Common Core Arithmetic for Teachers: the Essential Content

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Contents

Preface

The theory of the real numbers has two sources: the world of the discrete epitomized by the process of counting and the positive integers, and the world of the continuous epitomized by the process of geometric measurement and the real numbers. There is no conflict between these two approaches and all the numerical computations with integers or real numbers can be realized using the single real-world construct of the real line. The authors of the the Common Core State Standards in Mathematics (CCSS-M) clearly understood this and it was their intent that all children should have a fair understanding of how our number system arises from counting on the one hand, and measurement and geometry on the other. In addition, the CCSS-M expect that all children will be able to fluidly and accurately perform and apply the standard computations of arithmetic.

It will be argued by some, supported by existing test data, that to expect all, or even most, students to succeed at this level is fatuous. But consider the following fact:

> approximately 80% of children in high-performing countries achieve at the level reached by only 25% of North American students.

Clearly, the drafters were of the opinion:

> If they can do it, **why can't we?**

As discussed in Chapter 1 and again in Chapter 20, curricula in high-performing countries are **coherent and focused**. The CCSS-M are intended to produce a curriculum that is both coherent and focused. Comparative studies have shown that where states adopt curricula that are coherent and focused, children are much more successful and as noted above, approximately 80% of children in high-performing countries achieve what only 25% of North American students achieve. Thus, adopting a coherent and focused curricula should produce significantly higher levels of achievement for all students. Based on 40 years of experience teaching post-secondary mathematics, I believe the CCSS-M is a good approximation to a coherent and focused curricula.

For teachers to succeed in conveying these ideas to their students, they must thoroughly understand how these ideas fit together. As discussed in Chapter 1, many teachers know they need to upgrade their knowledge of arithmetic to succeed at the levels required by the new standards. Helping teachers acquire this knowledge in a comprehensive and thorough manner is what this book is about and it is my hope that the content will provide teachers with the information they need to succeed in implementing this curricula.

Acknowledgements

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Chapter 1

Teachers and the CCSS-M

The Common Core State Standards in Mathematics (CCSS-M) resulted from a process that began in 2009 under the auspices of the National Governors Association Center for Best Practices (NGA Center) and the Council of Chief State School Officers (CCSSO). The standards documents were released in June 2010 and after a careful review, the NGA Center and CCSSO assert that the standards are:[1]

- Reflective of the core knowledge and skills in English Language Arts and mathematics that students need to be college- and career-ready;
- Appropriate in terms of their level of clarity and specificity;
- Comparable to the expectations of other leading nations;
- Informed by available research or evidence;
- The result of processes that reflect best practices for standards development;
- A solid starting point for adoption of cross-state common core standards; and
- A sound basis for eventual development of standards-based assessments.

The positive sentiments expressed above can also be found in independent assessments as in the following taken from a working paper issued by the Education Policy Center at Michigan State University:

[1] See http://www.corestandards.org/about-the-standards/development-process/

> The adoption of the Common Core State Standards in Mathematics (CCSS-M) by nearly every state represents an unprecedented opportunity to improve U.S. mathematics education and to strengthen the international competitiveness of the American labor force.[2]

Successful implementation of the new standards does indeed represent a huge step in addressing the challenge of graduating students from high school that are both work and/or college ready. But, without doubt, successfully implementing the CCSS-M will be a challenge for teachers, for students, for administrators and for parents.[3]

In the remainder of this chapter we will review some of the data that led to the CCSS-M, the challenges that must be overcome to achieve successful implementation and the role of this book in that process.

1.1 The Data that Led to the Standards

The evidence that on graduation from high school and/or university many of our students have not learned what they need to know is substantial. For example:

1. Twenty-seven percent of Canadian university graduates are functionally illiterate as determined by an OECD standard.[4]

2. Fifty percent of US high school graduates will require remediation (usually in math) (UT 2013).[5]

3. According to ACT data, only one fourth of those tested were ready for college.[6]

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[2] *Implementing the Common Core State Standards for Mathematics: What We Know about Teachers of Mathematics in 41 States*, Leland Cogan, *et al.*, Education Policy Center at Michigan State University, **WP33**, 2013, available online. Hereafter referred to as WP33.

[3] *Implementing the Common Core State Standards for Mathematics: What Parents Know and Support*, Leland Cogan, *et al.*, Education Policy Center at Michigan State University, **WP34**, 2013, available online. Hereafter referred to as WP34.

[4] Most of the data discussed is associated with the US. However, this article about Canada is relevant: *Shocking Number Of Canadian University Grads Don't Hit Basic Literacy Benchmark*, **The Huffington Post Canada, Posted: 04/29/2014**.

[5] Uri Treisman, *Iris M Carl Equity Address: Keeping Our Eyes on the Prize*, NCTM, Denver, April 19, 2013. This address in a variety of formats can be found at: http://www.nctm.org

[6] See *The Common Core and the Common Good*, Charles M. Blow, NYT, 21 August, 2013. (CMB 2013)

The inescapable conclusion is that many students leaving high school in the US and Canada are simply not equipped to function in a work environment nor academically prepared for university. That this situation has existed for many years can be traced in the growing presence of remedial programs at various post-secondary institutions. Such programs have existed at some institutions for more than 40 years, but are now ubiquitous.

Aside from their profound effect on individuals, the results described above were also detected by international tests. In 2006, the US was ranked 25th by the OECD Programme for International Student Assessment (PISA) (UT 2013). Significantly, the list of higher ranked countries did not even include most of the highest rated Asian countries in terms of mathematics achievement! These results, together with other factors, led to the conclusion that to maintain its **international competitiveness**, the US would have to make changes in its school mathematics curriculum.[7] The result was the CCSS-M.[8] Clearly, a key question is:

> Can changes in the curriculum cure the problem?

