M 3001

ASSIGNMENT 10: SOLUTION

Due in class: Friday, April 1, 2011

Name	**M.U.N. Number**

1. Prove that $f(x) = |x|$ does not have a MacLaurin series.

   Solution. This function fails to have derivatives of any order. So it cannot have a MacLaurin series.

2. Prove that the inverse tangent function is analytic on $(-1, 1)$ and for each $x \in [-1, 1]$

$$\arctan x = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}.$$

   (Hint: Expand $(1 + t^2)^{-1}$ as a geometric series and integrate.)

   Solution. Write

$$1 - t^2 + t^4 - \cdots + (-1)^n t^{2n} = \frac{1 - (-1)^{n+1} t^{2n+2}}{1 + t^2},$$

   or

$$1 - t^2 + t^4 - \cdots + (-1)^n t^{2n} - (1 + t^2)^{-1} = \frac{(-1)^{n+1} t^{2n+2}}{1 + t^2}.$$

   For an arbitrary $x \in [-1, 1]$, integrate on $[0, x]$ to get

$$\left| x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} - \arctan x \right| = \left| (-1)^{n+2} \int_0^x \frac{t^{2n+2}}{1+t^2} dt \right| < \int_0^1 t^{2n+2} dt = \frac{1}{2n+3}.$$

   Letting $n \to \infty$, we just reach the desired formula.

3. Find the Maclaurin series for $f(x) = x^{-1} \sin x$ and show that f is analytic on $(-\infty, \infty)$.

   Solution. Note that

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}, \quad \forall x \in (-\infty, \infty).$$

   So

$$x^{-1} \sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k}, \quad \forall x \in (-\infty, \infty).$$

4. Find the Maclaurin series for $f(x) = \int_0^x \sin t^2 dt$.

   Solution. Note that

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}, \quad \forall x \in (-\infty, \infty).$$

   So, substituting t^2 for x in this formula, we have

$$\sin t^2 = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} (t^2)^{2k+1}, \quad \forall t \in (-\infty, \infty).$$

   Now integrating term-by-term, we get

$$\int_0^x \sin t^2 dt = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \int_0^x (t^2)^{2k+1} dt = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \cdot \frac{x^{4k+3}}{4k+3}, \quad \forall x \in (-\infty, \infty).$$

5. Suppose that α is continuous at a point $c \in [a,b]$. If

$$f(x) = \begin{cases} 0 & \text{if} \quad x \neq c \\ 1 & \text{if} \quad x = c, \end{cases}$$

prove that f is in $\mathcal{R}(\alpha)$ on $[a,b]$ and that $\int_a^b f\,d\alpha = 0$.

Solution. Because $f(x) = 0$ for any $x \neq c$, $L(P,f,\alpha) = 0$ for any $P \in \mathcal{P}[a,b]$. Consequently, if $f \in \mathcal{R}(\alpha)$ on $[a,b]$, then $\int_a^b f\,d\alpha = \sup\{L(P,f,\alpha) : P \in \mathcal{P}[a,b]\} = 0$. Let $P = \{x_j\}_{j=0}^n$ be in $\mathcal{P}[a,b]$. If $c = a$, then $U(P,f,\alpha) = M_1\Delta\alpha_1 = 1\Delta\alpha_1 = \alpha(x_1) - \alpha(c)$; if $c = b$, then $U(P,f,\alpha) = \Delta\alpha_n = \alpha(c) - \alpha(x_{n-1})$; if $a < c < b$, then we first refine P whenever necessary so that $c = x_j$ for some $j \in \{1,2,...,n-1\}$ and then $U(P,f,\alpha) = \Delta a_j + \Delta\alpha_{j+1} = (\alpha(c) - \alpha(x_{j-1})) + (\alpha(x_{j+1}) - \alpha(c))$.

