Quantum systems coupled to environments via mean field interactions

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ABSTRACT

We show that when a quantum system is coupled to an environment in a mean field way, then its effective dynamics is governed by a unitary group with a time-dependent Hamiltonian. The time-dependent modification of the bare system Hamiltonian is given by an explicit term involving the reservoir state. We show that entanglement within the system state is not changed during the dynamics. Our results hold for arbitrary strengths of the system-environment coupling, and for finite or infinite dimensional systems. As an application we show that the qualitative dynamical features of an N-body system can be altered drastically by the contact with the environment. For instance, a system having bound states only (point spectrum) can turn into a system with only scattering states (continuous spectrum) and vice-versa.

1. Introduction

Open quantum systems are often described as a bipartite complex of a quantum system S in contact with a quantum environment, also called a reservoir R. The system typically consists of one or several quantum particles, spins or qubits and the bath is commonly modeled by a collection of oscillatory degrees of freedom, such as a collection of quantum harmonic oscillators or a quantum field. The interaction between S and R is specified by an interaction operator in the total Hamiltonian governing the unitary dynamics of the *entire* complex S + R. The reservoir is typically considered very 'large' when compared to the system. This can be expressed for instance in terms of a sizeable spatial extension, a high density of the energy spectrum, or a large number of particles or other degrees of freedom in the reservoir. The influence of the reservoir on the system is often considered as 'noise' and one is interested in the resulting *effective* dynamics of the system alone, that is, an effective evolution equation for the system state. The latter must incorporate effects of the reservoir and it carries information about the state of the reservoir — such as its temperature in an equilibrium situation.

Finding the exact effective system dynamics is generally impossible except for very simple models. The *Markovian Master Equation* is a very popular approximation which can be justified under suitable assumptions. It is a relatively simple equation for the system state which approximates the true system dynamics well, for instance, if the SR interaction is weak and if the reservoir loses its memory quickly — notions which can be made physically and mathematically rigorous [1–8]. In the markovian approximation, the state (density matrix) of a system at time t is obtained by applying a dynamical semigroup (in t) to the initial system state. The mathematical structure of this approximate dynamics is simple, entirely encoded in the properties of the generator of the semigroup, and thus amenable to analytic and numerical analysis. It is routinely used to examine irreversible phenomena of open system dynamics, such as the approach of a stationary state (thermal equilibrium or non-equilibrium stationary states), as well as
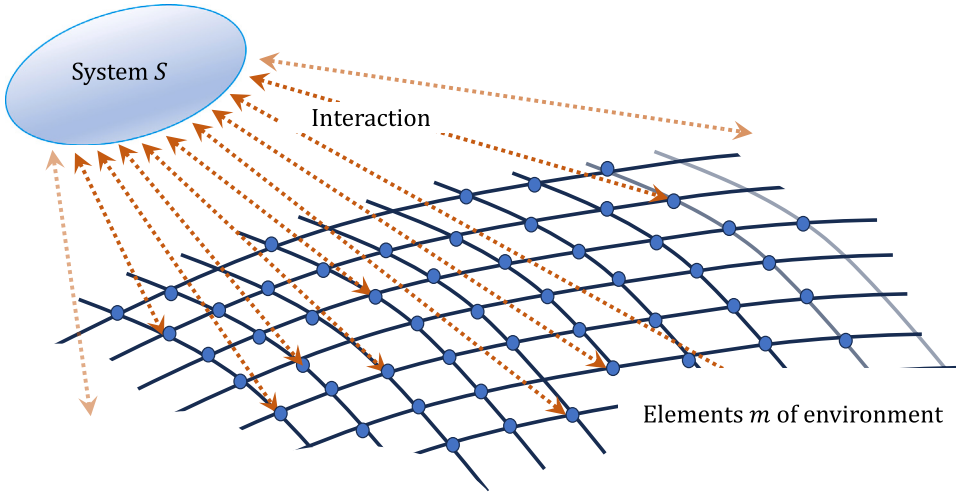

Fig. 1. The system S interacts with elements $m = 1, 2, \ldots, M \to \infty$ of the environment in a uniform way, the same for all m. In this rendition, m is interpreted as the label indicating the spatial location of a subsystem of the reservoir. An example for the reservoir is a lattice spin system, where a spin is located at each position m on a lattice.
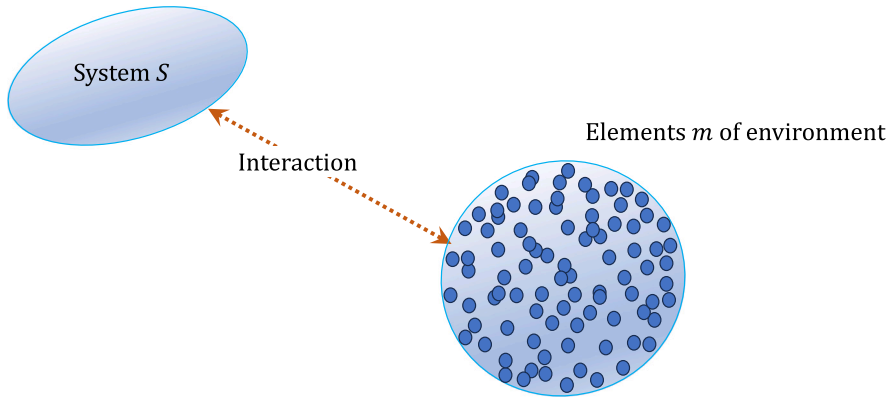

Fig. 2. The system S interacts with elements $m = 1, 2, \ldots, M \to \infty$ of the environment in a uniform way. In this rendition, m is labeling the units of a 'densely packed' environment. For instance, m may label the modes of oscillatory degrees of environment which may all lie spatially close together.

decoherence and the evolution of the entanglement between subparts of S. If the SR coupling is not small, though, or if the reservoir does not have a short memory, then the markovian approximation deviates substantially from the true dynamics. In particular, it obviously cannot capture any non-Markovian effects. In the *ultra-strong* coupling limit, where the SR interaction term dominates by far the system energy scales, some recent progress on the effective system evolution has been established with theoretical and numerical methods [9–13]. However, an approach which covers all scales of SR interaction strengths is not available [14]. In the current work we derive the effective system dynamics mathematically rigorously and without making any smallness assumption on the system-reservoir coupling, thus *covering the whole range, from weak to strong to ultra-strong couplings*. We are able to do this for a class of open systems in which the coupling to the reservoir is of *mean field* nature.

Figs. 1 and 2 give pictorial representations of the setup. We assume that R consists of a very large number M of identical subsystems, and that S is interacting with each reservoir subsystem in the same way. Each individual interaction term is scaled by a factor $1/M$, resulting in an SR interaction having the same order of magnitude as the bare system Hamiltonian H_S (no interaction). The symmetry of the interaction operator together with the mentioned scaling is the trademark of *mean field* theories [15,16].

Our **main results** are:

1. In the limit $M \to \infty$ the system evolves according to the Schrödinger equation with a generally time-dependent Hamiltonian $H_{S,\mathrm{eff}}(t) = H_S + W(t)$. The latter is obtained from the bare system Hamiltonian H_S by adding an explicit time-dependent
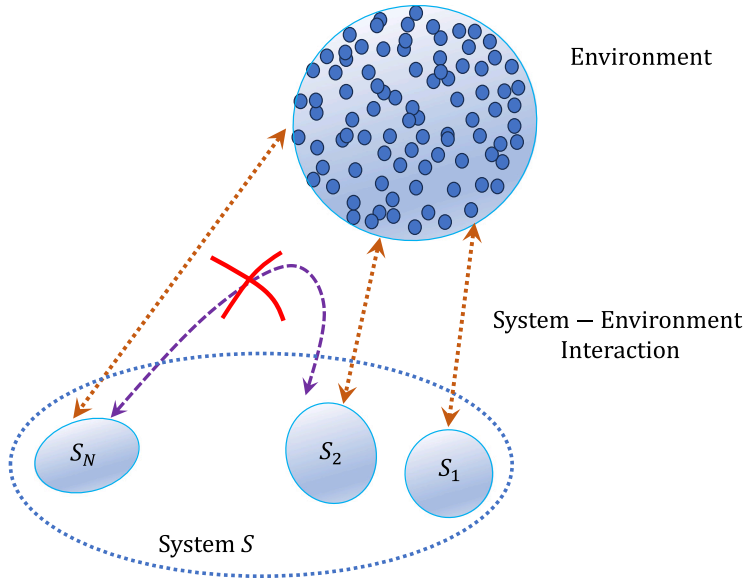
Fig. 3. The system S consists of N (not necessarily identical) subsystems $S_1, \ldots, S_N$, each of which interacts with the environment separately. Theorem 2 shows that the dynamics of $S_1, \ldots, S_N$ do not get coupled via the indirect interaction with the common reservoir. The mean-field type coupling does not change the entanglement among subparts of the system during the course of the dynamics. The entanglement is not degraded, it is protected.

    'potential' $W(t)$. The result is not perturbative, it holds for any SR interaction strength. This is the content of our Theorem 1 below.

2. The SR interaction does not affect the entanglement between the subparts of S. Consider a system $S = S_1 + \cdots + S_N$ composed of N subparts (which do not have to be of the same nature), where each subpart is coupled to R in a mean-field manner as described above, but the subparts are not interacting directly. Then the entanglement within the initial system state does not change in time even though the subparts are all coupled to the same environment. This is the content of our Theorem 2, which is illustrated in a picture in Fig. 3.

    Outline of the paper. In Section 2 we define the class of models and the main object of interest $\rho_S(t)$, the reduced system state at time t. We present and discuss our main results in Theorems 1 and 2. In the last section of the chapter we explain how the results apply to reservoirs with correlations. Section 3 contains concrete examples of systems, reservoirs and their interactions for which our general results hold. In Section 4 we present several extensions of Theorem 1. Section 5 contains proofs of the main theorems and other results presented in the other parts.

2. Main results

    Our two main results are Theorem 1 which shows that the effective system dynamics is unitary with an explicit time-dependent Hamiltonian, and Theorem 2 which implies that the intra-system entanglement is constant during the evolution. We present these results in Sections 2.2 and 2.3, after introducing the model in Section 2.1. In Section 2.4 we describe a class of admissible reservoir states with entanglement between different reservoir sites.

2.1. Model

    The Hilbert space of states of the system-reservoir complex is

$$\mathcal{H}_{\mathrm{SR},M} = \mathcal{H}_{\mathrm{S}} \otimes \mathcal{H}_{\mathrm{R},M}, \qquad \mathcal{H}_{\mathrm{R},M} = \mathcal{H}_{\mathrm{R}}^{\otimes M} \equiv \mathcal{H}_{\mathrm{R}} \otimes \cdots \otimes \mathcal{H}_{\mathrm{R}}, \tag{1}$$

where $\mathcal{H}_{\mathrm{S}}$ is the system Hilbert space (of dimension $\leq \infty$) and each of the M factors $\mathcal{H}_{\mathrm{R}}$ is the Hilbert space (of dimension $\leq \infty$) of a single reservoir subsystem. Given an operator X_{R} acting on $\mathcal{H}_{\mathrm{R}}$ we write

$$X_{\mathrm{R}}^{[m]} = \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes X_{\mathrm{R}} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1},$$

where X_{R} sits on the mth factor in the M-fold product. The total Hamiltonian is given by

$$H_M = H_{\mathrm{S}} + H_{\mathrm{R},M} + V_M, \tag{2}$$

where H_S is a self-adjoint operator on $\mathcal{H}_S$ and where the reservoir Hamiltonian is

$$H_{R,M} = \sum_{m=1}^{M} h_R^{[m]}, \tag{3}$$

with h_R a self-adjoint operator on $\mathcal{H}_R$. The interaction operator has the mean-field form

$$V_M = G \otimes \overline{V}_{R,M}, \qquad \overline{V}_{R,M} = \frac{1}{M} \sum_{m=1}^{M} v^{[m]}, \tag{4}$$

where G and v are self-adjoint operators on $\mathcal{H}_S$ and $\mathcal{H}_R$, respectively. The system is interacting in a mean field way with every one of the M subsystems of the reservoir. Denote the free Heisenberg dynamics of a system observable X_S and a ('single site') reservoir observable X_R as

$$X_S(t) = e^{itH_S} X_S e^{-itH_S}, \qquad X_R(t) = e^{ith_R} X_R e^{-ith_R} \tag{5}$$

and set

$$\overline{V}_{R,M}(t) = \frac{1}{M} \sum_{m=1}^{M} \left(v(t) \right)^{[m]}. \tag{6}$$

We consider initial system–reservoir states of the form

$$\rho_{SR,M} = \rho_S \otimes \rho_{R,M}, \tag{7}$$

where ρ_S is a density matrix of $\mathcal{H}_S$ and $\rho_{R,M}$ is a density matrix of $\mathcal{H}_{R,M}$. A density matrix ρ on either of $\mathcal{H}_R$, $\mathcal{H}_S$ or $\mathcal{H}_R^{\otimes M}$ gives rise to a 'state'

$$\omega(\cdot) = \mathrm{Tr}_H(\rho \cdot)$$

where the trace is taken over the Hilbert space H on which ρ acts. The $\omega(\cdot)$ is a positive linear functional on observables (operators), normalized as $\omega(\mathbb{1}) = 1$. We will use the notion of density matrix or linear functional on observables interchangeably. We consider reservoir states $\omega_{R,M}$ (equivalently, the density matrices $\rho_{R,M}$) which have certain symmetry and product properties as $M \to \infty$. This is central to the mean field approach. The main property we require is that there is a single site reservoir state ω_R (acting on operators on $\mathcal{H}_R$) such that for any integer n, any times $t_1, \ldots, t_n \geq 0$ we have

$$\lim_{M \to \infty} \omega_{R,M}\left(\overline{V}_{M,1}(t_1) \cdots \overline{V}_{M,n}(t_n) \right) = \omega_R(v(t_1)) \cdots \omega_R(v(t_n)). \tag{8}$$

The easiest state satisfying (8) is an M-fold product

$$\omega_{R,M} = \omega_R^{\otimes M} \equiv \omega_R \otimes \cdots \otimes \omega_R$$

of a single site reservoir state ω_R on $\mathcal{H}_R$. However, more complicated $\omega_{R,M}$ having correlations between different sites still satisfy (8). We discuss this in Section 2.4.