The good news here is that what and how students are taught, in other words, the **curriculum** can have a significant effect on adult problem-solving performance.[9] [10]

1.1.1 Other Math Test Data

The National Center for Educational Statistics keeps track of all the global data sets that bear on the quality of education.[11] Thanks to the Internet, this data is accessible to anyone with an interest in educational issues.[12] Information on the

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[7]Unfortunately, Canada was not so far down on this list as to provoke a response. Canadian scores went down in the last PISA round with still no response.

[8]More information of the generation of the CCSS can be found at http://www.corestandards.org/in-the-states

[9] *The Myth of Equal Content*, W. Schmidt and L. Cogan, 2009. Hereafter, S&C 2009. Available online at http://www.ascd.org/publications/educational-leadership/nov09/vol67/num03/The-Myth-of-Equal-Content.aspx

[10]Two articles on how math learning in childhood affects problem-solving skills in later life. http://www.nature.com/neuro/journal/vaop/ncurrent/full/nn.3788.html http://www.medicaldaily.com/math-skills-childhood-can-permanently-affect-brain-formation-later-life-298516

[11]This is a site all teachers and public school administrators should be aware of. For example, data archived here was analyzed and serves as the evidential basis for *The Public School Advantage*, C. Lubienski and S. Lubienski, 2014, available from Amazon. Hereafter PSA.

[12]See http://nces.ed.gov/ for the main site.

most recent PISA[13] is there as is data from the National Assessment of Educational Progress.[14] The NAEP is described at the site as

> the largest nationally representative and continuing assessment of what America's students know and can do in various subject areas.

Every two years students are assessed in Grades 4, 8 and 12. It is data from grades 4 and 8 that is relevant here.

According to data at the NAEP website, the average score on this assessment of mathematics achievement by Grade 8 students in 2011 was 284 . The score that represents proficiency is 299 (see UT 2013) and only 35% of all Grade 8 students were deemed proficient.[15] The reader may be tempted to conclude that this poor result is because the sample includes **all** students.[16] In fact, when the sample is restricted to private school students more than half the students still fail to achieve the proficiency score.[17]

As noted, assessments are also done in Grade 4.[18] The Grade 4 results for 2011 show only 40% of all students were found to be proficient. Although performance by students at private schools was better, still less than half are proficient.

Some may believe that the reason student performance appears dismal is due to the difficult nature of the test questions and that only mathematically gifted students could be expected to perform well. To address this possibility, let's look at some questions. While complete test instruments are not available, sample questions can be found for all test levels at the NAEP website.[19] The questions demand little more than recall and I would expect that every teacher of math at any level would agree that the questions are straightforward and that the only errors that should occur would be due to carelessness, and not a misunderstanding of the material being tested.

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[13] See http://nces.ed.gov/pubsearch/pubsinfo.asp?pubid=2014028

[14] See http://nces.ed.gov/nationsreportcard/

[15] See http://nationsreportcard.gov/math_2011/summary.aspx and click on Grade 8 in **Proficient** paragraph.

[16] *The Public School Advantage* (available from Amazon) completely destroys the myth that private schools provide better and more effective education. In fact, as shown in this work, American public schools do a more effective job educating the students that are placed in their care, a fact that every public school teacher and administrator should know.

[17] The situation in Canada is no better. On a curriculum assessment in the province of Newfoundland during the time I was head, more than half of all students in Grade 9 received a mark of less than 50%.

[18] See http://nationsreportcard.gov/math_2011/summary.aspx and click on Grade 4 in **Proficient** paragraph.

[19] See http://nationsreportcard.gov/ltt_2012/sample_quest_math.aspx

In briefest summary, the data show a majority of students begin accumulating knowledge deficits in mathematics prior to Grade 4 and this accumulation continues throughout the school career until graduation from high school. The OECD evidence from Canada shows these deficits are not repaired by the time of graduation from university.

In terms of the everyday work of classroom teaching in Grade 4 and above, these results show that in the course of a teaching year, there will certainly be topics which

> **the majority of students in the classroom are not ready to learn**.

Obviously, the presence of a large number of students not ready to learn what a teacher is trying to teach would have a substantial negative impact on that teacher's ability to teach that material effectively, even to students who are ready.[20] The question is:

> Is the CCSS-M a solution to the ready to learn year-by-year problem?

To answer this question, we need to examine how existing curricula contribute to this situation.

1.2 Mile Wide-Inch Deep

The Education Policy Center (EPC) at Michigan State University did a study of existing district math curricula in 41 states that have adopted the CCSS-M.[21] Among the key findings in respect to the primary curriculum were that

- there was little common agreement between districts as to when topics were taught;
- most topics were taught earlier than intended in the CCSS-M;
- many topics were taught in later grades than specified in the CCSS-M;
- for almost all topics, coverage extended over several grades;

[20] In respect to teacher effectiveness, *The Public School Advantage* bears witness to the incredibly fine job being done by public school math teachers all across America.

[21] *Implementing the Common Core State Standards for Mathematics: A Comparison of Current District Content in 41 States*, Leland Cogan, *et al.*, Education Policy Center at Michigan State University, **WP32**, 2013, available online. Hereafter referred to as WP32.

- many more topics were taught in each grade than were specified in the CCSS-M.