Given $\epsilon > 0$. Since α is continuous at c, there is a $\delta > 0$ such that if $x \in [a,b]$ with $|x-c| < \delta$, then $|\alpha(x) - \alpha(c)| < \epsilon/2$. Choose $P - \{x_j\}_{j=0}^n \in \mathcal{P}[a,b]$ with $c = x_j$ for some $j \in \{0,1,...,n\}$ and with $\|P\| < \delta$. Then $U(P,f,\alpha) - L(P,f,\alpha) = U(P,f,\alpha) < \epsilon$ in all three cases, and so $f \in \mathcal{R}(\alpha)$ on $[a,b]$.

6. Prove that if the sequence of functions $(f_n)_{n=1}^\infty$ converges uniformly to F on $[a,b]$ and if each f_n is in $\mathcal{R}(\alpha)$ on $[a,b]$, then F is in $\mathcal{R}(\alpha)$ on $[a,b]$ and $\int_a^b F\,d\alpha = \lim_{n\to\infty} \int_a^b f_n\,d\alpha$.

Solution. Without loss of generality, we may assume that α is not a constant function. Given $\epsilon > 0$. Choose $n \in \mathbf{N}$ such that $\sup_{x\in[a,b]} |f_n(x) - F(x)| < \epsilon(3(\alpha(b) - \alpha(a)))^{-1}$. For such an f_n choose $P = \{x_j\}_{j=0}^k \in \mathcal{P}[a,b]$ with $U(P,f_n,\alpha) - L(P,f_n,\alpha) < \epsilon/3$. Letting $M_j = \sup_{x\in[x_j,x_{j+1}]} F(x)$ and $M_{j,n} = \sup_{x\in[x_j,x_{j+1}]} f_n(x)$, we get

$$|U(P,F,\alpha) - U(P,f_n,\alpha)| \leq \sum_{j=1}^k |M_j - M_{j,n}|\Delta\alpha_j \leq \epsilon(3(\alpha(b) - \alpha(a)))^{-1} \sum_{j=1}^k \Delta\alpha_j = \epsilon/3.$$

Similarly, $|L(P,F,\alpha) - L(P,f_n,\alpha)| < \epsilon/3$. An application of the triangle inequality gives $|U(P,F,\alpha) - L(P,F,\alpha)| < \epsilon$, and so $F \in \mathcal{R}(\alpha)$ on $[a,b]$.

To prove $\int_a^b F\,d\alpha = \lim_{n\to\infty} \int_a^b f_n\,d\alpha$, let $\epsilon > 0$ and choose $N \in \mathbf{N}$ such that

$$\sup_{x\in[a,b]} |f_n(x) - F(x)| < \frac{\epsilon}{2(\alpha(b) - \alpha(a))} \quad \text{as} \quad n \geq N.$$

As a result, we get that under $n \geq N$,

$$\Big|\int_a^b F\,d\alpha - \int_a^b f_n\,d\alpha\Big| \leq \int_a^b |F - f_n|\,d\alpha \leq \frac{\epsilon}{2(\alpha(b) - \alpha(a))} \int_a^b d\alpha = \epsilon/2 < \epsilon,$$

as wanted.

7. Prove that if f and g are in $\mathcal{R}(\alpha)$ on $[a,b]$ then $|\int_a^b fg\,d\alpha|^2 \leq \int_a^b f^2\,d\alpha \int_a^b g^2\,d\alpha$.

Solution. Note that

$$0 \leq \int_a^b (xf + g)^2\,d\alpha = x^2 \int_a^b f^2\,d\alpha + 2x \int_a^b fg\,d\alpha + \int_a^b g^2\,d\alpha.$$

Letting

$$A = \int_a^b f^2\,d\alpha; \quad B = 2\int_a^b fg\,d\alpha; \quad C = \int_a^b g^2\,d\alpha,$$

we get $Ax^2 + Bx + C \geq 0$, whence $B^2 \leq 4AC$ that derives the required inequality.