Dynamical Localization of Quantum Walks in Random Environments

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Abstract

The dynamics of a one dimensional quantum walker on the lattice with two internal degrees of freedom, the coin states, is considered. The discrete time unitary dynamics is determined by the repeated action of a coin operator in $U(2)$ on the internal degrees of freedom followed by a one step shift to the right or left, conditioned on the state of the coin. For a fixed coin operator, the dynamics is known to be ballistic.

We prove that when the coin operator depends on the position of the walker and is given by a certain i.i.d. random process, the phenomenon of Anderson localization takes place in its dynamical form. When the coin operator depends on the time variable only and is determined by an i.i.d. random process, the averaged motion is known to be diffusive and we compute the diffusion constants for all moments of the position.

1 Introduction

The dynamics of Quantum Walks (QW for short) have become a popular topic in the Quantum Computing community as the simplest quantum generalization of classical random walks, see for example the reviews [3], [21], [24]. In the same way classical random walks play an important role in theoretical computer science, typically in search algorithms, QW provide a natural and fruitful extension in the study of quantum search algorithms, see e.g. [33], [4], [27] and the review [31] and references therein. On the other hand, QW also appeal to physicists interested in quantum dynamics. Indeed, QW can be considered as simple discrete dynamical systems governed by an effective unitary operator, not necessarily given as the exact exponential of i times a microscopic Hamiltonian. For a few models of this type, see e.g. [1], [28], [26], [10], [30], and [6], [9], [14], [17] for their mathematical analysis. Moreover, there are recent experimental realizations of QW dynamics: [19] showed that cold atoms trapped in optical lattices exhibit a QW for suitably monitored optical lattices and [35] show that the same is true for ions caught in monitored Paul traps.

While several types of QW have been defined and studied in different contexts, we will focus on the simplest one dimensional, discrete time QW on the lattice, defined in analogy with the classical random walk on the lattice. Consider a quantum walker on the lattice carrying two internal degrees of freedom (spin states), called the coin states in this context. The unit time step dynamics is defined by the action of a coin operator in $U(2)$ on the internal degrees of freedom followed by a one step shift to the right or left, conditioned on the state of the coin degree of freedom, see (2.5) below. As is well known, when the coin operator is identical at each time step, the QW typically exhibits ballistic dynamics

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due to translation invariance. Moreover, if the coin operator is chosen at random at each time step, which yields a non autonomous random dynamical system, then, typically, the averaged motion is diffusive, see *e.g.* [24], [25], see also [32].

In this paper we address the time-homogeneous case where the coin operator depends on the position of the walker on the lattice and is given by a sequence of random matrices in $U(2)$. This defines a QW in a *random environment*. Such situations were first considered numerically in [20], [34] for continuous time QW on graphs, that is, for an evolution operators generated by some random discrete Hamiltonian. The outcome of these numerical works provides evidence that Anderson localization, in a strong dynamical form, takes place. Discrete time QW in random environments were then considered in [22]. By contrast, the author establishes that a certain choice of random coin operators *does not* lead to Anderson localization. The random coin operators in [22] are given by a fixed unitary matrix whose diagonal elements carry random phases.

We consider here a family of i.i.d. random coin operators characterized by general requirements on the amplitude and transition probabilities to the right and to the left expressed as Assumptions **(a), (b)** and **(c)** below. This family is indexed by a real deterministic parameter and the randomness is determined by i.i.d. phases carried by all matrix elements of the coin operator, see (2.8). We prove that for all values of the deterministic parameter, dynamical localization takes place everywhere in the spectrum, for almost all realizations of coin operators.

The lack of Anderson localization in the model considered in [22] is explained by the existence of a gauge transformation that fully eliminates the randomness of the model. The set of spatially random unitary matrices we consider coincides with the one considered by [25] in the study of temporal randomness, in one space dimension.

For completeness, we briefly reconsider the study [25] of the non-autonomous case where the coin operators are random in time. We relate the averaged motion to that of a persistent random walk and provide a simple proof of the fact that for all $L \in \mathbb{N}$, the moments of order $2L$, at time n, behave as $D(2L)n^L$, for large n, with an explicit formula for the diffusion constants $D(2L) > 0$.

2 Setup and Main Results

The Hilbert space of pure states

$$\mathcal{H} = \mathbb{C}^2 \otimes l^2(\mathbb{Z}) \tag{2.1}$$

represents two internal degrees of freedom, also called a *coin*, with Hilbert space $\mathbb{C}^2$, and the *walker* whose position Hilbert space is $l^2(\mathbb{Z})$. We fix a canonical basis of $\mathbb{C}^2$ denoted by $|\uparrow\rangle, |\downarrow\rangle$, and the position basis consisting of vectors denoted by $|n\rangle$, $n \in \mathbb{Z}$ (eigenvectors of the position operator, the operator of multiplication by the variable n).

The dynamics of the system is composed of discrete steps, each step consisting of a unitary evolution of the coin (operator C on $\mathbb{C}^2$) followed by the motion of the walker, conditioned on the state of the coin. The latter step is determined by the action

$$|\uparrow\rangle \otimes |n\rangle \quad \mapsto \quad |\uparrow\rangle \otimes |n+1\rangle \tag{2.2}$$

$$|\downarrow\rangle \otimes |n\rangle \quad \mapsto \quad |\downarrow\rangle \otimes |n-1\rangle \tag{2.3}$$

extended by linearity to $\mathcal{H}$. This means that if the coin is pointing up the walker will move to the right one step, and if the coin points down the walker moves to the left. The action of (2.2), (2.3) is implemented by the unitary operator

$$S = \sum_{k\in\mathbb{Z}} \{P_\uparrow \otimes |k+1\rangle\langle k| + P_\downarrow \otimes |k-1\rangle\langle k|\}$$

where we have introduced the orthogonal projections

$$P_{\uparrow} = |\uparrow\rangle\langle\uparrow| \quad \text{and} \quad P_{\downarrow} = |\downarrow\rangle\langle\downarrow|. \tag{2.4}$$

The one step dynamics consists in tossing the quantum coin and then performing the coin dependent shift

$$U = S(C \otimes \mathbb{I}) \qquad \text{with} \qquad C = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \text{s.t.} \qquad C^* = C^{-1}. \tag{2.5}$$

Hence, if one starts form the state $|\uparrow\rangle \otimes |n\rangle$, the (quantum) probability to reach, in one time step, the site $|n+1\rangle$ equals $|a|^2$ whereas that to reach $|n-1\rangle$ equals $1-|a|^2$. Similarly, starting form $|\downarrow\rangle \otimes |n\rangle$, the probability to reach the site $|n-1\rangle$ equals $|a|^2$ and that to reach $|n+1\rangle$ is $1-|a|^2$. The evolution operator at time n reads U^n.

Despite the similarity of this dynamics with that of a classical random walk, there is nothing random in the quantum dynamical system at hand. The dynamics is invariant under translations on the lattice $\mathbb{Z}$, which hints at ballistic transport properties.

More precisely, let $X = \mathbb{I}\otimes x$ be the operator defined on its maximal domain in $\mathbb{C}^2 \otimes l^2(\mathbb{Z})$, where x is the position operator given by $x|k\rangle = k|k\rangle$, for all $k \in \mathbb{Z}$. For any $L > 0$ and any Ψ in the domain of X^L, we define

$$\langle X^L\rangle_\Psi(n) := \left\langle \Psi, U^{-n} X^L U^n \Psi \right\rangle. \tag{2.6}$$

The analog definition holds for $\langle |X|^L\rangle_\Psi(n)$. We recall in the appendix the proof that in our one dimensional setup we have

Lemma 2.1 (Deterministic walk) *Let Ψ belong to the domain of X^2. Then*

$$\lim_{n\to\infty} \frac{\langle X^2\rangle_\Psi(n)}{n^2} = B \geq 0$$

with $B = 0$ iff C is off diagonal.

Note: When C is off diagonal, one has complete localization, see the remarks following Theorem 2.2 below.

A QW in a non-trivial environment is characterized by coin operators that depends on the position of the walker: for every $k \in \mathbb{Z}$ we have a unitary C_k on $\mathbb{C}^2$, and the one step dynamics is given by

$$U = \sum_{k\in\mathbb{Z}} \{P_{\uparrow} C_k \otimes |k+1\rangle\langle k| + P_{\downarrow} C_k \otimes |k-1\rangle\langle k|\}. \tag{2.7}$$

We consider a *random environment* in which the coin operator C_k is a *random* element of $U(2)$, satisfying the following requirements:

Assumptions:
(a) $\{C_k\}_{k\in\mathbb{Z}}$ are independent and identically distributed $U(2)$-valued random variables.
(b) The quantum *amplitudes* of the transitions to the right and to the left are independent random variables.
(c) The quantum transition *probabilities* between neighbouring sites are deterministic and independent of the site.

The first assumption means that the sites $k \in \mathbb{Z}$ are independent. If the walker is localized at site n, then the state of the system is of the form $\varphi\otimes|n\rangle$ for some normalized $\varphi \in \mathbb{C}^2$. The probability amplitudes for transitions to states $\chi \otimes |n+1\rangle$ and $\chi \otimes |n-1\rangle$ ($\chi \in \mathbb{C}^2$

normalized) are $\langle \chi, \uparrow\rangle \langle \uparrow, C_n\varphi\rangle$ and $\langle \chi, \downarrow\rangle \langle \downarrow, C_n\varphi\rangle$, respectively. The second requirement says that these transitions are statistically independent. Finally, the last requirement says that randomness appears as phases (see after (2.5)) and that, in a certain sense, we remain close to a classical asymmetric random walk on the lattice.

We show in Lemma 2.5 below that under assumptions **(a), (b)** and **(c)**, we can consider without loss of generality

$$C_k = \begin{bmatrix} \mathrm{e}^{-\mathrm{i}\omega_k^{\uparrow}} t & -\mathrm{e}^{-\mathrm{i}\omega_k^{\uparrow}} r \\ \mathrm{e}^{-\mathrm{i}\omega_k^{\downarrow}} r & \mathrm{e}^{-\mathrm{i}\omega_k^{\downarrow}} t \end{bmatrix}, \qquad \text{with } 0 \le r, t \le 1 \text{ and } r^2 + t^2 = 1 \tag{2.8}$$

and with $\{\omega_k^{\uparrow}\}_{k\in\mathbb{Z}} \cup \{\omega_k^{\downarrow}\}_{k\in\mathbb{Z}}$ i.i.d. random variables. Thus, the action of the *random* operator U_ω is

$$\begin{aligned} U_\omega|\uparrow\rangle \otimes |n\rangle &= \mathrm{e}^{-\mathrm{i}\omega_n^{\uparrow}} t|\uparrow\rangle \otimes |n+1\rangle + \mathrm{e}^{-\mathrm{i}\omega_n^{\downarrow}} r|\downarrow\rangle \otimes |n-1\rangle \\ U_\omega|\downarrow\rangle \otimes |n\rangle &= -\mathrm{e}^{-\mathrm{i}\omega_n^{\uparrow}} r|\uparrow\rangle \otimes |n+1\rangle + \mathrm{e}^{-\mathrm{i}\omega_n^{\downarrow}} t|\downarrow\rangle \otimes |n-1\rangle. \end{aligned} \tag{2.9}$$

The elimination of the randomness in the model [22] is a consequence of Lemma 2.5 also.

To define the random phases properly, we introduce the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where $\Omega = \mathbb{T}^{\mathbb{Z}}$, $(\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z})$, $\mathcal{F}$ is the σ-algebra generated by cylinders of Borel sets, and $\mathbb{P} = \otimes_{k\in\mathbb{Z}}\mu$, where μ is a probability measure on $\mathbb{T}$. We note expectation values with respect to $\mathbb{P}$ by $\mathbb{E}$.

The random variables $\omega_k^{\uparrow}$ and $\omega_k^{\downarrow}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ are defined by

$$\omega_k^{\uparrow} : \Omega \to \mathbb{T}, \quad \omega_k^{\uparrow} = \omega_{2k}\ , \qquad \omega_k^{\downarrow} : \Omega \to \mathbb{T}, \quad \omega_k^{\downarrow} = \omega_{2k+1}. \tag{2.10}$$

A realization is thus denoted by $\omega = (\cdots, \omega_{-1}^{\uparrow}, \omega_{-1}^{\downarrow}, \omega_0^{\uparrow}, \omega_0^{\downarrow}, \omega_1^{\uparrow}, \omega_1^{\downarrow}, \cdots) \in \Omega$.

Main result. Our main result is Theorem 2.2 and its corollary. Let U_ω be the one step dynamics of a QW in a random environment defined by (2.7) with C_k, $k \in \mathbb{Z}$ given by (2.8), where $\{\omega_k^{\#}\}_{k\in\mathbb{N}, \#\in\{\uparrow,\downarrow\}}$ are the i.i.d. random variables defined in (2.10), distributed according to a measure μ on $\mathbb{T}$.