WP32 describes these curricula as being a *mile wide and an inch deep*, an assessment previously expressed about U.S. mathematics curricula, particularly in the primary years, in S&C 2009.[22]

Since addressing this descriptor of previous U.S. curricula is a critical feature the CCSS-M, we explore its meaning in greater detail. In a 2002 paper, Schmidt et al.[23] discuss an analysis of the Third International Math and Science Study (TIMSS) results on a country-by-country basis. The focus of their study is on whether and/or how an individual country's math curricula affects TIMSS performance. In their paper they make the following four observations about the then extant U.S. mathematics curricula (ACC, p. 3):

1. "Our intended content is not focused. If you look at state standards, you'll find more topics at each grade level than in any other nation. If you look at U.S. textbooks, you'll find there is no textbook in the world that has as many topics as our mathematics textbooks, bar none. ... And finally, if you look in the classroom, you'll find that U.S. teachers cover more topics than teachers in any other country.

2. Our intended content is highly repetitive. We introduce topics early and then repeat them year after year. To make matters worse, very little depth is added each time the topic is addressed because each year we devote much of the time to reviewing the topic.

3. Our intended content is not very demanding by international standards.

4. Our intended content is incoherent. Math, for example, is really a handful of basic ideas; but in the United States, mathematics standards are long laundry lists of seemingly unrelated, separate topics."

While this indictment was written in 2002, the analysis presented in WP32 shows it remains true about the various curricula being replaced in the states and districts that are in the process of adopting the CCSS-M.

[22] This descriptor can be traced in the literature back to at least 1997, e.g., *A splintered vision: An investigation of U.S. science and mathematics education*, W.H. Schmidt, *et al.* (available from Amazon). I believe I heard this descriptor applied to mathematics curricula in Canada in the early 1990's.

[23] *A Coherent Curriculum: The Case of Mathematics*, W. Schmidt, R. Houang and L. Cogan, **American Educator**, Summer 2002; available on-line. Hereafter ACC. This paper is a very worth-while read for all teachers and administrators engaged in implementing the CCSS-M. A table derived from their A+ curricula is presented below and again in Chapter 20 of this work.

In respect to the above observations about curricula in mathematics, ACC draws the following conclusion (p. 3):

> Our teachers work in a context that demands that they teach a lot of things, but nothing in-depth.

That such an assessment should have negative consequences for student success should not be surprising to educators in mathematics. At every level, learning requires sufficient time-on-task to permit the internal changes to occur in children's (and presumably adults') brain structures that are required as part of the learning process.[24]

Thus, one conclusion that might be drawn from the NAEP test data is that a *mile wide-inch deep* curricula simply will not permit a significant number of children to learn what they are being asked to learn.

1.3 The CCSS-M Response to Mile Wide-Inch Deep

To understand how the CCSS-M addresses the problem of *mile wide-inch deep*, we need to take a deeper look at the the Standards. We begin by reviewing some of what the developers wrote about their task:

> For over a decade, research studies of mathematics education in high-performing countries have pointed to the conclusion that the mathematics curriculum in the United States must become substantially more **focused and coherent** in order to improve mathematics achievement in this country. To deliver on the promise of common standards, the standards must address the problem of a curriculum that is a mile wide and an inch deep. These Standards are a substantial answer to that challenge (emphasis mine). CCSS-M, p. 3.[25]

In respect to what it means to be **focused**, the CCSS-M offer the following (p. 3):

> It is important to recognize that fewer standards are no substitute for focused standards. Achieving fewer standards would be easy to do by resorting to broad, general statements. Instead, these Standards aim for **clarity and specificity** (emphasis mine).

[24] See http://www.nature.com/neuro/journal/vaop/ncurrent/full/nn.3788.html

[25] The CCSS-M document is available at http://www.corestandards.org/wp-content/uploads/Math_Standards.pdf and can be obtained by anyone.

Reviewing the analysis of the CCSS-M in WP32 (see WP32 Display 1, p. 4) indicates that on a grade-by-grade basis, the CCSS-M concentrate on a narrow set of topics presented in an order from the particular to the complex. In comparison, extant state and district curricula still appear as a laundry-list[26] of topics. Moreover, in respect to individual topics, the learning objectives for children, as articulated by the CCSS-M, appear to be clear and specific (see CCSS-M, pp 9-84).

In respect to the notion of coherence, the CCSS-M turns to Schmidt *et al.*:

> We define content standards and curricula to be coherent if they are articulated over time as a sequence of topics and performances that are logical and reflect, where appropriate, the sequential or hierarchical nature of the disciplinary content from which the subject matter derives. That is, what and how students are taught should reflect not only the topics that fall within a certain academic discipline, but also the key ideas that determine how knowledge is organized and generated within that discipline.
>
> This implies that to be coherent, a set of content standards must evolve from particulars (e.g., the meaning and operations of whole numbers, including simple math facts and routine computational procedures associated with whole numbers and fractions) to deeper structures inherent in the discipline. This deeper structure then serves as a means for connecting the particulars (such as an understanding of the rational number system and its properties). The evolution from particulars to deeper structures should occur over the school year within a particular grade level and as the student progresses across grades (ACC, p. 9).

Again, deciding whether this objective was achieved is a matter of reviewing the analysis presented in WP32 (see Display 1, p. 4 of ACC and the A+ curricula in Table 1 below which is adapted from ACC) and comparing it to one's own understanding of the deep structure of the field of real numbers. On this basis, I conclude that the authors of the CCSS-M did indeed produce a coherent set of standards for Grades K-8.[27]

[26] ACC, p. 12.