2.2. Effective system dynamics

Our goal is to find an evolution equation for the dynamics of S alone, that is, for the reduced system state coupled to a reservoir in the limit $M \to \infty$,

$$\rho_S(t) = \lim_{M \to \infty} \mathrm{Tr}_{R,M}\left(e^{-itH_{SR,M}} \rho_{SR,M} e^{itH_{SR,M}} \right). \tag{9}$$

Here, $\mathrm{Tr}_{R,M}$ denotes the partial trace over the Hilbert space $\mathcal{H}_{R,M}$. We assume (c.f. (8)) that for any integer n, any $t_1, \ldots, t_n \geq 0$,

$$\sup_{M \in \mathbb{N}} \left| \omega_{R,M}\left(\overline{V}_{R,M}(t_1) \cdots \overline{V}_{R,M}(t_n) \right) \right| \leq C_{R,n}\, b_R(t_1) \cdots b_R(t_n), \tag{10}$$

and that (recall (5))

$$\left\| G(t_{\ell_1}) \cdots G(t_{\ell_L}) \rho_S G(t_{\ell_{L+1}}) \cdots G(t_{\ell_n}) \right\|_1 \leq C_{S,n}\, b_S(t_1) \cdots b_S(t_n). \tag{11}$$

Here, $C_{R,n}$, $C_{S,n}$ are some constants and $b_S(t)$, $b_R(t)$ are some functions. In (11), the t_{ℓ_j} are an arbitrary permutation of the $t_1, \ldots, t_n$ and $L = 0, \ldots, n$ divides the factors of G into two groups (to the left and the right of ρ_S; $L = 0$ corresponds to all the G being to the right of ρ_S, similarly for $L = n$). Also, $\|\cdot\|_1$ is the trace norm on bounded operators acting on $\mathcal{H}_S$, that is, for a bounded operator X on $\mathcal{H}_S$, $\|X\|_1 = \mathrm{Tr}(\sqrt{X^*X})$. The non-negative functions $b_S(t)$ and $b_R(t)$ must be such that for all $t \geq 0$,

$$B(t) \equiv \int_0^t b_S(s) b_R(s)\, ds < \infty. \tag{12}$$

We do allow for $B(t) \to \infty$ as $t \to \infty$ (so b_S and b_R are very general, it suffices that they be locally integrable). We further assume that for all $t \geq 0$,

$$\sum_{n \geq 0} \frac{(2B(t))^n}{n!} C_{S,n} C_{R,n} < \infty. \tag{13}$$

A simple example where all conditions (10)–(13) are satisfied is when $\|G\|, \|v\| < \infty$ (operator norm of linear operators acting on $\mathcal{H}_S$ and $\mathcal{H}_R$, respectively). Then we can take $C_{S,n} = \|G\|^n$, $C_{R,n} = \|v\|^n$, $b_S(t) = b_R(t) = 1$, $B(t) = t$ and (13) is satisfied for all $t \geq 0$. This case of bounded operators includes *all* situations where $\dim \mathcal{H}_S < \infty$ and $\dim \mathcal{H}_R < \infty$.

Theorem 1. *The system state* (9) *is given by*

$$\rho_S(t) = U(t)\rho_S(0)U(t)^*, \tag{14}$$

where $U(t)$ *is the unitary on* $\mathcal{H}_S$ *solving the equation*

$$i\partial_t U(t) = H_{S,\text{eff}}(t)U(t), \quad U(0) = \mathbb{1} \tag{15}$$

with the effective Hamiltonian

$$H_{S,\text{eff}}(t) = H_S + \omega_R(v(t))G. \tag{16}$$

The effective system equation is thus the Schrödinger equation with time-dependent Hamiltonian $H_{S,\text{eff}}(t)$. The interaction with the reservoir induces a time-dependent 'potential' $\omega_R(v(t))$. If ω_R is stationary (with respect to the single site reservoir dynamics generated by h_R) then the potential is constant in time. We note that the result is not a weak coupling result — there is no 'small coupling constant' entering the SR interaction.

Remark. The unitary group generated by a time-dependent generator $H(t)$ is a two-parameter group $U(t,s)$ satisfying $i\partial_t U(t,s) = H(t)U(t,s)$, $U(s,s) = \mathbb{1}$. It obeys the composition rule $U(t,t_0) = U(t,s)U(s,t_0)$. In this paper, we simply write $U(t) = U(t,0)$.

2.3. Protection of intra-system entanglement

Consider the system S to be made of N subsystems,

$$\mathcal{H}_S = \mathcal{H}_{S,1} \otimes \cdots \otimes \mathcal{H}_{S,N} \tag{17}$$

with Hamiltonian

$$H_S = H_{S,1} + \cdots + H_{S,N}, \tag{18}$$

where $H_{S,j}$ is the Hamiltonian of the jth subsystem, acting on $\mathcal{H}_{S,j}$. We do not assume that the N subsystems are of the same nature. In particular, the dimension of $\mathcal{H}_{S,j}$ may depend on j. We take the interaction between S and R to be

$$V_M = \sum_{j=1}^{N} G_j \otimes \overline{V}_{R,M,j}, \tag{19}$$

where G_j is a self-adjoint operator on $\mathcal{H}_{S,j}$ and we also write simply G_j for the operator acting trivially on all factors of $\mathcal{H}_S$ except on the jth one. Moreover,

$$\overline{V}_{R,M,j} = \frac{1}{M} \sum_{m=1}^{M} v_j^{[m]}, \tag{20}$$

with $v_j^{[m]} \equiv (v_j)^{[m]}$ and v_j an operator on $\mathcal{H}_R$. This means that each subsystem $j = 1, \ldots, N$ interacts with the reservoir independently via a mean-field interaction of the type (4). The total Hamiltonian is given by

$$H_{SR,M} = H_S + H_{R,M} + V_M, \tag{21}$$

where $H_{R,M}$ is as in (3). The reduced system state in the limit $M \to \infty$ is defined in (9). Analogously to (10), (11), we assume that

$$\sup_{M \in \mathbb{N}} \left| \omega_{R,M}\left(\overline{V}_{R,M,j_1}(t_1) \cdots \overline{V}_{R,M,j_n}(t_n) \right) \right| \leq C_{R,n} \, b_R(t_1) \cdots b_R(t_n), \tag{22}$$

$$\left\| G_{j_1}(t_{\ell_1}) \cdots G_{j_L}(t_{\ell_L}) \rho_S G_{j_{L+1}}(t_{\ell_{L+1}}) \cdots G_{j_n}(t_{\ell_n}) \right\|_1 \leq C_{S,n} \, b_S(t_1) \cdots b_S(t_n), \tag{23}$$

where ℓ_k is any permutations of $\{1, \ldots, n\}$ and $j_1, j_2, \ldots, j_n \in \{1, \ldots, n\}$. The constants $C_{S,n}$, $C_{R,n}$ and the functions $b_S(t)$, $b_R(t)$ are assumed be independent of $j_1, \ldots, j_n$ and to satisfy (13). Again, an example where the conditions are satisfied is when $\|G_j\|, \|v_j\| < \infty$. Then we can take $C_{S,n} = \max_{1 \leq j \leq N} \|G_j\|^n$, $C_{R,n} = \max_{1 \leq j \leq N} \|v_j\|^n$, $b_S(t) = b_R(t) = 1$ and $B(t) = t$. This includes all situations where $\dim \mathcal{H}_{S,j} < \infty$, $\dim \mathcal{H}_R < \infty$.

Theorem 2. *In the setup* (17)–(21), *the system state* (9) *is given by*

$$\rho_S(t) = U(t)\rho_S(0)U(t)^*, \tag{24}$$

where $U(t)$ *is the unitary on* $\mathcal{H}_S = \mathcal{H}_{S,1} \otimes \cdots \otimes \mathcal{H}_{S,N}$ *solving Eq.* (15) *with the effective Hamiltonian*

$$H_{S,\text{eff}}(t) = H_S + \sum_{j=1}^{N} \omega_R(v_j(t))G_j. \tag{25}$$

Theorem 2 shows that the mean-field reservoir does not entangle systems which are indirectly coupled by it, see also Fig. 3. It shows that the amount of entanglement within the system is preserved during the dynamics with the reservoir. Indeed, the propagator is a product of local unitaries, $U(t) = U_1(t) \otimes \cdots \otimes U_N(t)$, each given by

$$i\partial_t U_j(t) = \Big(H_{S,j} + \omega_R\big(v_j(t)\big) G_j \Big) U_j(t), \quad U_j(0) = \mathbb{1}.$$

Consequently, the amount of entanglement in the system state is independent of time. This is so since a basic property of any measure of entanglement is that entanglement does not change under local unitary operations [17]. In particular, if $\rho_S(0) = \rho_{S_1} \otimes \cdots \otimes \rho_{S_N}$ is initially of product form, then it will still be so at all later times $t > 0$, with independently evolving factors

$$\rho_S(t) = \rho_{S,1}(t) \otimes \cdots \otimes \rho_{S,N}(t), \qquad \rho_{S,j}(t) = U_j(t)\rho_{S,j}(0)U_j(t)^*.$$

Generally, systems coupled indirectly via an environment without mean-field coupling *do* get entangled among each other during the dynamics. The creation of entanglement between not directly interacting systems due to a common reservoir was first observed in [18] and intricate processes of sudden death and revival of intra-system entanglement in such models was exhibited in [19–22]. However, the mean-field nature of the coupling considered here prevents this capacity to generate (or alter in any way) entanglement within the system. In other words, subsystems which are interacting indirectly via a common reservoir by a mean-field coupling do not feel each other's presence — there is no effective subsystem interaction created by the reservoir. This may be seen as a classical feature — the same phenomenon of absence of the creation of intra-system entanglement occurs indeed when quantum systems are coupled indirectly by the interaction with a reservoir in the classical limit [23–25], which leads to a similar formal setting. In these references, the particular aspect was not addressed. The connection between mean field and semiclassical theories has been elucidated in [26,27].

Related work. In so-called central spin systems a 'central' spin 1/2 (the system) interacts with M independent spins 1/2 (the environment) in a symmetric way, as in (4) but with a scaling $\frac{1}{M} \to \frac{1}{M^z}$, for some $z > 0$. This model is used to investigate various physical systems exhibiting non local interactions leading to long coherence times and long range correlations. The author of [28] has analyzed this model analytically in the limit $M \to \infty$ for a specific interaction operator v and for an initially factorized, permutation invariant environment state. They show that for $z = 1$ (our case) the central spin dynamics is given by the Schrödinger equation with a time-dependent Hamiltonian | this is the same result as ours. Interestingly, the author also succeeds to solve this specific model for $z = 1/2$, a situation termed the *quantum fluctuation regime* (as opposed to the mean field regime resulting from $z = 1$). For this scaling, the dynamics as $M \to \infty$ is shown in [28] to reduce to that of an open single spin in contact with a harmonic oscillator which accounts for the bath quantum fluctuations.

In the work [29] the author considers M spins 1/2 in contact with a radiation field via a mean field coupling interaction (prefactor $1/M$ | the Dicke maser model). Translated to our setup, the system is the radiation field, while the atoms constitute the environment. The author derives a limiting ($M \to \infty$) Hamiltonian for the field. This Hamiltonian is time independent for their initial condition of the reservoir. It is further shown that the average number of photons in the field (which is initially in the vacuum state) either stays bounded in time t (off resonance) or grows in time as $\sqrt{t}$ (at resonance).

While the setup in the above works is similar to ours, the methods employed there are tailored to the specific models considered and are entirely different from our approach. Different models in which N quantum systems are in contact by a mean field interaction through a reservoir have also been considered in the literature. In this situation, non-linear, dissipative evolution equations arise in the limit $N \to \infty$ [30,31].

Mean field versus fluctuation, or mesoscopic coupling. The scaling $1/M$ in (4) is characteristic of mean field models. A different scaling, $1/\sqrt{M}$ is called the fluctuation, or mesoscopic regime and | for suitable initial states | also leads to a well defined limit as $M \to \infty$ [28,32–35]. Operators of the form $\overline{V}_{R,M}$, (4) with $1/M$ replaced by $1/\sqrt{M}$ are called *fluctuation observables*. They describe the variations of microscopic quantities around their averages, and they can retain quantum features even as $M \to \infty$ (while mean field observables characterized by $1/M$ become commutative in this limit). As discussed in the above references, fluctuation operators (when averaged over a reservoir state without long range correlations) behave as bosonic operators and their dynamics can be described in some models. The dynamical consequences of a *system* coupled to a reservoir via fluctuation observables has been analyzed in [28] for a specific solvable model. It is an interesting question whether the methods we develop in this work can be modified to include such interactions. We have not analyzed this question so far.

2.4. Initially correlated reservoir states

The purpose of this section is to exhibit some correlated states satisfying (8). We start by defining a correlation length L in a reservoir state.