Theorem 2.2 (Spatial disorder) *Suppose that μ is an absolutely continuous measure $d\mu(\omega) = \tau(\omega)d\omega$, with density $\tau \in L^\infty(\mathbb{T})$, and that the support of μ contains a non-empty open set. Then, for any $r \in (0,1)$, there exist $C < \infty$, $\alpha > 0$ such that for any $j, k \in \mathbb{Z}$ and any $\sigma, \tau \in \{\uparrow, \downarrow\}$*

$$\mathbb{E}\left[\sup_{f\in C(\mathbb{S}),\, \|f\|_\infty \le 1} \left|\langle \sigma \otimes j, f(U_\omega)\, \tau \otimes k\rangle\right|\right] \le Ce^{-\alpha|j-k|}. \tag{2.11}$$

Remarks:
1) The extreme cases $r = 1$, $t = 0$ and $r = 0$, $t = 1$ lead to deterministic results that are addressed in the remarks following Lemma 2.5. As seen from (2.9), when $r = 0$, the up and down components are independent and propagate respectively to the right and to the left. We get ballistic behaviour for any deterministic choice of phases. When $t = 0$, the walker oscillates back and forth, switching its coin state. Hence localization takes place for any deterministic choice of phases.
2) Specializing the result to the function $f(z) = z^n$, $z \in \mathbb{S}$, (2.11) implies the following almost sure result on the evolution in time of the quantum moments of the QW, see [17].

Corollary 2.3 *There exists a set Ω_0 of probability one, such that for any $\omega \in \Omega_0$, any $L > 0$ and for any $\Psi \in \mathbb{C}^2 \otimes l^2(\mathbb{Z})$ of finite support, there exists $C_\omega < \infty$ with*

$$\sup_{n\in\mathbb{Z}} \langle |X|^L \rangle_\Psi(n) < C_\omega. \tag{2.12}$$

Remarks:
1) Another Corollary of Theorem 2.2 is that the spectrum of U_ω is pure point, almost surely, see [17].
2) The proof of this localization result leans on the fact that the unitary operator U_ω can be viewed as a doubly infinite five-diagonal band matrix on $l^2(\mathbb{Z})$, a set of operators first considered in [9]. Moreover, under Assumptions **a), b), c)**, the randomness appears in such a way that it is possible to adapt the Aizenman-Molchanov method [2] developed for the study of Anderson localization in the self-adjoint case to this unitary setup, along the lines of [17]. Indeed, we show below that U_ω takes the form $D_\omega S$ in an adapted basis, where D_ω is a diagonal unitary operator whose elements are i.i.d. phases and S is a deterministic unitary band matrix. This is the starting point of [17]. Nevertheless, the random operator at hand differs from those considered in [9, 17] by the form of the deterministic matrix S. This forces us to revisit the arguments based on the analysis of products of random transfer matrices and the associated Lyapunov exponents, coupled with fractional moment estimates of finite volume restrictions of the associated resolvent.

For the sake of comparison, we compute the asymptotics of the expectation of all (integer) moments of the position operator for a dynamics that is random *in time*. We follow [25] and consider the random evolution operator at time n given by

$$U_\omega(n,0) = U_n U_{n-1} \cdots U_2 U_1, \quad \text{where} \quad U_k = S(C_k \otimes \mathbb{I}), \tag{2.13}$$

where the i.i.d. random variables $\{C_k\}_{k\in\mathbb{Z}}$ are defined by (2.8). The random phases are assumed in this case to satisfy

$$\mathbb{E}(e^{-i\omega_k^\uparrow}) = \mathbb{E}(e^{-i\omega_k^\downarrow}) = 0, \tag{2.14}$$

but their common probability measure μ is otherwise arbitrary. We extend the results of [25] on the diffusive behavior of the QW in this setup by proving the following

Proposition 2.4 (Temporal disorder) *Assume the evolution operator is given by (2.13) with random phases (2.10) distributed according to a measure $d\mu$ such that (2.14) holds. Let $\Psi_0 = \varphi_0 \otimes |0\rangle$ be of norm one. Then, for any $L \in \mathbb{N}$, we have*

$$\lim_{n\to\infty} \frac{\mathbb{E}(\langle X^L \rangle_{\psi_0}(n))}{n^{L/2}} = D(L) \tag{2.15}$$

where

$$D(L) = \begin{cases} 0 & \text{if } L \text{ is odd} \\ 1\cdot 3\cdot 5 \cdots (L-1)(t^2/r^2)^{L/2} & \text{if } L \text{ is even.} \end{cases} \tag{2.16}$$

Remark:
As observed in [25], assumption (2.14) allows one to express the averaged motion as a *persistent* classical random walk on $\mathbb{Z}$, which is known to display a diffusive behaviour. Actually, using results showing convergence of certain Markov chains to Brownian motion, e.g. [7], one can deduce the above result. Nevertheless, by considering the associated generating function and by analyzing its large times asymptotics, we provide a short elementary derivation of the values of all diffusion constants.

Before we turn to the proofs of these results, we briefly investigate the invariance properties of the model stemming from its structure (2.7) and assumptions **(a), (b)** and **(c)**.

2.1 Structure of U_ω

Lemma 2.5 *Under the assumptions* **(a), (b)** *and* **(c)**, *the operator U_ω defined by (2.7) is unitarily equivalent to the one defined by the choice*

$$\begin{bmatrix} \mathrm{e}^{-\mathrm{i}\omega_k^\uparrow} t & -\mathrm{e}^{-\mathrm{i}\omega_k^\uparrow} r \\ \mathrm{e}^{-\mathrm{i}\omega_k^\downarrow} r & \mathrm{e}^{-\mathrm{i}\omega_k^\downarrow} t \end{bmatrix} \qquad \textit{where } 0 \le t, r \le 1 \textit{ and } r^2 + t^2 = 1 \tag{2.17}$$

and $\{\omega_k^\uparrow\}_{k\in\mathbb{Z}} \cup \{\omega_k^\downarrow\}_{k\in\mathbb{Z}}$ are iid random variables defined by (2.10), up to multiplication by a global deterministic phase.

Proof. We first note that assumption **(a)** implies that we can consider one C_k at a time and **(b)** and **(c)** imply that the randomness appears as phases only (see after (2.5)). The independence of the right and left probability amplitudes implies that the rows of C_k are independent random variables.

As C_k must be unitary for all realizations, the scalar product of its columns equals zero. This shows that the random phases of the elements on the same line are identical, i.e.

$$C_k = \begin{bmatrix} \mathrm{e}^{-\mathrm{i}\omega_k^\uparrow} a & \mathrm{e}^{-\mathrm{i}\omega_k^\uparrow} b \\ \mathrm{e}^{-\mathrm{i}\omega_k^\downarrow} c & \mathrm{e}^{-\mathrm{i}\omega_k^\downarrow} d \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{-\mathrm{i}\omega_k^\uparrow} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\omega_k^\downarrow} \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \tag{2.18}$$

with $C \equiv \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ unitary and independent of k. Parametrizing C as

$$C = e^{-i\theta} \begin{bmatrix} te^{-\mathrm{i}\alpha} & \mathrm{i}re^{\mathrm{i}\gamma} \\ \mathrm{i}re^{-\mathrm{i}\gamma} & te^{\mathrm{i}\alpha} \end{bmatrix}, \qquad \text{with } 0 \le r, t \le 1 \text{ and } \alpha, \gamma, \theta \in \mathbb{T}, \tag{2.19}$$

one sees that at the cost of multiplying U_ω by $e^{\mathrm{i}\theta}$, we can assume $\theta = 0$. Moreover, with $\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & -\mathrm{i}e^{\mathrm{i}\gamma} \end{bmatrix}$, we compute

$$\Sigma C \Sigma^{-1} = \begin{bmatrix} te^{-\mathrm{i}\alpha} & -r \\ r & te^{\mathrm{i}\alpha} \end{bmatrix}. \tag{2.20}$$

Up to unitary equivalence, we can assume that C has the form of the r.h.s. of (2.20).

To get rid of the last phase α, we make use of the representation of U_ω in the ordered basis $\{\dots, |\uparrow\rangle \otimes |n-1\rangle, |\downarrow\rangle \otimes |n-1\rangle, |\uparrow\rangle \otimes |n\rangle, |\downarrow\rangle \otimes |n\rangle, \dots\}$ as the band matrix

$$U_\omega = D_\omega S, \quad \text{with} \quad S = \begin{bmatrix} \ddots & r & te^{\mathrm{i}\alpha} & & & & & \\ & 0 & 0 & & & & & \\ & 0 & 0 & r & te^{\mathrm{i}\alpha} & & & \\ & te^{-\mathrm{i}\alpha} & -r & 0 & 0 & & & \\ & & & 0 & 0 & r & te^{\mathrm{i}\alpha} & \\ & & & te^{-\mathrm{i}\alpha} & -r & 0 & 0 & \\ & & & & & 0 & 0 & \\ & & & & & te^{-\mathrm{i}\alpha} & -r & \ddots \end{bmatrix}, \tag{2.21}$$

where (upon relabeling the indices of the random phases) D_ω is diagonal with i.i.d. entries, $D_\omega = \mathrm{diag}(\dots, \mathrm{e}^{-\mathrm{i}\omega_k}, \mathrm{e}^{-\mathrm{i}\omega_{k+1}}, \dots)$, and the diagonal of S consists of zeroes. Our convention is to fix a labeling of the canonical basis e_n of that space so that the odd rows contain $r, te^{\mathrm{i}\alpha}$ and the even rows contain $te^{-\mathrm{i}\alpha}, -r$.

Following [9], we introduce the unitary operator V defined by $Ve_n = e^{i\zeta_n}e_n$, where $\zeta_n \in \mathbb{T}$, $n \in \mathbb{Z}$. We compute, for any $k \in \mathbb{Z}$,

$$\begin{aligned} V^{-1}U_\omega V e_{2k} &= e^{i(\zeta_{2k}-\zeta_{2k-1})}e^{-i\omega_{2k-1}}re_{2k-1} + e^{i(\zeta_{2k}-\zeta_{2k+2})}e^{-i\omega_{2k+2}}te^{-i\alpha}e_{2k+2} \\ V^{-1}U_\omega V e_{2k+1} &= e^{i(\zeta_{2k+1}-\zeta_{2k-1})}e^{-i\omega_{2k-1}}te^{i\alpha}e_{2k-1} - e^{i(\zeta_{2k+1}-\zeta_{2k+2})}e^{-i\omega_{2k+2}}re_{2k+2}. \end{aligned} \tag{2.22}$$

Choosing for any $k \in \mathbb{Z}$

$$\zeta_{2k+2} = \zeta_{2k+1} = -k\alpha, \tag{2.23}$$

yields the result. ■

Remarks:

1) The last argument of the proof above can be adapted to show that the randomness of the model of QW in random environment considered by [22] can be gauged away, which explains the absence of localization. Indeed, this model is characterized by random coin matrices of the form (with $t = r = 1/\sqrt{2}$)

$$\tilde{C}_k = \begin{bmatrix} te^{i\omega_k} & r \\ r & -te^{-i\omega_k} \end{bmatrix}, \quad \text{where } \{\omega_k\}_{k\in\mathbb{Z}} \text{ are i.i.d. random variables on } \mathbb{T}. \tag{2.24}$$

The corresponding operator $\tilde{U}_\omega$ then satisfies

$$\begin{aligned} V^{-1}\tilde{U}_\omega V e_{2k} &= e^{i(\zeta_{2k}-\zeta_{2k-1})}re_{2k-1} + e^{i(\zeta_{2k}-\zeta_{2k+2})}e^{-i\omega_{2k}}te_{2k+2} \\ V^{-1}\tilde{U}_\omega V e_{2k+1} &= -e^{i(\zeta_{2k+1}-\zeta_{2k-1})}e^{i\omega_{2k}}te_{2k-1} + e^{i(\zeta_{2k+1}-\zeta_{2k+2})}re_{2k+2}. \end{aligned} \tag{2.25}$$

Choosing $\zeta_0 = 0$, $\zeta_{-1} = 0$ and, for any $k \in \mathbb{Z}^+$,

$$\begin{aligned} \zeta_{2k+2} &= -(\omega_{2k} + \omega_{2k-2} + \cdots + \omega_0), \\ \zeta_{-2k} &= \omega_{-2k} + \omega_{-2k-2} + \cdots + \omega_{-2}, \\ \zeta_{2k+1} &= -(\omega_{2k} + \omega_{2k-2} + \cdots + \omega_0), \\ \zeta_{-(2k+1)} &= \omega_{-2k} + \omega_{-2k-2} + \cdots + \omega_{-2}, \end{aligned} \tag{2.26}$$

we see that $\tilde{U}_\omega$ is unitarily equivalent to the deterministic operator $\tilde{U}_0$, characterized by the the deterministic coin operator $\tilde{C} = \begin{bmatrix} t & r \\ r & -t \end{bmatrix}$.

2) We will consider from now on U_ω to be defined by (2.21) (with $\alpha = 0$) acting on the Hilbert space $l^2(\mathbb{Z})$, with canonical basis $e_{2k} = |\uparrow\rangle \otimes |k\rangle$, $e_{2k+1} = |\downarrow\rangle \otimes |k\rangle$. We note that the position operator x on $l^2(\mathbb{Z})$ is related to X defined by (2.6) on $\mathbb{C}^2 \otimes l^2(\mathbb{Z})$ by $(x-1)/2 \le X \le x/2$, so that dynamical localization is equivalent in both representations.

