M 3001

ASSIGNMENT 8: SOLUTION

Due in class: Wednesday, March 16, 2011

Name	**M.U.N. Number**

1. Prove: if $\sum_{n=1}^{\infty} f_n$ is convergent on D, then $\lim_{n\to\infty} f_n(x) = 0$ for each $x \in D$.

   Solution. This follows from

$$f_n(x) = \sum_{k=1}^{n} f_k(x) - \sum_{k=1}^{n-1} f_k(x), \quad \forall x \in D.$$

2. Prove that $\sum_{k=1}^{\infty} f_k$ is uniformly convergent on the given domain D for which:

   (i) $f_k(x) = \frac{\sin kx}{k^2}$, $D = (-\infty, \infty)$;

   (ii) $f_k(x) = \left(\frac{\tan x}{2}\right)^k$, $D = [0, \frac{\pi}{4}]$;

   (iii) $f_k(x) = kx^k$, $D = [-2^{-1}, 2^{-1}]$;

   (iv) $f_k(x) = \frac{k+1}{ke^{kx}}$, $D = [1, \infty)$.

   Solution. (i) Since $|\sin kx| \leq 1$ for all $x \in (-\infty, \infty)$, take $M_k = k^{-2}$ and apply the M-Test.

   (ii) Since $|\tan x| \leq 1$ for all $x \in [0, \frac{\pi}{4}]$, take $M_k = 2^{-k}$ and apply the M-Test.

   (iii) Since $|kx^k| \leq k2^{-k}$ for all $x \in [-2^{-1}, 2^{-1}]$, take $M_k = k2^{-k}$ and apply the Ratio Test to show that $\sum_{k=1} k2^{-k}$ is convergent, and then use the M-Test to get the result.

   (iv) If $x \geq 1$, then $\frac{k+1}{ke^{kx}} \leq 2e^{-k}$. Take $M_k = 2e^{-k}$; $\sum_{k=1}^{\infty} e^{-k}$ is a convergent geometric series, and apply the M-Test.

3. By considering $f(x) = x^{-1}$ on $(0, 1)$, prove that the Weierstrass Approximation Theorem would not hold if $[a, b]$ were replaced by (a, b).

   Solution. If we take $\epsilon = 1$, we wish to show that no polynomial P can satisfy

$$|P(x) - x^{-1}| < 1, \quad \forall x \in (0, 1). \tag{1}$$

   Since the polynomial P is continuous, it is bounded on $[0, 1]$. Consequently, it is also bounded on $(0, 1)$, say $\max_{x\in[0,1]} |P(x)| \leq M$. But this inequality and (1) together yield $x^{-1} < 1 + M$ for every $x \in (0, 1)$, which contradicts the unboundedness of x^{-1} on $(0, 1)$.