Definition (*Reservoir Correlation Length*). We say the state $\omega_{R,M}(\cdot) = \text{Tr}_{R,M}(\rho_{R,M} \cdot)$ on $\mathcal{H}_R^{\otimes M}$ has correlation length $L \geq 1$ if the following property holds. There is a state ω_R on $\mathcal{H}_R$ such that for any integer n and any operators $x_1, \ldots, x_n$ on $\mathcal{H}_R$ we have

$$\lim_{M \to \infty} \sup_{\substack{m_1, \ldots, m_n = 1, \ldots, M \\ \forall i \neq j, |m_i - m_j| > L}} \left| \omega_{R,M}\big(x_1^{[m_1]} \cdots x_n^{[m_n]}\big) - \omega_R(x_1) \cdots \omega_R(x_n) \right| = 0. \tag{26}$$

The property (26) is in line with the construction of the 'thermodynamic limit' of quasi-local, translation invariant systems. For 'local' observables $x_j^{[m_j]}$ (located at fixed 'sites' m_j) the state is not changing if M is large enough (growing volume), and it factorizes into position independent factors for sufficiently separated observables.

The next result shows that the property (26) implies the wanted asymptotic factorization, relation (8).

Proposition 1. *Suppose $\omega_{\mathrm{R},M}$ satisfies (26) and let x_j, $j \geq 1$, be operators on $\mathcal{H}_{\mathrm{R}}$ such that all the moments in the state $\omega_{\mathrm{R},M}$ are finite: For any n there is an $M_0(n)$ and a $C(n)$ such that for $M \geq M_0(n)$,*

$$\omega_{\mathrm{R},M}\big(x_1^{[m_1]} \cdots x_n^{[m_n]}\big) \leq C(n). \tag{27}$$

Define the operators $\overline{X}_{M,j} = \frac{1}{M} \sum_{m=1}^{M} x_j^{[m]}$ on $\mathcal{H}_{\mathrm{R},M}$. Then we have, for any $n \in \mathbb{N}$,

$$\lim_{M \to \infty} \omega_{\mathrm{R},M}\big(\overline{X}_{M,1} \cdots \overline{X}_{M,n}\big) = \omega_{\mathrm{R}}(x_1) \cdots \omega_{\mathrm{R}}(x_n). \tag{28}$$

We present a proof of Proposition 1 in Section 5.3.

Examples:

- A product state $\omega_{\mathrm{R},M} = \omega_{\mathrm{R}} \otimes \cdots \otimes \omega_{\mathrm{R}}$ has correlation length $L = 1$.
- Let E_L be a quantum channel acting on $L \geq 2$ subsystems of R, that is, acting on states of the L-fold product $\mathcal{H}_{\mathrm{R}} \otimes \cdots \otimes \mathcal{H}_{\mathrm{R}}$. For $j = 0, \ldots, M - L$, define the state

$$\omega_j = \omega_{\mathrm{R}}^{\otimes j} \otimes E_L(\omega_{\mathrm{R}}^{\otimes L}) \otimes \omega_{\mathrm{R}}^{\otimes M - L - j}, \tag{29}$$

  where ω_{R} is a given state on $\mathcal{H}_{\mathrm{R}}$. In the state ω_j the cluster of L subsystems with indices $j+1, \ldots, j+L$ are modified (entangled) by the action of E_L, while the remaining subsystems are left in the state ω_{R}. By averaging over all positions j we obtain a state

$$\omega_{\mathrm{R},M} = \frac{1}{M - L + 1} \sum_{j=0}^{M-L} \omega_j \tag{30}$$

  which satisfies (26) with correlation length L. To see this, we fix any $m_1, \ldots, m_n$ such that $|m_k - m_l| > L$ for $k \neq l$. Then $\omega_j(x_1^{[m_1]} \cdots x_n^{[m_n]})$ equals $\omega_{\mathrm{R}}(x_1) \cdots \omega_{\mathrm{R}}(x_n)$ for *most* of the indices $j = 0, \ldots, M - L$. Indeed, this is not the case only for values of j such that the set $\{j, j+1, \ldots, j+L\}$, where the operator E_L acts, contains one of the $m_1, \ldots, m_n$. There are nL such values of j in the sum (30). Thus $\omega_{\mathrm{R},M}(x_1^{[m_1]} \cdots x_n^{[m_n]}) = \frac{M-L+1-nL}{M-L+1} \omega_{\mathrm{R}}(x_1) \cdots \omega_r(x_n) + T(M,n)$, where $|T(M,n)| \leq \frac{nL}{M-L+1} C(n)$, provided the ω_j satisfy the finite moment condition (27).
- As a concrete example, consider $\mathcal{H}_{\mathrm{R}} = \mathbb{C}^2$ (reservoir made of spins 1/2) and take $\omega_{\mathrm{R}} = \mathrm{Tr}(|0\rangle\langle 0| \cdot) = \langle 0| \cdot |0\rangle$ to be the ground state (relative to σ_z), with density matrix $|0\rangle\langle 0|$. Then consider a quantum channel E_2 which produces a Bell state, $E_2|00\rangle\langle 00| = \frac{1}{2}(|00\rangle + |11\rangle)(\langle 00| + \langle 11|)$. In the corresponding state $\omega_{\mathrm{R},M}$, (30), consecutive elements of the reservoir are maximally entangled.

De finetti type states. The Quantum De Finetti theorem describes the interplay between symmetry of a state of an infinite system and the absence of entanglement in its finite parts. It was shown in [36] in a form similar to what we discuss here, and later more refined versions were developed in view of applications to mean-field theory, for example in [37–39].

For $M \geq 1$, let $\omega_{\mathrm{R},M}$ be a state defined on $\mathcal{H}_{\mathrm{R}}^{\otimes M}$ given by a density matrix $\rho_{\mathrm{R},M}$, with the following properties:

(i) (Consistency) Given any bounded operators $X_1, \ldots, X_M$ on $\mathcal{H}_{\mathrm{R}}$, we have $\omega_{\mathrm{R},M+1}(X_1 \otimes \cdots \otimes X_M \otimes \mathbb{1}_{\mathcal{H}_{\mathrm{R}}}) = \omega_{\mathrm{R},M}(X_1 \otimes \cdots \otimes X_M)$,

(ii) (Symmetry) For any bounded operators $X_1, \ldots, X_M$ on $\mathcal{H}_{\mathrm{R}}$ and any permutation π of $\{1, \ldots, M\}$, we have $\omega_{\mathrm{R},M}(X_1 \otimes \cdots \otimes X_M) = \omega_{\mathrm{R},M}(X_{\pi(1)} \otimes \cdots \otimes X_{\pi(M)})$.

A (sequence of) states satisfying these properties obey the hypotheses of the Quantum De Finetti theorem appearing as Theorem 2.1 of [38]:

Suppose $\omega_{\mathrm{R},M}$ satisfies the consistency and symmetry properties above. Then there is a unique Borel probability measure μ on the set of states $S_{\mathrm{R}} = \{\omega \in \mathcal{B}(\mathcal{H}_{\mathrm{R}})^* \ : \ \omega(\mathbb{1}) = 1, \ \omega \geq 0\}$ (the set of normalized, positive linear functionals on $\mathcal{H}_{\mathrm{R}}$), such that for all M,

$$\omega_{\mathrm{R},M} = \int_{S_{\mathrm{R}}} d\mu(\omega) \, \omega^{\otimes M}, \tag{31}$$

where $\omega^{\otimes M} = \omega \otimes \cdots \otimes \omega$ is the M-fold product state acting on bounded operators of $\mathcal{H}_{\mathrm{R}}^{\otimes M}$.

The key property here is that for De Finetti type states, instead of (28) we have

$$\lim_{M \to \infty} \omega_{\mathrm{R},M}\big(\overline{X}_{M,1} \cdots \overline{X}_{M,n}\big) = \int_{S_{\mathrm{R}}} d\mu(\omega) \, \omega(x_1) \cdots \omega(x_n). \tag{32}$$

By linearity of the integral, this allows us to establish the following result.

Proposition 2. *Suppose $\omega_{R,M}$ satisfies the consistency and symmetry conditions above, so that the De Finetti theorem applies and gives (31). Suppose that G and v are bounded operators. Then the reduced system dynamics is given by*

$$\rho_S(t) = \int_{S_R} d\mu(\omega) U(\omega,t) \rho_S(0) U^*(\omega,t), \tag{33}$$

where for all $\omega \in S_R$,

$$i\partial_t U(\omega,t) = H_{S,eff}(\omega,t) U(\omega,t) \qquad \text{with} \qquad H_{S,eff}(\omega,t) = H_S + \omega(v(t))G.$$

We give a proof of Proposition 2 in Section 5.4. It is very similar to the proof in the case the reservoir is in a product state, leading to the result of Theorem 1. However, in contrast to that result, here the effective system dynamics is no longer given by a unitary group. Indeed, by taking the integral in (33), the product property of the propagators is broken and the evolution is no longer necessarily Markovian. An analogous situation is described in [25]. There the dynamics of an open system is derived to be given by (33) with a measure emerging from the classical limit of the reservoir. In that reference, decoherence and non-Markovianity of the system dynamics are investigated.

3. Applications

In Section 3.1 we describe some reservoir models which fit our assumptions. Then we present a discussion on the effective system dynamics in Section 3.2.

3.1. Examples of reservoirs

We give some examples of reservoir models which satisfy the condition (10).

1. The reservoir as a spin lattice. The index $m = 1, \ldots, M$ labels the position on a lattice (in any dimension) and $M \to \infty$ corresponds to the infinite size lattice limit (see also Fig. 1 depicting a two-dimensional lattice). At each lattice site m is located a spin (generally, any quantum system with Hilbert space $\mathcal{H}_R$, $\dim \mathcal{H}_R = d_R < \infty$). The state ω_R can be taken as an arbitrary state of a d_R-level system, determined by an arbitrary density matrix ρ_R on $\mathcal{H}_R$ and the Hamiltonian H_R is any hermitian operator on $\mathcal{H}_R$. $\omega_{R,M}$ can then be taken as the M-fold product of ω_R, or it may contain correlations, such as in (30). Let X be an operator on $\mathcal{H}_R \otimes \cdots \otimes \mathcal{H}_R$ (M-fold product). As $\omega_{R,M}$, (30) is a state, that is, a linear functional of norm one, we have

$$|\omega_{R,M}(X)| \le \|X\| \tag{34}$$

where $\|X\|$ is the operator norm of X. In view of (10) we take $X = \overline{V}_{R,M}(t_1) \cdots \overline{V}_{R,M}(t_n)$, so

$$|\omega_{R,M}(X)| \le \frac{1}{M^n} \sum_{m_1=1}^{M} \cdots \sum_{m_n=1}^{M} \|v(t_1)^{[m_1]}\| \cdots \|v(t_n)^{[m_n]}\| = \|v\|^n,$$

where we used that $\|v(t)^{[m]}\| = \|v\|$. Consequently (10) holds with $C_{R,n} = \|v\|^n$ and $b_R(t) = 1$.

What kind of time dependent potentials in the system Hamiltonian can we get through the mean-field coupling for $d_R = \dim \mathcal{H}_R < \infty$? Denote the (single site) reservoir Hamiltonian by $h_R = \sum_{l=1}^{d_R} \lambda_l |e_l\rangle\langle e_l|$ and denote matrix elements of a single site operator X as $[X]_{k,l} = \langle e_k | X | e_l \rangle$. Then

$$\omega_R(v(t)) = \mathrm{Tr}_R(\rho_R v(t)) = \sum_{l,k=1}^{d_R} [\rho_R]_{kl} [v]_{lk} e^{it(\lambda_l - \lambda_k)},$$

which gives a quasi-periodic time dependent potential in the effective system Hamiltonian $H_{eff}(t)$, (16).

2. The reservoir as an oscillator field. Consider $\mathcal{H}_R$ to be the Hilbert space of a quantum oscillator with creation and annihilation operators $a^\dagger, a$, satisfying the commutation relation $[a, a^\dagger] = \mathbb{1}$. For the single oscillator Hamiltonian (*c.f.* after (3)) take

$$h_R = \Omega_R a^\dagger a \tag{35}$$

where $\Omega_R > 0$. As a first choice, take the interaction operator to be the field operator,

$$v_1 = \varphi = \frac{a^\dagger + a}{\sqrt{2}}, \quad v_1(t) = \varphi(t) = e^{ith_R} \varphi e^{-ith_R} = \frac{e^{i\Omega_R t} a^\dagger + e^{-i\Omega_R t} a}{\sqrt{2}}. \tag{36}$$

If ω_R is chosen to be a gauge-invariant state of the oscillator, meaning that the expectation of a product of an odd number of a or $a^\dagger$ vanishes (such as the equilibrium state, for example), then $\omega_R(v(t)) = 0$. This means that the effective evolution of S is not affected by the mean field coupling to the reservoir. We discuss two other models, in which the effective dynamics does feel the coupling.

(2a) Coherent state: Suppose $\rho_{R,M} = |\alpha\rangle\langle\alpha| \otimes \cdots \otimes |\alpha\rangle\langle\alpha|$ is the product of M coherent states $|\alpha\rangle$, $\alpha \in \mathbb{C}$ defined by $a|\alpha\rangle = \alpha|\alpha\rangle$. Let v_1, (36) be the interaction operator.

Proposition 3. *The bound* (10) *holds with*

$$C_{\mathrm{R},n} = (2\lfloor n/2 \rfloor)!! \left(\tfrac{1}{2} + |\alpha| \right)^n, \qquad b_{\mathrm{R}}(t) = 1, \tag{37}$$

where for any even integer k, we set $k!! = (k-1)(k-3) \cdots 3 \cdot 1$.

Note that $2\lfloor n/2 \rfloor = n$ if n is even and $2\lfloor n/2 \rfloor = n-1$ if n is odd. By using the root test for the convergence of the series, one shows that (13) is satisfied if for some $\xi \geq 0$ we have $C_{\mathrm{S},n} \leq e^{\xi n}$ (which is the case if $\|G\| < \infty$ in particular). We prove Proposition 3 in Section 5.5.

