Applied Mathematics 2130
Lab 2009F–1: Elliptic coordinates

The goal of this laboratory is twofold. On the one hand, you will explore various mathematical facts about elliptic coordinates. On the other hand, you will learn how to produce mathematical graphics by means of programming and LaTeX.

Mathematical description

To explain what the elliptic coordinates are, we will begin with more familiar coordinate systems: Cartesian coordinates (x, y) and polar coordinates (r, θ). The relationship between the Cartesian and polar coordinates is

$$x = r\cos\theta, \qquad y = r\sin\theta. \tag{1}$$

To make the correspondence one-to-one, let us impose constraints: $r \geq 0$ and $0 \leq \theta < 2\pi$. The origin, whose Cartesian coordinates are $(0, 0)$, is exceptional: it corresponds to infinitely many pairs ($r = 0$, any θ) of polar coordinates.

In any coordinate system, *grid lines* are those where one of the coordinates has the same value and the other one is allowed to vary over the whole admissible range. In Cartesian coordinates, fixing the value of x, we obtain vertical lines. Every vertical line is described by the equation $x = c$, where c is a constant. The value of y is unrestricted. Different lines correspond to different values of c. Similarly, any horizontal line is also a Cartesian grid line; it is described by the equation $y = c$, where c is a constant. On a horizontal line, the value of x is unrestricted.

In polar coordinates, grid lines described by equations of the form $r = c$, where $c > 0$ is a constant, are concentric circles with their commont centre at the origin. The other family of grid lines is described by equations of the form $\theta = c$, where c is a constant in $[0, 2\pi)$. These lines are rays starting at the origin, with slopes determined by the value of c. (Why are they rays and not lines infinite in both directions? — Because points given by Equations (1) with the same value of θ always lie on the same side from the origin as r varies — recall our convention that $r \geq 0$.)

The elliptic coordinates (σ, ν) are related to the Cartesian coordinates by the equations

$$x = \frac{1}{2}\left(\sigma + \frac{1}{\sigma}\right)\cos\nu, \qquad y = \frac{1}{2}\left(\sigma - \frac{1}{\sigma}\right)\sin\nu. \tag{2}$$

The values of σ and ν are subject to the constraints $\sigma \geq 1$ and $0 \leq \nu < 2\pi$.

Visualization of the elliptic grid

The most important part of this assignment is **to visualize** the elliptic coordinate grid, that is, to produce a plot showing a family of lines described by equations $\sigma = c$ with several different constant values of c, as well as a family of lines described by equations $\nu = c$ with several different constant values of c. You should choose these constant values within the admissible ranges for σ and ν respectively; make an effort to choose them so as to obtain a grid that is neither too dense nor too sparse and as eye-pleasing as you can.

To produce the required plot, you must complete the following two steps:

(1) Write a program in a programming language of your choice to generate data files for the grid lines. Section 4.1.3 in Course Manual can be helpful. To compute the set of points on a grid line of the type $\sigma =$ const, fix a value of σ and run a loop over a varying value of ν, obtaining the coordinates of the points according to Eqs. (2). Generate several data sets of this type. Proceed similarly to generate data sets for grid lines of the type $\nu =$ const.

(2) Use one of the methods described in the Course Manual (Gnuplot, Postscript, or `join` command of enhanced LaTeX picture environment) to produce your graphics from the data. In this assignment you are not allowed to use Maple. All graphics produced should either be LaTeX or `eps`. The `gif`, `jpeg`, `pgn` etc. formats are not allowed. Borrowing graphics for your paper from the Internet is prohibited.

Further explorations

As a mandatory research part of the assignment, you should elaborate on the following statement:

The grid lines in the elliptic coordinates are ellipses and hyperbolas.

Define all the necessary concepts accurately; formulate a theorem; prove it.

There are many mathematical questions related to the elliptic coordinates, and many applications. An open-ended part of this assignment is to explore at least one of of them if you target a top mark for the research component of your project. Some possible topics to explore are suggested below:

- Why are the elliptic coordinates also known as *confocal* coordinates?
- The Cartesian, polar and elliptic coordinate systems are all known to be *orthogonal*. What precisely does it mean? How are angles between curves defined? Why are our ellipses and hyperbolas mutually orthogonal?
- Let us consider the following transformation of plane. Take any point P with Cartesian coordinates (x, y). Solving Eqs. (1) for r and θ, we find the polar coordinates of the point P. Now substitute $\sigma = r$, $\nu = \theta$ to Eqs. (2) to find the new point P', the image of the original point P. Knowing how to map points, we can map straight lines and curves. It is especially interesting to plot the images of lines and circles passing through the point $(1, 0)$ (in the Cartesian coordinates). A circle will be mapped into the so called *Joukowski airfoil*, an airplane-wing shaped curve!
- If you are comfortable with complex numbers, you may try to understand the elliptic coordinate grid as the polar coordinate grid subjected to the transformation $z \mapsto \frac{1}{2}(z + \frac{1}{z})$ of the complex plane.