3) In case $r = 0, t = 1$, (2.21) shows that U_ω is equivalent to a direct sum of two shifts, which, in other words, corresponds to $C_k = \mathbb{I}$, $k \in \mathbb{Z}$. This leads to ballistic behaviour. In case $r = 1, t = 0$, the two-dimensional subspaces $\text{span}\{e_{2k-1}, e_{2k}\}$, $k \in \mathbb{Z}$, are invariant, which forbids any kind of transport.

4) The appealing five diagonal representation (2.21) of QW on $\mathbb{N}$ is also related so called CMV matrices associated with certain orthogonal polynomials on the unit circle, when restricted to $l^2(\mathbb{Z}^+)$. This fact is used in [11] to study certain properties of deterministic QW.

Finally note that the structure of S can be described also via the two unitary operators ("even" and "odd")

$$B_e = \begin{bmatrix} r & t \\ t & -r \end{bmatrix} \quad \text{and} \quad B_o = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{2.27}$$

S is the product of S_oS_e, where S_e is block-diagonal, with blocks B_e placed so that the $(1,1)$ entry of B_e lies on even indices of the diagonal, and S_o is block-diagonal with blocks B_o placed so that the $(1,1)$ entry of B_o lies on odd diagonal elements. Obviously S_e and S_o are unitary, and thus so is S and hence U_ω.

3 Proofs

We make use of the structure just described to define finite volume approximations of S and thus U_ω.

3.0.1 Finite and semifinite volume truncation

On $l^2(\{2n_0, 2m_0\})$ we define the matrices

$$\bar{S}_o = \begin{bmatrix} 1 & & & \\ & B_o & & \\ & & \ddots & \\ & & & B_o \end{bmatrix}, \qquad \bar{S}_e = \begin{bmatrix} B_e & & & \\ & \ddots & & \\ & & B_e & \\ & & & 1 \end{bmatrix}, \tag{3.1}$$

where each 1 is a 1×1 block, and the 2×2 blocks B_o, B_e are given in (2.27). The finite volume propagator is

$$\bar{U}_\omega = \bar{D}_\omega \bar{S}, \quad \text{with} \quad \bar{S} = \bar{S}_o \bar{S}_e = \begin{bmatrix} r & t & & & & & & & & \\ 0 & 0 & r & t & & & & & & \\ t & -r & 0 & 0 & & & & & & \\ & & 0 & 0 & r & t & & & & \\ & & t & -r & 0 & 0 & & & & \\ & & & & 0 & 0 & & & & \\ & & & & t & -r & & & & \\ & & & & & & \ddots & & & \\ & & & & & & & r & t & 0 \\ & & & & & & & 0 & 0 & 0 \\ & & & & & & & 0 & 0 & 1 \\ & & & & & & & t & -r & 0 \end{bmatrix}. \tag{3.2}$$

Here, $\bar{D}_\omega = \mathrm{diag}(\mathrm{e}^{-\mathrm{i}\omega_{2n_0}}, \ldots, e^{-\mathrm{i}\omega_{2m_0}})$.

Similarly we define $U_\omega^\pm = D_\omega^\pm S^\pm$ acting on the Hilbert spaces $l^2(\{2n_0, \ldots\})$ and $l^2(\{\ldots, 2m_0\})$. The matrix S^+ is defined as in (3.2), but where the 2×4 blocks are repeated indefinitely towards the bottom right. Similarly we define S^-, and the $D_\omega^\pm$ are simply the corresponding semifinite truncations of the diagonal matrix D_ω. Analogous definitions hold for the Hilbert spaces $l^2(\{2n_0 + 1, \ldots\})$, $l^2(\{\ldots, 2m_0 + 1\})$.

3.0.2 Transfer matrix

Infinite volume. For $z \in \mathbb{C}\backslash\{0\}$, consider the equation $(U - z)\psi = 0$ (dropping the subscript ω in the notation). In what follows we always consider $z \neq 0$; an analysis for $z = 0$ can be done in a similar way, but is not of any use in this work. Due to the band structure of U, we obtain a recurrence relation for the components of vectors solving the above equation. Denote by ψ_n the component of ψ along the basis element e_n. A vector ψ

solves $(U-z)\psi=0$ if and only if its components satisfy

$$\left[\begin{array}{c}\psi_{2n+1}\\ \psi_{2n}\end{array}\right]=T_z(\omega_{2n},\omega_{2n-1})\left[\begin{array}{c}\psi_{2n-1}\\ \psi_{2n-2}\end{array}\right], \tag{3.3}$$

where the transfer matrix is given by

$$T_z(\omega_{2n},\omega_{2n-1})=\frac{\mathrm{e}^{-\mathrm{i}\omega_{2n}}}{zt}\left[\begin{array}{cc}z^2\mathrm{e}^{\mathrm{i}(\omega_{2n-1}+\omega_{2n})}+r^2 & -rt\\ -rt & t^2\end{array}\right]. \tag{3.4}$$

We have

$$\det T_z=\mathrm{e}^{-\mathrm{i}(\omega_{2n}-\omega_{2n-1})}, \tag{3.5}$$

so equation (3.3) can be inverted. It follows that if we choose ψ_1 and ψ_0 then all components of ψ are fixed, by (3.3). A priori therefore, each eigenvalue could be doubly degenerate; but this does not happen as we see from the following argument. Suppose that ψ and χ are two eigenvectors with the same eigenvalue z. Then we have

$$\left[\begin{array}{cc}\psi_{2n+1} & \chi_{2n+1}\\ \psi_{2n} & \chi_{2n}\end{array}\right]=T_z(\omega,n)\left[\begin{array}{cc}\psi_1 & \chi_1\\ \psi_0 & \chi_0\end{array}\right], \tag{3.6}$$

where

$$T_z(\omega,n):=T_z(\omega_{2n},\omega_{2n-1})\cdots T_z(\omega_2,\omega_1). \tag{3.7}$$

The determinant of the matrix on the left side of (3.6) tends to zero as $n\to\infty$, since ψ and χ are eigenvectors and hence belong to l^2. On the other hand, $|\det T_z(\omega,n)|=1$ for all n. We conclude that the columns of the matrix to the right in (3.6) are multiples of each other, and hence so are the vectors ψ and χ.

Solutions $\varphi^\pm$. Take $0\neq z\in\mathbb{C}$ from the resolvent set of U. Then there are unique (up to multiplication by a scalar) solutions $\varphi^\pm\in l^2_\pm$ to $(U-z)\psi=0$. Here, $l^2_\pm=\{\psi\in l^2 : \sum_{n\geq 1}|\psi_{\pm n}|^2<\infty\}$. To see existence we set $\psi_0=(U-z)^{-1}e_0$, so that $(U-z)\psi_0=e_0$, and we choose $\varphi^+_k=[\psi_0]_k$ for $k\geq 1$, $\varphi^-_k=[\psi_0]_k$ for $k\leq -1$. We then define $\varphi^\pm_k$ for the remaining components k by applying the transfer matrix. Uniqueness of $\varphi^\pm$ is shown just as in the above argument following (3.6).

Similarly to (3.6), we have

$$\left[\begin{array}{cc}\varphi^+_{2n} & \varphi^-_{2n}\\ \varphi^+_{2n-1} & \varphi^-_{2n-1}\end{array}\right]=T(\omega,z,n)\left[\begin{array}{cc}\varphi^+_2 & \varphi^-_2\\ \varphi^+_1 & \varphi^-_1\end{array}\right], \tag{3.8}$$

for a matrix satisfying $|\det T(\omega,z,n)|=1$. The columns of the matrix to the right are linearly independent, for otherwise φ^+ and φ^- would be multiples of each other. This in turn would mean that $\varphi^\pm\in l^2$, which is not the case since z is not an eigenvalue of U. By taking the determinant in (3.8) we see that for all $n\in\mathbb{Z}$,

$$\varphi^+_{2n}\varphi^-_{2n-1}-\varphi^+_{2n-1}\varphi^-_{2n}\neq 0. \tag{3.9}$$

Semifinite volume. Consider U^+ acting on $l^2(\{2n_0,\ldots\})$. We solve the equation $(U^+-z)\psi=0$ recursively and obtain for $n\geq n_0+2$

$$\left[\begin{array}{c}\psi_{2n+1}\\ \psi_{2n}\end{array}\right]=T^+_z(\omega,n)\left[\begin{array}{cc}-\frac{r}{t} & \frac{z}{t^2}\mathrm{e}^{\mathrm{i}\omega_{2n_0+1}}(z\mathrm{e}^{\mathrm{i}\omega_{2n_0}}-r)\\ 1 & 0\end{array}\right]\left[\begin{array}{c}\psi_{2n_0+2}\\ \psi_{2n_0}\end{array}\right], \tag{3.10}$$

and

$$\psi_{2n_0+1}=\frac{\mathrm{e}^{\mathrm{i}\omega_{2n_0}}}{t}(z-r\mathrm{e}^{-\mathrm{i}\omega_{2n_0}})\psi_{2n_0}. \tag{3.11}$$

Here, we have set $T_z^+(\omega, n) = T_z(\omega_{2n}, \omega_{2n-1}) \cdots T_z(\omega_{2n_0+4}, \omega_{2n_0+3})$. A similar argument as in the infinite volume case shows that each eigenvalue has multiplicity one.

Lyapunov exponent. Products of transfer matrices yield generalized eigenvectors ψ solutions to $(U-z)\psi = 0$ and their asymptotic behaviour at infinity is related to the associated Lyapunov exponents. The following theorem is an important ingredient in the proof of Lemma 3.8 below (see [15], Appendix A, for a proof of that lemma, which we do not reproduce in the present paper.

Theorem 3.1 *Let μ be absolutely continuous with density $\tau \in L^\infty(\mathbb{T})$ and having support with nonempty interior. Then, there is an $\epsilon > 0$ such that for every nonzero $z \in \mathbb{C}$ with $1 - \epsilon < |z| < 1 + \epsilon$, the limit*

$$\lim_{n\to\infty} \frac{1}{n} \log \| T_z(\omega_{2n}, \omega_{2n-1}) \cdots T_z(\omega_2, \omega_1) \| = \gamma(z) \tag{3.12}$$

exists almost surely, $\gamma(z)$ is deterministic and strictly positive. Moreover, $z \mapsto \gamma(z)$ is continuous.

Remark: The limit $\gamma(z)$ is called the *Lyapunov exponent.*
The proof of this theorem is given in the appendix.

3.0.3 Green's function

Infinite volume. Let $z \neq 0$ be in the resolvent set of U, and let $\varphi^\pm$ be the unique solutions of $(U-z)\varphi^\pm = 0$ in $l^2_\pm$. We define the resolvent by

$$G_z^\omega = (U_\omega - z)^{-1} \tag{3.13}$$

(leaving sometimes out the superscript ω). Using the explicit form of U we solve the equation $[(U-z)G_z e_n]_k = \delta_{k,n}$ (Kronecker symbol) for $G_z e_n$ and hence obtain the matrix elements of the resolvent. The following is the result. For all $n \in \mathbb{Z}$, we have

$$\langle e_k, G_z^\omega e_{2n} \rangle = \frac{1}{z} \frac{1}{\varphi^+_{2n-1}\varphi^-_{2n} - \varphi^+_{2n}\varphi^-_{2n-1}} \left\{ \begin{array}{ll} \varphi^+_{2n-1}\, \varphi^-_k, & k \leq 2n-1 \\ \varphi^+_k\, \varphi^-_{2n-1}, & k \geq 2n \end{array} \right. \tag{3.14}$$

and

$$\langle e_k, G_z^\omega e_{2n-1} \rangle = \frac{1}{z} \frac{1}{\varphi^+_{2n-1}\varphi^-_{2n} - \varphi^+_{2n}\varphi^-_{2n-1}} \left\{ \begin{array}{ll} \varphi^+_{2n}\, \varphi^-_k, & k \leq 2n-1 \\ \varphi^+_k\, \varphi^-_{2n}, & k \geq 2n \end{array} \right. \tag{3.15}$$

Notice that these matrix elements are well defined due to (3.9).

Semifinite volume. One can easily construct a vector φ^a with components φ^a_k, $k \in \mathbb{Z}$, satisfying $(U-z)\varphi^a = 0$ and $[(U^+ - z)\varphi^a]_k = 0$ for all $k \geq a := 2n_0$. Here, U^+ is acting on $l^2(\{2n_0, \ldots\})$. The components of φ^a are obtained by the transfer matrix for a suitable 'initial condition' $\varphi^a_{2n_0}$ and $\varphi^a_{2n_0+1}$.

In the same way, one can construct φ^b with components φ^b_k, $k \in \mathbb{Z}$, satisfying $(U-z)\varphi^b = 0$ and $[(U^- - z)\varphi^b]_k = 0$ for all $k \leq b := 2m_0$. Here, U^- acts on $l^2(\{\ldots, 2m_0\})$.

The Green's functions on the semi-finite spaces are defined by

$$G_z^\pm = (U^\pm - z)^{-1}, \tag{3.16}$$

for $|z| \neq 1$ and where the operator is acting on the l^2 space over the appropriate half-line of integers. Proceeding as in the infinite-volume case, one shows that $\langle e_k, G_z^+ e_{2n} \rangle$ and $\langle e_k, G_z^+ e_{2n-1} \rangle$, for $k \geq 2n_0$, are given by the r.h.s. of (3.14) and (3.15) respectively, where

φ^- is replaced by φ^a. Furthermore, $\langle e_k, G_z^- e_{2n}\rangle$ and $\langle e_k, G_z^- e_{2n-1}\rangle$, for $k \leq 2m_0$, are given by the r.h.s. of (3.14) and (3.15) respectively, where φ^+ is replaced by φ^b.