[27] My focus is on primary and elementary because, in my view and experience, K-6 has always been the critical area. Knowledge of real numbers and arithmetic are key. Get that right and the rest will almost take care of itself.

TOPIC & GRADE:	1	2	3	4	5	6
Whole Number Meaning	•	•	•	○	○	
Whole Number Operations	•	•	•	○		
Common Fractions			□	•	•	○
Decimal Fractions				○	•	○
Relationship of Common & Decimal Fractions				○	•	○
Percentages					○	○
Negative Numbers, Integers & Their Properties						□
Rounding & Significant Figures				○	○	
Estimating Computations				○	○	○
Estimating Quantity & Size				□	□	
TOPIC & GRADE:	1	2	3	4	5	6
Equations & Formulas			□	○	○	○
Properties of Whole Number Operations				□	○	
Properties of Common & Decimal Fractions					○	○
Proportionality Concepts					○	○
Proportionality Problems					○	○
TOPIC & GRADE:	1	2	3	4	5	6
Measurement Units	□	•	•	•	•	•
2-D Geometry: Basics			□	○	○	○
Polygons & Circles				○	○	○
Perimeter, Area & Volume				○	○	○
2-D Coordinate Geometry					○	○
Geometry: Transformations						○
TOPIC & GRADE:	1	2	3	4	5	6
Data Representation & Analysis			□	□	○	○

Table 1. This table is adapted from ACC. Topics have been reorganized to reflect domains identified in the CCSS-M and only grades 1-6 are shown. Topics identified with a • are in the intended curricula of all the A+ countries in the grade shown; ○ identify 80% of A+ countries; □ identify 67% of A+ countries. Topics not on this list are **not** in the intended curricula of A+ countries in Grades 1-6!

To summarize, the Standards for K-8 are:

- narrowly focused in respect to topics covered on a year-by-year basis;
- reflect the natural development of mathematics as a discipline;

- expect individual standards to be ”introduced and taught to mastery all during a single school year” (WP32, p. 11);
- demand the fluidity with computations necessary for the work place and/or further academic work;
- develop mathematical ideas on a hierarchical basis from simple to complex;
- require a deeper knowledge of our numeration system and its role in computations than in previously extant U.S. curricula;
- intend that computations and principles presented in the curricula to be learned sufficiently well that they will be usable on a life-long basis;
- drive the development of learning on the basis of key mathematical ideas.

In short, students who achieve competency as specified in the CCSS-M will succeed from a mathematical perspective.

1.4 How Teachers Enact Curricula

In their analysis of TIMSS data, Schmidt *et al.* looked for effects of curricula on student performance.

> One of the most important findings from TIMSS is that the differences in achievement from country to country are related to what is taught in different countries (ACC, p. 2).

In analyzing the TIMSS data, the authors distinguished between *intended* curricula — what is in curriculum documents — and *enacted* curricula — what teachers actually teach in their classrooms (ACC, p. 3). Their analysis found:

> ... that in most countries studied, the intended content that is formally promulgated (at the national, regional, or state level) is essentially replicated in the nation’s textbooks. We can also say that in most countries studied, teachers follow the textbook. By this we mean that they cover the content of the textbook and are guided by the depth and duration of each topic in the textbook. From this knowledge, we can say with statistical confidence that what is stated in the intended content (be it a national curriculum or state standards) and in the textbooks is, by and large, taught in the classrooms of most TIMSS countries (ACC, p. 3).

Since teachers are the ultimate arbiters of whether the CCSS-M will be enacted, the EPC surveyed some 12,000 teachers in CCSS-M states. The results of this survey form the database studied in WP33.

In respect to how teachers determine what to teach, WP33 asserts

> Perhaps as a result of emphasis on standards in the past decade or more teachers reported that their classroom teaching was primarily influenced by standards rather than their textbook ..., (WP33, p. 2).

After noting that *For the most part, textbooks still embody the distinctive "mile wide-inch deep" curriculum* ..., WP33 makes the further observation that is relevant in this context:

> The triage required in deciding among the competing curriculum vision presented by the CCSSM and textbooks is particularly problematic for primary grades teachers as they are the least well prepared mathematically and, consequently, to make these critical decisions (WP33, p. 3).[28]

These observations provide a clear message to primary school administrators at the state and local level: textbooks consistent with the CCSS-M will be necessary for successful implementation. Where such textbooks are not available, other means for supporting teachers in the process of implementation must be provided.

Because the data at the NAEP site[29] has such profound consequences for teachers and students, we go through it again. Sixty percent of Grade 4 students are not proficient in math. By Grade 8, the number has reached 65%.

How should we, as educators, respond to this situation? In discussions with classroom teachers at every grade level I have heard the following view expressed:

> **there is no point in responding because these children are incapable of learning math.**

Those who hold this view communicate it to the children in their classroom about whom it is held. Worse yet, I have heard this view expressed by senior administrators and math consultants. It can be properly expressed as:

> Blame the child![30]

[28]WP33 is quoting *Foundations for Success: The Final Report of the National Mathematics Advisory Panel*, U.S. Department of Education, 2008, 120pp.

[29]See http://nces.ed.gov/nationsreportcard/

[30]Or an alternative: Some kids just can't do math. All these are excuses for a failure to teach.