(2b) Instead of the $v_1 = \varphi$ considered in the previous point, now consider the interaction operator

$$v_2 = v a^\dagger a,$$

where $v \in \mathbb{R}$. This represents a 'scattering' interaction in which field quanta are not created or destroyed, as v_2 commutes with the number operator $a^\dagger a$.

Proposition 4. *The bound* (10) *is satisfied provided* $\omega_{\mathrm{R},M}$ *is product state* $\omega_{\mathrm{R},M} = \omega_{\mathrm{R}} \otimes \cdots \otimes \omega_{\mathrm{R}}$ *satisfying*

$$\omega_{\mathrm{R}}\left((a^\dagger a)^k \right) \leq c^k, \qquad k \in \mathbb{N}, \tag{38}$$

for some constant c. In (10) *we can take* $C_{\mathrm{R},n} = (cv)^n$ *and* $b_{\mathrm{R}}(t) = 1$.

The condition (38) holds in particular if the density matrix of ω_{R} is a mixture of states all of which have at most c excitations. We prove Proposition 4 in Section 5.6.

The operator v_2 is stationary under the Heisenberg dynamics generated by h_{R}, (35). The effective Hamiltonian for the system dynamics becomes (see (15)),

$$H_{\mathrm{eff}} = H_{\mathrm{S}} + v \omega_{\mathrm{R}}(a^\dagger a) G. \tag{39}$$

3. The reservoir as a quantum field. Instead of having a single oscillator at each 'site' m we may have a collection of them. We now consider $\mathcal{H}_{\mathrm{R}} = \mathcal{F}(L^2(\mathbb{R}^d, d^d k))$, the (bosonic) Fock space over $L^2(\mathbb{R}^d, d^d k)$. It carries creation and annihilation operators $a_k^\dagger, a_k$ and smoothed out versions,

$$a^\dagger(f) = \int_{\mathbb{R}^d} f(k) a_k^\dagger \, d^d k, \quad a(f) = \int_{\mathbb{R}^d} \overline{f(k)} a_k^\dagger \, d^d k,$$

where $f \in L^2(\mathbb{R}^d, d^d k)$ is a 'test function'. The canonical commutation relations are $[a(f), a^\dagger(g)] = \langle f, g \rangle$, where the right side is the inner product in $L^2(\mathbb{R}^d, d^d x)$. The field Hamiltonian is given by the Hamiltonian

$$h_{\mathrm{R}} = d\Gamma(\omega) \equiv \int_{\mathbb{R}^d} \omega(k) a_k^\dagger a_k \, d^d k, \tag{40}$$

where $\omega(k) \geq 0$ is the 'dispersion relation' of the field. It generates the Bogoliubov transformation,

$$e^{ith_{\mathrm{R}}} a^\dagger(f) e^{-ith_{\mathrm{R}}} = a^\dagger(e^{i\omega t} f).$$

As in the previous oscillator case, the mean field interaction with an operator

$$v_1 = \varphi(g) \equiv \frac{1}{\sqrt{2}} (a^\dagger(g) + a(g)) \tag{41}$$

which is linear in creation and annihilation operators, does not affect the system dynamics if the reservoir state ω_{R} is a gauge-invariant state (such as an equilibrium state), because for those states we have $\omega_{\mathrm{R}}(e^{ith_{\mathrm{R}}} \varphi(g) e^{-ith_{\mathrm{R}}}) = 0$. We then consider again the cases (a) and (b) as in the single oscillator model.

(3a) Coherent state: Suppose $\rho_{\mathrm{R},M} = |\Psi_f\rangle\langle\Psi_f| \otimes \cdots \otimes |\Psi_f\rangle\langle\Psi_f|$ is the product of M coherent states $\Psi_f = W(f)|\Omega\rangle$, where $|\Omega\rangle$ is the vacuum state, $f \in L^2(\mathbb{R}^d, d^d k)$ is a fixed function and $W(f) = e^{i\varphi(f)}$ is the Weyl operator. We take the interaction operator v_1, (41). As in the oscillator case, we obtain the following result:

Proposition 5. *The bound* (10) *holds with* $C_{\mathrm{R},n} = (2\lfloor n/2 \rfloor)!!(\tfrac{1}{2} + \|f\|)^n \|g\|^n$ *and* $b_{\mathrm{R}}(t) = 1$.

A proof of Proposition 5 can be obtained by following exactly the lines of the proof of Proposition 3, see at the end of Section 5.5.

(3b) In view of a 'scattering' interaction, consider the field interaction operator to be given by

$$v_2 = a^\dagger(g)a(g), \quad v_2(t) = a^\dagger(e^{i\omega t} g)a(e^{i\omega t} g)$$

for a fixed $g \in L^2(\mathbb{R}^d, d^d k)$.

Proposition 6. *The bound* (10) *holds provided* $\omega_{R,M}$ *is the M-fold tensor product of* ω_R *satisfying (c.f.* (38)*)*

$$\omega_R(\hat{N}^k) \leq c^k, \qquad k \in \mathbb{N}, \tag{42}$$

for some constant c, and where $\hat{N} = \int_{\mathbb{R}^d} a_k^\dagger a_k d^d k$ is the number operator. For the constants in (10) *we can take $C_{R,n} = (2c\|g\|^2)^n$ and $b_R(t) = 1$.*

We give a proof of Proposition 6 in Section 5.7. The effective Hamiltonian of the system is

$$H_{\text{eff}}(t) = H_S + \omega_R\big(a^\dagger(e^{i\omega t}g)a(e^{i\omega t}g)\big)G. \tag{43}$$

As an example, consider $\omega_R = |a^\dagger(h)\Omega\rangle\langle a^\dagger(h)\Omega|$ for a function $h \in L^2(\mathbb{R}^d, d^d k)$, where $|\Omega\rangle$ is the vacuum state. This is a state having one excitation (in the form of the wave function h). Then

$$\omega_R\big(a^\dagger(e^{i\omega t}g)a(e^{i\omega t}g)\big) = \big|\langle h, e^{i\omega t}g\rangle\big|^2. \tag{44}$$

Typically, the quantity (44) tends to zero for large times: Take for instance $d = 3$, $\omega(k) = |k|$, then (in spherical coordinates $(\Sigma, r = |k|)$ of $\mathbb{R}^3$),

$$\langle h, e^{i\omega t}g\rangle = \int_0^\infty r^2 e^{irt}\overline{h'(r)}g'(r)dr, \tag{45}$$

where $g'(r) = \int_{S^2} g(r, \Sigma)d\Sigma$ (angles integrated out). By the Riemann–Lebesgue theorem we have $\lim_{t\to\infty}\langle h, e^{i\omega t}g\rangle = 0$ at a speed which depends on the regularity and the infra-red behavior (small $|k|$) of the functions g, h. Compared the case of a single oscillator in the previous paragraph (or a finite amount of oscillators), the continuous modes of the oscillator field here generates decay in the reservoir dynamics. As a consequence the Hamiltonian $H_{\text{eff}}(t)$, (43) converges to the free Hamiltonian H_S as $t \to \infty$ (while it is a quasi-periodic function of time for finitely many oscillators).

3.2. Examples of the effective system dynamics

If ω_R is stationary with respect to the free reservoir dynamics then the effective system Hamiltonian $H_{S,\text{eff}} = H_S + \omega_R(v)G$ is time-independent, and we have

$$\rho_S(t) = e^{-it(H_S + \omega_R(v)G)}\rho_S(0)e^{it(H_S + \omega_R(v)G)},$$

yielding a unitary dynamics with renormalized energies. If the average of the operator v in the state ω_R vanishes, then the system dynamics is not affected by the coupling to the reservoir. For finite-dimensional systems S in contact with a stationary reservoir, the coupling to the reservoir thus simply causes a shift of the energy eigenvalues. If the term $\omega_R(v)G$ is small relative to H_S, then the uncoupled energies are modified little by the interaction with the reservoir. However, our result is not of perturbative nature, in the sense that we do not make any assumptions on the relative sizes of H_S and $\omega_R(v)G$. The interaction term $\omega_R(v)G$ may be of the same size as H_S, or it may exceed it arbitrarily.

In case $\dim \mathcal{H}_S < \infty$, the spectrum of $H_{S,\text{eff}}(t)$ consists only of eigenvalues ('bound states'). However, in the infinite dimensional case, the contact with the reservoir can modify the nature of the spectrum: it can turn bound states into scattering states and *vice versa*. In other words, the effective potential which S is subjected to can trap or 'localize' particles or on the contrary, it can 'ionize' trapped particles. This is what we explore now.

Turning bound states into scattering states and vice versa. One of the prime examples of continuous systems is that of $N = 1, 2, \ldots$ quantum particles in position space $\mathbb{R}^d$ ($d = 1, 2, 3, \ldots$). The Hilbert space of a single particle is $\mathcal{H}_S = L^2(\mathbb{R}^d, d^d x)$ and the free Hamiltonian is the Laplacian, $H_S = -\Delta = -\sum_{j=1}^d \partial_{x_j}^2$. Consider an interaction operator $G = V(x)$ to be a bounded, real valued function on $\mathbb{R}^d$ (G is a multiplication operator). Assuming the reservoir to be in a stationary state, Theorem 1 shows that the effective Hamiltonian of the particle is given by

$$H_{\text{eff}} = -\Delta + W(x), \tag{46}$$

which is a Schrödinger operator with potential $W(x) = \omega_R(v)V(x)$. Let $W(x)$ be of the form of a potential well. The spectrum of the Laplacian on $\mathbb{R}^d$ is purely absolutely continuous, that is, it does not contain any eigenvalues in the sense that there are no (normalizable) eigenstates. Generalized (not normalizable) eigenstates are called scattering states. In contrast, it is well known that (depending on the depth of the potential well), H_{eff} (46) can have an arbitrary number of *bound states*, by which we mean normalized eigenvectors associated to eigenvalues (discrete spectrum). Bound states are localized in space ($\mathbb{R}^d$) and they stay localized during the evolution, while scattering states escape any bounded fixed region in space if one waits long enough | hence the nomenclature. Our results thus show that the mean field coupling to the reservoir can create bound states, that is, it can cause localization in the particle system. Conversely, one could start with a system Hamiltonian $H_S = -\Delta + W(x)$ which does have bound states — then the mean field coupling will alter the potential and can create additional bound states or remove some or all of them.

In the above examples we have considered bounded potentials, $\|G\| = \sup_{x\in\mathbb{R}^d}|V(x)| < \infty$, so that the condition (11) is easily seen to be correct (see also the explanation after (13)). For such potentials, the interaction with the reservoir can change the number of bound states only by a finite amount. We discuss now an interaction with the field which produces an unbounded potential.

Consider a particle in $\mathbb{R}^d$, with $\mathcal{H}_S = L^2(\mathbb{R}^d, d^d x)$ with free Hamiltonian $H_S = -\Delta$ and an interaction operator

$$G = \hat{E} \cdot x, \tag{47}$$

the operator of multiplication by $\hat{E} \cdot x$ (unbounded potential), where $\hat{E} \in \mathbb{R}^d$ is a fixed vector of unit length and x is the operator of multiplication by $x \in \mathbb{R}^d$. We assume that the reservoir state ω_S and interaction operator v satisfy $\omega_R(v(t)) = \alpha \in \mathbb{R}$ (independent of time) but do not further specify the reservoir characteristics. The resulting effective system Hamiltonian is

$$H_{\text{eff}} = -\Delta + \alpha \hat{E} \cdot x, \tag{48}$$

which is the 'Stark Hamiltonian' describing a free charged particle in a constant electric field $\hat{E}$.

Proposition 7. *Suppose G is given in (47) and that $\rho_S = \sum_{\alpha \geq 0} p_\alpha |\psi_\alpha\rangle\langle\psi_\alpha|$ is a density matrix on $\mathcal{H}_S = L^2(\mathbb{R}^d, d^d x)$, with probabilities p_α and normalized vectors ψ_α. Suppose that there is a constant c such that for all $k \in \mathbb{N}$,*

$$\left\| \left(\hat{E} \cdot (x - \nabla_x) \, \hat{E} \cdot (x + \nabla_x) \right)^k \psi_\alpha \right\| \leq c^k. \tag{49}$$

Take a reservoir satisfying (10) with $C_{R,n} \leq (c')^n$ for some constant c' and $b_R(t)$ a locally integrable function of $t \geq 0$. Then the system dynamics is given by the effective Hamiltonian

$$H_{\text{eff}}(t) = -\Delta + \omega_R(v(t)) \, \hat{E} \cdot x. \tag{50}$$

In dimension $d = 1$, the condition (49) is $\|(-\Delta + x^2 - 1)^k \psi_\alpha\| \leq c^k$. The eigenfunctions of $-\Delta + x^2 - 1$ are the Hermite functions Ψ_ν, $\nu = 0, 1, 2, \ldots$, satisfying $(-\Delta + x^2 - 1)\Psi_\nu = 2\nu\Psi_\nu$. They are an orthonormal basis of $L^2(\mathbb{R}, dx)$. Let ν_* be any fixed integer and for each α, take ψ_α to be a linear combination of Ψ_ν with $\nu \leq \nu_*$. Then (49) holds with $c = \nu_*$. One may carry out a similar analysis in dimension $d \geq 2$. The condition (49) can be given directly on the density matrix by requiring that

$$\text{Tr}_{\mathcal{H}_S}\left(\rho_S \left(\hat{E} \cdot (x - \nabla_x) \, \hat{E} \cdot (x + \nabla_x) \right)^{2k} \right) \leq c^{2k}, \quad k \in \mathbb{N}.$$

For each t fixed, $H_{\text{eff}}(t)$, (50) has absolutely continuous spectrum and no eigenvalues (the electric field drives the charged particle indefinitely in one direction). One may consider the problem on a half line, $\mathcal{H}_S = L^2(\mathbb{R}_+, dx)$ with Hamiltonian $H_S = -\Delta$ with Dirichlet boundary conditions [40]. The result of Proposition 7 is still valid (our proof directly carries through to this case). It is well known that the spectrum of the resulting effective Hamiltonian (50), for $\omega_R(v(t))E > 0$, is purely discrete [41] (Theorem XIII.16). As a consequence, when the free particle on the half-line is coupled to the reservoir, the nature of its dynamics changes entirely, from having only scattering states to having only bound states.

