Bose – Einstein Condensate of Ultra-Light Axions as a Candidate for the Dark Matter Galaxy Halos

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Abstract

We suggest that the dark matter halo in some of the spiral galaxies can be described as the ground state of the Bose-Einstein condensate of ultralight self-gravitating axions. We have also developed an effective "dissipative" algorithm for solution of non-linear integro-differential Schrödinger equation describing self-gravitating Bose-Einstein condensate. The mass of an ultra-light axion is estimated.

Keywords: mean-field, dark matter, axions, ground state, density distribution

I. Introduction

The content and properties of the dark matter (DM) is one of the most pressing problems in contemporary physics. According to the Planck 2018 data the mass of the DM is more than five times greater than the mass of the baryonic matter in the visible Universe [1]. In particular, DM is expected to form a huge halo around the spiral galaxies (see, for example, [2, 3]). There are various theoretical models describing the DM. We restrict ourselves with models suggesting that the DM is a Bose-Einstein condensate (BEC) of the ultralight axions (see, for example, [4,5]). In particular, we do not consider a core-envelope model where part of the ultralight axions forms a dense BEC core, while the other part forms a low density quasi-classical envelope (see, for example, [6]). In our work, we suggest that the DM in some spiral galaxies can be described as the ground state of the ultralight self-gravitating axions, and estimate the mass of the axion.

The non-linear integro-differential equation, considered in our work, is equivalent to the Schrödinger-Newton equations studied in many publications (e.g. [7-9]). The standard methods of numerical computations allow one to compute the stable ground state and the unstable excited states of these equations [9]. We have developed an independent numerical method (the "dissipation algorithm") which we have used for verification of the solution of the Schrödinger-Newton equations. Our method can also be used for studying the two- and many-component BECs. This work is now in progress.

II. Non-Linear Integro-Differential Schrödinger Equation and its Ground State

***N*-particle Schrödinger equation**. We start with the non-relativistic Schrödinger equation, $i\hbar\partial\psi\left(\vec{r}_1,...,\vec{r}_N;t\right)/\partial t = H_N\psi\left(\vec{r}_1,...,\vec{r}_N;t\right)$, for N gravitationally interacting axions, described by the Hamiltonian, $H_N = -\left(\hbar^2/2m\right)\sum_{i=1}^{N}\Delta_{\vec{r}_i} - Gm^2\sum_{i<j=1}^{N}\left(1/\left|\vec{r}_i-\vec{r}_j\right|\right)$, where m, $\vec{r}_i$, G are the axion mass, coordinate, and the gravitational constant, correspondingly. Introduce the dimensionless variables: $\vec{x}_i = \vec{r}_i/r_0$, $\tau = \left(\hbar/mr_0^2\right)t$. Then, the dimensionless Schrödinger equation becomes: $i\,\partial\psi\left(\vec{x}_1,...,\vec{x}_N;\tau\right)/\partial\tau = \mathrm{H}_N\psi\left(\vec{x}_1,...,\vec{x}_N;\tau\right)$, where the dimensionless Hamiltonian takes the form, $\mathrm{H}_N \equiv \left(mr_0^2/\hbar^2\right)H_N = -\left(1/2\right)\sum_{i=1}^{N}\Delta_{\vec{x}_i} - \frac{1}{N}\sum_{i<j=1}^{N}\left(1/\left|\vec{x}_i-\vec{x}_j\right|\right)$, and we have chosen: $r_0 = \left(\hbar^2/Gm^2M\right)$, $M = mN$.

Mean-field approach. The dimensionless Hamiltonian, H_N, coincides with the dimensionless Hamiltonian, H_N in [10], where it was shown that the mean-field (MF) approach can be used when $N\to\infty$. For N interacting axions, we can consider N very large and $M = mN$ fixed (as the axion mass is tiny). In the MF approach, one considers the initially disentangled state of N identical particles, with $\psi\left(\vec{x}_1,...,\vec{x}_N;0\right) = \psi\left(\vec{x}_1,0\right)\otimes\cdots\otimes\psi\left(\vec{x}_N,0\right)$. It is shown in [10], that for any fixed τ and $N\to\infty$, the wave function is: $\psi\left(\vec{x}_1,...,\vec{x}_N;\tau\right) = \psi\left(\vec{x}_1,\tau\right)\otimes\cdots\otimes\psi\left(\vec{x}_N,\tau\right)$. The equation for a single-particle wave function, $\psi\left(\vec{r},t\right)$, back in the dimensional variables, becomes the nonlinear integro-differential equation:

$$i\hbar\frac{\partial\psi\left(\vec{r},t\right)}{\partial t} = \left(-\frac{\hbar^2}{2m}\Delta_{\vec{r}} - GmM\int\frac{|\psi(r',t)|^2\,d^3\vec{r}'}{|\vec{r}-\vec{r}'|}\right)\psi\left(\vec{r},t\right).$$