Finite volume. On $l^2(\{2n_0, \ldots, 2m_0\})$ the reduced unitary is given by (3.2). The matrix elements of the Green's function $G_z^{[2n_0,2m_0]} = (U^{[2n_0,2m_0]} - z)^{-1}$ is easily obtained in terms of the vectors φ^a, φ^b defined in the previous paragraph. We mention explicitly only the formula

$$\left\langle e_{2n_0}, G_z^{[2n_0,2m_0]} e_{2m_0} \right\rangle = -\frac{1}{z} \frac{1}{\varphi^a_{2m_0-1}\varphi^b_{2m_0} - \varphi^b_{2m_0-1}\varphi^a_{2m_0}} \varphi^b_{2m_0-1}\varphi^a_{2n_0}. \tag{3.17}$$

3.1 Bound on fractional moments of Green's function

For $|z| \neq 1$ denote the matrix elements of the resolvent of U_ω by

$$\langle e_k, (U_\omega - z)^{-1} e_l \rangle = G_z^\omega(k,l). \tag{3.18}$$

Theorem 3.2 *Assume that the random variables ω_k, $k \in \mathbb{Z}$, are iid with distribution $\mathrm{d}\mu(\omega_0) = \tau(\omega_0)\mathrm{d}\omega_0$ with $\tau \in L^\infty(\mathbb{T})$. Let $0 < s < 1$ and $0 < \epsilon < 1$. Then there exists a constant $0 < C(s,\epsilon) < \infty$ such that*

$$\mathbb{E}\big[|G_z^\omega(k,l)|^s\big] \leq C(s,\epsilon), \tag{3.19}$$

for all $z \in \{\zeta \in \mathbb{C} \ : \ 1 - \epsilon < |\zeta| < \infty, |\zeta| \neq 1\}$ and all $k, l \in \mathbb{Z}$.

Remark: The result holds for the semifinite volume restrictions $G_z^\pm(k,l)$ as well.
The proof of this result is given in Hamza's thesis (Theorem 6.1) and [17]. It only depends on the fact that U_ω is the product $D_\omega S$, where S is unitary and $D_\omega = \mathrm{diag}(\ldots, \omega_k, \omega_{k+1}, \ldots)$ has the specified random properties.

3.2 Exponential decay of fractional moments of Green's function

Theorem 3.3 *Assume that the random variables ω_k, $k \in \mathbb{Z}$, are iid with distribution $\mathrm{d}\mu(\omega_0) = \tau(\omega_0)\mathrm{d}\omega_0$ with $\tau \in L^\infty(\mathbb{T})$ and that the support of μ contains a non-empty open set. Let $0 < \epsilon < 1$. There exists an s with $0 < s < 1/2$, and there exist constants $0 < C < \infty$, $\alpha > 0$, such that*

$$\mathbb{E}\big[|G_z^\omega(k,l)|^s\big] \leq C\mathrm{e}^{-\alpha|k-l|}, \tag{3.20}$$

for all $z \in \mathbb{C}$ satisfying $|z| \neq 1$ and $\frac{1}{1+\epsilon} < |z| < 1 + \epsilon$, and for all $k, l \in \mathbb{Z}$.

The proof is done in three steps that we now describe.

3.2.1 Reduction to even matrix elements

Theorem 3.4 *Suppose that $|k - l| \geq 4$ and let m, n be the unique integers s.t. $k \in \{2m, 2m+1\}$ and $l \in \{2n-1, 2n\}$. Then we have*

$$|G_z^\omega(k,l)| \leq \frac{|z| + t}{r} \left[\frac{|z|}{r} + \frac{|z|^2 + r^2}{rt}\right] \sum_{p_1,\, p_2 \in \{0,1\}} |G_z^\omega(2m - 2p_1, 2n + 2p_2)|. \tag{3.21}$$

It follows that for any $s>0$

$$|G_z^\omega(k,l)|^s \le C(z,t)^s \sum_{p_1,\,p_2\in\{0,1\}} |G_z^\omega(2m-2p_1,2n+2p_2)|^s, \tag{3.22}$$

where the constant C is that in front of the sum in (3.21) and where we have we used that $(a+b)^s \le [a^s+b^s]$ for $a,b\ge 0$ and $0<s<1$.

Proof of Theorem 3.4. We write simply G for G_z^ω in this proof. We first show that for $k\neq n$,

$$G(2k+1,2n) = \frac{z}{r}\mathrm{e}^{\mathrm{i}\omega_{2k-1}}G(2k-2,2n) - \frac{z^2\mathrm{e}^{-\mathrm{i}(\omega_{2k}+\omega_{2k-1})}+r^2}{rt}G(2k,2n), \tag{3.23}$$

and that for $k\ge 2n+2$ or $k\le 2n-1$,

$$G(k,2n-1) = \frac{z}{r}\mathrm{e}^{+\mathrm{i}\omega_{2n-1}}G(k,2n) - \frac{t}{r}\frac{d_{n+1}}{d_n}G(k,2n+2), \tag{3.24}$$

where $d_n = [\varphi^+_{2n-1}\varphi^-_{2n} - \varphi^+_{2n}\varphi^-_{2n-1}]^{-1}$ (c.f. (3.9)). Combining (3.23) and (3.24), and using the fact that $|\frac{d_n}{d_{n+1}}|=1$, the bound (3.21) is easily obtained. The latter fact follows from

$$d_{n+1}^{-1} = -\det\begin{bmatrix}\varphi^+_{2n+2} & \varphi^-_{2n+2}\\ \varphi^+_{2n+1} & \varphi^-_{2n+1}\end{bmatrix} = -\det\left(T\begin{bmatrix}\varphi^+_{2n} & \varphi^-_{2n}\\ \varphi^+_{2n-1} & \varphi^-_{2n-1}\end{bmatrix}\right) = d_n^{-1}\det T,$$

where $|\det T| = 1$.

Let us now derive (3.24) for $k\le 2n-1$. We have $G(k,2n-1) = \frac{1}{z}d_n\varphi^+_{2n}\varphi^-_k$, and φ^+_{2n} is given, using the transfer matrix as in (3.3), by $\varphi^+_{2n} = \frac{\mathrm{e}^{-\mathrm{i}\omega_{2n}}}{zt}\left(-rt\varphi^+_{2n-1}+t^2\varphi^+_{2n-2}\right)$. In this relation, we replace φ^+_{2n-2} by solving (3.3) (using the inverse transfer matrix), yielding the expression $\varphi^+_{2n-2} = \frac{r}{z}\mathrm{e}^{-\mathrm{i}\omega_{2n-1}}\varphi^+_{2n+1} + \frac{\mathrm{e}^{-\mathrm{i}\omega_{2n-1}}}{zt}[z^2\mathrm{e}^{\mathrm{i}(\omega_{2n-1}+\omega_{2n})}+r^2]\varphi^+_{2n}$. It follows that

$$\begin{aligned} G(k,2n-1) &= -\frac{r}{z}\mathrm{e}^{-\mathrm{i}\omega_{2n}}G(k,2n) + \frac{rt}{z^2}\mathrm{e}^{-\mathrm{i}(\omega_{2n}+\omega_{2n-1})}\frac{d_{n+1}}{d_n}G(k,2n+2)\\ &\quad +\frac{1}{z^2}\mathrm{e}^{-\mathrm{i}(\omega_{2n}+\omega_{2n-1})}[z^2\mathrm{e}^{\mathrm{i}(\omega_{2n-1}+\omega_{2n})}+r^2]G(k,2n-1),\end{aligned}$$

which yields (3.24). If $k\ge 2n$, the same arguments with φ^-_{2n} in place of φ^+_{2n} give the result. Expression (3.23) is obtained in an analogous manner. ∎

3.2.2 Reduction to finite volume

Theorem 3.5 *Assume the hypotheses of Theorem 3.2, with $0<s<1/2$. For any pair of integers $m\neq n$ let $k_+ = \max\{m,n\}$ and $k_- = \min\{m,n\}$. There exists a constant $C_\mu(s,t,\epsilon)<\infty$ s.t. we have*

$$\mathbb{E}\big[|G_z(2m,2n)|^s\big]^2 \le C_\mu(s,t,\epsilon)\mathbb{E}\big[|G_z^{[2k_-,2k_+]}(2k_-,2k_+)|^{2s}\big], \tag{3.25}$$

for all z satisfying $|z|\neq 1$ and $1-\frac{\epsilon}{1+\epsilon}<|z|<1+\epsilon$.

Proof. For a given m, we decompose $l^2(\mathbb{Z}) = l^2(\{\ldots,2m-1\})\oplus l^2(\{2m,\ldots\})$, and set $U = U^-\oplus U^+ + \Gamma$, where $U^- = D_\omega S_o^{(-\infty,2m-1]}S_e^{(-\infty,2m-1]}$ (of course, the diagonal D_ω is restricted to the correct half-space) and where $U^+ = D_\omega S_o^{[2m,\infty)} S_e^{[2m,\infty)}$. The operator Γ is explicitly given by

$$\langle e_k, \Gamma e_l \rangle = \Gamma(k,l) = \begin{cases} r\mathrm{e}^{-\mathrm{i}\omega_{2m-1}}, & (k,l) = (2m-1, 2m-1), (2m-1, 2m) \\ -r\mathrm{e}^{-\mathrm{i}\omega_{2m}}, & (k,l) = (2m, 2m-1), (2m, 2m) \\ -t\mathrm{e}^{-\mathrm{i}\omega_{2m-1}}, & (k,l) = (2m-1, 2m-2) \\ t\mathrm{e}^{-\mathrm{i}\omega_{2m-1}}, & (k,l) = (2m-1, 2m+1) \\ t\mathrm{e}^{-\mathrm{i}\omega_{2m}}, & (k,l) = (2m, 2m-2) \\ -t\mathrm{e}^{-\mathrm{i}\omega_{2m}}, & (k,l) = (2m, 2m+1) \end{cases} \tag{3.26}$$

and $\Gamma(k,l) = 0$ for all other values k, l. Denote

$$G_z^m := (U^- \oplus U^+ - z)^{-1} = G_z^{(-\infty, 2m-1]} \oplus G_z^{[2m,\infty)}, \tag{3.27}$$

then the second resolvent identity gives $G_z = G_z^m - G_z \Gamma G_z^m$. Suppose that $m \leq n-1$. Then $G_z^m(2m, 2n) = G_z^{[2m,\infty)}(2m, 2n)$ and we have

$$G_z(2m, 2n) = G_z^{[2m,\infty)}(2m, 2n) - \sum_{k,l} G_z(2m, k) \Gamma(k,l) G_z^m(l, 2n). \tag{3.28}$$

According to (3.26), the matrix elements $\Gamma(k,l)$ vanish unless $l = 2m, 2m-1, 2m-2, 2m+1$. However, for $l = 2m-1, 2m-2$ we have $G_z^m(l, 2n) = 0$ (by the block diagonal form (3.27)), so the only terms in the sum in (3.28) are with $l = 2m, 2m+1$. It follows that (still $m < n$)

$$\begin{aligned} &|G_z(2m, 2n)| \\ &\quad \leq \Big\{ 1 + 2[1 + (|z| + r)t^{-1}] \max_{k=2m,2m-1} |G_z(2m, k)| \Big\} |G_z^{[2m,\infty)}(2m, 2n)|. \end{aligned} \tag{3.29}$$

To arrive at this bound we use in (3.28) the relation

$$G_z^m(2m+1, 2n) = \frac{\mathrm{e}^{\mathrm{i}\omega_{2m}}}{t} [z - r\mathrm{e}^{-\mathrm{i}\omega_{2m}}] \; G_z^m(2m, 2n)$$

which follows readily from the explicit expressions of the semi-finite volume Green's function, see (3.11).

In a next step, we estimate $|G_z^{[2m,\infty)}(2m, 2n)|$ in (3.29) from above by the finite-volume Green's function $|G_z^{[2m,2n]}(2m, 2n)|$. We split the space as $l^2(\{2m, \ldots \infty\}) = l^2(\{2m, 2n\}) \oplus l^2(\{2n+1, \ldots \infty\})$, so that $U^{[2m,\infty)} = U^{[2m,2n]} \oplus U^{[2n+1,\infty)} + \widetilde{\Gamma}$, with

$$\left\langle e_k, \widetilde{\Gamma} e_l \right\rangle = \widetilde{\Gamma}(k,l) = \begin{cases} (r-1)\mathrm{e}^{-\mathrm{i}\omega_{2n-1}}, & (k,l) = (2n-1, 2n), \\ t\mathrm{e}^{-\mathrm{i}\omega_{2n-1}}, & (k,l) = (2n-1, 2n+1), \\ t\mathrm{e}^{-\mathrm{i}\omega_{2n+2}}, & (k,l) = (2n+2, 2n) \\ -(r+1)t\mathrm{e}^{-\mathrm{i}\omega_{2n+2}}, & (k,l) = (2n+2, 2n+1) \end{cases} \tag{3.30}$$

and $\widetilde{\Gamma}(k,l) = 0$ for all other values k, l. By the second resolvent identity, we have $G_z^{[2m,\infty)} = (G_z^{[2m,2n]} \oplus G_z^{[2n+1,\infty)})(1 - \widetilde{\Gamma} G_z^{[2m,\infty)})$. Taking the matrix elements, with $m < n$, we obtain

$$\begin{aligned} &G_z^{[2m,\infty)}(2m, 2n) \\ &\quad = G_z^{[2m,2n]}(2m, 2n) - \sum_{k,l} \left\langle e_{2m}, G_z^{[2m,2n]} \oplus G_z^{[2n+1,\infty)} e_k \right\rangle \widetilde{\Gamma}(k,l) G_z^{[2m,\infty)}(l, 2n). \end{aligned}$$

Note that due to (3.30), the sum over k is really only over $k = 2n-1, 2n+2$, and the term with $k = 2n+2$ is absent since the scalar product in the sum vanishes for this value of k.

