Why **Less is More** for Children Learning Math: How Parents Can Help Their Child Succeed by Concentrating on Essential Topics

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This book is dedicated to the children who need to learn arithmetic and to the mentors willing to help.

Preface

During the period I was Head of the Mathematics and Statistics Department at Memorial University of Newfoundland, the problem of how to deal with entering students who lacked basic math skills reached unacceptable levels. Unfortunately, in spite of changes in Newfoundland K-12 math curriculum, this problem still exists and was the essential reason I started working on an arithmetic book for parents in 2013. The book was based on the premise, derived from 30 years of teaching, that there was a short list of topics that children needed to learn to succeed in the rest of their math schooling. Thanks to advice from a colleague in the US, I became aware of the Common Core State Standards in Mathematics (CCSS-M) which were, according to their stated guiding principles, aimed at producing a focused and coherent curriculum. The standards articulated by the CCSS-M for K-6 appeared to achieve my goals and I adopted the CCSS-M as a guide with the end result being *Parents' Guide to Common Core Arithmetic* jointly authored with my wife. The story could have ended there.

However, in response to the *Parents' Guide* some questions about K-6 curricula arose which caused me to undertake a serious review of the underlying rationale for the CCSS-M. The principal study on which the CCSS-M is based is an analysis of The International Math and Science Study data from the late 1990s. The lead researcher was Wm. Schmidt of the Education Policy Center at Michigan State University. The analysis performed by the Schmidt group showed that successful countries teach a narrowly focused coherent curriculum. The sad part, as far as I am concerned is that the CCSS-M did not, in my view, carefully follow the prescription for achieving focus set out by the Schmidt group in respect to K-6.[1] As a result, the K-6 Standards, although an improvement, still include too many topics.

The purpose of this book is to remedy that by carefully following Schmidt's prescription which produces a narrow list of essential topics on which parents can focus to ensure their child is able to succeed no matter what math curricula their child's school is using.

The book is also designed to be used as a guide for developing a Schmidt type curriculum for K-6. Such curricula would improve learning outcomes on both sides of the US/Canada border.

Finally, many, if not most, of us believe math is hard. This is a tragedy and it doesn't have to be so. We can make math hard by choosing to teach the poorly designed curricula used in the under-performing countries identified by Schmidt. Or, we can make it straightforward and fun by teaching narrowly focused and coherent curricula in which most children succeed as used in the high-performing countries identified by Schmidt.

[1]Links to all supporting documents can be found at:
www.mun.ca/math/people/ppl-faculty/retired/

Acknowledgements

The essential difficulty in writing a book like this is in making sure it is accessible to its intended audience. To the degree that this book achieves that goal is entirely due to the efforts of my partner Catherine Gaskill. She has been patient and painstaking in her approach and I am truly grateful for help on this project.

I would also like to thank the current head of the Department of Mathematics and Statistics at Memorial, Chris Radford who has supported all my projects.

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Chapter 1

Why Less Really is More in K-6 Math

On the face of it, the assertion of the title *Less is More for Children Learning Math* sounds absurd, particularly in the context of math. Nevertheless, this assertion when applied to K-6 math curricula is true and the reader needs to understand why.

1.1 The Concept of an A+ Curriculum

We all know what it means to get an A+ in school, although for most of us, receiving that grade on anything was probably not a common experience. So a K-6 math curriculum that could be described as being **A+** would certainly have to have the following properties:

- almost all children would succeed in mastering the learning tasks set by an A+ math curriculum;
- children learning from an A+ math curriculum would perform better than children learning under all alternative curricula;
- children learning from an A+ math curriculum would be able to continue learning math topics to mastery at higher levels.

The notion of an A+ math curriculum with these properties sounds too good to be true. Moreover, if such a curriculum did exist, it would seem that the children studying under it must do nothing but math all day. In actuality A+ math curricula do exist. By focussing on a narrow list of key topics, these curricula produce successful learning outcomes at high levels for almost all children.

1.2 The Research Identifying A+ Curricula

The concept of an A+ math curriculum was developed by researchers working at the Education Policy Center (EPC) at Michigan State University.[1] The particular body of work that identified the criteria for an A+ curriculum was derived from an analysis of the data set that was collected in 1997 during the Third International Math and Science Study (TIMSS). This analysis was performed by a group led by Wm. Schmidt and published in 2002.[2]

The Schmidt group found that the education systems in various countries have a similar hierarchical structure. At the top is a department of education whose jurisdiction may be nation-wide in a small country like Singapore, or state- or province-wide in a large countries like the U.S and Canada. At the bottom are classroom teachers.

The principal activity of a department of education is to specify what is to be taught to the nation's children. This is accomplished through promulgating an **intended curriculum**[3] consisting of lists of topics to be taught, grades in which these topics are to be taught, and the amount of time allocated to each topic. Of course, for various reasons what is actually taught in a particular classroom may differ from the intended curriculum and for this reason ACC refers to what is actually taught as the **enacted curriculum**. Among the important findings of this TIMSS study were[4]

1. in all countries, what students learn from the enacted curricula is what is in the intended curriculum;
2. some intended curricula are much more effective than others in respect to measurable student learning outcomes;
3. effective intended curricula have common features in respect to the number of topics presented each year, the order and duration of presentation of individual topics, and so forth;
4. successful intended curricula are narrowly focused and coherent;
5. North American intended curricula extant in 2002 tended to be ineffective in respect to learning outcomes precisely because they lacked focus and coherence; hence, they were termed *a mile-wide and an inch-deep.*

To understand the import of the findings for parents and children we need to dig a little deeper.

[1]For more information on the work of the EPC visit: http://education.msu.edu/epc/.

[2]*A Coherent Curriculum: The Case of Mathematics*, Wm. Schmidt, H. Houang and L. Cogan, **American Educator**, Summer 2002, available on-line. Hereafter, ACC.

[3]See ACC.

[4]See ACC.