The corresponding eigenvalue problem, in the MF approximation, can be formulated as follows. We assume that all axions are in the same spherically symmetrical ground state which is described by the real wave function, $\psi(r)$. The wave function satisfies the non-linear stationary Schrödinger equation,

$$\left(-\frac{\hbar^2}{2m}\Delta_{\vec{r}} + mV(\vec{r})\right)\psi = E\psi\,, \tag{1}$$

where m is the axion mass, and $V\left(\vec{r}\right)$ is the gravitational potential produced by all the axions of the galaxy,

$$V(\vec{r}) = -GM\int\frac{|\psi(r')|^2\,d^3\vec{r}'}{|\vec{r}-\vec{r}'|}\,. \tag{2}$$

Here M is the total mass of the axions. (We assume that the mass of the baryonic matter is much smaller than M, and it can be ignored in the first approximation.)

Similar to above, we introduce the following natural parameters,

$r_0 = \dfrac{\hbar^2}{Gm^2M}$ – characteristic size,

$\vec{x} = \dfrac{\vec{r}}{r_0}$ – dimensionless coordinate,

$\tau = \dfrac{\hbar}{mr_0^2}t$ – dimensionless time,

$\phi(x) = r_0^{3/2}\psi(r)$ – dimensionless wave function,

$\varepsilon = \dfrac{mr_0{}^2}{\hbar^2}E$ - dimensionless energy

Now, we obtain the dimensionless equation,

$$-\left\{\frac{1}{2}\Delta_{\vec{x}} + \int d\vec{x}'\frac{\phi^2(x')}{|\vec{x}-\vec{x}'|}\right\}\phi(x) = \varepsilon\phi(x). \tag{3}$$

Integrating over polar and azimuthal angles and taking into account that,

$$\int_0^{\pi} d\theta \cdot \frac{\sin\theta}{\sqrt{x^2 - 2xx'\cos\theta + x'^2}} = \begin{cases} 2/x, \ x>x', \\ 2/x', \ x<x', \end{cases} \tag{4}$$

we obtain the integro-differential equation of the Hartree-Fock type,

$$-\frac{1}{2x}\frac{\partial^2}{\partial x^2}\left(x\phi(x)\right) - \frac{4\pi}{x}\phi(x)\int_0^x s^2\phi^2(s)ds - 4\pi\phi(x)\int_x^{\infty} s\phi^2(s)ds = \varepsilon\phi(x). \tag{5}$$

Also, we add two standard conditions,

$$\phi(x=\infty) = 0, \quad 4\pi\int_0^{\infty} x^2\phi^2(x)dx = 1. \tag{6}$$

Using a substitution,

$$\varphi(x) = \sqrt{4\pi}\,x\phi(x), \tag{7}$$

we obtain the following equations,

$$
\begin{aligned}
&\varepsilon\varphi(x) = -\frac{1}{2}\frac{\partial^2\varphi(x)}{\partial x^2} + U(x)\varphi(x),\\
&U(x) = -\frac{1}{x}\int_0^x \varphi^2(s)ds - \int_x^\infty \frac{\varphi^2(s)}{s}ds,\\
&\varepsilon = \frac{1}{2}\int_0^\infty \left(\frac{\partial\varphi(x)}{\partial x}\right)^2 dx + \int_0^\infty U(x)\varphi^2(x)dx,\\
&\varphi(0) = \varphi(\infty) = 0,\\
&\int_0^\infty \varphi^2(x)dx = 1.
\end{aligned} \tag{8}
$$

III. Numerical solution

We have developed an effective computer algorithm which can be used for solving equation (8), and other similar problems. The solution has been found on the basis of a discrete iterative procedure which is a proper "dissipative protocol" (DP) of the effective dynamical Schrödinger equation,

$$
i\frac{\partial\varphi(x,\tau)}{\partial\tau} = -\frac{1}{2}\frac{\partial^2\varphi(x,\tau)}{\partial x^2} + \left[U(x) - \varepsilon\right]\varphi(x,\tau). \tag{9}
$$