There are multiple grounds on which to confront the notion that a large percentage of children are unable to succeed with the math curriculum. For example, based on my 40 years experience as a teacher of math, at the level of arithmetic, indeed of calculus, there is no **math gene** that makes some kids successful while others are not. In my experience, what determines success is whether a student knows and can use the prerequisite material. Clearly evidence based on my personal experience is anecdotal. However, there is TIMSS data that bears on the question. Consider the following:

> ... a comparison of mathematics scores in 22 countries revealed that U.S. eighth-graders who scored at the 75th percentile were actually far below the 75th percentile in 19 of the other countries. The most dramatic results were in comparison to Singapore — a score at the 75th percentile in the U.S. was below the 25th percentile in Singapore (ACC, p. 2).

ACC concludes from this that what is considered above average in the U.S. is far below average in high-performing countries. But think about this as information about the children in high-performing countries. What this is telling us is that their children — 80% for Singapore — achieve at a standard of which every North American parent would be proud.[31]

1.4.1 The CCSS-M and Individual Competency Issues

As already noted, the CCSS-M is narrowly focused and expects most individual standards to be introduced and taught to mastery in one school year (WP32). From a teaching perspective, this has the effect of making much more time available for Core topics, in particular, numbers and computations with numbers. Indeed, this narrowed focus of the CCSS-M in comparison to their own state's extant standards was one of the features that surveyed teachers gave as a reason for liking the CCSS-M (WP33, p. 4).

By narrowing the focus and treating fewer topics each year, the CCSS-M provides the time necessary for children to achieve mastery of these topics. That the CCSS-M is the same across states and districts means that teachers can expect a more uniform group of students in their classrooms, further raising the probability

[31] I've had students from Singapore in class and they are a delight to teach. One such student was in a 2nd year calculus class and was easily the best student in the class. He told me his grades did not qualify him to get into a Singapore university, so he came to N. America. The point is, this student was considered second rank in Singapore by his own description.

that each child will achieve at the standard identified. This is another positive feature of the CCSS-M identified by teachers (WP33, p. 4).

The analysis of TIMSS data in ACC shows definitively that **what students learn is what is in the curriculum**.[32] Thus, successful implementation of a coherent and focused curriculum like the CCSS-M will have a positive effect on student performance, both on assessments like TIMSS and also on preparation for the work-place and/or college.

1.5 What Parents Say

In WP34 the EPC reports on a 2011 survey of parental attitudes in respect to the CCSS-M. This paper also includes information from earlier surveys. In briefest summary, parents support education of their children and the teachers and schools that educate them. They understand the importance of education and want education to be protected in times of budget stringency.

In respect to statements about math that could be applied directly to their own children, the survey reported levels of agreement of more than 75% with the statements. For example (WP34, p. 6):

- Any child can learn math if they have a good teacher (87%).
- Any child can learn math if they have a good curriculum (85%).
- All children in grades 1-8 should study the same mathematics (79%).

The first two statements suggest parents have high expectations in respect to their own children. That these expectations are not unreasonable is shown by the fact that in countries identified by ACC that have A+ curricula and well-trained primary math teachers, 80% of students perform at levels achieved by only 25% of North American students. The last of the statements, together with similar statements, are clear expressions of support for a common curriculum on the part of parents. However, WP34 expresses concern about continued support as follows:

> However, what happens when their children find the math harder and more fail, especially on the first CCSSM assessment, remains to be seen (WP34, p. 6).

Seeing that such concerns are well-founded is only a matter of keeping track of discussion in the public media. To enhance a child's chance of success with the higher standards, parents need to be engaged as participants in their child's education which we discuss further below.

[32] ACC p. 3.

1.6 Rote-learning and the CCSS-M

We have referred several times to research on cognitive development. How human beings learn and represent learning internally in the brain are major research areas for modern psychologists. In respect to mathematics, a key focus is how children learn to solve problems and how problem-solving methods evolve as the brain matures. Cognitive psychologists distinguish two problem-solving methods, procedure-based and memory-based.[33] To give an example, consider finding $3+5$. There are many procedure-based methods; for example, one could count on ones fingers, or consult an addition table, or use the procedures described in Chapter 3. On the other hand, there is only one memory-based procedure; one simply recalls the answer.

It is recognized that memory-based problem-solving is far more efficient and that children naturally transition from procedure-based methods to memory-based methods. Clearly, in order to apply a memory-based method, the required fact base must be incorporated into the long-term memory of the problem solver — in the case above the fact is: $3+5=8$. Acquiring the required fact bases involves rote learning. If acquiring the fact bases merely enhanced problem-solving skills, we could perhaps leave natural development to itself, but the story doesn't end there.

The important thing the research on brain development shows is that

> rote-learning of math in childhood creates long-term changes in brain structures that are critical to memory-based problem-solving skills in later life (see MED).

This fact is certainly one of the underlying reasons explaining why the coherent curricula described in ACC are so effective. Recall that coherent curricula expect most topics to be taught to mastery in a single year. To achieve this the number of topics in each year is vastly reduced. Moreover, for a student to achieve mastery of a topic, the essential facts associated with that topic must become stored in the student's brain as part of long-term memory. As QIN shows, this creates structural changes in the student's brain which in turn make problem solving more efficient. In other words, there is a self-reinforcing feed-back loop operating here.

The CCSS-M quite clearly intends for children to transition to memory-based problem-solving. For example, the CCSS-M expect children to achieve fluidity with the standard computational algorithms. Fluidity can only be achieved as part of

[33]See Qin, 2014, http://www.nature.com/neuro/journal/vaop/ncurrent/full/nn.3788.html (hereafter, QIN). For a general discussion of the research in relation to rote-learning of math by the Stanford group see: http://www.medicaldaily.com/math-skills-childhood-can-permanently-affect-brain-formation-later-life-298516 (hereafter, MED).

a memory-based solution process. This is but one example of the expectation that the procedures and knowledge embodied in the Standards will become part of a memory-based repertoire. Learning the required facts to achieve fluidity is but one example in the learning process where parents can play an important supporting role. And as noted above, incorporating this knowledge into memory will benefit a child's problem-solving skills as an adult, a benefit to all possible career choices.