1.3 When Students Fail, Who or What Gets the Blame?

When a country's rankings sink on an international study, assigning blame becomes an important activity. Since the delivery system for the intended curricula is the teacher, it is easy to understand why, as noted by Schmidt *et al.*, there is a tendency to blame the teachers.[5] That such a response is typical may be seen from the following quote excerpted from an editorial on the math curriculum in the Province of Newfoundland and Labrador:

> *The fact that marks have dropped almost universally in tandem with this new math is explained away as a matter of teacher training alone, not curriculum.*[6]

Now consider the problem of poor performance in math from the perspective of a parent. The critical fact to remember is that what a child learns has two components: the teacher (as presenter of the enacted curriculum) and the intended curriculum (as a list of topics promulgated by the central authority). What most of us who were not trained as curriculum professionals assume is that the intended curriculum must be good, since it was prepared by our best trained educators. Thus, if a particular child is failing, it is very easy for a parent to blame the teacher for not getting the job done. Simultaneously, from the teacher's perspective, it is very easy to blame the child, or the home environment, etc. The point is, the curriculum is almost never seen as the fundamental cause of failure.

Now consider the first conclusion of ACC to the effect that universally teachers are teaching children what is in the intended curriculum set out by your state/province's department of education, and that a given teacher is likely to be doing a good job of teaching that curriculum as measured by tests. That's the upside.

The downside is that some curricula are much less effective in producing positive student learning outcomes, so that even though a child has a great teacher, the constraints imposed by an ineffective intended curriculum lead to reduced learning outcomes.[7] As Schmidt *et al.* observed (ACC, p. 3), teachers using typical North American curricula are instructed:

[5] ACC, p. 3.

[6] *I used to be good at math. Now I find it hard.*, by Peter Jackson, **The Telegram**, April 29, 2015.

[7] The EPC surveyed teachers on exactly this point and found the principal determinant of what that teacher did was what was in the intended curriculum. More information can be found at the EPC website in Working Paper 33.

Teach everything you can. Don't worry about depth. Your goal is to teach 35 things briefly, not 10 things well.[8]

In short, the curriculum set out by your state/province's department of education really does matter when it comes to your child's long-term success because that is what your child's teacher is going to teach. The question for parents is whether an ineffective math curriculum really will have a measurable negative impact on their child's future success. The fact that at every post-secondary institution in North America students are regularly denied access to programs because of missing math skills attests to the negative impacts. Moreover there are still entry-level jobs that require an ability to correctly perform arithmetic computations as a condition of employment.[9] Being denied access to a program or losing a job because of poor math skills is devastating to a young adult.[10]

The other side of this is whether there really is a curriculum under which essentially all students succeed?

1.4 The Effect of an A+ Curriculum on Student Success

We need to make explicit what the difference is in terms of student outcomes between a coherent curriculum and a typical mile-wide North American curriculum. Arriving at such an assessment starts with asking students from different countries having different math curricula to perform the same, or equivalent, mathematical tasks. This is easy to do in a discipline like mathematics because computations don't change when borders between countries are crossed. Let's look at how a data set like TIMSS is analyzed.

Within each country individual students are rank ordered by percentile. Thus, if a child is ranked in the 100 th percentile in the US, it means no US student answered more questions correctly than that child. If a child was ranked in the 75 th percentile in Canada, it means 25 % of Canadian students answered more questions correctly,

[8] Unfortunately, in the opinion of this writer, the Common Core State Standards in Math (CCSS-M) also suffer from too many topics in K-6.

[9] I recently had breakfast at a Waffle House. The waitress was great at her job until it came time to add up the bill. Her computation contained $12.00 + $1.38 = $12.38. I suggested she check her arithmetic which she did and fixed it. She thanked me and assured me that an error like that could result in her dismissal.

[10] This remark is obvious on its face. But as head of a math department having to witness the pain experienced by such students provides an entirely different perspective, particularly when it does not have to happen.

while 74 % answered less correctly. Finally, if a child was ranked in the 25 th percentile it means that 75 % of students answered more questions correctly. Percentile rankings provide no information as to how many actual questions were correctly answered by any student. They only indicate how a particular test score by one child ranks relative to the test scores generated by all children in the given country. Percentile data in one form or another is generally what is available on public websites such as the one maintained by the National Center for Educational Statistics in the US.[11]

Countries can also be rank ordered based on the relative performance of that country's students. The simplest way to do this is to average the scores of all students within a country and then list the averages in descending order to obtain a ranking in comparison to other countries. As an example, the US was ranked 25 th on PISA[12] (UT 2013).[13] This ranking is not a percentile, nor does it tell us anything about actual student performance. It simply means that the students of 24 other countries were judged to perform better on this series of tests, perhaps by the method outlined above. However, within each country there are still percentile/raw-score tables which inform researchers exactly how many correct answers were required to achieve a given percentile ranking in that country. Because the same test instrument is used by each country, the table can be used to determine exactly what an individual raw score in one country means as a percentile in any other country. Thus for example, the researchers at EPC found that the raw score of a US student in the 75 th percentile on TIMSS was ranked **below** the 25 th percentile in Singapore (ACC)! Another way to think of this is that more than 75 % of students being taught under Singapore's curriculum achieve at levels that only 25 % of US students achieve at using a US curricula! In essence, the overwhelming majority of students in Singapore are achieving results every North American parent would be proud of. This is the functional difference in individual student outcomes between learning from a coherent curriculum, such as Singapore's and the typical mile-wide curricula extant in North America prior to 2012.[14] This difference is the reason why parents need to understand exactly what constitutes an A+ curriculum.

[11]For example, data from the National Assessment of Educational Progress (NAEP) is kept here: http://nationsreportcard.gov/math_2011/.

[12]PISA is the Program for International Student Assessment run by the Organization for Economic Co-operation and Development (OECD).

[13]Uri Treisman, *Iris M Carl Equity Address: Keeping Our Eyes on the Prize*, NCTM, Denver, April 19, 2013. The address in a variety of formats can be found at: http://www.nctm.org

[14]As I have already noted, the CCSS-M in K-6 still contain far too many topics, although they are certainly better.