4. Extensions of Theorem 1

In this section we present some generalizations of Theorem 1.

1. Many-site (many-body) system-reservoir interaction. Consider a ν-body reservoir operator v^ν, that is an operator acting on the ν-fold tensor product $\mathcal{H}_R \otimes \cdots \otimes \mathcal{H}_R$ and let the interaction operator V_M in (2) be given by

$$V_M = G \otimes \binom{M}{\nu}^{-1} \sum_{\Lambda \in \mathcal{P}_{\nu,M}} (v^\nu)^{[\Lambda]}, \tag{51}$$

where Λ is drawn from the set $\mathcal{P}_{\nu,M}$ of all clusters of ν sites (indices) of the M total reservoir sites, and where $(v^\nu)^{[\Lambda]}$ operates on the cluster Λ. In (51), we have replaced the normalization factor M^{-1} in (4) by $\binom{M}{\nu}^{-1}$ which divides by the total number of ways to choose clusters of size ν among M elements. Suppose that $\omega_{R,M}$ is of product form $\omega_{R,M} = \omega_R \otimes \cdots \otimes \omega_R$. Then (14) is valid with the unitary solving

$$i\partial_t U(t) = \left(H_S + \omega_R^{\otimes \nu}(v^\nu(t)) \, G \right) U(t), \quad U(0) = \mathbb{1}, \tag{52}$$

where $\omega_R^{\otimes \nu} = \omega_R \otimes \cdots \otimes \omega_R$ (ν factors). We give the proof of this result within the proof of Theorem 1.

2. Interaction as a sum of terms. Take an interaction operator given by a sum of terms of the form (4),

$$V_M = \sum_{j=1}^{J} G_j \otimes \overline{V}_{R,M,j}, \qquad \overline{V}_{R,M,j} = \frac{1}{M} \sum_{m=1}^{M} (v_j)^{[m]}, \tag{53}$$

where G_j and v_j are operators such that (10), (13) are satisfied. Then (14) is valid with the unitary being the solution of

$$i\partial_t U(t) = \left(H_S + \sum_{j=1}^{J} \omega_R(v_j(t)) G_j \right) U(t), \quad U(0) = \mathbb{1}. \tag{54}$$

A proof of this result is obtained along the lines of the proof of Theorem 2 below, where we show that the result (54) is also valid when S is composed of N subsystems and the operators G_j are given by $G_j \equiv \mathbb{1}_{\mathcal{H}_{S,1}} \otimes \cdots \otimes G_j \otimes \cdots \otimes \mathbb{1}_{\mathcal{H}_{S,N}}$, acting on the jth subsystem of S.

3. Macroscopically varying reservoir state. Suppose the state $\rho_{R,M}$ is a product state in which M_k factors are in given states $\sigma_{R,k}$ on $\mathcal{H}_R$, for $k = 1, \ldots, K$. We have $M_1 + \cdots + M_K = M$ (so the M_k depend on M) and we set

$$\lambda_k = \lim_{M \to \infty} \frac{M_k}{M} \in [0, 1].$$

As $\sum_{k=1}^{K} \lambda_k = 1$ the λ_k are probabilities and we set

$$\omega_{\text{R}}(\cdot) = \text{Tr}(\rho_{\text{R}} \cdot), \qquad \rho_{\text{R}} = \sum_{k=1}^{K} \lambda_k \, \sigma_{\text{R},k}. \tag{55}$$

Then Theorem 1 holds with ω_{R} given in (55).

To show this, we proceed as in the proof of Theorem 1 with the following modification. The condition (26) does not hold for the state described here: Even though $\rho_{\text{R},M}$ is a product state the factors are not the same. We thus need an alternative proof of the result (28) of Proposition 1. Namely, we show now that (using the notation of Proposition 1)

$$\lim_{M \to \infty} \text{Tr}_{\text{R},M} \left(\rho_{\text{R},M} \overline{X}_{M,1} \cdots \overline{X}_{M,n} \right) = \omega_{\text{R}}(x_1) \cdots \omega_{\text{R}}(x_n), \tag{56}$$

where ω_{R} is given in (55). The proof of Theorem 1 is then the same as before.

To show (56) we write the left side as

$$\lim_{M \to \infty} \frac{1}{M^n} \sum_{m_1, \ldots, m_n = 1}^{M} \text{Tr}_{\text{R},M} \left(\rho_{\text{R},M} x_1^{[m_1]} \cdots x_n^{[m_n]} \right) = \lim_{M \to \infty} \frac{1}{M^n} \sum_{\mathcal{M}_n} \text{Tr}_{\text{R},M} \left(\rho_{\text{R},M} x_1^{[m_1]} \cdots x_n^{[m_n]} \right), \tag{57}$$

where $\mathcal{M}_n$ is the set of all distinct indices, as in (79). For each $(m_1, \ldots, m_n) \in \mathcal{M}_n$, there are l_1 of the indices which 'point to' a site where the state $\rho_{\text{R},M}$ has the factor $\sigma_{\text{R},1}$. There are l_2 indices corresponding to $\sigma_{\text{R},2}$, and so on. This defines numbers $0 \le l_k \le n$, $k = 1, \ldots, K$, satisfying $l_1 + \cdots + l_K = n$. For $l_1, \ldots, l_K$ fixed, the expectation value (trace in (57)) is given by

$$\sum_{\pi \in \mathcal{P}_{l_1, \ldots, l_K}} \prod_{q=1}^{n} \text{Tr}_{\text{R}} \left(\sigma_{\text{R},\pi(q)} x_q \right),$$

where

$$\mathcal{P}_{l_1, \ldots, l_K} = \left\{ \pi : \{1, \ldots, n\} \to \{1, \ldots, K\} \text{ such that } |\pi^{-1}(k)| = l_k \text{ for } k = 1, \ldots, K \right\}. \tag{58}$$

Additionally, when distributing the $m_1, \ldots, m_n$ into the groups defined by $l_1, \ldots, l_K$, their 'location' within the group creates a combinatorial factor. For the first group this factor is $M_1(M_1 - 1) \cdots (M_1 - l_1 + 1) = \frac{(M_1)!}{(M_1 - l_1)!}$, and similarly for the other groups. We thus obtain

$$\frac{1}{M^n} \sum_{\mathcal{M}_n} \text{Tr}_{\text{R},M} \left(\rho_{\text{R},M} x_1^{[m_1]} \cdots x_n^{[m_n]} \right) = \sum_{\substack{l_1, \ldots, l_K \ge 0 \\ l_1 + \cdots + l_K = n}} \frac{\prod_{k=1}^{K} \frac{(M_k)!}{(M_k - l_k)!}}{M^n} \sum_{\pi \in \mathcal{P}_{l_1, \ldots, l_K}} \prod_{q=1}^{n} \text{Tr}_{\text{R}} \left(\sigma_{\text{R},\pi(q)} x_q \right).$$

Next, $\lim_{M \to \infty} M^{-l_k} \frac{(M_k)!}{(M_k - l_k)!} = \lambda_k^{l_k}$ and therefore,

$$\lim_{M \to \infty} \frac{1}{M^n} \sum_{\mathcal{M}_n} \text{Tr}_{\text{R},M} \left(\rho_{\text{R},M} x_1^{[m_1]} \cdots x_n^{[m_n]} \right) = \sum_{\substack{l_1, \ldots, l_K \ge 0 \\ l_1 + \cdots + l_K = n}} \lambda_1^{l_1} \cdots \lambda_K^{l_K} \sum_{\pi \in \mathcal{P}_{l_1, \ldots, l_K}} \prod_{q=1}^{n} \text{Tr}_{\text{R}} \left(\sigma_{\text{R},\pi(q)} x_q \right).$$

The double sum over the decomposition $l_1 + \cdots + l_K = n$ and then over the functions $\pi \in \mathcal{P}_{l_1, \ldots, l_K}$ constraint to have the sizes of the inverse images as dictated by (58) is the same as the sum over all functions $f : \{1, \ldots, n\} \to \{1, \ldots, K\}$. Thus, distributing the product of the λ_k over the states $\sigma_{\text{R},k}$, we get

$$\sum_{\substack{l_1, \ldots, l_K \ge 0 \\ l_1 + \cdots + l_K = n}} \lambda_1^{l_1} \cdots \lambda_K^{l_K} \sum_{\pi \in \mathcal{P}_{l_1, \ldots, l_K}} \prod_{q=1}^{n} \text{Tr}_{\text{R}} \left(\sigma_{\text{R},\pi(q)} x_q \right) = \sum_{f : \{1, \ldots, n\} \to \{1, \ldots, K\}} \prod_{q=1}^{n} \text{Tr}_{\text{R}} \left(\lambda_{f(q)} \sigma_{\text{R},f(q)} x_q \right). \tag{59}$$

Finally, we note that the right side of (59) is just $\prod_{q=1}^{n} \left(\sum_{j=1}^{K} \lambda_j \text{Tr}_{\text{R}} \left(\sigma_{\text{R},j} x_q \right) \right)$. Here is a way of seeing this: Set $a_{q,j} := \lambda_j \text{Tr}_{\text{R}} \left(\sigma_{\text{R},j} x_q \right)$ and write down the product

$$(a_{1,1} + \cdots + a_{1,K}) \cdot (a_{2,1} + \cdots + a_{2,K}) \cdots (a_{n,1} + \cdots + a_{n,K}).$$

By multiplying out the terms in the product, we have to pick one of the $a_{q,j}$ from each of the parenthesis indexed by q. But this is exactly constructing a function f: we assign $f(q) = j$ by picking the jth element from the qth parenthesis. We have thus shown that

$$\lim_{M \to \infty} \text{Tr}_{\text{R},M} \left(\rho_{\text{R},M} \overline{X}_{M,1} \cdots \overline{X}_{M,n} \right) = \prod_{q=1}^{n} \left(\sum_{j=1}^{K} \lambda_j \text{Tr}_{\text{R}} \left(\sigma_{\text{R},j} x_q \right) \right), \tag{60}$$

that is, (56) holds with ω_{R} given in (55).

5. Proofs

We collect the proofs of our main results, Theorems 1 and 2 and some Propositions.

5.1. Proof of Theorem 1

We start by using the Dyson series expansion,

$$\mathrm{Tr}_{\mathrm{R},M}\left(e^{itH_M^0}e^{-itH_{\mathrm{SR},M}}\rho_{\mathrm{SR},M}\,e^{itH_{\mathrm{SR},M}}e^{-itH_M^0}\right)=\sum_{n\geq 0}(-i)^n\int_{0\leq t_n\leq\cdots\leq t_1\leq t}dt_1\cdots dt_n\mathrm{Tr}_{\mathrm{R},M}\left(\left[V_M(t_1),\left[\cdots\left[V_M(t_n),\rho_{\mathrm{S}}\otimes\rho_{\mathrm{R},M}\right]\ldots\right]\right]\right),\qquad(61)$$

where we set

$$H_M^0=H_{\mathrm{S}}+H_{\mathrm{R},M}\tag{62}$$

and the integrand is the multiple commutator of the initial state with the interaction operator at different times. We first show that the series converges in trace norm of system operators, for all t, and uniformly in M. Expanding the multiple commutator yields 2^n terms, each of the form

$$G(t_{\ell_1})\cdots G(t_{\ell_L})\rho_{\mathrm{S}}G(t_{\ell_{L+1}})\cdots G(t_{\ell_n})\otimes\overline{V}_{\mathrm{R},M}(t_{\ell_1})\cdots\overline{V}_{\mathrm{R},M}(t_{\ell_L})\rho_{\mathrm{R},M}\overline{V}_{\mathrm{R},M}(t_{\ell_{L+1}})\ldots\overline{V}_{\mathrm{R},M}(t_{\ell_n})\tag{63}$$

for some L and some permutation $j\mapsto\ell_j$ of the n indices. Due to the cyclicity of the trace and (10), we have

$$\left|\mathrm{Tr}_{\mathrm{R},M}\left(\overline{V}_{\mathrm{R},M}(t_{\ell_1})\cdots\overline{V}_{\mathrm{R},M}(t_{\ell_L})\rho_{\mathrm{R},M}\overline{V}_{\mathrm{R},M}(t_{\ell_{L+1}})\ldots\overline{V}(t_{\ell_n})\right)\right|\leq C_{\mathrm{R},n}\,b_{\mathrm{R}}(t_1)\cdots b_{\mathrm{R}}(t_n).\tag{64}$$

Further, according to (11),

$$\|G(t_{\ell_1})\cdots G(t_{\ell_L})\rho_{\mathrm{S}}G(t_{\ell_{L+1}})\cdots G(t_{\ell_n})\|_1\leq C_{\mathrm{S},n}\,b_{\mathrm{S}}(t_1)\cdots b_{\mathrm{S}}(t_n)\tag{65}$$

Using the bounds (64) and (65) we obtain

$$\left\|\mathrm{Tr}_{\mathrm{R},M}\left(\left[V_M(t_1),\left[\cdots\left[V_m(t_n),\rho_{\mathrm{S}}\otimes\rho_{\mathrm{R},M}\right]\ldots\right]\right]\right)\right\|_1\leq C_{\mathrm{S},n}C_{\mathrm{R},n}b_{\mathrm{S}}(t_1)\cdots b_{\mathrm{S}}(t_n)b_{\mathrm{R}}(t_1)\cdots b_{\mathrm{R}}(t_n).$$

