation on the GNS space. This should not lead to any confusion, as the context makes it clear which operator we mean (and it has the advantage that we do not have to introduce further notation for the representation map).

This finishes the first step of the proof, showing that
\[
\lim_{n \to \infty} \frac{1}{n} \mathbb{E} \left[ \text{number of } k \in \{1, \ldots, n\} \text{ s.t. } X_k = m \right] = \nu,
\]
\[\text{(56)}\]
where \( \nu := \omega_{\text{in}}(E_m) + \lambda f'_m + O(\lambda^2) \) and \( f'_m \) is defined in Theorem 4.

**Step 2.** We upgrade (56) to almost sure convergence, by using a classical fourth moment method. Introduce the random variable

\[
Z_n = \{ \text{number of } k \in \{1, \ldots, n\} \text{ s.t. } X_k = m \}.
\]

We are going to show below that, for any \( \epsilon > 0 \),

\[
\sum_{n \geq 1} P(|Z_n/n - E[Z_n]/n|^2 \geq \epsilon) < \infty. \tag{57}
\]

Then by the (first) Borel-Cantelli lemma, \( P(|Z_n/n - E[Z_n]/n|^2 \geq \epsilon \text{ i.o.}) = 0 \), i.e., there is a set \( \Sigma \) of measure one, s.t. for all \( \omega \in \Sigma \), there exists a \( k \) with

\[
|Z_n/n - E[Z_n]/n|^2 < \epsilon, \quad \forall n \geq k. \tag{58}
\]

From (56) we know that \( E[Z_n]/n \) converges to \( \nu \), so (58) implies that \( Z_n/n \) converges to \( \nu \) almost everywhere. It remains to prove the summability (57).

By Chebyshev’s inequality,

\[
P(|Z_n/n - E[Z_n]/n|^2 \geq \epsilon) \leq \frac{1}{\epsilon^2 n^4} E[|Z_n - E[Z_n]|^4]. \tag{59}
\]

We get an upper bound on the r.h.s. Set

\[
Z_n - E[Z_n] = \sum_{i=1}^n \{ \chi(X_i = m) - P(X_i = m) \} =: \sum_{i=1}^n Y_i,
\]

so that \( E[Y_i] = 0 \). Here, \( \chi(X_i = m) = 1 \) if \( X_i = m \) and \( \chi(X_i = m) = 0 \) otherwise. We have

\[
E[|Z_n - E[Z_n]|^4] \leq \sum_{i,j,k,l=1}^n E[|Y_i||Y_j||Y_k||Y_l|]. \tag{60}
\]

The idea is to control the sum by using that if the indices \( i, j, k, l \) are far apart from each other, then the expectation is small due to the decay of correlations. Thus only a few terms in the sum contribute to its value. Let \( \Lambda \geq 0 \) be a given integer “length” scale. All combinations of values of the four indices \( i, j, k, l \) belong to exactly one of the following five cases: (1) all indices lie inside an interval of length \( \Lambda \), (2) three indices lie within \( \Lambda \), the
fourth does not, (3) two pairs of indices are separated by more than $\Lambda$, but within each pair, the indices are apart at most by $\Lambda$, (4) one pair lies within $\Lambda$, the other two indices are apart from each other and from the close pair by more than $\Lambda$, (5) all four indices are apart from each other by at least $\Lambda$.