The numerical solution of Eq. (9) is defined in some finite region, $0 \le x \le R$, such that: $\varphi(x=0,\tau) = \varphi(x=R,\tau) = 0$. It is important to note that, for a given initial condition, $\varphi(x,\tau=0)$, we are looking not for an "exact" numerical solution based, for example, on the well-known Crank–Nicolson finite difference method [11], which is a second-order method in time and space. (In this case, the errors of the finite difference scheme of the left part is $\sim(\Delta\tau)^2$, and for the right part is $\sim(\Delta x)^2$, where $\Delta\tau$ and Δx are the discretization steps in time and space, correspondingly.) We apply the implicit finite-difference method of the first order in time,

$$
i\frac{\varphi_i^{j+1} - \varphi_i^j}{\Delta\tau} = -\frac{1}{2}\frac{\varphi_{i+1}^{j+1} - 2\varphi_i^{j+1} + \varphi_{i-1}^{j+1}}{(\Delta x)^2} + \left[U(x_i) - \varepsilon\right]\varphi_i^{j+1}, \tag{10}
$$

where $\varphi_i^j \equiv \varphi(x_i,\tau_j)$. As it is shown below, the scheme (10) generally does not conserve the probability, $\int_0^R \varphi^2(x,\tau)dx$. However, it is a valuable advantage, as compared to the conservative Crank–Nicolson method because, on the basis the difference scheme (10), an iterative algorithm can be built to reach the steady state for Eq. (9), $\partial\varphi(x,\tau)/\partial\tau = 0$, or, in other words, to obtain the solution of the eigenvalue problem (8).

The point is that for given potential, $U(x)$, and energy, ε, the procedure (10) exhibits the "filter" properties in the dynamics of an arbitrary initial condition, $\varphi(x,\tau=0)$. Indeed, decomposing the initial condition into eigenfunctions, $\phi_k(x)$, of the operator, $\hat{H} = -\frac{1}{2}\frac{\partial^2}{\partial x^2} + U(x)$, we obtain,

$$\varphi(x,\tau=0)=\sum_k c_k\phi_k(x). \tag{11}$$

Then, the exact solution of Eq. (9) is,

$$\varphi(x,\tau)=\sum_k c_k \exp(i\omega_k\tau)\phi_k(x), \tag{12}$$

where $\omega_k=\varepsilon-\varepsilon_k\equiv\Delta\varepsilon_k$ and ε_k is the eigenvalue of $\hat{H}$. However, following the difference scheme (10),

$$i\phi_{k,i}\frac{e^{i\omega_k(\tau+\Delta\tau)}-e^{i\omega_k\tau}}{\Delta\tau}=(\varepsilon_k-\varepsilon)\phi_{k,i}e^{i\omega_k(\tau+\Delta\tau)}, \tag{13}$$

we obtain another relation for determining the frequency, ω_k:

$$\omega_k\approx\Delta\varepsilon_k+i\frac{(\Delta\varepsilon_k)^2\Delta\tau}{2}. \tag{14}$$

That is, instead of the non-dissipative time-dependent solution, we get the "filter",

$$\varphi(x,\tau)=\sum_k c_k e^{i\Delta\varepsilon_k\tau}e^{-\gamma_k\tau}\phi_k(x),\ \gamma_k=(\Delta\varepsilon_k)^2\Delta\tau/2, \tag{15}$$

that extracts from the entire set of the eigenfunction, $\{\phi_k(x)\}$, the modes with the lowest values of $|\Delta\varepsilon_k|$. Thus, if at a given potential, $U(x)$, the eigenvalue of energy, ε_m, is known, then the scheme (10), which only imitates the solution of Eq. (9), transforms over time an arbitrary initial function, $\varphi(x,\tau=0)$, into a stationary state wave function,

$$\frac{\partial\varphi(x,\tau)}{\partial\tau}=0,\ \varphi(x,\tau)=f(x)e^{i\beta}, \tag{16}$$

where the phase, β, does not depend on x, and either the function $\mathrm{Re}[\varphi(x,\tau)]$ or $\mathrm{Im}[\varphi(x,\tau)]$ can be identified with the eigenfunction, $\phi_m(x)$, as if it is initially unknown.