1.7 Formative Assessment

It is a simple fact that the only way to determine whether someone knows something is to ask them. In the context of education, this means testing. Tests are the only way we have to determine whether a child knows what is required by the CCSS-M, or indeed, any curricula.

The importance of assessment/feedback in respect to the CCSS is described in a position paper on **formative assessment** at the NCTM website.[34] In particular, the paper recommends:

1. The provision of effective feedback to students
2. The active involvement of students in their own learning
3. The adjustment of teaching, taking into account the results of the assessment
4. The recognition of the profound influence that assessment has on the motivation and self-esteem of students, both of which are crucial influences on learning
5. The need for students to be able to assess themselves and understand how to improve

(quoted from NCTM-FA).

The intention of formative assessment is that students would immediately know whether they have successfully acquired a body of material and, if not, information would be immediately available enabling a response to successfully complete the learning process. Responding to incomplete learning is essential if students are to be **ready to learn** future items in the curriculum. This is particularly true of a well-designed curricula like the CCSS-M that develops knowledge from the simple to the complex in a manner that reflects the true structure of the discipline.

[34] See **Formative Assessment** A position of the National Council of Teachers of Mathematics, found at http://www.nctm.org/uploadedFiles/About_NCTM/Position_ Statements/Formative%20Assessment1.pdf (NCTM-FA).

This critical feature of the CCSS-M, that **lack of prerequisite knowledge will impede future learning**, is the reason why immediate corrective action must be taken as soon as deficits are identified. Providing time in instructional plans implementing the CCSS-M for individual assessment followed by corrective action where necessary is essential for successful implementation of the Standards.

In respect to self-esteem issues, it is essential that students, parents and teachers come to view assessment as one of the key tools for success and not as a punitive device. In short, assessment should be seen as providing the answer to one and only one question:

> **Does this student need more time on this task?**

To successfully meet the new standards, children must have feedback from assessment and respond to that feedback in effective ways.

1.7.1 Responding to Formative Assessments

The implied theory of formative assessment as described in the five points above is that teachers will identify difficulties, communicate those difficulties in a suitable manner to learners, and that the **learners will take responsibility for fixing the problem** (see NCTM-FA). The question that must be posed is:

> **Is it reasonable to expect children aged 5-11 to take responsibility for fixing the problem?**

It seems unlikely that the expectations in respect to students described in the Formative Assessment paper (NCTM-FA) will be met without serious adult intervention in the corrective process. Although the CCSS-M appear to contemplate additional time for this process, it will be labor intensive. Further, it seems unlikely that governments will provide additional resources in the form of money and qualified personnel beyond what is already present, and you can already find evidence of this fact on Internet news sites.[35]

Given these realities, it seems plausible that the educational system may continue to fail for many children unless additional sources of adult support are found. The most plausible source lies in parents and is the underlying reason why C.M. Gaskill and I wrote a book on arithmetic for parents.[36] In the view of the author, it

[35] A search of Huffington Post has more than 50 pages of articles on Common Core. Some focus on resource/training requirements and whether such resources/training will be available in particular states. See, for example, http://www.huffingtonpost.com/stephen-chiger/to-improve-teaching-get-s_b_3655190.htm

[36] **Parents' Guide to Common Core Arithmetic**, 2014. Available from Amazon.

is only by viewing parents as an essential resource and actively enlisting their help in educating their children that teachers will be able to succeed with the CCSS-M and seriously increase the 40% proficiency rates in arithmetic that we currently measure in Grade 4.

Understandably, teachers may have concerns about whether parents should be enlisted.[37] Nevertheless, because learning to the CCSS-M standard must become memory-based, parents need to be seen by teachers as allies and parents say they are ready. WP34 (p. 1) states: *survey responses suggest most parents are ready to provide support for the CCSSM both in the public arena and at home.* Surely we can all agree that at a minimum, every parent could successfully help their child with rote learning issues such as mastering the tables and achieving fluidity with standard computations.

1.8 The Elephants in the Room

We began this chapter by quoting a statement of support for the CCSS-M by qualified experts:

> The adoption of the Common Core State Standards in Mathematics (CCSS-M) by nearly every state represents an unprecedented opportunity to improve U.S. mathematics education and to strengthen the international competitiveness of the American labor force (WP33).

These judgements lead directly to the conclusion that after adopting a proper implementation of the CCSS-M, students in primary and elementary should be more successful and this success should propagate into higher grades. On the one hand, it would seem that parents and the public in general should be pleased with this prospect. And on the other, it would seem that enhanced success for their students would be enough to garner the enthusiastic support of an overwhelming proportion of teachers. Why then is there sufficient conflict about the Common Core initiative that would lead a state to revoke its adoption,[38] or a teachers' union to vote

[37] A colleague attending a math education conference in Singapore was asked the purpose of his visit by a customs officer. When he answered that he was attending a math-ed conference, the customs officer pulled him aside and spent so much time seeking pointers as to how he might help his child with math that my colleague always spoke of this feature of Singapore culture — that every parent expects to be involved in their child's education — as one of the reasons underlying Singapore's success at math.