1.5 A Model Coherent (A+) K-6 Curriculum

The following table is derived from the effective curricula identified by Schmidt **et al.** in their 2002 paper (ACC) and which they term A+ and illustrates the shared features that make all effective curricula effective. The table faithfully reproduces the data reported in ACC for A+ curricula but has been truncated to omit the data from Grades 7 & 8 because our focus is elementary school. As well, the table has been rearranged so that like topics are presented as a group, e.g., numbers and operations on numbers are presented together.

TOPIC & GRADE:	1	2	3	4	5	6
Whole Number Meaning	●	●	●	○	○	
Whole Number Operations	●	●	●	○		
Common Fractions			□	●	●	○
Decimal Fractions				○	●	○
Relationship of Common & Decimal Fractions				○	●	○
Percentages					○	○
Negative Numbers, Integers & Their Properties						□
Rounding & Significant Figures				○	○	
Estimating Computations				○	○	○
Estimating Quantity & Size				□	□	
TOPIC & GRADE:	1	2	3	4	5	6
Equations & Formulas			□	○	○	○
Properties of Whole Number Operations				□	○	
Properties of Common & Decimal Fractions					○	○
Proportionality Concepts					○	○
Proportionality Problems					○	○
TOPIC & GRADE:	1	2	3	4	5	6
Measurement Units	□	●	●	●	●	●
2-D Geometry: Basics			□	○	○	○
Polygons & Circles				○	○	○
Perimeter, Area & Volume				○	○	○
2-D Coordinate Geometry					○	○
Geometry: Transformations						○
TOPIC & GRADE:	1	2	3	4	5	6
Data Representation & Analysis			□	□	○	○

This table is adapted from the A+ curricula identified by ACC. Topics have been reorganized to reflect domains identified in the CCSS-M and only grades 1-6 are shown. Topics identified with a • are in the intended curricula of all the A+ countries in the grade shown; ◦ identify 80% of A+ countries; □ identify 67% of A+ countries. Topics not on this list are **not** in the intended curricula of A+ countries in Grades 1-6!

Solid circles (•) indicate that the topic at the left was taught in the grade at the top by 100% of high-performing countries, i.e., countries that met Schmidt's criteria for being A+ (ACC). A blank indicates the topic was taught by fewer than two thirds of the A+ countries; most likely, the topic was not taught. A careful read of the Schmidt study leads to the conclusion that what is not taught in a particular grade is as important as what is. In other words, **more is not better**. Rather, fewer, carefully selected topics in the right order need to be taught in a concentrated fashion so that students learn to mastery and retain what they have learned over the long-term. This is the essence of **less is more** when it comes to K-6 math.

Focus for a moment on the first group of topics in the table related to numbers and operations. Notice that within this group each consecutive line of the table represents more difficulty and/or sophistication. For example, the first line of the table is concerned with the meaning of whole numbers and reflects the fact that the computational procedures of arithmetic depend on the place-value system of notation. The intention is for children to learn to count and through this process learn about the Arabic (place-value) notation system (see Chapter 7). On the next line the topic is whole number operations. The first operation studied here is single-digit addition for which there is a finger counting procedure that is based on counting. More advanced topics like two-column addition are founded on the simpler topic combined with a deeper knowledge of place-value. A+ countries expect children to have mastered operations with whole numbers by the end of Grade 4.[15] Indeed, 20 % of the A+ curricula complete the learning of operations by the end of Grade 3. To ensure children succeed at this, A+ curricula limit the topics taught in Grades 1 and 2 to whole numbers, addition and subtraction and making measurements, as with a ruler.

In A+ countries there is little or no study of common fractions before Grade 3, and a full third of the A+ curricula delay any discussion of fractions until Grade 4. In making this decision, A+ countries recognize the children must master operations with whole numbers first because operations with fractions utilize whole-number operations. Again, introducing more topics in earlier grades does not produce better results. It simply detracts from mastering the important concepts and procedures associated with whole number arithmetic.

Another feature of the A+ curricula is the avoidance of topics that derive from

[15] My memory of my own public schooling is that we had finished long division by the end of Grade 4.

algebra and data analysis in the early grades. Rather, the entire focus in Grades 1-3 is on number, operations with numbers and measurements related to the geometry of the line. The reader may wonder, if students are dealing with fewer topics, why don't they learn less? The answer is that students learning from an A+ curriculum are expected to have a more extensive knowledge of what they do learn. For example, the place-value system of notation (§7.1.2) for whole numbers connects to our system of computations (§8.6) and A+ curricula expect that students will come to terms with these relationships and be able to execute the procedures fluidly.[16]

In summary, the best way to help your child is to provide learning support for the limited number of topics identified in the A+ curriculum. Thus, for example, in Grade 1 the issue for your child is coming to terms with the number system, the meaning of addition and subtraction, and how to add and subtract counting numbers less than 20 . So the purpose of this book is to enable parents to focus on ensuring that your child masters the critical computational skills that are essential to continued success. In the long run, ensuring your child masters the short list of topics identified on the A+ list is what will provide the best foundation for success in the workplace and/or post-secondary.[17]

[16] A detailed analysis of the difference between an A+ curriculum and North American curricula was performed by Liping Ma: *A Critique of the Structure of U.S. Elementary School Mathematics*, found at www.ams.org/notices/201310/fea-ma.pdf This paper is essential reading for anyone who wants to understand how North American curricula in K-6 math went seriously astray.

[17] Recently, a friend asked my advice about a child who was having trouble with the math taught at the start of Grade 4. I told him to make sure his child learned her addition and times tables well. At the end of the year his child got the prize as the best math student in her class. The point is: small amounts of knowledge learned really well go a long way in arithmetic. There is also huge gratification to a child who knows they know the answer.

Chapter 2

The Nature of Whole Number Arithmetic

A familiar complaint of today's parents is:

Math has changed so much that I am unable to help my child.

This complaint would not be so meaningful if today's children were generally successful with the material being taught in our schools. That our children are having difficulty dealing with today's math curricula is evidenced by the number of services that provide tutoring in math.[1] The point is that parents are frustrated by a perceived inability to help their children. And children are frustrated by a curriculum that makes learning difficult for all but a very few, and even those few do not perform at the levels achieved by most students in A+ countries. Clearly we ought to be able to do better on both scores!