Let $n_j, j = 1, \ldots, 5$, be the number of terms in the sum that satisfy cases (1) to (5) above. We have

$$n_1 \leq 4! n^3, n_2 \leq 4! n^2 \Lambda^2, n_3 \leq 4! n^2 \Lambda^2, n_4 \leq 4! n^3 \Lambda, n_5 \leq 4! n^4.$$ 

Next, due to the separation of indices and Theorem 1, each term of case (5) is of the form

$$E[|Y_i||Y_j||Y_k||Y_l|] \leq E[|Y_i|] E[|Y_j|] E[|Y_k|] E[|Y_l|] + O(e^{-\gamma \Lambda}) = O(e^{-\gamma \Lambda}),$$

as $E[|Y_i|] = 0$ for all $i$. Similarly, each term of cases (4), (3) and (2) have the same upper bound. Each term of case (1) is bounded above by one. We conclude that

$$E[|Z_n - E[Z_n]|^4] \leq C(n_2 + n_3 + n_4 + n_5)e^{-\gamma \Lambda} + n_1.$$ 

Choose now $\Lambda = n^\alpha$, with $0 < \alpha < 2/3$. Then using the above bounds on $n_j$, we have that $n^{-4} E[|Z_n - E[Z_n]|^4]$ is summable over $n \geq 1$, i.e., by (59), the inequality (57) holds. This completes the proof.

### 4.4 Proof of Theorem 5

We prove that $\lim_{n \to \infty} E[\overline{X}_n - \mu_\infty] = 0$, with

$$\mu_\infty = \langle \psi_S, \Pi PB^* Be^{i\tau K} MP \psi_S \rangle. \quad (61)$$

By proceeding as in step 2 of the proof of Theorem 4 above, one upgrades this to almost everywhere convergence, $\overline{X}_n \to \mu_\infty$ a.e. Since

$$E[\overline{X}_n] = \frac{1}{n} \sum_{m_1, \ldots, m_n} (m_1 + \cdots + m_n) P(X_1 = m_1, \ldots, X_n = m_n)$$

and $\frac{1}{n} \sum_{j=1}^n T_j^{-1} \to \Pi = \Pi$ as $n \to \infty$, we obtain the law of large numbers with (61). To obtain the expansion in $\lambda$ for $\lambda$ small, we find the Taylor series of $\Pi = \Pi(\lambda)$ and $e^{i\tau K(\lambda)}$, as in (55) and (53). Note that this calculation is the same as that carried out in Section 4.3 to calculate the expansion of $F_m$, under the replacement of $E_m$ with $M$. The result follows.
A Ergodic and Riesz projections

Let $V$ be a power bounded operator on a Hilbert space $\mathcal{H}$, i.e., such that $\|V^n\| \leq M$ for a constant $M$ independent of $n \in \mathbb{N}$. Set $\mathcal{F} = \{ \varphi \in \mathcal{H} : V\varphi = \varphi \}$ and $\mathcal{R} = \text{Ran}(1 - V)$.

Lemma 19 We have $\mathcal{H} = \mathcal{F} + \overline{\mathcal{R}}$ (closure) with $\mathcal{F} \cap \overline{\mathcal{R}} = \{0\}$. Moreover, the projection onto $\mathcal{F}$ in this decomposition is the ergodic projection, $\tilde{\Pi} = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} V^n$.

Proof. Clearly $\frac{1}{N} \sum_{n=1}^{N} V^n \varphi = \varphi$ for all $N$ and all $\varphi \in \mathcal{F}$. Also, $\frac{1}{N} \sum_{n=1}^{N} V^n (1 - V) \psi = \frac{1}{N} V^n [\psi - V^n \psi] \to 0$ as $N \to \infty$, for any $\psi \in \mathcal{H}$. Thus, if $\varphi \in \mathcal{F} \cap \mathcal{R}$ then $\varphi = \frac{1}{N} \sum_{n=1}^{N} V^n \varphi \to 0$ as $N \to \infty$. This shows $\mathcal{F} \cap \mathcal{R} = \{0\}$. Similarly one shows that $\mathcal{F} \cap \overline{\mathcal{R}} = \{0\}$: let $\varphi \in \mathcal{F} \cap \overline{\mathcal{R}}$. Then $\varphi = \lim_k \varphi_k$, with $\varphi_k \in \mathcal{R}$. We have

$$\varphi = \frac{1}{N} \sum_{n=1}^{N} V^n \varphi = \frac{1}{N} \sum_{n=1}^{N} V^n \varphi_k - \frac{1}{N} \sum_{n=1}^{N} V^n (\varphi_k - \varphi).$$