By using additionally that $G^{[2m,2n]}(2m,2n-1) = z\mathrm{e}^{\mathrm{i}\omega_{2n-1}}G^{[2m,2n]}(2m,2n)$, which follows from the explicit formulas for the Green's function, we obtain the bound

$$|G_z^{[2m,\infty)}(2m,2n)| \leq \left\{1 + 3|z| \max_{k=2n,2n+1} |G_z^{[2m,\infty)}(k,2n)|\right\} |G_z^{[2m,2n]}(2m,2n)|. \tag{3.31}$$

Combining (3.31) with (3.29) yields that for $m \leq n-1$,

$$\begin{aligned} |G_z(2m,2n)| \leq & \left\{1 + 2(1 + (|z|+r)t^{-1}) \max_{k=2m,2m-1} |G_z(2m,k)|\right\} \\ \times \quad & \left\{1 + 3|z| \max_{k=2n,2n+1} |G_z^{[2m,\infty)}(k,2n)|\right\} |G_z^{[2m,2n]}(2m,2n)|. \end{aligned} \tag{3.32}$$

We take the expectation of the inequality (3.32), use Hölder's inequality and Theorem 3.2 to arrive at the bound ($m \leq n-1$)

$$\mathbb{E}\big[|G_z(2m,2n)|^s\big]^2 \leq C_\mu(s,t,\epsilon,z)\ \mathbb{E}\big[|G_z^{[2m,2n]}(2m,2n)|^{2s}\big], \tag{3.33}$$

for $0 < s < 1/2$, $0 < \epsilon < 1$, $|z| \neq 1$ s.t. $1-\epsilon < |z| < \infty$, and where $C_\mu(s,\epsilon,z)$ is bounded uniformly in compact regions of z.

Next we deal with matrix elements of the resolvent with $m \geq n+1$. One can proceed as in the above argument, or instead use the following path. Since U is unitary, $|G_z(k,l)| = |\langle e_k, (U-z)^{-1}e_l\rangle| = |\langle e_l, (U^{-1}-\bar{z})^{-1}e_k\rangle|$, and $(U^{-1}-\bar{z})^{-1} = -1/\bar{z} - (1/\bar{z})^2(U-1/\bar{z})^{-1}$. Hence we have for $m \geq n+1$

$$|G_z(2m,2n)| = \frac{1}{|z|^2}|G_{1/\bar{z}}(2n,2m)|. \tag{3.34}$$

We want to use the bound (3.33) on the right hand side of (3.34). The condition $1-\epsilon < 1/|\bar{z}| < 1+\epsilon$ is equivalent to $1 - \frac{\epsilon}{1+\epsilon} < |z| < 1 + \frac{\epsilon}{1-\epsilon}$. It follows that under the last condition on $|z|$ we have, for $m \geq n+1$,

$$\mathbb{E}\big[|G_z(2m,2n)|^s\big]^2 \leq C_\mu(s,t,\epsilon,1/\bar{z})\ \frac{1}{|z|^{4s}}\mathbb{E}\big[|G_{1/\bar{z}}^{[2n,2m]}(2n,2m)|^{2s}\big]. \tag{3.35}$$

Combining (3.33) and (3.35) yields the bound (3.25), provided $|z| \neq 1$, $1-\epsilon < |z| < 1+\epsilon$ and $1 - \frac{\epsilon}{1+\epsilon} < |z| < 1 + \frac{\epsilon}{1-\epsilon}$. ■

3.2.3 Exponential decay in finite volume

We assume that the support of the measure μ contains a non-empty open set in $[0, 2\pi)$. This implies positivity of the Lyapunov exponent, see Theorem 3.1, a result we use implicitly in this section.

Theorem 3.6 *There are numbers α, s satisfying $\alpha > 0$, $0 < s < 1$ such that for all $z \in \mathbb{C}$, $|z| \neq 0, 1$ and $n < m$, we have*

$$\mathbb{E}\left[|G_z^{[2n,2m]}(2n,2m)|^s\right] \leq C\mathrm{e}^{-\alpha(m-n)}. \tag{3.36}$$

The constant C depends on s, z, t and μ, but not on m, n. It is furthermore uniform in z, for $|z| \neq 1$ restricted to compact sets of $\mathbb{C} \setminus \{0\}$ (see explicit bound in proof).

Proof of Theorem 3.6. The proof is based on the following two Lemmas.

Lemma 3.7 *Let $0 < s < 1$. Then there is a constant $0 < C_\mu(s) < \infty$ s.t. for all $|z| \neq 0, 1$,*

$$\mathbb{E}\left[|G_z^{[2n,2m]}(2n, 2m)|^s\right] \leq |z|^{-s}(1 + |z|^{-s})(1 + 2^s C_\mu(s))\ \mathbb{E}\left[\left\| \begin{array}{c} \varphi^a_{2m} \\ \varphi^a_{2m-1} \end{array} \right\|^{-s}\right]. \tag{3.37}$$

Here, φ^a is the solution satisfying the left boundary condition at $a = 2n$, see also before (3.16).

We present a proof of this Lemma at the end of the present section. The following a priori bound has been adapted from the self-adjoint Anderson model situation (see [12], Lemma 5.1), and is given in Appendix A of [15]. The proof uses strict positivity and continuity of the Lyapunov exponent, which we prove in Theorem 3.1 below. We omit the proof of this lemma which is very similar to the one given for the self-adjoint Anderson model in [12], see Appendix A in [15] for details.

Lemma 3.8 *Let $\Lambda \subset \mathbb{C}$ be a compact set not containing the origin. There are numbers $\alpha > 0$ and $0 < s < 1$ (depending on Λ), such that the product of transfer matrices (3.4) satisfy, for all $m > n$,*

$$\mathbb{E}\left[\|T_z(\omega_{2m}, \omega_{2m-1}) \cdots T_z(\omega_{2n}, \omega_{2n-1})v\|^{-s}\right] \leq \bar{C}\mathrm{e}^{-\alpha(m-n)}, \tag{3.38}$$

for any normalized vector $v \in \mathbb{C}^2$. The constant $\bar{C}$ is independent of $z \in \Lambda$ and v.

A combination of these two Lemmas yields a proof of Theorem 3.6 as follows. Solving the equation (3.3) for φ^a_{2m-1} gives the relation $\varphi^a_{2m-1} = \mathrm{e}^{-\mathrm{i}\omega_{2m-1}}[t\varphi^a_{2m+1} + r\varphi^a_{2m}]/z$, and therefore

$$\left[\begin{array}{c} \varphi^a_{2m+1} \\ \varphi^a_{2m} \end{array}\right] = \frac{1}{t}\left[\begin{array}{cc} -r & z\mathrm{e}^{\mathrm{i}\omega_{2m-1}} \\ t & 0 \end{array}\right]\left[\begin{array}{c} \varphi^a_{2m} \\ \varphi^a_{2m-1} \end{array}\right] \equiv \tilde{T}\left[\begin{array}{c} \varphi^a_{2m} \\ \varphi^a_{2m-1} \end{array}\right].$$

We have $\|\tilde{T}\| \leq \widetilde{C}(1+|z|)/t$, for some $\widetilde{C}$ independent of z, r, t and any of the phases. Using this expression and definition of the transfer matrices (3.3) we obtain

$$\left\|\left[\begin{array}{c} \varphi^a_{2m} \\ \varphi^a_{2m-1} \end{array}\right]\right\| \geq \left\|T_z(\omega_{2m}, \omega_{2m-1}) \cdots T_z(\omega_{2n+2}, \omega_{2n+1})v^a\right\| \frac{t}{\widetilde{C}(1+|z|)} \tag{3.39}$$

where we chose $\left[\begin{array}{c} \varphi^a_{2n+1} \\ \varphi^a_{2n} \end{array}\right]$ to be the normalized vector

$$v^a = \frac{1}{\sqrt{t^2 + |z\mathrm{e}^{\mathrm{i}}\omega_{2n} - r|^2}}\left[\begin{array}{c} z\mathrm{e}^{\mathrm{i}\omega_{2n}} - r \\ t \end{array}\right].$$

Combining (3.39) with (3.37) and (3.38) proves the bound (3.36) and hence Theorem 3.6. It remains to give the

Proof of Lemma 3.7. The components of φ^b, with $b = 2m$, satisfy $-z\varphi^b_{2m-1} + \mathrm{e}^{-\mathrm{i}\omega_{2m-1}}\varphi^b_{2m} = 0$, see before (3.16). Moreover, since φ^b, is defined modulo a multiplicative factor only, we set $\varphi^b_{2m-1} = 1$ (note that $\varphi^b_{2m-1} = 0$ would imply that $\varphi^b = 0$, so the normalization $\varphi^b_{2m-1} = 1$ is possible). Therefore we obtain from (3.17) the following expression for Green's function:

$$G_z^{[2n,2m]}(2n, 2m) = \frac{1}{z}\frac{1}{\varphi^a_{2m} - z\mathrm{e}^{\mathrm{i}\omega_{2m-1}}\varphi^a_{2m-1}}. \tag{3.40}$$

Note that φ^a_{2m} and φ^a_{2m-1} cannot both vanish, since otherwise we would have $\varphi^a = 0$. It follows from (3.3) that the components of φ^a satisfy

$$\begin{bmatrix} \varphi^a_{2m-1} \\ \varphi^a_{2m-2} \end{bmatrix} = T_z(\omega_{2m-2}, \omega_{2m-3}) \begin{bmatrix} \varphi^a_{2m-3} \\ \varphi^a_{2m-4} \end{bmatrix}, \tag{3.41}$$

and furthermore, that $\varphi^a_{2m} = \frac{\mathrm{e}^{-\mathrm{i}\omega_{2m}}}{z}\left[t\varphi^a_{2m-2} - r\varphi^a_{2m-1}\right]$. Consequently, φ^a_{2m} depends only on ω_{2m} and ω_j, with $j \leq 2m-2$. We have

$$\begin{aligned} &\mathbb{E}\left[\left|G_z^{[2n,2m]}(2n,2m)\right|^s\right] \\ &\quad = |z|^{-s}\widehat{\mathbb{E}}\left[\int_0^{2\pi} \mathrm{d}\mu(\omega_{2m-1}) \frac{1}{|\varphi^a_{2m} - z\mathrm{e}^{\mathrm{i}\omega_{2m-1}}\varphi^a_{2m-1}|^s}\right], \end{aligned} \tag{3.42}$$

where $\widehat{\mathbb{E}}$ is the expectation over all ω_j, $j = 2n, \ldots, 2m-2$ and $j = 2m$. The dependence on ω_{2m-1} of the integrand in

$$I := \int_0^{2\pi} \mathrm{d}\mu(\omega_{2m-1}) \frac{1}{|\varphi^a_{2m} - z\mathrm{e}^{\mathrm{i}\omega_{2m-1}}\varphi^a_{2m-1}|^s} \tag{3.43}$$

is concentrated exclusively in $\mathrm{e}^{\mathrm{i}\omega_{2m-1}}$. Let us define the vector

$$v_m := \begin{bmatrix} \varphi^a_{2m} \\ \varphi^a_{2m-1} \end{bmatrix}. \tag{3.44}$$

If $\varphi^a_{2m-1} = 0$ then we have $I = |\varphi^a_{2m}|^{-s} = \|v_m\|^{-s}$. If $\varphi^a_{2m} = 0$ then we have $I = |z|^{-s}|\varphi^a_{2m-1}|^{-s} = |z|^{-s}\|v_m\|^{-s}$. Next suppose that both φ^a_{2m-1} and φ^a_{2m} are nonzero. We distinguish two cases: either $|\varphi^a_{2m}| \geq |\varphi^a_{2m-1}|$ or $|\varphi^a_{2m}| < |\varphi^a_{2m-1}|$. In the former case we have $\|v_m\| \leq |\varphi^a_{2m}| + |\varphi^a_{2m-1}| \leq 2|\varphi^a_{2m}|$, and

$$\begin{aligned} I &= |\varphi^a_{2m}|^{-s} \int_0^{2\pi} \mathrm{d}\mu(\omega_{2m-1}) \left|\mathrm{e}^{-\mathrm{i}\omega_{2m-1}} - z\varphi^a_{2m-1}/\varphi^a_{2m}\right|^{-s} \\ &\leq C_\mu(s)|\varphi^a_{2m}|^{-s} \leq 2^s C_\mu(s)\|v_m\|^{-s}. \end{aligned} \tag{3.45}$$

Here, we have used that for all $0 < s < 1$ there exists $0 < C_\mu(s) < \infty$ such that for all $\beta \in \mathbb{C}$

$$\int_0^{2\pi} \mathrm{d}\mu(\omega)|\mathrm{e}^{\pm i\omega} - \beta|^{-s} \leq C_\mu(s),$$

see [18]. Next we consider the case $|\varphi^a_{2m}| < |\varphi^a_{2m-1}|$, in which case we have $\|v_m\| \leq |\varphi^a_{2m}| + |\varphi^a_{2m-1}| \leq 2|\varphi^a_{2m-1}|$, and

$$\begin{aligned} I &= |\varphi^a_{2m-1}|^{-s}|z|^{-s} \int_0^{2\pi} \mathrm{d}\mu(\omega_{2m-1}) \left|\mathrm{e}^{\mathrm{i}\omega_{2m-1}} - z^{-1}\varphi^a_{2m}/\varphi^a_{2m-1}\right|^{-s} \\ &\leq |z|^{-s}C_\mu(s)|\varphi^a_{2m-1}|^{-s} \leq |z|^{-s}2^s C_\mu(s)\|v_m\|^{-s}. \end{aligned} \tag{3.46}$$

Combining these estimates, we see that in any event,

$$I \leq (1 + |z|^{-s})(1 + 2^s C_\mu(s))\|v_m\|^{-s}. \tag{3.47}$$

This bound, together with (3.42), yields (3.37) with $\widehat{\mathbb{E}}$ instead of $\mathbb{E}$. But both expressions are the same since v_m does not depend on ω_{2m-1}.