The multiple time integral in (61) is bounded in the trace norm by

$$\left\|\int_{0\leq t_n\leq\cdots\leq t_1\leq t}dt_1\cdots dt_n\mathrm{Tr}_{\mathrm{R},M}\left(\left[V_M(t_1),\left[\cdots\left[V_M(t_n),\rho_{\mathrm{S}}\otimes\rho_{\mathrm{R},M}\right]\ldots\right]\right]\right)\right\|_1$$

$$\leq 2^nC_{\mathrm{S},n}C_{\mathrm{R},n}\int_{0\leq t_n\leq\cdots\leq t_1\leq t}dt_1\cdots dt_nb_{\mathrm{S}}(t_1)b_{\mathrm{R}}(t)\cdots b_{\mathrm{S}}(t_n)b_{\mathrm{R}}(t_n)=\frac{2^nC_{\mathrm{S},n}C_{\mathrm{R},n}}{n!}\left[\int_0^tb_{\mathrm{S}}(s)b_{\mathrm{R}}(s)ds\right]^n=\frac{C_{\mathrm{S},n}C_{\mathrm{R},n}(2B(t))^n}{n!},\tag{66}$$

where we used the symmetry of the integrand to transform the integral over the simplex into an integral over a hypercube. The factor 2^n comes from expanding the multiple commutator into 2^n terms. The bound (66) together with (13) shows that the series (61) converges for any value of and t, and uniformly in M. As the series (and the integral over times) converges uniformly in M, we have

$$\lim_{M\to\infty}\mathrm{Tr}_{\mathrm{R},M}\left(e^{itH_M^0}e^{-itH_{\mathrm{SR},M}}\rho_{\mathrm{SR},M}\,e^{itH_{\mathrm{SR},M}}e^{-itH_M^0}\right)$$

$$=\sum_{n\geq 0}(-i)^n\int_{0\leq t_n\leq\cdots\leq t_1\leq t}dt_1\cdots dt_n\lim_{M\to\infty}\mathrm{Tr}_{\mathrm{R},M}\left(\left[V_M(t_1),\left[\cdots\left[V_M(t_n),\rho_{\mathrm{S}}\otimes\rho_{\mathrm{R},M}\right]\ldots\right]\right]\right).\tag{67}$$

We next simplify the value of the limit in (67) using Proposition 1, an argument which is central to many arguments in mean field theory. Namely, we use the result (28) in the expression (67), where the role of the $\overline{X}_j$ is played by the factors $\overline{V}_{\mathrm{R},M}(t_j)$ in the operators $V_M(t_j)=G(t_j)\otimes\overline{V}_{\mathrm{R},M}(t_j)$. By expanding out the multi-commutators, as explained above, we obtain 2^n terms of the form (63). Upon taking the partial trace $\mathrm{Tr}_{\mathrm{R},M}$ and sending $M\to\infty$, the term (63) becomes

$$\lim_{M\to\infty}\mathrm{Tr}_{\mathrm{R},M}\left(G(t_{\ell_1})\cdots G(t_{\ell_L})\rho_{\mathrm{S}}G(t_{\ell_{L+1}})\cdots G(t_{\ell_n})\otimes\overline{V}_{\mathrm{R},M}(t_{\ell_1})\cdots\overline{V}_{\mathrm{R},M}(t_{\ell_L})\rho_{\mathrm{R},M}\overline{V}_{\mathrm{R},M}(t_{\ell_{L+1}})\ldots\overline{V}(t_{\ell_n})\right)$$

$$=G(t_{\ell_1})\cdots G(t_{\ell_L})\rho_{\mathrm{S}}G(t_{\ell_{L+1}})\cdots G(t_{\ell_n})\omega_{\mathrm{R}}(v(t_1))\cdots\omega_{\mathrm{R}}(v(t_n))=G'(t_{\ell_1})\cdots G'(t_{\ell_L})\rho_{\mathrm{S}}G'(t_{\ell_{L+1}})\cdots G'(t_{\ell_n}),\tag{68}$$

where we set

$$G'(t)=\omega_{\mathrm{R}}(v(t))G(t).\tag{69}$$

This shows that by taking $\lim_{M\to\infty}\mathrm{Tr}_{\mathrm{R},M}$ in the Dyson series (67) we obtain the following Dyson series involving the system alone,

$$\lim_{M\to\infty}\mathrm{Tr}_{\mathrm{R},M}\left(e^{itH_M^0}e^{-itH_{\mathrm{SR},M}}\rho_{\mathrm{SR},M}\,e^{itH_{\mathrm{SR},M}}e^{-itH_M^0}\right)=\sum_{n\geq 0}(-i)^n\int_{0\leq t_n\leq\cdots\leq t_1\leq t}dt_1\cdots dt_n\left[G'(t_1),\left[\cdots\left[G'(t_n),\rho_{\mathrm{S}}\right]\ldots\right]\right].\tag{70}$$

The left side of (70) equals (see (9), (62) and note that the conjugation $e^{itH_{\mathrm{R},M}}\cdot e^{-itH_{\mathrm{R},M}}$ falls away due to the cyclicity of the trace)

$$\lim_{M\to\infty}\mathrm{Tr}_{\mathrm{R},M}\left(e^{itH_M^0}e^{-itH_{\mathrm{SR},M}}\rho_{\mathrm{SR},M}\,e^{itH_{\mathrm{SR},M}}e^{-itH_M^0}\right)=e^{itH_{\mathrm{S}}}\rho_{\mathrm{S}}(t)e^{-itH_{\mathrm{S}}}.\tag{71}$$

We take the derivative with respect to t of both sides in (70) to obtain

$$e^{itH_{\mathrm{S}}}\left(i[H_{\mathrm{S}},\rho_{\mathrm{S}}(t)]+\partial_t\rho_{\mathrm{S}}(t)\right)e^{-itH_{\mathrm{S}}}=\sum_{n\geq 1}(-i)^n\int_{0\leq t_n\leq\cdots\leq t_2\leq t}dt_2\cdots dt_n\left[G'(t),\left[G'(t_2),\left[\cdots\left[G'(t_n),\rho_{\mathrm{S}}\right]\ldots\right]\right]\right]=-i[G'(t),e^{itH_{\mathrm{S}}}\rho_{\mathrm{S}}(t)e^{-itH_{\mathrm{S}}}].$$

$$(72)$$

We used the formulas (70) and (71) to arrive at the last equality. It follows from (72) that

$$i\partial_t \rho_S(t) = [H_S, \rho_S(t)] + [e^{-itH_S} G'(t) e^{itH_S}, \rho_S(t)].$$

$$(73)$$

From (69), (5) we have $e^{-itH_S} G'(t) e^{itH_S} = \omega_R(v(t))G$, so

$$i\partial_t \rho_S(t) = [H_S + \omega_R(v(t))G, \rho_S(t)].$$

$$(74)$$

This shows (14) and completes the proof of Theorem 1. $\qquad\square$

We now give a proof of (52) for the interaction V_M as in (51). To do so, we repeat the above argument and only need to modify Proposition 1 to the following.

Proposition 8. *Suppose $\omega_{R,M}$ satisfies an analogue condition to (26) of the form*

$$\lim_{M\to\infty} \sup_{\substack{\Lambda_1,\dots,\Lambda_n\in\mathcal{P}_{\nu,M}\\ \forall i\neq j,\Lambda_i\cap\Lambda_j=\emptyset}} \left| \omega_{R,M}\left((x_1^\nu)^{[\Lambda_1]}\cdots(x_n^\nu)^{[\Lambda_n]}\right) - \omega_R^{\otimes\nu}(x_1^\nu)\cdots\omega_R^{\otimes\nu}(x_n^\nu) \right| = 0$$

and let x_j^ν, $j\geq 1$, be operators on $\mathcal{H}_{R,\nu}$ such that all mixed moments in the state $\omega_{R,M}$ are finite: For any n there is an $M_0(n)$ and $C(n)$ such that for $M \geq M_0(n)$,

$$\omega_{R,M}\left((x_1^\nu)^{[\Lambda_1]}\cdots(x_n^\nu)^{[\Lambda_n]}\right) \leq C(n).$$

Define the operators $\overline{X}_{M,j} = \binom{M}{\nu}^{-1}\sum_{\Lambda\in\mathcal{P}_{\nu,M}}(x_j^\nu)^{[\Lambda]}$ on $\mathcal{H}_{R,M}$. Then we have, for any $n\in\mathbb{N}$,

$$\lim_{M\to\infty}\omega_{R,M}\left(\overline{X}_{M,1}\cdots\overline{X}_{M,n}\right) = \omega_R^{\otimes\nu}(x_1^\nu)\cdots\omega_R^{\otimes\nu}(x_n^\nu),$$

where $\omega_R^{\otimes\nu}(x^\nu) = (\omega_R\otimes\cdots\otimes\omega_R)(x^\nu) = \mathrm{Tr}_{\mathcal{H}_{R,\nu}}\left((\rho_R\otimes\cdots\otimes\rho_R)x^\nu\right)$.

Proof of Proposition 8. We introduce the set $\mathcal{M}_n$ consisting of all collections $(\Lambda_1,\dots,\Lambda_n)\in(\mathcal{P}_{\nu,M})^n$ such that $\Lambda_k\cap\Lambda_\ell=\emptyset$ whenever $k\neq\ell$. The cardinality of $\mathcal{M}_n$ is (for $M\geq n\nu$)

$$|\mathcal{M}_n| = \binom{M}{\nu}\binom{M-\nu}{\nu}\cdots\binom{M-(n-1)\nu}{\nu}.$$

It satisfies

$$\lim_{M\to\infty}\frac{|\mathcal{M}_n|}{\binom{M}{\nu}^n} = 1.$$

Proceeding exactly as in the proof of Proposition 1 the sum

$$\frac{1}{\binom{M}{\nu}^n}\sum_{\Lambda_1,\dots,\Lambda_n\in\mathcal{M}_n}\omega_{R,M}\left((x_1^\nu)^{[\Lambda_1]}\cdots(x_n^\nu)^{[\Lambda_n]}\right)$$

converges to $\omega_R^{\otimes\nu}(x_1^\nu)\cdots\omega_R^{\otimes\nu}(x_n^\nu)$ in the limit $M\to\infty$. The remaining part of the sum converges to zero,

$$\lim_{M\to\infty}\frac{1}{\binom{M}{\nu}^n}\sum_{\Lambda_1,\dots,\Lambda_n\in\mathcal{M}_n^c}\omega_{R,M}\left((x_1^\nu)^{[\Lambda_1]}\cdots(x_n^\nu)^{[\Lambda_n]}\right) = 0,$$

which is again shown as in Proposition 1. $\quad\square$

5.2. Proof of Theorem 2

In the Dyson series (61) we now have

$$V_M(t) = e^{it(H_S+H_{R,M})}V_M e^{-it(H_S+H_{R,M})} = \sum_{j=1}^N G_j(t)\otimes\overline{V}_{R,M,j}(t),$$

where $G_j(t) = e^{itH_S^{(j)}}G_j e^{-itH_S^{(j)}}$ and $\overline{V}_{R,M,j}(t) = e^{itH_{R,M}}\overline{V}_{R,M,j}e^{-itH_{R,M}}$ (see also (5)). Therefore,

$$\mathrm{Tr}_{R,M}\left(e^{itH_M^0}e^{-itH_{SR,M}}\rho_{SR,M}\,e^{itH_{SR,M}}e^{-itH_M^0}\right)$$

$$= \sum_{n\geq 0}(-i)^n\sum_{j_1=1}^N\cdots\sum_{j_n=1}^N\int_{0\leq t_n\leq\cdots\leq t_1\leq t}dt_1\cdots dt_n\mathrm{Tr}_{R,M}\left(\left[G_{j_1}(t_1)\otimes\overline{V}_{R,M,j_1}(t_1),[\cdots[G_{j_n}(t_n)\otimes\overline{V}_{R,M,j_n}(t_n),\rho_S\otimes\rho_{R,M}]\cdots]\right]\right).$$

$$(75)$$

For $j_1,\dots,j_n$ fixed, we estimate in trace norm (for operators on $\mathcal{H}_S$),

$$\left\|\mathrm{Tr}_{R,M}\left(\left[G_{j_1}(t_1)\otimes\overline{V}_{R,M,j_1}(t_1),[\cdots[G_{j_n}(t_n)\otimes\overline{V}_{R,M,j_n}(t_n),\rho_S\otimes\rho_{R,M}]\cdots]\right]\right)\right\|_1 \leq 2^n C_{S,n}C_{R,n}\,b_S(t_1)\cdots b_S(t_n)b_R(t_1)\cdots b_R(t_n),$$

where we used the bounds (22) and (23) and the factor 2^n comes from expanding the multiple commutator in 2^n terms of the form (compare with (63))

$$G_{j_1}(t_{\ell_1}) \cdots G_{j_L}(t_{\ell_L}) \rho_S G_{j_{L+1}}(t_{\ell_{L+1}}) \cdots G_{j_{L_n}}(t_{\ell_{L_n}}) \otimes \overline{V}_{R,M,j_1}(t_{\ell_1}) \cdots \overline{V}_{R,M,j_L}(t_{\ell_L}) \rho_{R,M} \overline{V}_{R,M,j_{L+1}}(t_{\ell_{L+1}}) \dots \overline{V}_{R,M,j_n}(t_{\ell_n}). \qquad (76)$$