The first equality holds since $\varphi \in \mathcal{F}$. Since $V$ is power bounded, the norm of the second sum on the r.h.s. is bounded above by $M \| \varphi - \varphi_k \|$ for some $M$ independent of $N, k$. The first sum on the r.h.s. converges to zero as $N \to \infty$, since $\varphi_k \in \mathcal{R}$. Thus, upon taking first $k$, then $N$ large enough, we see that $\varphi = 0$. This shows that $\mathcal{F} \cap \overline{\mathcal{R}} = \{0\}$.

The equality $\mathcal{H} = \mathcal{F} + \overline{\mathcal{R}}$ is equivalent to $\mathcal{H} = \overline{\mathcal{F} + \mathcal{R}}$. We have

$$\mathcal{F}^\perp = \overline{\text{Ran}(1 - V^*)}, \quad \mathcal{R}^\perp = \{ \varphi \in \mathcal{H} : V^* \varphi = \varphi \}$$

Let $\varphi \in (\mathcal{F} + \mathcal{R})^\perp$. Then $\varphi \in \mathcal{F}^\perp \cap \mathcal{R}^\perp$. However, $V^*$ is power bounded and thus, as above, $\mathcal{F}^\perp \cap \mathcal{R}^\perp = \{0\}$. This shows that $\mathcal{F} + \mathcal{R}$ is dense in $\mathcal{H}$. ■

We have $\{ \Pi = 0 \Leftrightarrow \mathcal{F} = \{0\} \} \Leftrightarrow \{1 \text{ is not an eigenvalue of } V \}$. Assume that there is a neighbourhood $U$ of 1 in the complex plane which does not contain any spectrum of $V$, except possibly the point 1. Let $\Pi$ be the Riesz spectral projection, $\Pi = \frac{1}{2\pi i} \oint_{\Gamma} (z - V)^{-1} dz$, where $\Gamma \subset U$ is a simple closed curve encircling 1. $\Pi$ acts as the identity on $\mathcal{F}$, as does $\Pi$. Let $\varphi = (1 - V) \chi \in \mathcal{R}$. Then $\Pi \varphi = (1 - V) \Pi \chi$. The operator $(1 - V)\Pi$ is the eigen-nilpotent associated to the eigenvalue 1 of $V$. If 1 is a semisimple eigenvalue of $V$, then
\( \Pi \varphi = 0 \) for all \( \varphi \in \mathcal{R} \), and by continuity, \( \Pi \mathcal{R} = \{0\} \), which coincides with the action of \( \tilde{\Pi} \) on \( \mathcal{R} \). If the eigenvalue 1 is not semisimple (the nilpotent is nonzero), then the action of \( \Pi \) on \( \mathcal{R} \) does not coincide with that of \( \tilde{\Pi} \). This shows the following result.

**Lemma 20** \( \Pi = \tilde{\Pi} \) if and only if 1 is a semisimple eigenvalue of \( V \) (or if 1 is not an eigenvalue of \( V \), in which case \( \Pi = \tilde{\Pi} = 0 \)).

**Lemma 21** Suppose that \( \dim \mathcal{H} < \infty \). If 1 is an eigenvalue of \( V \) then it is semisimple. We thus have \( \Pi = \tilde{\Pi} \).