At the societal level, doing better would require completely revising the curricula. The US has tried to do this with the Common Core State Standards (CCSS) initiative which began in 2009[2] and the process was still on-going in 2014. In consequence, expecting an immediate societal response that will help children and their parents is simply not possible.

The purpose of this book is to make it possible for parents to help their children in spite of the nature of modern K-6 math curricula. The book accomplishes this by providing a complete explanation of the A+ portions of the curriculum and instructions on how to communicate those explanations to children. I can appreciate

[1] A Google search under the phrase *math tutor* turned up more that 26 pages of listings. Exactly how many I don't know because I stopped looking at 26.

[2] See http://www.corestandards.org/about-the-standards/development-process/#timeline for the exact timeline.

that there are many who might like to do this but suffer from the *I've never been good at math.* syndrome. The fact is that if you can count, you can master both whole number arithmetic and the arithmetic of common fractions. Not only that, but you can **understand** it at the level intended by those who drafted the CCSS-M. I know this is hard to believe for some, but remember, in A+ countries, essentially all students achieve at high levels, so why not all of us?

2.1 The Nature of Arithmetic and Mathematics

The purpose of mathematics in general and arithmetic in particular is to answer questions. So the first thing to consider when confronted with a piece of arithmetic is:

What question is being answered by this computation?

For example, consider $5+4$. The implicit question is:

> Given a collection of five objects and a second collection of four objects, if the two collections are combined into a single collection, how many objects will be in the combined collection?

It is essential that every child know that providing answers to questions like this one is the primary purpose of the addition operation applied to counting numbers.

The next important fact to recognize is that knowledge of what question is being posed is fairly useless without a procedure for finding the answer. The key fact that makes the remainder of this book possible is that for every arithmetical question there is a straightforward procedure that provides the answer. Indeed, for every question that is posed in math and/or physics up until at least the end of first year university, there is a procedure that yields the answer. Thus, the second critical fact that every child must know is:

Every question I will be asked is solved by using a procedure.

Consider again finding $5+4$. The first procedure a child should learn for solving this problem is:

> count out five objects to form a collection and then count out four different objects to form a second collection, then combine the two collections and count the result.

This process provides a concrete method for solving all addition problems and the process tells the child the question that is being addressed.[3] Later the child learns a second, more effective procedure, namely, count-on starting at the larger number, and finally, the child will find the sum by recall. But always there is a beginning method that guarantees a successful conclusion.

2.2 Rote Learning vs Understanding in K-6 Math Curricula

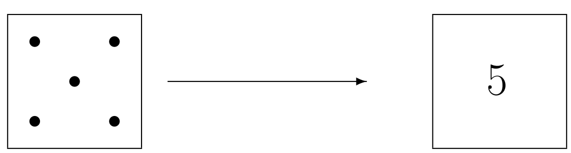
Rote learning has a bad name. In part this was due to the fact that in the period prior to the introduction of the post-Sputnik New Math curricula, the focus of K-6 school math curricula was on a seemingly endless repetition of computations in which almost no theory was taught. So we need to ask the question: Does rote learning serve a useful purpose?

2.2.1 The Two Functions of Rote Learning

The most important fact about rote learning is that there are some things that can only be learned by rote. The most important examples are the meanings attached to symbols. For example, the symbols

$$dog \quad \text{and} \quad 5$$

have specific meanings.[4] We cannot sensibly question the meanings assigned to these symbols. They are definitions and simply are. So to ask for an explanation as to why when someone says *dog* we think of a four-legged pet like a spaniel is ridiculous. Similarly, to ask a child to explain why the numeral 5 is associated with a diagram like



[3] Clearly this procedure is not efficient for large numbers, but for the moment we are not concerned with efficiency issues.

[4] A frustrated parent called a local radio station to complain that her daughter had been asked: $2 \times 2 =?$ Explain. Since any explanation comes down to saying this is a definition, it is no wonder the parent was frustrated.

is absurd. These are definitions and definitions have no explanation. But they must be learned exactly by rote. For this reason rote learning has an important place in mathematics education.

The second function of rote learning relates to the transition from external problem solving to internal problem solving. The process of transition between the two methods has been extensively studied by cognitive psychologists. A brief summary of recent work follows.

Cognitive psychologists identify two kinds of problem-solving. The first is **procedure-based** which simply means the problem solver uses an external procedure to solve the problem, e.g., as described above in respect to $5 + 4$. Since, arithmetic is completely procedure-based, learning these procedures is the first step in a successful math education.

The second kind is **memory-based** problem-solving which means the problem solver recalls the solution from memory. Obviously, an internal-to-the-brain memory-based problem-solving process is much a more efficient means of problem solving.

Until recently, cognitive psychologists knew that children learning math transition from procedure-based to memory-based problem-solving, but they did not understand how. That changed when the most recent research on learning found that the transition to memory-based problem-solving involved changes in a child's brain and these changes continue to enhance the individual's skill at problem-solving as they become adults.[5] We repeat: these changes in a child's brain **last a lifetime** and continue to enhance problem-solving skills. This is why the effect of rote learning of math is so important. And, as we all know, rote learning is accomplished through **practice**.

While memory-based problem solving is the ideal, actual arithmetic problems often require a combination of internal and external procedures. For example consider, 357×924. To perform this computation a child would use recall to produce the set up on paper, use recall to find products like 7×4, use recall of the process to properly place partial results on paper, and so forth. The product example also illustrates the hierarchical nature of arithmetic. New procedures are developed using old procedures as a base which is why it is essential that children learn the procedures for solving problems to the point of mastery. Then they able to successfully incorporate old procedures into new ones. As well, they will always have the most basic procedures, e.g., the procedure for single digit addition, as a fall-back and a means to check their memories.