[38] See for example: http://www.newsobserver.com/2014/09/22/4174322_common-core-review-begins.html?rh=1 Hereafter RNO2014.

no-confidence in the standards?[39]

1.8.1 National Assessments and Standardized Tests

It is clear that national and international data on student performance will continue to be generated by programs such as NAEP and PISA and collected at sites like the NCES.[40]

The NAEP assessments are supposed to be designed to reflect the entirety of the standards. In the future, this means, assessments will test the entire CCSS-M. Clearly new test instruments need to be created. Achieving this is a major and expensive task. One group engaged in this task is PARCC.[41] Visitors to the website can view sample test instruments and get a fair idea of what students are expected to master. I have worked all the sample tests for Grades 3-6. Out of around 150 questions on these sample exams, I had wording and/or clarity issues with less than ten. Even so, all appropriately reflected the focus of the Standards and were consistent with my understanding of its intent. I would suggest that every teacher visit the site and work the problems to enhance their understanding of what the test designers consider to be appropriate emphasis on various topics.

Given that the nature of the tests being developed is appropriate, and that national assessments provide useful data, what is the problem that would lead to a teachers union voting no confidence in the Standards?[42] The answer lies in the political use that such data can be put to, namely, to indict schools for failing to achieve on a relative basis.

For years, private schools have been touted as out-performing public schools based on this data. In the last year however, a careful analysis showed that public schools were actually doing a better job of educating children than private schools.[43] The point is that proper analysis of NCES data demonstrates that the American public school system is doing a superb job in comparison to other American schools and teachers should welcome national tests as a means for generating the data to continue to demonstrate that fact.

[39] See: http://www.wbez.org/news/education/chicago-teachers-union-votes-oppose-common-core-110152 Hereafter, WBEZ.

[40] NEAP, PISA and NCES are the National Assessment of Educational Progress, the Program for International Student Assessment and the National Center for Educational Statistics, respectively.

[41] See www.parcconline.org/parcc-assessment

[42] See WEBZ.

[43] *The Public School Advantage*, 2014.

1.8.2 Testing Understanding

We have already pointed out that there is conflict over standardized testing. Because the Standards make an issue of **understanding**, it is evident that questions testing understanding will be part of assessments. But what is understanding? Unless there is clarity on what it means to test understanding, conflict over testing can only grow.

The notion that American curricula were deficient in that students did not *understand* became prevalent in the 1960's with the advent of the **New Math**. There is substantial literature on this subject.[44] What is clear from Usiskin's paper is that even today, there is no universal agreement on what it means to **understand a given mathematical idea**. Thus, until we can all agree on what exactly it means to understand and how to demonstrate that understanding, we will have difficulty assessing the **understanding component** of the CCSS-M, particularly at the primary and elementary level. So we are clear, ask yourself:

> What does it mean to say a child understands the **Distributive Law**?
> How would a child demonstrate her understanding to your satisfaction?

Answering these questions is difficult and quite likely idiosyncratic.

With the above caveats in mind, we note that in the 2012 paper, Z. Usiskin[45] suggests that there are four independent components to mathematical understanding:

1. procedural understanding — how to correctly perform a computation;
2. use-application understanding — being able to recognize when a computation should be applied;
3. proof understanding — how to derive the formula underlying a computation;
4. representational understanding — how to pictorially represent a computation.

Bleiler and Thompson[46] argue that these components should be used as a basis for testing understanding. This division makes sense but the components are certainly not of equal importance or appropriate to children of all ages. For example, being able to correctly perform computations with fluidity is critical to all children not

[44]See Z. Usiskin, (2012) http://www.icme12.org/upload/submission/1881_F.pdf (hereafter, ZU 2012)

[45]See ZU 2012.

[46]*Multidimentional Assessment of the CCSS-M*, **Teaching Children Mathematics**, Dec., 2012, 292-300.

only because it is the foundation on which the others rest, but because the internal processes associated with the development of these skills have life-long effects (see QIN and MED). Expecting children to associate operations with physical processes — addition as combining or subtraction as taking away — is critical because it is the key to knowing which particular operation should be applied in practical situations.

The example of proof discussed by Usiskin is Rule 13 of §13.9.1 which states:

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}.$$

There may be a point at which students should be expected to reproduce this proof, but it is not in elementary school. However, the CCSS-M makes the point that children should know that

$$\frac{a}{b} = a \times \frac{1}{b}$$

and that knowledge of this fact together with the Associative and Commutative Laws for multiplication enables one to explain why we expect this Rule 13 to be valid.

The degree of importance assigned to pictorial representations is problematic because in many instances these are individual constructs as opposed to natural representations. To be clear what I mean, in §6.2.2 I discuss a concrete realization of the Arabic System of numeration. It is a construct in the sense that I created it for explanatory purposes. Alternatively, to picture a real-world collection (see Chapter 3) and say that its cardinal number is the abstraction that arises by counting its contents is a fundamental natural representation. For a variety of reasons, I might believe my construct to be the best such representation of the Arabic System so that it, or something like it, might even be valuable for teaching children why the standard computational algorithms work. But I would certainly not think that children should ever actually use it in a computation as a substitute for the standard algorithm which is what is suggested when we ask children to use a number table to solve:

$$57 - x = 24.$$

That said, every child should know how the notion of numbers arise from counting real-world collections and/or measuring lengths. These are fundamental and the basis for our thinking.