It follows by the same argument given after (66), that the series (75) converges uniformly in M due to condition (13). Next, we have, for each $j_1, \dots, j_n$ fixed,

$$\lim_{M \to \infty} \mathrm{Tr}_{R,M}\left(\overline{V}_{R,M,j_1}(t_{\ell_1}) \cdots \overline{V}_{R,M,j_L}(t_{\ell_L}) \rho_{R,M} \overline{V}_{R,M,j_{L+1}}(t_{\ell_{L+1}}) \dots \overline{V}_{R,M,j_n}(t_{\ell_n}) \right) = \omega_R\left(v_{j_1}(t_{\ell_1})\right) \cdots \omega_R\left(v_{j_n}(t_{\ell_n})\right).$$

Therefore, in view of (75), when taking the trace of the reservoir and $M \to \infty$ and summing over the $j_1, \dots, j_n$, the term (76) becomes (c.f. (68))

$$\lim_{M \to \infty} \sum_{j_1=1}^{N} \cdots \sum_{j_n=1}^{N} \mathrm{Tr}_{R,M} \Big(G_{j_1}(t_{\ell_1}) \cdots G_{j_L}(t_{\ell_L}) \rho_S G_{j_{L+1}}(t_{\ell_{L+1}}) \cdots G_{j_{L_n}}(t_{\ell_{L_n}})$$
$$\otimes \overline{V}_{R,M,j_1}(t_{\ell_1}) \cdots \overline{V}_{R,M,j_L}(t_{\ell_L}) \rho_{R,M} \overline{V}_{R,M,j_{L+1}}(t_{\ell_{L+1}}) \dots \overline{V}_{R,M,j_n}(t_{\ell_n}) \Big) = G''(t_{\ell_1}) \cdots G''(t_{\ell_L}) \rho_S G''(t_{\ell_{L+1}}) \cdots G''(t_{\ell_{L_n}}), \qquad (77)$$

where

$$G''(t) = \sum_{j=1}^{N} \omega_R\left(v_j(t)\right) G_j(t).$$

As in (70), we see that

$$\rho_S(t) = \lim_{M \to \infty} \mathrm{Tr}_{R,M}\left(e^{itH_M^0} e^{-itH_{SR,M}} \rho_{SR,M} e^{itH_{SR,M}} e^{-itH_M^0} \right) = \sum_{n \geq 0} (-i)^n \int_{0 \leq t_n \leq \cdots \leq t_1 \leq t} dt_1 \cdots dt_n [G''(t_1), [\cdots [G''(t_n), \rho_S] \dots]], \qquad (78)$$

from which (25) follows. This completes the proof of Theorem 2. $\qquad \square$

5.3. Proof of Proposition 1

We have

$$\omega_{R,M}\left(\overline{X}_{M,1} \cdots \overline{X}_{M,n}\right) = \frac{1}{M^n} \sum_{m_1=1}^{M} \cdots \sum_{m_n=1}^{M} \omega_{R,M}\left(x_1^{[m_1]} \cdots x_n^{[m_n]}\right) = \frac{1}{M^n} \Big(\sum_{\mathcal{M}_n} + \sum_{\mathcal{M}_n^c} \Big) \omega_{R,M}\left(x_1^{[m_1]} \cdots x_n^{[m_n]}\right), \qquad (79)$$

where $\mathcal{M}_n$ is the set of all indices $m_1, \dots, m_n$ with the constraint that $|m_j - m_k| > L$ whenever $j \neq k$ and $\mathcal{M}_n^c$ is the collection of all remaining indices. 'Most' indices are such that all the m_j are apart from each other by at least L: The cardinality of $\mathcal{M}_n$ satisfies

$$\lim_{M \to \infty} \frac{|\mathcal{M}_n|}{M^n} = 1 \qquad (80)$$

and $|\mathcal{M}_n^c| = M^n - |\mathcal{M}_n| \to 0$ as $M \to \infty$. We have

$$\frac{1}{M^n} \sum_{\mathcal{M}_n} \omega_{R,M}\left(x_1^{[m_1]} \cdots x_n^{[m_n]}\right) - \omega_R(x_1) \cdots \omega_R(x_n) = \sum_{\mathcal{M}_n} \frac{\omega_{R,M}\left(x_1^{[m_1]} \cdots x_n^{[m_n]}\right) - \omega_R(x_1) \cdots \omega_R(x_n)}{M^n} + \Big(\frac{|\mathcal{M}_n|}{M^n} - 1 \Big) \omega_R(x_1) \cdots \omega_R(x_n). \qquad (81)$$

Let $\epsilon > 0$. Due to (26) there is an $M_0 > 0$ such that if $M > M_0$ then

$$\left| \omega_{R,M}\left(x_1^{[m_1]} \cdots x_n^{[m_n]}\right) - \omega_R(x_1) \cdots \omega_R(x_n) \right| < \epsilon,$$

uniformly in the $m_1, \dots, m_n$. The absolute value of the sum over $\mathcal{M}_n$ on the right side of (81) is thus bounded above by $\epsilon \frac{|\mathcal{M}_n|}{M^n}$ for $M > M_0$. So as $M \to \infty$ the upper bound is ϵ by (80), so that this limit is zero. Again by (80) the second term on the right side of (80) vanishes as well for $M \to \infty$. We conclude that

$$\lim_{M \to \infty} \frac{1}{M^n} \sum_{\mathcal{M}_n} \omega_{R,M}\left(x_1^{[m_1]} \cdots x_n^{[m_n]}\right) = \omega_R(x_1) \cdots \omega_R(x_n). \qquad (82)$$

Next by the estimate (27),

$$\left| \frac{1}{M^n} \sum_{\mathcal{M}_n^c} \omega_{R,M}\left(x_1^{[m_1]} \cdots x_n^{[m_n]}\right) \right| \leq \frac{M^n - |\mathcal{M}_n|}{M^n} C(n).$$

The left hand side thus vanishes in the limit $M \to \infty$. Combining this with (79) and (82) yields the result (28). This completes the proof of Proposition 1. $\qquad \square$

5.4. Proof of Proposition 2

For bounded x_j, we have (c.f. (28))

$$\lim_{M \to \infty} \omega_{R,M}\left(\overline{X}_{M,1} \cdots \overline{X}_{M,n}\right) = \int_{S_R} d\mu(\omega)\,\omega(x_1)\cdots\omega(x_n). \tag{83}$$

The proof of (83) is as in Proposition 1, where in the sum over $\mathcal{M}_n$ in (79) we use the property (31) and the sum over $\mathcal{M}_n^c$ times M^{-n} vanishes as $M \to \infty$. The right hand side of (68) is then $\int_S d\mu(\omega) G'_\omega(t_{\ell_1}) \cdots G'_\omega(t_{\ell_L}) \rho_S G'_\omega(t_{\ell+1}) \cdots G'_\omega(t_{\ell_n})$, where $G'_\omega(t) = \omega(v(t))G(t)$. Continuing in the same way as in the proof of Theorem 1 we arrive at (compare with (70)),

$$e^{itH_S}\rho_S(t)e^{-itH_S} = \int_{S_R} d\mu(\omega) \sum_{n \ge 0}(-i)^n \int_{0 \le t_n \le \cdots \le t_1 \le t} [G'_\omega(t_1), [\cdots [G'_\omega(t_n), \rho_S]\cdots]]. \tag{84}$$

It follows that

$$e^{itH_S}\rho_S(t)e^{-itH_S} = \int_{S_R} d\mu(\omega)\, U'(\omega,t)\rho_S U'(\omega,t)^*, \tag{85}$$

where for $\omega \in S_R$ fixed,

$$i\partial_t U'(\omega,t) = \omega\big(v(t)\big)G(t)\,U'(\omega,t), \qquad U'(\omega,0) = \mathbb{1}.$$

To show that the right side of (85) equals that of (84), it suffices to note that both quantities solve the same differential equation (obtained upon taking ∂_t) and they have the same initial condition for $t = 0$ (namely, ρ_S).

Thus (33) holds with $U(\omega,t) = e^{-itH_S}U'(\omega,t)$. $\qquad\square$

5.5. Proof of Proposition 3

We show (37). The coherent state $|\alpha\rangle$ of a single mode (oscillator) is defined by $|\alpha\rangle = W(\alpha)|0\rangle$, where $|0\rangle$ is the vacuum state and the Weyl operator is defined by $W(\alpha) = \exp[i\frac{1}{\sqrt{2}}(\alpha a^* + \bar{\alpha}a)]$ and satisfies the commutation relation

$$\varphi(t)W(\alpha) = W(\alpha)\big(\varphi(t) - \mathrm{Im}(e^{i\Omega_R t}\alpha)\big). \tag{86}$$

We now obtain a bound of the type (10) for $\rho_{R,M} = |\alpha\rangle\langle\alpha| \otimes \cdots \otimes |\alpha\rangle\langle\alpha|$. We estimate,

$$\mathrm{Tr}_{R,M}\left((|\alpha\rangle\langle\alpha|)^{\otimes M}\overline{V}_{R,M}(t_1)\cdots\overline{V}_{R,M}(t_n)\right)$$

$$= M^{-n}\sum_{f:\{1,\dots,n\}\to\{1,\dots,M\}}\prod_{m=1}^{M}\langle\alpha|\mathcal{T}\prod_{q\in f^{-1}(m)}\varphi(t_q)\alpha\rangle$$

$$= M^{-n}\sum_{f:\{1,\dots,n\}\to\{1,\dots,M\}}\prod_{m=1}^{M}\sum_{S\subset f^{-1}(m)}\big(\prod_{s'\in S^C}\mathrm{Im}(-e^{i\Omega_R t_{s'}}\alpha)\big)\langle 0|\mathcal{T}\prod_{s\in S}\varphi(t_s)0\rangle$$

$$= M^{-n}\sum_{f:\{1,\dots,n\}\to\{1,\dots,M\}}\prod_{m=1}^{M}\sum_{\substack{S\subseteq f^{-1}(m),\\|S|\ \text{even}}}\prod_{s'\in S^C}\mathrm{Im}(-e^{i\Omega_R t_{s'}}\alpha)\sum_{p\in\mathcal{P}_S}\prod_{(s_1,s_2)\in p}e^{-i\Omega_R(t_{s_1}-t_{s_2})}. \tag{87}$$

Here, $\mathcal{T}$ is the time ordering operator and we sum over all M^n functions f from the set $\{1,\dots,n\}$ to the set $\{1,\dots,M\}$. In the second step we use the commutation relation (86). In the last step we use Wick's theorem to express the average over a product of field operators by a sum over two-point correlation functions $\langle 0|\varphi(t)\varphi(s)|0\rangle = e^{-i\Omega_R(t-s)}$. $\mathcal{P}_S$ is the set of all pairings with indices in S. The number of such pairings is $|\mathcal{P}_S| = |S|!!$, where

$$k!! \equiv \frac{k!}{2^{k/2}(k/2)!} = (k-1)\cdot(k-3)\cdots 3\cdot 1, \qquad k \text{ even}. \tag{88}$$

We have,

$$\Big|\sum_{\substack{S\subseteq f^{-1}(m),\\|S|\ \text{even}}}\prod_{s'\in S^C}\mathrm{Im}(-e^{i\Omega_R t_{s'}}\alpha)\sum_{p\in\mathcal{P}_S}\prod_{(s_1,s_2)\in p}e^{-i\Omega_R(t_{s_1}-t_{s_2})}\Big| \le \sum_{\substack{S\subseteq f^{-1}(m),\\|S|\ \text{even}}}|\alpha|^{|f^{-1}(m)|-|S|}\,(|S|!!) \le (1+|\alpha|)^{|f^{-1}(m)|}\,(S_{f,m})!!, \tag{89}$$

where

$$S_{f,m} = 2\lfloor|f^{-1}(m)|/2\rfloor \tag{90}$$

is the cardinality of the biggest subset of $f^{-1}(m)$ with an even number of elements. (Here, $\lfloor x\rfloor$ is the floor function; e.g. $\lfloor\pi\rfloor = 3$.) The first factor on the right side of (89) is due to

$$\sum_{S\subset f^{-1}(m)}|\alpha|^{|f^{-1}(m)|-|S|} = \sum_{k=0}^{|f^{-1}(m)|}\binom{|f^{-1}(m)|}{k}|\alpha|^{|f^{-1}(m)|-k} = (1+|\alpha|)^{|f^{-1}(m)|}.$$

We use (89) in (87),

$$\left| \mathrm{Tr}_{\mathrm{R},M} \left((|\alpha\rangle\langle\alpha|)^{\otimes M} \overline{V}_{\mathrm{R},M}(t_1) \cdots \overline{V}_{\mathrm{R},M}(t_n) \right) \right| \le M^{-n} \sum_{f:\{1,\dots,n\}\to\{1,\dots,M\}} \prod_{m=1}^{M} (1+|\alpha|)^{|f^{-1}(m)|} (S_{f,m})!! . \tag{91}$$

We use $\sum_{m=1}^{M} |f^{-1}(m)| = n$ to estimate the product of the factors containing the α by $(1+|\alpha|)^n$. Observe that for any integers n_1, n_2,

$$(2(n_1+n_2))!! \ge (2n_1)!! \cdot (2n_2)!! . \tag{92}$$

To see this, we write down the product explicitly and estimate,

$$\begin{aligned}
(2(n_1+n_2))!! &= (2n_1 - 1 + 2n_2) \cdot (2n_1 - 3 + 2n_2) \cdots (3 + 2n_2) \cdot (1 + 2n_2) \cdot (2n_2)!! \\
&\ge (2n_1 - 1) \cdot (2n_1 - 3) \cdots (3) \cdot (1) \cdot (2n_2)!! \\
&= (2n_1)!! \cdot (2n_2)!! .
\end{aligned}$$

It follows from (90), (92) and $\sum_{m=1}^{M} |f^{-1}(m)| = n$ that

$$\prod_{m=1}^{M} (S_{f,m})!! = \prod_{m=1}^{M} (2\lfloor |f^{-1}(m)|/2 \rfloor)!! \le \left(2 \sum_{m=1}^{M} \lfloor \tfrac{|f^{-1}(m)|}{2} \rfloor \right)!! \le (2\lfloor n/2 \rfloor)!! . \tag{93}$$

Finally, as the number of terms of the sum over the f equals M^n, we combine (91) and (93) to arrive at

$$\left| \mathrm{Tr}_{\mathrm{R},M}\left((|\alpha\rangle\langle\alpha|)^{\otimes M} \overline{V}_{\mathrm{R},M}(t_1) \cdots \overline{V}_{\mathrm{R},M}(t_n) \right) \right| \le (2\lfloor n/2 \rfloor)!! \, (1+|\alpha|)^n . \tag{94}$$

This shows the validity of the expression (37). □

We now indicate how to modify this proof to arrive at Proposition 5. Analogously to (86), we have the commutation relations $\varphi(g)W(f) = W(f)(\varphi(g) - \mathrm{Im}\langle g|f\rangle)$ and we can replicate exactly the combinatorial argument as in the oscillator case. The two-point correlation functions in Wick's theorem is now $\langle \Omega | \varphi(e^{i\omega t} g) \varphi(e^{i\omega s} g) \Omega \rangle = \frac{1}{2} \langle g | e^{i\omega(t-s)} g \rangle$, which is bounded in modulus by $\frac{1}{2}\|g\|^2$.