Successfully transitioning to memory-based knowledge of the facts and procedures of arithmetic is critical because unlearning a wrong memory-based skill is next to

[5]See http://www.nature.com/neuro/journal/vaop/ncurrent/full/nn.3788.html and http://www.medicaldaily.com/math-skills-childhood-can-permanently-affect-brain-formation-later-life-298516

impossible at a later point in their education.[6]

So here are two simple things that every parent/mentor can do that are guaranteed to help a child succeed. First, ensure the child knows the addition table from memory by the start of Grade 2, and second, ensure the child knows the multiplication table from memory by the end of Grade 3. Of course there are many other things parents can do, but these two are guaranteed to have immediate substantive benefit to your child's success in math.

Skills requiring rote learning are only effectively acquired by practice. Our presentation will focus on achieving fluidity with computational procedures because these same procedures apply at all levels. For example, the simple procedure we will give for adding common fractions applies equally well to algebraic fractions. This is another reason why these skills have to **last a lifetime**: they will be used over and over due to the hierarchical nature of mathematics.

2.2.2 What is Meant by 'Understanding' in K-6 Curricula

The notion that North American curricula were deficient in that students did not *understand* became prevalent in the 1960's with the advent of the **New Math**. There is substantial literature on this subject but we focus on the discussion in a paper by Z. Usiskin.[7] What is clear from Usiskin's paper is that even today, there is no universal agreement on what it means to **understand a given mathematical idea**. Thus, until we can all agree on what exactly it means to understand and how to demonstrate that understanding, educators will have difficulty assessing the **understanding component** of math curricula, particularly at the primary and elementary level.

With the above caveat in mind, we note that in the 2012 paper, Z. Usiskin[8] suggests that there are four independent components to mathematical understanding:

1. procedural understanding — how to correctly perform a computation;
2. use-application understanding — being able to recognize when a computation should be applied;
3. representational understanding — how to pictorially, or otherwise, represent a computation;

[6] I make this last statement based on working with post-secondary students who had weak math kills. Their problems were almost always associated with wrongly learned methods for performing arithmetic computations.

[7] See Z. Usiskin, (2012) http://www.icme12.org/upload/submission/1881_F.pdf (hereafter, ZU 2012)

[8] See ZU 2012.

4. proof understanding — how to derive the formula underlying a computation.

The meaning of **procedural understanding** is the most transparent. We have a procedure and to find out if a child meets the procedural understanding criteria, we would ask the child to execute the relevant computation. For example: Find 26×42 without using a calculator.

The meaning of **use-application understanding** is also fairly transparent. A child is presented with a problem which can be solved by performing a computation. Consider the following:

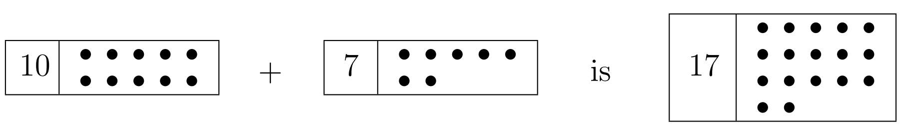
> A child has a marble collection containing 13 marbles. If a friend gives this child 8 marbles, how many total marbles will the child have?

To solve this problem the child has to know that the correct tool to apply is addition and the required computation is: $13 + 8$. Clearly, this is a higher level of knowledge and illustrates one aspect of the hierarchical nature of mathematics. A child may be unable to solve this problem for two fundamentally different reasons. First, the child may be able to execute $13 + 8$ but may not know that this is the required computation because he/she does not know what question addition is supposed to answer. Alternatively, the child may know that finding $13 + 8$ will answer the question, but not know how to perform the computation. Readers should know that this type of understanding is a major focus of all modern curricula.

Representational understanding is also a major focus of modern curricula. A typical question designed to test this form of understanding is the following:

> Find $10 + 7$ and explain your thinking by drawing a picture.[9]

What would be expected is a diagram something like the following:



Asking students to explain their work with diagrams like this is ubiquitous in K-2. Producing such a diagram demands that the child know the meaning of each counting number on the left and that they can represent this counting number visually as in §2.2.1. In addition, they must know the operational meaning of $+$ and be able to represent that visually. The diagram shown also makes use of $=$ which represents an additional level of abstraction. Many of the websites containing worksheets have

[9] A+ curricula avoid the early introduction of algebra, i.e., prior to Grade 3. Thus, we do not use 'equation' as a descriptor. The CCSS-M does not adhere to this admonition.

questions of this type on which children can get practice. We also note that producing a solution is procedural.[10]

In an A+ curriculum we would expect students in Grade 3 to respond to the following more abstract form of the same question:

> Formulate an equation that expresses the sum of 10 and 7 as a quantity to be found.

by writing

$$10 + 7 = \boxed{?}$$

and to understand the meaning of the equality symbol and the $\boxed{?}$ as being an unknown.

The last type is proof understanding. We consider the question of whether proof understanding is, or ought to be, a component of K-6 math curricula.[11] Usiskin equates proof understanding with the knowledge of how to derive a formula (ZU, 2012). Certainly the ability to derive theorems and formulae is basic to understanding. But the ability to reproduce a derivation is not equivalent to *understanding that derivation* as the phrase is used by mathematicians. That type of understanding implies complete knowledge of the ideas and concepts on which the derivation is based, an ability to explain each step of the derivation and a further ability to apply the ideas to discover related theorems and construct proofs for same. In short, proof understanding is highly complex and requires what mathematicians refer to as **mathematical sophistication.** The complexity described suggests that working with proofs would appear to have little place in K-6 and runs counter to the idea that mathematics is hierarchical and is best learned from the particular to the general. The designers of the A+ curricula were clear on this when they made the choice to avoid all algebra before grade 3. Thus, their answer was that proof understanding ought not to be within the province of K-6, a decision with which I completely agree. For this reason, **proof understanding can, and will, be ignored**. We will revisit proof understanding briefly when we discuss discovery in the next section.

To summarize, of the types understanding identified by Usiskin, only three are important for children in K-6. These are procedural, use-application and representational. All questions designed to test one of these three types of understanding are amenable to solution using a recalled procedure because the required information on

[10] Step 1: The child must diagram each counting number on the left. Step 2: The child produces a diagram on the right containing one item for each item appearing in their diagrams on the left. Step 3: The plus sign and the equality relation have to be introduced.