**Proof.** Suppose 1 is an eigenvalue of \( V \). It suffices to show that \( z \mapsto (z - V)^{-1} \) has a simple pole at \( z = 1 \). We have \( (z - V)^{-1} = \frac{1}{z-1} \tilde{\Pi} + (z - \tilde{V})^{-1}(1 - \tilde{\Pi}) \), where \( \tilde{V} = (1 - \tilde{\Pi}) V (1 - \tilde{\Pi}) \mid_{\text{Ran}(1 - \tilde{\Pi})} \), so we only need to show that \( 1 \notin \text{spec}(\tilde{\Pi}) \). Suppose that \( 1 \in \text{spec}(\tilde{\Pi}) \) and take \( \varphi \in \text{Ran}(1 - \tilde{\Pi}) \) satisfying \( \tilde{V} \varphi = \varphi, \|\varphi\| = 1 \). Since \( \tilde{\Pi} V = V \tilde{\Pi} \) we have \( \tilde{V} \varphi = V \varphi \), so \( (V^n - 1) \varphi = 0 \) for all \( n = 1, 2, \ldots \). By applying \( \frac{1}{N} \sum_{n=1}^{N} \) and taking \( N \to \infty \), we obtain \( (\tilde{\Pi} - 1) \varphi = 0 \). This is in contradiction to \( \varphi \in \text{Ran}(1 - \tilde{\Pi}) \) with \( \|\varphi\| = 1 \).

\[ \square \]

**B Logarithmic moment generating function**

It is possible to give general conditions ensuring that the logarithmic moment generating function (36) exists. However, these conditions will be rather abstract (see before (66) below). They amount to knowing that a certain operator \( (R_{\lambda}(a)) \) given in (64) below) has a unique eigenvalue of largest modulus, with corresponding eigenprojection satisfying a non-vanishing overlap condition. We show in this section how perturbation theory can be applied to analyze the spectrum of the operator in question. The formulas established here can be used in the analysis of concrete systems. (An example being the Jaynes-Cummings model of section 3.)

To calculate the logarithmic moment generating function, (36), we write

\[
\mathbb{E}[e^{\alpha(X_1 + \cdots + X_n)}] = \sum_{m_1, \ldots, m_n} e^{\alpha(m_1 + \cdots + m_n)} P(X_1 = m_1, \ldots, X_n = m_n)
\]

\[
= \omega(e^{aM})^n \langle \psi, R_{\lambda}(a)^n \psi \rangle , \quad (63)
\]