This completes the proof of Lemma 3.7 and with that the proof of Theorem 3.6. ∎

3.3 Proof of Theorem 3.3

Let s, α, C be as in Theorem 3.6. Combining the latter theorem with Theorem 3.5, we obtain the bound

$$\mathbb{E}\left[|G_z(2n,2m)|^{s/2}\right] \leq C_1 \mathrm{e}^{-2\alpha|m-n|}, \tag{3.48}$$

for all $|z| \neq 0, 1$ satisfying the bound indicated in Theorem 3.5. The constant C_1 and all further constants C_j introduced in this proof depend on z, s, ϵ and t and are uniform in z in compacts of $\mathbb{C} \setminus \{0\}$. We use the inequality (3.22) after Theorem 3.4 to arrive at the following bound for $|k-l| \geq 4$,

$$\mathbb{E}\left[|G_z(k,l)|^{s/2}\right] \leq C_2 \sum_{p_1,p_2 \in \{0,1\}} \mathbb{E}\left[|G_z(2m-2p_1, 2n+2p_2)|^{s/2}\right]. \tag{3.49}$$

Next, since $|m-p_1-(n+p_2)| \geq |m-n|-2$, and $|k-l| \leq |2m-2n|+2$, combining (3.48) and (3.49) gives

$$\mathbb{E}\left[|G_z(k,l)|^{s/2}\right] \leq C_3 \mathrm{e}^{-\alpha|k-l|}, \tag{3.50}$$

provided $|k-l| \geq 4$ and $|z| \neq 1$, $\frac{1}{1+\epsilon} < |z| < 1+\epsilon$. Finally, if $|k-l| < 4$, the bound (3.50) is implied by Theorem 3.2. This completes the proof of Theorem 3.3. ■

3.4 Proof of Theorem 2.2

Exponential decay of fractional moments of Green's function implies dynamical localization for band matrices of the type $U_\omega = D_\omega S$, as shown in the following result.

Theorem 3.9 ([17], Theorem 3.2) *Assume that the random variables ω_k satisfy the conditions of Theorem 2.2 and that for some $s \in (0,1)$, $C < \infty$, $\alpha > 0$ and $\epsilon > 0$,*

$$\mathbb{E}(|G_z(k,l)|^s) \leq C\mathrm{e}^{-\alpha|k-l|}$$

for all $k, l \in \mathbb{Z}$ and all $z \in \mathbb{C}$ s.t. $1-\epsilon < |z| < 1$. Then there is a $\widetilde{C} < \infty$ such that

$$\mathbb{E}\left[\sup_{f \in C(\mathbb{S}), \|f\|_\infty \leq 1} |\langle e_k, f(U_\omega) e_l \rangle|\right] \leq \widetilde{C} \mathrm{e}^{-\alpha|k-l|/4}$$

for all $k, l \in \mathbb{Z}$.

With the identification of the basis elements $e_{2k} = |\uparrow\rangle \otimes |k\rangle$, $e_{2k+1} = |\downarrow\rangle \otimes |k\rangle$, see after (2.21), we immediately obtain that Theorems 3.3 and 3.9 imply Theorem 2.2. ■

4 Temporal disorder

For the sake of comparison, we briefly consider in this section the case where the disorder is introduced in the model through the time variable. This means that at each time step, the evolution of the coin variable is randomly chosen within a set of iid unitary 2×2 matrices $\{C_k\}_{k \in \mathbb{Z}}$. Therefore, the random evolution after n steps reads

$$U(n,0) = U_n U_{n-1} \cdots U_2 U_1, \quad \text{where} \quad U_k = S(C_k \otimes \mathbb{I}). \tag{4.1}$$

As we will see, our choice of random unitary matrices $\{C_k\}_{k \in \mathbb{Z}}$

$$C_k = \begin{bmatrix} \mathrm{e}^{-\mathrm{i}\omega_k^+} t & -\mathrm{e}^{-\mathrm{i}\omega_k^+} r \\ \mathrm{e}^{-\mathrm{i}\omega_k^-} r & \mathrm{e}^{-\mathrm{i}\omega_k^-} t \end{bmatrix} \tag{4.2}$$

with $\{\omega_k^+\}_{k\in\mathbb{Z}}\cup\{\omega_k^-\}_{k\in\mathbb{Z}}$ iid subjected to the condition

$$\mathbb{E}(e^{i\omega_k^+}) = \mathbb{E}(e^{i\omega_k^-}) = 0, \tag{4.3}$$

naturally leads to the study of (classical) *persistent random walks.* For notational reasons which will be clear below, we change notations to $\uparrow = +1, \downarrow = -1$ so that

$$|\uparrow\rangle = |+1\rangle, \quad |\downarrow\rangle = |-1\rangle, \quad P_\uparrow = P_{+1}, \quad P_\downarrow = P_{-1}. \tag{4.4}$$

We first state a deterministic result dealing with expectation values of position operators at time n.

Let X denote the position operator on $\mathbb{C}^2\otimes l^2(\mathbb{Z})$ defined by $(X\psi)(x) = (\mathbb{I}\otimes x)\psi(x)$, $x\in\mathbb{Z}$, on its maximal domain

$$D = \left\{\psi\in\mathbb{C}^2\otimes l^2(\mathbb{Z}),\ \text{s.t.} \sum_{x\in\mathbb{Z}}\sum_{\sigma\in\{+1,-1\}} \|(P_\sigma\otimes x)\psi(x)\|^2_{\mathbb{C}^2} < \infty\right\}.$$

For $f:\mathbb{Z}\mapsto\mathbb{C}$, we define the operator $F(X)$ on $\mathbb{C}^2\otimes l^2(\mathbb{Z})$ by $(F\psi)(x) = (\mathbb{I}\otimes f(x))\psi(x)$ on its maximal domain D_F via the spectral Theorem. For any $\Psi_0\in\mathbb{C}^2\otimes l^2(\mathbb{Z})$ such that $U(n,0)\psi_0\in D_F$, we note

$$\langle\Psi_0, U(n,0)^*F(X)U(n,0)\psi_0\rangle \equiv \langle F(X)\rangle_{\psi_0}(n). \tag{4.5}$$

Explicit computations with $P_{\pm1} = |\pm1\rangle\langle\pm1|$ yield the following Lemma:

Lemma 4.1 *With the notations above,*

$$\begin{aligned} U_nU_{n-1}\cdots U_1 &= \sum_{\substack{\sigma_n,\cdots,\sigma_1\\ \sigma_j\in\{-1,1\}}}\sum_{x\in\mathbb{Z}} P_{\sigma_n}C_n\cdots P_{\sigma_1}C_1\otimes|x+\sum_{j=1}^n\sigma_j\rangle\langle x| \\ &= \sum_{x\in\mathbb{Z}}\sum_{k=-n}^{n} J_k(n)\otimes|x+k\rangle\langle x| \end{aligned} \tag{4.6}$$

where

$$J_k(n) = \sum_{\substack{\sigma_n,\cdots,\sigma_1\\ \sum_{j=1}^n\sigma_j=k}} P_{\sigma_n}C_n\cdots P_{\sigma_1}C_1 \in M_2(\mathbb{C}) \tag{4.7}$$

satisfies $J_k(n)=0$ if k and n have different parities or if $k>n$ or $k<-n$.

Moreover, if $\Psi_0 = \varphi_0\otimes|0\rangle$, we have for any $f:\mathbb{Z}\mapsto\mathbb{C}$ and all $n\in\mathbb{N}$,

$$\langle F(X)\rangle_{\psi_0}(n) = \sum_{k=-n}^{n} f(k)\,\langle\varphi_0, J_k^*(n)J_k(n)\varphi_0\rangle_{\mathbb{C}^2}\,, \tag{4.8}$$

where $W_k(n)\equiv\langle\varphi_0, J_k^*(n)J_k(n)\varphi_0\rangle_{\mathbb{C}^2}$ satisfies

$$W_k(n)\geq 0 \quad \textit{and} \quad \sum_{k=-n}^{n} W_k(n) = \|\varphi_0\|^2. \tag{4.9}$$

Remarks:
i) If the initial coin vector φ_0 is normalized, which we assume from now on, then the quantum mechanical expectation value of the operator $F(X)$ at time n coincides with the expectation value of the function f with respect to the classical discrete probability distribution $\{W_k(n)\}_{k\in\{-n,\dots,n\}}$ on $\{-n,\dots,n\}\subset\mathbb{Z}$. The quantity $W_k(n)$ is interpreted as the probability to reach site $k\in\mathbb{Z}$ in n steps.
ii) The probabilities $\{W_k(n)\}_{k\in\{-n,\dots,n\}}$ are actually φ_0-dependent random variables for randomly chosen coin operators $\{C_k\}_{k\in\mathbb{Z}}$ with arbitrary distribution. Taking expectation with respect to the distribution of the C_k's, we get a new discrete probability distribution on $\{-n,\dots,n\}\subset\mathbb{Z}$ given by $\{w_k(n)\}_{k\in\{-n,\dots,n\}}$ with

$$w_k(n)=\mathbb{E}(W_k(n)),\quad k\in\{-n,\dots,n\},\ \ \forall\ n\in\mathbb{N}, \tag{4.10}$$

with same interpretation in terms of a classical random walk on $\mathbb{Z}$.

We shall focus on the expectation value of the quantum mechanical moments of the position operator for our choice (4.2) of $\{C_k\}_{k\in\mathbb{Z}}$ under condition (4.3), for a normalized initial condition of the form $\psi_0=\varphi_0\otimes|0\rangle$, i.e. on

$$\mathbb{E}(\langle X^L\rangle_{\psi_0}(n))=\sum_{k=-n}^{n}k^L w_k(n)\quad\text{as}\quad n\to\infty. \tag{4.11}$$

We remind the reader that a *persistent* or *correlated random walk* on $\mathbb{Z}$ is determined by a probability p that the walker moves one unit in the same direction as in the previous step, and a probability $1-p$ that the walker moves one unit in the opposite direction to the last step. See e.g. [29].

We let $S_n=\sum_{j=1}^n\sigma_j$ with $\sigma_j\in\{-1,+1\}$ be the random walk defined by $\mathbb{P}(S_n=k)=w_k(n)$. This random walk is characterized as follows.

Proposition 4.2 *Assume the matrices $\{C_k\}_{k\in\mathbb{Z}}$ are given by (4.2) and (4.3) holds. Take $\psi_0=\varphi_0\otimes|0\rangle$ with $\varphi_0=\alpha|+\rangle+\beta|-\rangle\in\mathbb{C}^2$ normalized. Then S_n is a* persistent random walk *with parameters*

$$\begin{aligned}
&\mathbb{P}(\sigma_1=+1)=|\alpha|^2t^2+|\beta|^2r^2-2\Re(\overline{\alpha}\beta)rt:=a\\
&\mathbb{P}(\sigma_1=-1)=|\alpha|^2r^2+|\beta|^2t^2+2\Re(\overline{\alpha}\beta)rt:=b=1-a\\
&\mathbb{P}(\sigma_j=+1|\sigma_{j-1}=+1)=P(\sigma_j=-1|\sigma_{j-1}=-1)=t^2\\
&\mathbb{P}(\sigma_j=-1|\sigma_{j-1}=+1)=P(\sigma_j=+1|\sigma_{j-1}=-1)=r^2=1-t^2,
\end{aligned}$$

for $j\geq 2$.