The filtration properties (see Eq. (15)) of the scheme (10) can be used as basis for algorithms of different purposes. For instance, the problem (8) is solved using the following simple iterative procedure. Determine from the physical considerations the initial/trial "eigenfunction", $\varphi(x,\tau=0)$, energy, $\varepsilon^{(0)}$, and evaluate the corresponding potential $U^{(0)}(x)$ by using the relations (8). In accordance with scheme (10), calculate n time steps so that $\Delta\tau\times(n\Delta\tau)>>1$ (see Eq. (15)). Then recalculate the iteration values, $U^{(1)}(x)$ and $\varepsilon^{(1)}$, using in (8) the current normalized function $abs[\varphi^{(1)}(x,n\Delta\tau)]$ as if it is the genuine eigenfunction, and run the next series of n time steps. By running N such successive cycles, we obtain converging sequence not only for the energy, $\varepsilon^{(N)}$, but for the function $\varphi^{(N)}(x, N\times n\Delta\tau)$, too. As an example, for the trial wave function, $\varphi(x,\tau=0)=xe^{-x^2}$, $0\le x\le 40$, $\varepsilon^{(0)}=0$, $\Delta\tau=0.5$, $n=500$, the reliable stationary ground state, $\phi(x)=abs[\varphi^{(N)}(x)]/(\sqrt{4\pi}x)$, is achieved in $N=20$ large cycles: $\varepsilon_0\approx -0.1628$.

The attenuation of the expansion terms in Eq. (15) occurs with the decrement, γ_k. For the effective convergence of iterations, the quantity, $\gamma_k t$, should be large, $\gamma_k t \gg 1$ $(t=n\Delta\tau)$, which for small $\Delta\varepsilon_k$ requires the fulfillment of the condition $\Delta\tau(n\Delta\tau)\gg 1$. However, too large values of n lead to a strong attenuation of the wave function and a loss of accuracy of calculations, since the number of bits, representing numbers in the computer memory, is limited. Our solution is the same as that obtained in ref. [7], where the authors solved the coupled system of Schrödinger and Poisson equations. Thus, our computation confirms the result obtained in ref. [7].

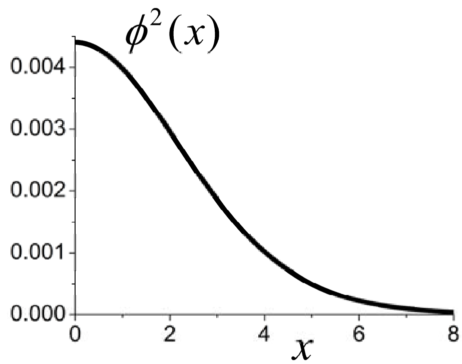

Fig. 1: The probability density, $\phi^2(x)$.

In Fig. 1, we present the probability density, $\phi^2(x)$. One can see that the "diameter of the function", $\phi^2(x)$, is about 16. Thus, according to our assumption the diameter, D, of a galactic halo is, $D\approx 16r_0$. Taking the typical diameter, D, and the mass, M, of a spherical halo from Ref. [2], $D=300$ kpc and $M=1.8\cdot 10^{12}M_{\odot}$, we obtain an estimation for the axion mass, m,

$$m=\frac{\hbar}{\sqrt{r_0 GM}}\sim 1.6x10^{-25}\,\text{eV}. \quad (17)$$

This estimation correlates with the current proposals (e.g. [4,12,13]) suggesting that the DM particles are the ultra-light axions. Our value of the axion mass is greater than the value, $m\sim 10^{-26}$ eV, estimated in [12], but smaller than $m\sim 10^{-22}$ eV estimated in [4,13]. We should note here that recent observations restrict the values of the mass of the DM axions (see, for example, [14-19]). In particular, the data obtained in [14-18], rule out axions with mass of the order of 10^{-25} eV as candidates for the universal DM. However, we believe that DM does not have to be a unique single-component substance across the Universe. The content of DM can be different in the galaxies of different type or even in the galaxies of the same type. Thus, we should not exclude the possibility that some of the spiral galaxies have the DM halo consisting of the axions of the mass estimated in our work.

Conclusion

We suggest that the DM halo of some of the spiral galaxies may consist of ultra-light self-gravitating axions in the ground BEC state. We have developed an effective "dissipative" numerical algorithm for computation of the ground state for non-linear integro-differential Schrödinger equation. We have estimated the mass of an ultra-light axion as, $m \sim 1.6x10^{-25}\,\mathrm{eV}$, which is comparable with many other proposals.

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