34
where
\[
R_\lambda(\alpha) = \sum_m e^{\alpha m} T_{(m)} = PB^* B e^{i\tau K} A(\alpha) P
\]
(64)
\[
A(\alpha) = \frac{e^{\alpha M}}{\omega(e^{\alpha M})}.
\]
(65)
The scaling of \(A(\alpha)\) by \(1/\omega(e^{\alpha M})\) gives the normalization
\[
R_0(\alpha) = e^{i\tau L} S.
\]
The existence of the limit (36) is guaranteed if \(R_\lambda(\alpha)\) is diagonalizable and has a
unique eigenvalue \(\rho_+ (\lambda, \alpha)\) of largest modulus, and the corresponding Riesz
projection \(P_+(\lambda, \alpha)\) satisfies \(\langle \psi, P_+(\lambda, \alpha)\psi \rangle \neq 0\). In this case (36) and (63)
give that
\[
\Lambda(\alpha) = \log \omega(e^{\alpha M}) + \log \rho_+ (\lambda, \alpha).
\]
(66)
In concrete examples, it is usually not possible to explicitly evaluate \(\rho_+\)
(however, it is possible in the example considered in Section 3!), so pertur-
bation theory is in order. We analyze the analyticity properties of \(R_\lambda(\alpha)\) in
\(\lambda\) and \(\alpha\). Let \(P_j\) be the orthogonal spectral projections of \(M\). For \(\alpha \in \mathbb{C}\) we have
\[
\|A(\alpha) P\| = \frac{\sqrt{\sum_j e^{2m_j \text{Re} \alpha} \|P_j \psi_P\|^2}}{\|\sum_j e^{m_j \alpha} \|P_j \psi_P\|^2}.
\]
For \(\alpha \in \mathbb{R}\) there are constants \(0 < c \leq C < \infty\), independent of \(\alpha \in \mathbb{R}\), s.t.
\(c \leq \|A(\alpha) P\| \leq C\). This cannot be extended to all complex \(\alpha\), since otherwise
\(A(\alpha) P\) would have to be constant in \(\alpha\) by Liouville’s theorem of complex
analysis. However, if the imaginary part of \(\alpha\) is small, then the weighted
superposition of the \(\|P_j \psi_P\|^2\) of the denominator is still bounded away from
zero. The growth of the numerator and denominator as \(\text{Re} \alpha \to \pm \infty\) is the
same. Thus there is an \(\alpha_0 > 0\) s.t. if \(|\text{Im} \alpha| < \alpha_0\), then \(c' \leq \|A(\alpha) P\| \leq C'\)
for some \(0 < c' \leq C' < \infty\).
The operator \(R_\lambda(\alpha)\) is holomorphic in \((\lambda, \alpha) \in \mathbb{C} \times \{z : |\text{Im} z| < \alpha_0\}\),
and has the expansion
\[
R_\lambda(\alpha) = \sum_{n \geq 1} \lambda^n R^{(n)}(\alpha),
\]
(67)
with
\[
R^{(n)}(\alpha) = i^n e^{i\tau L} \int_0^\tau ds_1 \cdots \int_0^{s_{n-1}} ds_n P B^* B I(s_n) \cdots I(s_1) A(\alpha) P,
\]
(68)
and where $I(s) = e^{isL_0} I e^{-isL_0}$. The bound $\| R^{(n)}(\alpha) \| \leq \frac{\tau^n \|I\| n}{n!} \|B^* B\| \|A(\alpha)P\|$ implies that
\[
\sup_{|\text{Im } \alpha| < \alpha_0} \| R^{(n)}(\alpha) \| \leq C' \frac{\tau^n \|I\|^n}{n!}
\] (69)
for some $C' < \infty$. Thanks to this bound we can perform perturbation theory in $\lambda$ uniformly in $\alpha$ s.t. $|\text{Im } \alpha| < \alpha_0$.

**Proposition 22** There are constants $C$, $\lambda_1$, both independent of $\alpha \in \mathbb{C}$ with $|\text{Im } \alpha| < \alpha_0$ and of $\tau \geq 0$, s.t. if $|\lambda| < \lambda_1$, then
\[
\text{dist}\left( \text{spec}(R_\lambda(\alpha)), \text{spec}(e^{i\tau L}) \right) \leq C |\lambda| \tau.
\] (70)
Moreover, the group of eigenvalues associated to any two distinct eigenvalues $e^{i\tau_1}$, $e^{i\tau_2}$ of $R_0(\alpha) = e^{i\tau L}$ belong to disjoint balls centered at $e^{i\tau_1}$ and $e^{i\tau_2}$.

A proof is obtained from a straightforward estimate of the resolvent $(R_\lambda(\alpha) - z)^{-1}$ using the Neumann series and the fact that $\|(R_0(\alpha) - z)^{-1}\| = [\text{dist}(z, \text{spec}R_0(\alpha))]^{-1}$ since $R_0(\alpha)$ is normal. Due to (69), the motion of eigenvalues of $R_\lambda(\alpha)$ under variation of $\lambda$ is estimated for $|\lambda| < \lambda_1$, uniformly in $\alpha \in \mathbb{C}$, $|\text{Im } \alpha| < \alpha_0$, see [26] Section II §3.

**Acknowledgements.** We are grateful to Shannon Starr for pointing out to us the fourth moment method that we use in the proof of step 2 of Theorem 4.3. We have profitted from insightful discussions about this project with Walid Abou Salem, Laurent Bruneau, Alain Joye, Bruno Nachtergaele, Yoshiko Ogata, Shannon Starr. We also thank a referee for carefully reading our work and giving numerous helpful suggestions. In particular, we thank the referee for pointing out references [2, 10].

**References**