Clearly, both state and national agencies intend to test understanding. One major developer is the PARCC consortium and sample tests can be found at their website.[47] Examination of the types of questions in PARCC sample tests leads to

[47]See: www.parcconline.org/parcc-assessment

the conclusion that, for the most part, the questions asked are fairly straight forward, although they do demand a substantial facility with reading and interpreting what is read. Teachers should visit the site for a sense of PARCC's view as to what is appropriate.

1.9 The Content of this Book

The remainder of this book is devoted to what I call arithmetic. In terms of the CCSS-M, it is the underlying material on which the **Domain** containing:

number, operations on numbers and the naming system for numbers

is based. The relevant standards are found in the K-8 sections of the CCSS-M. I focus on this material because my teaching experience in post-secondary has convinced me that students who learn this body of material well will succeed at Algebra 1 which is the keystone course leading to post-secondary success. In this respect, my experience appears totally consistent with the experts who designed the curriculums in high-performing countries.

The CCSS-M raise the standard of learning for students in respect to arithmetic. In his paper on understanding, Usiskin (ZU 2012) notes that teachers need a substantially deeper level of knowledge than that demanded of the students. In WP34, the teacher survey data indicate that teachers, particularly at the primary level, understand this fact and want help. WP33 tells us that primary school teachers are the least likely to comfortably triage among the competing curriculum visions presented by the CCSS-M and previously extant textbooks. The remainder of this book is devoted to providing a deep knowledge of arithmetic that will enable teachers to confidently make the required choices.

The mathematical content of the book falls between that in *Parents' Guide to Common Core Arithmetic* and the initial chapter of *Elements of Real Analysis*.[48]

Unlike traditional mathematics books, arithmetic is developed here as an experimental science which ultimately can be turned into a completely logical construct. So readers are clear, we take real-world collections as basic objects of study. We attach to each collection a numerical measure, namely the cardinal number that tells us how many belong to the collection. While no one can be sure how human beings arrived at numbers and mathematics, it seems likely counting collections played a major role. Since the notion of physical collection differs from the mathematicians notion of set, we spend a brief chapter elucidating the difference.

[48]H.S. Gaskill and P.P. Narayanaswami, *Elements of Real Analysis*, Prentice Hall, 1998. (ERA)

Chapter 3 develops the notion of cardinal number as an abstract property associated with collections. To achieve this, we use pairing which the reader may recognize as one-to-one correspondence. In Chapter 4, collections are studied from the perspective of determining their properties in respect to the notion of cardinal number. Specifically, we are given a collection and its cardinal number and we want to know what happens to the cardinal number of that collection as we put more elements into the collection, or take elements out of the collection. Studying collections in this way is much like what early scientists did in respect to studying the motion of physical objects. We are looking for general principles that apply to counting as a process. In Chapter 5 we turn the principles discovered in Chapter 4 into mathematical statements about the set of counting numbers. This chapter transitions from experiment to the abstract mathematical model.

Chapter 6 explains why numbers are useless without a system of notation and that the one we have is special because of its ability to support computations. Introducing a system of notation requires a zero and it is in this chapter that zero is discussed. The remarkable fact is that while the Arabic System required thousands of years of human intellectual development, a seven-year old can not only master it, but use it to perform computations that would astound all but the very few 1000 years ago.

Chapters 7-9 and 12 develop the operations of addition, subtraction, multiplication and division on the set of counting numbers. The definitions of the operations are sourced in our understanding of counting and collections. As such, it is required that these operations acting on counting numbers model the behavior of counting in respect to real-world collections. This behavior is used to identify the key properties that each operation must have. As well, it is shown how computations involving the operations are supported by the Arabic System.

Chapter 10 introduces the set of whole numbers (integers). Introducing this set involves negative numbers and the algebraic concept of additive inverse. Since the inverse concept may well be new to some readers, considerable explanation is provided. The important algebraic properties that apply to arithmetic are identified and proofs given from a set of axioms — the standard ring axioms. Why we should take these as axioms is explained and proofs are provided for the important facts.

Chapter 11 deals with the order properties of the integers. The critical definition relating order to algebra is given. How the number line is constructed is discussed. How addition and subtraction are represented on the line is presented.

Chapter 13 develops the real numbers. This development again requires new numbers and a new concept, namely, the multiplicative inverse. The importance of unit fractions is discussed. The usual field axioms are given. The key ideas related to nomenclature and notation are discussed. The algebraic facts of arithmetic of importance to school children are derived from the axioms. The chapter concludes with a discussion of how the facts are applied in practical situations.

Chapters 14-16 focus on fractions. Explanatory material is given that supports the CCSS-M standards for this material. Traditionally this is the most difficult topic in the primary curriculum. Chapter 14 deals with topics from K-4. It is all about what common fractions are. Two key ideas are presented. Knowledge of these two ideas makes the arithmetic of fractions simple. Chapter 15 deals with the basic arithmetic of multiplying and adding fractions. All the usual rules are discussed and explanations of why the operations work as they do are given in terms of material from Chapter 14. Chapter 16 deals with the most advanced topics from fractions which are found in grades 4-8.

Chapter 17 deal with the order properties of the reals. It begins by discussing why the field axioms are not enough to specify the real numbers. The axiom scheme is based on the notion of positive and the order relations are then defined. The standard rules governing order are derived and the key topics from the CCSS-M, e.g., placing fractions on the line, are discussed.

Chapters 18-19 cover decimals and decimal arithmetic. A higher level of understanding of the Arabic System using exponents is useful for coming to terms with why the computations work as they do. So Chapter 18 begins by treating exponents and the rules governing their behavior. Chapter 19 treats decimal arithmetic and the relation between fractions and decimals.