[11] My comments are based on more that 30 years of teaching university math courses in which understanding proofs was a significant course requirement.

which answers are based is the factual knowledge of what defines particular operations on numbers. The question above about $10+7$ illustrates this perfectly. For this reason, none of these questions should ever require guessing on the part of a child.

2.3 Discovery Learning in K-6 Math Curricula

Many modern math curricula contain a component of 'discovery'. The idea is that students will discover the principles for themselves with minimal guidance.[12] They are expected to achieve this by completing exercises that lead them to the principle. One of the greatest thinkers ever was Albert Einstein and this is what he said about the task of finding the principles:

> ... *But the former task, namely to establish these principles which can serve as the basis for his deductions, is one of a completely different kind. Here there is no learnable, systematically applicable method which would lead to the objective.* ..., p. 24[13]

The procedures and principles on which our mathematics is based took tens of thousands of years to develop. It is a fact that these procedures and principles can be mastered by most children. However asking children to develop these, even in the presence of a qualified mentor, is problematic for the reason identified by Einstein. To do so is wasteful of a child's time, a source of major discouragement for many children and destructive for many in the form of lost potential. The question is what to do about it. The answer provided by this book is to focus on the topics in the A+ curriculum and ignore the rest. Ensuring your child learns these topics to mastery will enable your child to succeed in K-6 and master algebra which is the key to post-secondary and full participation in our technologically based society. That is the goal.

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[12] An analysis of this instructional paradigm by the eminent cognitive scientists Paul Kirschner, John Sweller and Richard Clark can be found at:
projects.ict.usc.edu/itw/vtt/Constructivism_Kirschner_Sweller_Clark_EP_06.pdf
This paper explains why instructional methods based that expect novice learners to teach themselves do not work and may be counter productive. The paper is informed by the neuroscience of the brain and just what it means for learning to occur. It is essential reading for all who want to know about this method of instruction.

[13] **Einstein's Unification**, J. Van Dongen, Cambridge University Press, 2010, 212pp.

Chapter 3

How to Use This Book to Help Your Child

3.1 Learning the 3 R's

In the primary and elementary grades children learn three essential life skills: reading, writing and arithmetic. This learning is of a profoundly different nature than much of that which occurs in most other subjects later in their schooling. The difference is that the 3 R's all involve

doing[1]

which means being able to perform a **new behavior**.

Consider what it means to have learned to read. After a successful conclusion, we expect a child to be able to read and understand written **material they have never seen before**, even if the material contains new words. This is what it means to **do reading**.

Similarly, for a child who has learned how to add whole numbers, we expect the child to correctly find a sum, **even if they have never seen that particular sum before**.[2] This is what it means to be able to **do addition**. And, we could make the same observation about writing.

In short, children who have mastered the 3 R's are able to continually perform new behaviors *on the fly*. This is remarkable because the number of possible sums is infinite.

[1] Learning a new language would be a further example of a subject that requires **doing**.
[2] Fluidly execute the standard algorithm in the words of the CCSS-M.

Learning new behaviors continues to be an essential component of math classes in university.[3] However, this component is most intense in primary and elementary school. What is important to realize here is that successfully acquiring new behaviors requires practice. In this sense, learning the behaviors required to do reading, writing and arithmetic are just like learning the skills required to succeed at sports or playing a musical instrument. They require practice and helping children get the required practice is an area where parents can provide valuable support.

To succeed in K-6 children must have feedback and respond to that feedback in effective ways. This process is essential to a child's success and because it is labor intensive, it will require co-operation between parents and teachers. So this is what we consider next.

3.1.1 The Concept of Formative Assessment

The importance of assessment/feedback as a component of the K-6 math curricula is described in a position paper on formative assessment at the National Council of Teachers of Mathematics (NCTM)[4] website.[5] In particular, the paper recommends:

1. The provision of effective feedback to students
2. The active involvement of students in their own learning
3. The adjustment of teaching, taking into account the results of the assessment
4. The recognition of the profound influence that assessment has on the motivation and self-esteem of students, both of which are crucial influences on learning
5. The need for students to be able to assess themselves and understand how to improve

(quoted from NCTM-FA).

The purpose of formative assessment is that students would immediately know whether they have successfully acquired a body of material and, if not, that information would be available in a timely fashion enabling an immediate response to correct deficits. Without doubt, using assessments to correct deficits in the manner described is a desirable outcome for all students, parents and teachers, since it would prevent

[3] For example in a calculus class a new behavior might take the form of learning to differentiate a new class of functions.

[4] The NCTM is the primary professional organization for teachers of K-12 mathematics in the US.

[5] See **Formative Assessment** A position of the NCTM, found at: www.nctm.org/ uploadedFiles/About_NCTM/Position_Statements/Formative%20Assessment1.pdf (NCTM-FA).

the kind of knowledge deficits evidenced by the standardized math tests in Grade 4 that found less than 40% of students proficient in math.[6]

3.1.2 How the Process Works

The implied theory of formative assessment as described in the five points involves three steps: teachers/mentors will identify difficulties, communicate those difficulties in a suitable manner to learners, and the learners will take responsibility for fixing the problem. Consider the following example from Grade 2.

Step 1

A child computes five sums of two digit numbers and gets the first two questions right and the last three wrong. Specifically, the child gets this, and similar problems right:

$$\begin{array}{r} 23 \\ +45 \\ \hline \end{array}$$

but gets this and similar problems wrong:

$$\begin{array}{r} 27 \\ +45 \\ \hline \end{array}$$

The fact that the child gets the first and similar problems correct suggests two things to the teacher: the child knows all single digit sums less than 10, and the child knows the basics of the standard algorithm for addition (see §8.6).

The fact that the child gets problems of the second type wrong suggests a problem either with single digit sums more than 10, or with the carrying procedure. To determine which, the teacher considers the answers given. For example, here are three possible wrong answers: 73, 62 and 61. Compared with the correct answer, 72, the first answer is wrong in the *ones* place but correct in the *tens* place. Thus, the child is able to correctly execute carrying; but the error suggests the student believes $7 + 5 = 13$. The second answer, 62, suggests that the student has an incomplete knowledge of place value and carrying while the third answer suggests the student has problems with single digit sums that have a two digit answer and also has difficulty carrying. Confirming the exact source of each error may require asking the child:

> Exactly how did you arrive at this answer?