Proof: For $n\in\mathbb{N}$, let $\sigma=(\sigma_1,\sigma_2,\dots,\sigma_n)\in\{-1,+1\}^n$. We need to consider

$$w_k(n)=\sum_{\substack{\sigma,\sigma'\in\{-1,+1\}^n\\ \sum_{j=1}^n\sigma_j=\sum_{j=1}^n\sigma'_j=k}}\mathbb{E}(\langle\varphi_0,C_1^*P_{\sigma'_1}C_2^*P_{\sigma'_2}\cdots C_n^*P_{\sigma'_n}P_{\sigma_n}C_n\cdots P_{\sigma_1}C_1\varphi_0\rangle), \tag{4.12}$$

where $\sigma_n=\sigma'_n$. With $P_\sigma=|\sigma\rangle\langle\sigma|$, $\sigma=\pm1$, and reorganizing the product, the scalar product under the sum equals

$$\overline{\langle\sigma'_1,C_1\varphi_0\rangle}\,\langle\sigma_1,C_1\varphi_0\rangle\prod_{j=2}^{n}\overline{\langle\sigma'_j,C_j\sigma'_{j-1}\rangle}\,\langle\sigma_j,C_j\sigma_{j-1}\rangle\,,$$

where the first two factors can be further expanded as

$$\Big(\bar\alpha\,\overline{\langle\sigma_1', C_1+\rangle}+\bar\beta\,\overline{\langle\sigma_1', C_1-\rangle}\Big)\Big(\alpha\langle\sigma_1, C_1+\rangle+\beta\langle\sigma_1, C_1-\rangle\Big).$$

By independence, the expectation factorizes and, with the notation $\langle\sigma, C\tau\rangle = C_{\sigma,\tau}$, $\sigma,\tau\in\{+1,-1\}$, we immediately get from (4.2) and (4.3)

$$\mathbb{E}(\overline{C_{\sigma_j',\sigma_{j-1}'}}C_{\sigma_j,\sigma_{j-1}})=0 \ \text{ if } \ \sigma_j'\neq\sigma_j, \ \ \forall\, j\geq 2.$$

This, together with the conditions $\sigma_n=\sigma_n'$ and $\sum_{j=1}^n\sigma_j=\sum_{j=1}^n\sigma_j'=k$ in (4.12), imposes $\sigma_j=\sigma_j'$ for all $j\geq 1$. Hence, (4.12) reduces to

$$w_k(n)= \tag{4.13}$$
$$\sum_{\substack{\sigma\in\{-1,+1\}^n\\ \sum_{j=1}^n\sigma_j=k}}\mathbb{E}(|\alpha|^2|C_{\sigma_1,+1}|^2+|\beta|^2|C_{\sigma_1,-1}|^2+2\Re(\bar\alpha\beta C_{\sigma_1,-1}\overline{C_{\sigma_1,+1}}))\prod_{j=2}^n\mathbb{E}(|C_{\sigma_j,\sigma_{j-1}}|^2),$$

where

$$\begin{aligned}&\mathbb{E}(|C_{+1,+1}|^2)=\mathbb{E}(|C_{-1,-1}|^2)=t^2\\ &\mathbb{E}(|C_{+1,-1}|^2)=\mathbb{E}(|C_{-1,+1}|^2)=r^2\\ &\mathbb{E}(C_{-1,-1}\overline{C_{-1,+1}})=-\mathbb{E}(C_{+1,-1}\overline{C_{+1,+1}})=rt.\end{aligned} \tag{4.14}$$

The result then follows with the definition of $S_n=\sum_{j=1}^n\sigma_j$. ■

Persistent random walks or correlated random walks are well known and have been studied in many details and greater generality, ours being the simplest instance. See e.g. [29], [13] and the references therein. In particular, when $r=t=1/\sqrt{2}$, the persistent and symmetric random walks are equivalent. Also it is known that the first moment is finite and that the second moment is proportional to n for n large. This leads to the following diffusive behaviour:

$$\begin{aligned}&\lim_{n\to\infty}\frac{\mathbb{E}(\langle X\rangle_{\psi_0}(n))}{n}=0\\ &\lim_{n\to\infty}\frac{\mathbb{E}(\langle X^2\rangle_{\psi_0}(n))}{n}=t^2/r^2.\end{aligned} \tag{4.15}$$

For completeness, we provide below a simple proof of the fact that all moments of the persistent random walk display a diffusive behaviour, a statement that we couldn't find as such in the literature, although certainly well known.

Proof of Proposition 2.4. We assume the hypotheses of Proposition 4.2 and use the familiar setup of generating functions together with a classical scaling argument.
We consider only the situation $r\neq t$ which differs from the usual symmetric random walk. Let $w_k^{\pm}(n)$ be the conditional probabilities

$$w_k^{\pm}(n)=\mathbb{P}(S_n=k|\sigma_n=\pm 1)\quad \text{s.t.}\quad w_k^+(n)+w_k^-(n)=w_k(n).$$

Thus we have, for $n\geq 1$ and $|k|\leq n$

$$\begin{aligned}w_k^+(n+1)&=r^2w_{k-1}^-(n)+t^2w_{k-1}^+(n)\\ w_k^-(n+1)&=t^2w_{k+1}^-(n)+r^2w_{k+1}^+(n)\end{aligned} \tag{4.16}$$

with

$$w_1^+(1) = a, \quad w_1^-(1) = b. \tag{4.17}$$

Moreover, $w_k^\pm(n) = 0$ if $|k| > n$. We introduce the generating functions $\Phi_n^\pm$ and Ψ_n by

$$\begin{aligned} \Phi_n^\pm(z) &= \sum_{k=-n}^{n} e^{izk} w_k^\pm(n), \\ \Psi_n(z) &= \Phi_n^+(z) + \Phi_n^-(z), \quad \forall z \in \mathbb{C}. \end{aligned} \tag{4.18}$$

As a consequence of (4.16), introducing $\Phi_n(z) = (\Phi_n^+(z), \Phi_n^-(z))^T$, we have

$$\begin{aligned} &\Phi_{n+1}(z) = M(z)\Phi_n(z), \quad \text{with} \\ &M(z) = \begin{bmatrix} t^2 e^{iz} & r^2 e^{iz} \\ r^2 e^{-iz} & t^2 e^{-iz} \end{bmatrix} \quad \text{and} \quad \Phi_1(z) = \begin{bmatrix} ae^{iz} \\ be^{-iz} \end{bmatrix}. \end{aligned} \tag{4.19}$$

This allows to determine explicitly $\Phi_n(z)$ and

$$\Psi(z) = \left\langle \begin{bmatrix} 1 \\ 1 \end{bmatrix}, M^{n-1}(z)\Phi_1(z) \right\rangle, \quad n \geq 1, \tag{4.20}$$

and, in turn, all moments of the probability distribution $\{w_k(n)\}_{k\in\mathbb{Z}}$.

We consider now the diffusive scaling introducing the macroscopic time variable $N = n\tau$, where $\tau >> 1$, $\tau \in \mathbb{N}$ and the macroscopic space variable $K = \sqrt{\tau}k$, such that $K/\sqrt{N} = k/\sqrt{n}$ remains finite. Expecting a probability distribution $\{w_k(n)\}_{k\in\mathbb{Z}}$ asymptotically invariant under this scaling, we are led to the study of the generating function at $z = y/\sqrt{\tau}$ in the limit $\tau \to \infty$:

Lemma 4.3

$$\lim_{\substack{\tau\to\infty \\ \tau\in\mathbb{N}}} \sum_{k=-\tau n}^{\tau n} e^{i\frac{y}{\sqrt{\tau}}k} w_k(\tau n) = e^{-n\frac{t^2}{2r^2}y^2}, \tag{4.21}$$

uniformly in y in any compact set of $\mathbb{C}$.

Since the functions of y involved are entire and the convergence is uniform in compact sets, we can differentiate the above identity w.r.t. y as many times as we wish. In particular, differentiating L times with $n = 1$ and setting $y = 0$ immediately yields the following Corollary which ends the proof of the Proposition 2.4:

Corollary 4.4

$$\lim_{\substack{\tau\to\infty \\ \tau\in\mathbb{N}}} \sum_{k=-\tau}^{\tau} \frac{(ik)^L}{\tau^{L/2}} w_k(\tau) = \left(\frac{t^2}{r^2}\right)^{L/2} H_L(0)(-1)^L, \tag{4.22}$$

where H_L denotes the Hermite polynomial $H_L(z) = (-1)^L e^{z^2/2} \left(\frac{d}{dz}\right)^L e^{-z^2/2}$.

Proof of Lemma 4.3. Note that the left side of (4.21) equals

$$\lim_{\tau\to\infty} \Psi_{\tau n}(y/\sqrt{\tau}) = \lim_{\tau\to\infty} \left\langle \begin{bmatrix} 1 \\ 1 \end{bmatrix}, M^{\tau n-1}(y/\sqrt{\tau})\Phi_1(y/\sqrt{\tau}) \right\rangle \tag{4.23}$$

so that we are interested in small values of $z = y/\sqrt{\tau}$. For $|z|$ small enough, we compute the spectrum of the analytic matrix $M(z)$

$$\sigma(M(z)) = \{\lambda_1(z), \lambda_2(z)\} \tag{4.24}$$

with distinct eigenvalues

$$\lambda_{\frac{1}{2}}(z) = t^2\cos(z) \pm \sqrt{t^4\cos^2(z) + 1 - 2t^2}. \tag{4.25}$$

Hence, for $|z|$ small, there exists an invertible matrix $R(z)$, analytic in z, such that

$$M^n(z) = R^{-1}(z) \begin{bmatrix} \lambda_1^n(z) & 0 \\ 0 & \lambda_2^n(z) \end{bmatrix} R(z). \tag{4.26}$$

For $|z|$ small enough, we have

$$\begin{aligned} \lambda_1(z) &= 1 - z^2\frac{t^2}{2r^2} + O(z^4) \\ \lambda_2(z) &= (t^2 - r^2) + z^2\frac{t^2(t^2 - r^2)}{2r^2} + O(z^4), \end{aligned} \tag{4.27}$$

whereas

$$M^{-1}(0) = M^{-1}(0)^* = \frac{1}{t^2 - r^2}\begin{bmatrix} t^2 & -r^2 \\ -r^2 & t^2 \end{bmatrix} \quad \text{satisfies} \quad M^{-1}(0)\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

and

$$R(0) = R^*(0) = R^{-1}(0) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Since (4.27) yields for any $y \in \mathbb{C}$ and τ large enough

$$\begin{aligned} \lambda_1(y/\sqrt{\tau})^{\tau n} &= \mathrm{e}^{-ny^2\frac{t^2}{2r^2}} + O(ny^4/\tau) \\ \lambda_2(y/\sqrt{\tau})^{\tau n} &= (t^2 - r^2)^{\tau n}\mathrm{e}^{ny^2\frac{t^2}{2r^2}} + O(ny^4/\tau), \end{aligned} \tag{4.28}$$

we eventually obtain, since $|r^2 - t^2| < 1$ and $a + b = 1$,

$$\lim_{\tau\to\infty}\left\langle \begin{bmatrix} 1 \\ 1 \end{bmatrix}, M^{\tau n-1}(y/\sqrt{\tau})\Phi_1(y/\sqrt{\tau})\right\rangle = \tag{4.29}$$

$$\left\langle \begin{bmatrix} e^{-ny^2\frac{t^2}{2r^2}} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} a+b \\ a-b \end{bmatrix}\right\rangle = \mathrm{e}^{-ny^2\frac{t^2}{2r^2}}$$

■

A Appendix

A.1 Proof of Lemma 2.1

Introducing the discrete Fourier transform $\mathcal{F} : L^2([0, 2\pi), \mathbb{C}^2) \to \mathbb{C}^2 \otimes l^2(\mathbb{Z})$ by

$$(\mathcal{F}f)(k) = \hat{f}(k) = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-ikx}dx, \tag{A.1}$$

we get that U is unitarily equivalent to a multiplication operator by a matrix $V(x)$.

$$\mathcal{F}^{-1}U\mathcal{F} = \text{mult}V(x) = \text{mult}\begin{bmatrix} e^{-ix} & 0 \\ 0 & e^{ix} \end{bmatrix} C. \tag{A.2}$$

The matrix $V(x)$ is entire in x, and unitary for x real, which means that its eigenvalues $\{\alpha_1(x), \alpha_2(x)\}$ and eigenprojectors $\{P_1(x), P_2(x)\}$ are also analytic functions of $x \in \mathbb{R}$, even at the possible crossings of eigenvalues for $x \in \mathbb{R}$. Moreover, for any $n \in \mathbb{Z}$,

$$V^n(x) = P_1(x)\alpha_1^n(x) + P_2(x)\alpha_2^n(x). \tag{A.3}$$

Similarly, the position operator $K = (\mathbb{I} \otimes k)$ is unitarily equivalent to differentiation w.r.t. x (on its natural domain):

$$\mathcal{F}^{-1}K\mathcal{F} = -i\partial_x. \tag{A.4}$$

In particular, for $\Psi \in \mathbb{C}^2 \otimes l^2(\mathbb{Z})$ such that $\mathcal{F}^{-1}\Psi = f \in L^2([0, 2\pi), \mathbb{C}^2)$ and Ψ in the domain of $K^2 = (\mathbb{I} \otimes k)^2$, we have for all $n \in \mathbb{Z}$,

$$\langle K^2 \rangle_\Psi(n) := \left\langle \Psi, U^{-n}K^2U^n\Psi \right\rangle = \int_0^{2\pi} \|\partial_x(V^n(x)f(x))\|^2_{\mathbb{C}^2} dx. \tag{A.5}$$

Now, it is not difficult to see by explicit computations that this quantity behaves as n^2 for $n \to \infty$, unless the analytic eigenvalues $\{\alpha_1(x), \alpha_2(x)\}$ of $V(x)$ are independent of x. We have

$$\alpha_1(x)\alpha_2(x) = \det V(x) = \det C \quad \text{and} \quad \alpha_1(x) + \alpha_2(x) = \text{tr } V(x) = e^{-ix}a + e^{ix}d. \tag{A.6}$$

Hence the eigenvalues $\alpha_j(x)$ are independent of x iff $a = d = 0$. ∎

A.2 Positivity and continuity of Lyapunov exponent

This Section is devoted to the proof of Theorem 3.1.