5.6. Proof of Proposition 4

As v_2 is stationary under the reservoir evolution, we have

$$\omega_{\mathrm{R},M}\left(\overline{V}_{\mathrm{R},M}(t_1) \cdots \overline{V}_{\mathrm{R},M}(t_n) \right) = \frac{1}{M^n} \sum_{m_1=1}^{M} \cdots \sum_{m_n=1}^{M} \omega_{\mathrm{R},M}\left(v_2^{[m_1]} \cdots v_2^{[m_n]} \right) = \frac{1}{M^n} \omega_{\mathrm{R},M}\left(\Big(\sum_{m=1}^{M} v_2^{[m]} \Big)^n \right) = \frac{v^n}{M^n} \omega_{\mathrm{R},M}\left(\Big(\sum_{m=1}^{M} \widehat{N}_m \Big)^n \right), \tag{95}$$

where $\widehat{N}_m = (a^\dagger a)^{[m]}$ is the number operator acting on the mth oscillator. Using the multinomial expansion for $(\sum_{m=1}^{M} \widehat{N}_m)^n$ in (95),

$$\omega_{\mathrm{R},M}\left(\overline{V}_{\mathrm{R},M}(t_1) \cdots \overline{V}_{\mathrm{R},M}(t_n) \right) = \frac{v^n}{M^n} \sum_{\substack{k_1+\cdots+k_M=n \\ k_1,\dots,k_M \ge 0}} \binom{n}{k_1,\dots,k_M} \omega_{\mathrm{R},M}\left((\widehat{N}_1)^{k_1} \cdots (\widehat{N}_M)^{k_M} \right). \tag{96}$$

Due to (38), we have $\omega_{\mathrm{R},M}((\widehat{N}_1)^{k_1} \cdots (\widehat{N}_M)^{k_M}) \le c^{k_1+\cdots+k_M} = c^n$ and we obtain from (96),

$$\omega_{\mathrm{R},M}\left(\overline{V}_{\mathrm{R},M}(t_1) \cdots \overline{V}_{\mathrm{R},M}(t_n) \right) \le \frac{(cv)^n}{M^n} \sum_{\substack{k_1+\cdots+k_M=n \\ k_1,\dots,k_M \ge 0}} \binom{n}{k_1,\dots,k_M} = (cv)^n . \tag{97}$$

It follows that (10) holds with $C_{\mathrm{R},n} = (cv)^n$ and $b_{\mathrm{R}}(t) = 1$. □

5.7. Proof of Proposition 6

The number operator of the 'site' m is defined by

$$\widehat{N}_m = \int_{\mathbb{R}^d} (a_k^\dagger a_k)^{[m]} d^d k . \tag{98}$$

It commutes with the interaction, $\left[\widehat{N}_m, (v_2(t))^{[m']} \right] = 0$. Furthermore, we have the well known bound

$$\left\| (v_2(t))^{[m]} (\widehat{N}_m + 1)^{-1} \right\| = \left\| v_2^{[m]} (\widehat{N}_m + 1)^{-1} \right\| \le 2\|g\|^2 , \tag{99}$$

where $\|g\|^2 = \int_{\mathbb{R}^d} |g(k)|^2 d^d k$ is the L^2 norm of the function g. We use (99) and the fact that $\widehat{N}_m$ commutes with $(v_2(t))^{[m']}$ to get,

$$\left| \omega_{\mathrm{R},M}\left(v_2(t_1)^{[m_1]} \cdots v_2(t_n)^{[m_n]} \right) \right| = \left| \omega_{\mathrm{R},M}\left(w_{m_1,\dots,m_n}^{1/2} v_2(t_1)^{[m_1]} (\widehat{N}_{m_1} + 1)^{-1} \cdots v_2(t_n)^{[m_n]} (\widehat{N}_{m_n} + 1)^{-1} w_{m_1,\dots,m_n}^{1/2} \right) \right| , \tag{100}$$

where $w_{m_1,\dots,m_n} = (\widehat{N}_{m_1} + 1) \cdots (\widehat{N}_{m_n} + 1)$. Next, using that for any state ω, any self-adjoint operator $w \ge 0$ and any bounded operator A, we have $|\omega(w^{1/2} A w^{1/2})| \le \|A\| \, \omega(w)$, we obtain from (99), (100),

$$\left| \omega_{\mathrm{R},M}\left(v_2(t_1)^{[m_1]} \cdots v_2(t_n)^{[m_n]} \right) \right| \le \left(2\|g\|^2 \right)^n \omega_{\mathrm{R},M}\left((\widehat{N}_{m_1} + 1) \cdots (\widehat{N}_{m_n} + 1) \right). \tag{101}$$

It follows that

$$\left| \omega_{R,M}\big(\overline{V}_{R,M}(t_1)\cdots\overline{V}_{R,M}(t_n)\big)\right| \le \frac{(2\|g\|^2)^n}{M^n}\sum_{m_1=1}^{M}\cdots\sum_{m_n=1}^{M}\omega_{R,M}\big((\hat{N}_{m_1}+1)\cdots(\hat{N}_{m_n}+1)\big) = \frac{(2\|g\|^2)^n}{M^n}\,\omega_{R,M}\Big(\big(\sum_{m=1}^{M}\hat{N}_m+1\big)^n\Big). \tag{102}$$

We may now proceed as after (95) to conclude that if $\omega_{R,M}$ is the M-fold tensor product of ω_R satisfying (42), then (10) holds with $C_{R,n} = (2c\|g\|^2)^n$ and $b_R(t) = 1$. □

5.8. Proof of Proposition 7

We start by noticing that for operators A, B and a normalized state ψ,

$$\|A|\psi\rangle\langle\psi|B\|_1 = \mathrm{Tr}\sqrt{B^*|\psi\rangle\langle\psi|A^*A|\psi\rangle\langle\psi|B} = \|A\psi\|\,\|B^*\psi\|. \tag{103}$$

We are then led to find a bound on

$$\|G(t_1)\cdots G(t_n)\psi\| = \|e^{it_1(-\Delta)}(E\cdot x)e^{-it_1(-\Delta)}\cdots e^{it_n(-\Delta)}(E\cdot x)e^{-it_n(-\Delta)}\psi\|. \tag{104}$$

Let us introduce the creation and annihilation operators in the direction $\hat{E}$

$$a_E = \frac{1}{\sqrt{2}}\,\hat{E}\cdot(x+i(-i\nabla_x)), \qquad a_E^\dagger = \frac{1}{\sqrt{2}}\,\hat{E}\cdot(x-i(-i\nabla_x)). \tag{105}$$

They satisfy the canonical commutation relations $[a_E, a_E^\dagger] = 1$. We define the field operator

$$\varphi_E(\alpha) = \frac{\alpha a_E^\dagger + \bar{\alpha}a_E}{\sqrt{2}}, \qquad \alpha \in \mathbb{C}. \tag{106}$$

It follows from (105) that $\hat{E}\cdot x = \varphi_E(1)$ and $\hat{E}\cdot(-i\nabla_x) = \varphi_E(i)$. By passing to the Fourier transform, one readily sees that

$$e^{is(-\Delta)}(\hat{E}\cdot x)e^{-is(-\Delta)} = \hat{E}\cdot x + 2s\hat{E}\cdot(-i\nabla_x) = \varphi_E(1+2is).$$

We further define the number operator

$$N_E = a_E^\dagger a_E$$

and we have the commutation rules

$$N_E a_E = a_E(N_E - 1) \quad \text{and} \quad N_E a_E^\dagger = a_E^\dagger(N_E + 1). \tag{107}$$

To estimate (104) we express the operators in terms of the fields and insert suitable weights,

$$\|G(t_1)\cdots G(t_n)\psi\| = \|\varphi_E(1+2it_1)\ldots\varphi_E(1+2it_n)\psi\| \tag{108}$$

$$= \|\varphi_E(1+2it_1)\,(N_E+1)^{-\frac{1}{2}}(N_E+1)^{\frac{1}{2}}\varphi_E(1+2it_2)(N_E+2)^{-1}(N_E+2)\cdots(N_E+(n-1))^{\frac{n-1}{2}}\varphi_E(1+2it_n)(N_E+n)^{-\frac{n}{2}}(N_E+n)^{\frac{n}{2}}\psi\|.$$

We now need to estimate the norm of the weighted field operators, $(N_E+l)^{\frac{l}{2}}\varphi_E(1+2it)(N_E+(l+1))^{-\frac{l+1}{2}}$, for $l \in \mathbb{N}\cup\{0\}$. It follows from (107) that

$$(N_E+l)^{\frac{l}{2}}\varphi_E(1+2it) = \frac{1}{\sqrt{2}}(N_E+l)^{\frac{l}{2}}\big((1+2it)a_E^\dagger + (1-2it)a_E\big) = \frac{1}{\sqrt{2}}\big((1+2it)a_E^\dagger(N_E+l+1)^{\frac{l}{2}} + (1-2it)a_E(N_E+l-1)^{\frac{l}{2}}\big). \tag{109}$$

We combine this with the standard estimate [42]

$$\|a_E(N_E+1)^{-1/2}\| \le 1, \quad \|a_E^\dagger(N_E+1)^{-1/2}\| \le 1$$

to obtain,

$$\|(N_E+l)^{\frac{l}{2}}\varphi_E(1+2it)\big(N_E+l+1\big)^{-\frac{l+1}{2}}\| \le \sqrt{2(1+4t^2)}. \tag{110}$$

Using the bound (110) in (108) gives

$$\|G(t_1)\cdots G(t_n)\psi\| \le 2^{\frac{n}{2}}\Big(\prod_{j=1}^{n}\sqrt{1+4t_j^2}\Big)\|(N_E+n)^{\frac{n}{2}}\psi\| \le 2^{\frac{n}{2}}\Big(\prod_{j=1}^{n}\sqrt{1+4t_j^2}\Big)(1+n)^{\frac{n}{2}}\|N_E^{\frac{n}{2}}\psi\|, \tag{111}$$

where we used $\|(N_E+n)^{\frac{n}{2}}\psi\|^2 = \langle\psi, (N_E+n)^n\psi\rangle \le \langle\psi, (1+n)^n N_E^n\psi\rangle = (1+n)^n\|N_E^{\frac{n}{2}}\psi\|^2$. The bounds (103) and (111) give for a normalized ψ,

$$\left\|G(t_{\ell_1})\cdots G(t_{\ell_L})|\psi\rangle\langle\psi|G(t_{\ell_{L+1}})\cdots G(t_{\ell_n})\right\|_1 \le (2(1+n))^{\frac{n}{2}}\Big(\prod_{j=1}^{n}\sqrt{1+4t_j^2}\Big)\|N_E^{\frac{L}{2}}\psi\|\,\|N_E^{\frac{n-L}{2}}\psi\|. \tag{112}$$

The bound (49) implies that for all $k \in \mathbb{N}$,

$$\|N_E^k\psi\| \le c^k. \tag{113}$$

Then (112) shows that (10) holds with

$$C_{S,n} = \left(2c(1+n)\right)^{\frac{n}{2}}, \qquad b_S(t) = \sqrt{1+4t^2}. \tag{114}$$

As we are assuming that $C_{R,n} \leq (c')^n$ and $b_R(t) = 1$ locally integrable, the condition (13) reads

$$\sum_{n \geq 0} \frac{\delta^n (1+n)^{\frac{n}{2}}}{n!} < \infty, \qquad \delta = c' \sqrt{2c} \int_0^t b_R(s) \sqrt{1+4s^2} ds.$$

The ratio test shows that the series converges absolutely for all values of the parameters. Finally, suppose $\rho_S = \sum_{\alpha \geq 0} p_\alpha |\psi_\alpha\rangle\langle\psi_\alpha|$ such that each ψ_α satisfies (113). Then

$$\left\| G(t_{\ell_1}) \cdots G(t_{\ell_L}) \rho_S G(t_{\ell_{L+1}}) \cdots G(t_{\ell_n}) \right\|_1 \leq \sum_{\alpha \geq 0} p_\alpha \left\| G(t_{\ell_1}) \cdots G(t_{\ell_L}) |\psi_\alpha\rangle\langle\psi_\alpha| G(t_{\ell_{L+1}}) \cdots G(t_{\ell_n}) \right\|_1 \leq C_{S,n} b_S(t_1) \cdots b_S(t_n)$$

for the same choice of constants (114). This completes the proof of Proposition 7. □

CRediT authorship contribution statement

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Data availability

No data was used for the research described in the article.

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