[6] These deficits were identified as part of the 2011 National Assessment of Educational Progress. See http://nationsreportcard.gov/math.2011/summary.aspx and click on Grade 4 in the **Proficient** paragraph.

At the conclusion of evaluating the child's responses, the teacher/mentor should have a fair idea of what the child knows, what the child doesn't know and where more practice is needed.

Step 2

Communicating results to children is complicated because these results are loaded with success/failure connotations.[7] Point 4 above recognizes that discussing results with children is delicate. We all need to recognize that the purpose of evaluation is to determine those areas in which a child needs more practice and to act on test results accordingly. Testing is not about finding fault or failure. It is about identifying future learning actions.

Step 3

The intended response to test results is that after the error is explained, the child will undertake sufficient additional practice in the identified area to correct the difficulty. This step is critical because once a wrong fact becomes fixed in memory, e.g. $7+5 = 13$, it becomes almost impossible to unlearn.[8]

For a successful response to be effected, all parties, teachers, parents and students, must believe that essentially every child has the intellectual capacity to learn arithmetic to the standard expected by the A+ curriculum. Without such a belief, the non-believer will subvert the learning process. The only party that can conceivably be excused is the child who is likely to be frustrated by the need for additional work. The point is that all must believe that success can be achieved.

The reason I say that all children can achieve at the required level is that the skills of arithmetic that the A+ curriculum expects children to master are algorithmic. By this I mean there is a fixed solution procedure which may also require knowledge of a small data base, e.g., the addition table in the example above. And even the items in that data base can be found by a procedure, namely **counting-on**, so if a required fact is not remembered, it can be constructed on the spot by a child who knows the counting-on procedure. For this reason, achieving mastery is one of practicing procedures and/or recalling facts.

Once we all agree that the issue is one of practice, we have to consider whether it is reasonable to expect a child in Grade 2 to act responsibly in this matter. My answer is that ensuring an individual child in elementary school undertakes the practice requires

[7] Test results are, and have been, wrongly used to make judgements about student abilities, teacher abilities, school quality, and the list goes on. We will keep our focus on the child.

[8] This is why having a simple procedure available for checking results as part of the recall process is an essential component of a child's math toolbox (see §8.6).

adult supervision[9] and in providing this supervision, teachers and parents should be allies.[10]

3.1.3 Meeting the Standard

As is evident from the last section, ensuring children achieve mastery of the A+ topics will be **labor intensive**. Further, it seems unlikely that governments will provide the necessary resources in the form of money and qualified personnel to ensure all children meet the goal. Indeed you can already find evidence of this on Internet news sites.[11]

Given these realities, it seems plausible that even if the A+ curriculum were universally adopted, the educational system may continue to fail for many children due to lack of resources. The remedy for this circumstance is that responsibility for ensuring a child's success is taken on by that child's parents. What is truly encouraging in this respect is that survey data by the EPC of parents in CCSS adopting states indicate that parents are willing to undertake this responsibility.[12] This book exists to help parents who want to act as mentors to help their children. It does this by providing the detailed knowledge of procedures mentors will need in a framework they can understand. The book has been written with the specific intent that it should be accessible to all parents, even those who consider themselves weak at math.

3.2 What a Mentor Needs to Know

An adult who contemplates undertaking this responsibility may be reluctant on the grounds that it is absurd to expect the average person to be able to help a child through the entirety of the school math curriculum. Moreover, recent studies seem

[9]In my experience, there are many members of extended families who are willing to take this responsibility on.

[10]I recall an article in *Scientific American* on why the children of immigrants tended to out-perform local children in their school work. The answer was that such children did homework around the kitchen table after supper as a regular practice. In other words, there was parental supervision on a regular basis. This practice was dropped in the 2nd generation and as a result the grandchildren of immigrants performed like local North American children.

[11]A search of Huffington Post has more than 50 pages of articles on Common Core. Some focus on resource/training requirements and whether such resources/training will be available in particular states. See, for example, http://www.huffingtonpost.com/stephen-chiger/to-improve-teaching-get-s_b_3655190.htm

[12]See the EPC Working Paper 34, on-line.

to confirm this idea.[13] These studies suggest that when a child enters middle school, parental efforts can become counter productive and actually result in lowering their child's test scores. An identifiable cause is that

> parents may have forgotten, or never truly understood, the material their children learn in school (see Goldstein, p. 85).

However, based on more than forty years of teaching in universities, the only portion of the curriculum you need to concern yourself with to ensure your child will succeed is primary and elementary.

> During primary and elementary children are at an age most amenable to help from parents and if a child completes this portion of the curriculum with no deficits, the child will do just fine on the rest of the school math curriculum.

So in terms of math content, this book confines itself to topics from arithmetic identified in the A+ curriculum. As already noted, all of math at this level is procedural and the book presents procedures in an easily understood manner. For this reason, with the aid of this book, even parents who consider themselves weak at math will be able use this book to help their children.

In respect to helping your child, you need to know various things:

1. Is my child performing up to standard on a grade-by-grade basis?

2. You need knowledge of the underlying content to be able to intervene and help your child where necessary.

3. You need to know where to find additional practice materials to support your intervention.

Information regarding each of these points is included.

3.3 The Content

The remainder of this book is about numbers, their representation and the operations on numbers. These topics are found in the first two sections of the A+ curriculum table on p. 6.[14] Topics related to proportionality are not treated and geometry is

[13]... *And Don't Help Your Kids With Their Homework*, Dana Goldstein, **The Atlantic**, April, 2014, p. 85

[14]They comprise the **number strand** in the CCSS-M.

treated only in respect to length and area. Data analysis is not treated and to the degree that it should contribute topics in a K-6 curriculum, it should be in the context of science.