Remember that we take μ to be absolutely continuous with density $\tau \in L^\infty(\mathbb{T})$ having support with nonempty interior and that the transfer matrices $T_z(\theta, \eta)$ are given in (3.4) (with $\omega_{2n} = \theta$ and $\omega_{2n-1} = \eta$).

To prove Theorem 3.1, we use Furstenberg's theorem, which is developed for real square matrices. We thus map $T_z \in \mathbb{M}_2(\mathbb{C})$ (square 2×2 matrices with complex entries) into $\xi(T_z) \in \mathbb{M}_4(\mathbb{R})$ using the bijection

$$\xi : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} I\text{Re}a + J\text{Im}a & I\text{Re}b + J\text{Im}b \\ I\text{Re}c + J\text{Im}c & I\text{Re}d + J\text{Im}d \end{bmatrix}, \tag{A.7}$$

where

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

We refer to [9] for more detail on this transformation. In particular, we have

$$\|\xi(A)\| = \sqrt{2}\|A\|, \tag{A.8}$$

where the norms are given by $\|X\|^2 = \text{Tr}(X^*X)$, for $X \in \mathbb{M}_2(\mathbb{C})$ or $X \in \mathbb{M}_4(\mathbb{R})$, and that if $A \in \mathbb{M}_2(\mathbb{C})$ satisfies $|\det(A)| = 1$, then $|\det \xi(A)| = 1$. Due to (3.5), we have $|\det \xi(T_z(\theta, \eta))| = 1$, for all $z \neq 0$ and all θ, η.

Relation (A.8) together with the fact that $\xi(AB) = \xi(A)\xi(B)$ shows that the statement of Theorem 3.1 is equivalent to $\lim_{n\to\infty} \frac{1}{n}\|\xi(T_z(\theta_n, \eta_n)) \cdots \xi(T_z(\theta_1, \eta_1))\| = \gamma$ almost surely, for the same deterministic $\gamma > 0$ as in Theorem 3.1.

The measure μ on $\mathbb{T}$ induces a measure on $\mathbb{M}_4(\mathbb{R})$, supported on the subset

$$\mathcal{M} := \{\xi(T_z(\theta, \eta)), \theta, \eta \in \text{supp}\mu\}. \tag{A.9}$$

We call the induced measure again μ. Let $Y_1, Y_2, \dots$ be iid random matrices in $\mathbb{M}_4(\mathbb{R})$ with common distribution μ. If the integrability condition

$$\mathbb{E}\left[\max\{\log \|Y_1\|, 0\}\right] < \infty \tag{A.10}$$

is satisfied, then the upper Lyapunov exponent $\gamma \in \mathbb{R} \cup \{-\infty\}$ is defined as

$$\gamma := \lim_{n\to\infty} \frac{1}{n}\mathbb{E}[\log \|Y_n \cdots Y_1\|].$$

The theorem of Furstenberg and Kesten ([8], Theorem 4.1) states that if in addition the matrices Y_j are invertible ($Y_j \in \mathrm{GL}_4(\mathbb{R})$), then

$$\lim_{n\to\infty} \frac{1}{n} \log \|Y_n \cdots Y_1\| = \gamma \tag{A.11}$$

almost surely. In our case, $Y_j = \xi(T_z(\theta_j, \eta_j))$ is invertible, and $\|Y_1\| \le C$ is uniformly bounded in θ_1, η_1 (see (3.4)), so that (A.10) is trivially satisfied. This implies that (3.12) holds almost surely, with a deterministic γ. Moreover, since $S = T_z(\theta_n, \eta_n) \cdots T_z(\theta_1, \eta_1)$ is a 2×2 invertible matrix, we have $\|S\| = \|S^{-1}\| \ge 1$, and therefore $\gamma \ge 0$. The remaining part of the proof of Theorem 3.1 consists in proving that γ is *strictly* positive and continuous.

Let $\mathcal{G}_\mu \subset \mathrm{GL}_4(\mathbb{R})$ be the (multiplicative) group of matrices generated by the transfer matrices $T_z(\theta, \eta)$, where θ, η vary throughout the support of the measure μ. Here, z is fixed and not displayed in $\mathcal{G}_\mu$.

Theorem A.1 (Furstenberg, [8] Thm. 6.3) *Suppose that $\mathcal{G}_\mu$ is strongly irreducible and non-compact. Then the upper Lyapunov exponent associated with any sequence of random matrices $Y_1, Y_2, \dots$ in $\mathrm{SL}_4(\mathbb{R}) \cap \operatorname{supp}\mu$, iid with common distribution μ, is strictly positive. This means that (A.11) holds with $\gamma > 0$.*

Now we show that $\mathcal{G}_\mu$ is strongly irreducible and non-compact.

Lemma A.2 *If $\operatorname{supp}\mu$ contains two distinct points, then $\mathcal{G}_\mu$ is not compact for all $z \neq 0$.*

Proof. For fixed $z = R\mathrm{e}^{\mathrm{i}\alpha}$, we have

$$T_z(\theta, \eta) = \frac{\mathrm{e}^{-\mathrm{i}(\theta+\alpha)}}{R} \begin{bmatrix} R^2 \mathrm{e}^{\mathrm{i}(\theta+\alpha+\eta+\alpha)} + r^2 & -rt \\ -rt & t^2 \end{bmatrix},$$

and a direct calculation shows that

$$T_z(\eta, \theta) T_z(\eta, \eta)^{-1} = (T_z(\theta, \theta)^{-1} T_z(\theta, \eta))^* = \begin{bmatrix} \mathrm{e}^{\mathrm{i}(\theta-\eta)} & \frac{r}{t}(\mathrm{e}^{\mathrm{i}(\theta-\eta)} - 1) \\ 0 & 1 \end{bmatrix}.$$

Both operators $T_z(\eta, \theta) T_z(\eta, \eta)^{-1}$ and $T_z(\theta, \theta)^{-1} T_z(\theta, \eta)$ belong to $\mathcal{G}_\mu$, and hence so does their positive-definite product,

$$0 \le M := T_z(\eta, \theta) T_z(\eta, \eta)^{-1} T_z(\theta, \theta)^{-1} T_z(\theta, \eta) = \mathbb{1} + \frac{r}{t} \begin{bmatrix} \frac{r}{t}|\mathrm{e}^{\mathrm{i}(\theta-\eta)} - 1| & \mathrm{e}^{\mathrm{i}(\theta-\eta)} - 1 \\ \mathrm{e}^{-\mathrm{i}(\theta-\eta)} - 1 & 0 \end{bmatrix}.$$

Note that the r.h.s. does not depend on z at all. Since the determinant of the last matrix is $-\frac{r^2}{t^2}|\mathrm{e}^{\mathrm{i}(\theta-\eta)} - 1|^2 < 0$, $\mathcal{G}_\mu$ contains a non-negative matrix M having an eigenvalue strictly larger than one. It follows that $\|M^n\| \to \infty$ as $n \to \infty$. Thus the group generated by the $\xi(T_z(\theta, \eta))$, $\theta, \eta \in \operatorname{supp}\mu$, is not compact. ∎

Finally we prove strong irreducibility of $\mathcal{G}_\mu$. Let J be a non-empty open interval, $J \subset \operatorname{supp}\mu$. Since the subset of matrices $\mathcal{M}'$ obtained from $\mathcal{M}$, (A.9), by restricting $\theta, \eta \in J$, is a continuous image of the connected set $J \times J \subset \mathbb{R}^2$, $\mathcal{M}'$ is a connected set in $\mathbb{M}_4(\mathbb{R})$. Strong irreducibility of $\mathcal{M}'$ (which implies strong irreducibility of $\mathcal{M}$) is then equivalent to irreducibility of $\mathcal{M}'$, see [8] Exercise IV.2.9.

Lemma A.3 *The only subspaces $V \subseteq \mathbb{R}^4$ invariant under the action of $\mathcal{M}'$ are $V = \{0\}$ and $V = \mathbb{R}^4$. Hence $\mathcal{M}'$ is strongly irreducible. Since $\mathcal{M}' \subset \mathcal{G}_\mu$, we have that $\mathcal{G}_\mu$ is strongly irreducible.*

Proof. We just need to show that $V = \{0\}$ and $V = \mathbb{R}^4$ are the only invariant subspaces of $\mathcal{M}'$. Let $z = Re^{i\alpha} \neq 0$ be fixed. One easily finds that

$$\xi(T_z) = \frac{1}{R}\left[T_1 \cos\varphi + T_2 \sin\varphi + T_3 \cos\chi + T_4 \sin\chi\right], \tag{A.12}$$

where $\varphi = \alpha + \eta$, $\chi = \alpha + \theta$ vary in the open interval $\alpha + J$, and the matrices T_j are given by

$$T_1 = R^2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad T_2 = R^2 \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$T_3 = \begin{bmatrix} r^2 & 0 & -rt & 0 \\ 0 & r^2 & 0 & -rt \\ -rt & 0 & t^2 & 0 \\ 0 & -rt & 0 & t^2 \end{bmatrix}, \qquad T_4 = \begin{bmatrix} 0 & -r^2 & 0 & rt \\ r^2 & 0 & -rt & 0 \\ 0 & rt & 0 & -t^2 \\ -rt & 0 & t^2 & 0 \end{bmatrix}.$$

By taking derivatives in the angles, we see that if V is invariant under $\mathcal{M}'$, then V is invariant also under the action of $-T_1 \sin\varphi + T_2 \cos\varphi$ (and $-T_3 \sin\chi + T_4 \cos\chi$), and hence (by again differentiating) V is as well invariant under the action of $T_1 \cos\varphi + T_2 \sin\varphi$ (and $T_3 \cos\chi + T_4 \sin\chi$). Therefore, $-T_1 \sin^2\varphi + T_2 \sin\varphi\cos\varphi$ and $T_1 \cos^2\varphi + T_2 \sin\varphi\cos\varphi$ leave V invariant, and hence so does T_1 (the difference) and consequently T_2 as well. Similarly one sees that T_3, T_4 leave V invariant, too. Hence any subspace V invariant under $\mathcal{M}'$ must be invariant separately under T_1, T_2, T_3 and T_4.

Consider first V with $\dim V = 1$, i.e., $V = \langle v \rangle$ (real span). It is easy to see that the only possibility for v resulting in a V invariant under $T_1 + T_2$ and $T_1 - T_2$ is $v = \alpha e_3 + \beta e_4$ (canonical basis elements of $\mathbb{R}^4$). But either such V is not left invariant under the action of T_3. Consequently, no one-dimensional subspace is invariant under $\mathcal{M}$.

Consider next V with $\dim V = 2$. Since T_3 is real symmetric and must leave V invariant, V must be spanned by two eigenvectors of T_3. One easily finds that T_3 has eigenvalues $\{0, 1\}$, both twice degenerate, that a basis for the kernel of T_3 is $\{[1, 0, r/t, 0]^t, [0, 1, 0, r/t]^t\}$ (transpose), and a basis of the eigenspace with eigenvalue 1 is $\{[1, 0, -t/r, 0]^t, [0, 1, 0, -t/r]^t\}$. There are three possible cases: 1. both eigenvectors spanning V belong to the kernel of T_3, 2. both eigenvectors spanning V belong to the eigenspace with eigenvalue 1 of T_3, or 3. one eigenvector belongs to the kernel of T_3 and the other one belongs to the other spectral subspace. Either of these three cases can be analyzed separately, and one finds that none of the thus formed spaces V with $\dim V = 2$ is invariant under all of the T_j, $j = 1, 2, 3, 4$. In conclusion, no two-dimensional subspace is invariant under $\mathcal{M}$.

Consider now V with $\dim V = 3$. Then $V^\perp$ has dimension 1 and is invariant under T_3, since the latter is real symmetric. Hence $V^\perp$ is spanned by one of the eigenvectors of T_3. In the same way, one sees that $V^\perp$ must also be invariant under T_1, and one finds easily that this implies that $V^\perp = \{0\}$, a contradiction to $\dim V = 3$. This shows that there is no three-dimensional subspace invariant under $\mathcal{M}$. Lemma A.3 follows. ∎

This proves all assertions of Theorem 3.1, except for the continuity of $z \mapsto \gamma(z)$. However, the latter has been shown to hold in Section VII of [16] (Theorem 7.1). ∎

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