We focus on topics related to arithmetic because the factual evidence based on testing is that students who master this limited set of topics out perform students taught under any expanded curriculum. Moreover, in the experience of the author, students who can do and understand these arithmetic topics have no difficulty getting through courses in calculus and statistics of the type required for a career in business, the sciences, engineering or economics. It's also the case that if a student can't do arithmetic they are not only blocked from the above, but also from many industrial trades having a substantial math component. Finally, the focus of the book is on topics children learn in primary and elementary because the test data tells us that this is where deficits begin to accumulate.

3.3.1 Using the Content

In previous sections we have made the point that the computations of arithmetic can all be performed using straight-forward mechanical procedures. For example, I think most would accept that the most feared topic in elementary school arithmetic is **fractions**. In §14.3.3 a three-step procedure is presented for adding fractions that for common fractions involves only computations with whole numbers and in all cases always produces a correct answer, even for algebraic fractions. Once a child has learned this procedure, that child is set for life when it comes to adding fractions. At a minimum, every parent can familiarize themselves with this procedure and make sure that their child knows how to use it.

At this point you may be saying something like: If it's that simple, why is there a problem with math? The key here is that the nature of mathematical knowledge is hierarchical — new procedures use and incorporate previous procedures. Thus, unless a child has mastered the previous procedures, that child will be unable to effectively learn the dependent procedure. As test results show, a mile-wide, inch-deep curriculum makes this type of mastery learning impossible for all but a few.

The following point by point outline presents the key procedures of arithmetic in a fashion that illustrates their hierarchical relationship. The only knowledge that is not procedural is the definitions of abstract notions and symbols related to number. These items must be mastered in the order shown:

1. the concept of counting number as measure of the relative size of a collection and conservation of counting numbers — Chapters 4-7;

    (a) using pairing to show which collection has more;

(b) the process of counting by rote;
(c) meanings of the numerals;
(d) 1 is the least counting number;
(e) for every counting number there is a next counting number;
(f) the process of counting-on beginning at a specific number;
(g) the place-value naming system for counting numbers;
(h) ordering of counting numbers using numerals;

2. addition of counting numbers — Chapter 8;
   (a) understanding addition as finding total number in combined collections;
   (b) procedure for single-digit addition based on counting-on — addition table;
   (c) representing single digit sums visually (see diagram for $10 + 7$ in Chapter 2);
   (d) procedure for two two-digit sums using addition table and counting-on;
   (e) procedure for up to four two-digit sums using addition table;
   (f) procedure for two multi-digit sums using addition table;

3. the equality symbol, $=$ — Chapter 8;
   (a) essential properties of the equality relation are discussed in relation to addition;

4. subtraction of counting numbers — Chapter 9;
   (a) understand subtraction as removing members from a collection;
   (b) procedure for subtraction as take-away;
   (c) visually represent subtraction process;
   (d) procedure for subtracting two-digit numbers using standard method;
   (e) borrowing;

5. length as a numerical attribute — Chapter 10;
   (a) units of length;
   (b) making measurements;
   (c) the half-line as a way of representing counting numbers;

   (d) using lengths to model the addition of counting numbers;

6. multiplication — Chapter 11;
   (a) defining multiplication of counting numbers as repetitive addition;
   (b) visually representing products of two using area;
   (c) commutative law from multiplication from area;
   (d) distributive law from multiplication from area;
   (e) visually representing products of three using volume;
   (f) multiplication of single-digit numbers as repetitive addition — multiplication table;
   (g) multiplication of two-digit numbers by single-digit numbers using standard procedure;
   (h) multiplication of multi-digit numbers by single-digit numbers using standard procedure;
   (i) multiplication of two-digit numbers by two-digit numbers using standard procedure;
   (j) multiplication of multi-digit numbers by two-digit numbers using standard procedure;

7. division — Chapter 12;
   (a) defining division as splitting counting numbers into equal groups;
   (b) concepts of multiples and factors;
   (c) skip-counting to generate multiples;
   (d) standard procedure for division;

8. concept of fractions — Chapter 13;
   (a) making measurements that are not counting numbers;
   (b) concept of unit fractions and dividing the unit interval into equal parts;
   (c) naming unit fractions;
   (d) the Fundamental Equation governing the behavior of unit fractions;
   (e) repetitive addition and common fractions;
   (f) notation for common fractions and the Notation Equation;

9. basic computations with fractions — Chapter 14;
   (a) procedure for multiplication of fractions;
   (b) procedure for addition of fractions with same denominator;
   (c) procedure for addition of fractions with different denominator;
   (d) three-step procedure for adding fractions that always works;
10. advanced computations with fractions — Chapter 15;
   (a) subtraction of common fractions;
   (b) dividing of common fractions;
   (c) visual representation of multiplication of common fractions as scaling;
   (d) putting fractions in lowest terms;
   (e) mixed numbers;
11. ordering fractions — Chapter 16;
   (a) using computational procedures to order common fractions;
   (b) placing common fractions on the line;
12. common fractions and decimals — Chapter 17;
   (a) understanding decimal notation;
   (b) the notion of decimal fraction;
   (c) placing decimals on the line;
13. operations with decimals — Chapter 18;
14. negative whole numbers —Chapter 19;
   (a) the notion of additive inverse;
   (b) computational definition of negative numbers;
   (c) algebraic properties of the integers;
   (d) essential properties of arithmetic;
   (e) the whole number line;
15. the arithmetic of real numbers — Appendix A;
   (a) real numbers from geometry;

   (b) the Fundamental Equation and notion of multiplicative inverse;
   (c) the rules of arithmetic;

16. ordering the real numbers — Appendix B;
   (a) arithmetic and $<$;
   (b) procedure for finding length;
   (c) absolute value;
   (d) addition and the real line;

17. the arithmetic of exponents — Appendix C.

The most important thing to take from this outline is how each procedure builds on and/or incorporates previous procedures. This is why mastery of prerequisite procedures is essential both for you as mentor and for your child who is learning. It is also why children need to focus their attention on learning the specific procedures that are supported by and mesh with our system of numeration. In my experience, mastery of the topics outlined above and the laws on which they are based will enable children to succeed in higher level math courses precisely because knowing how to do the underlying computations will not be an issue. Finally, thorough knowledge of these procedures is the foundation on which **mathematical understanding** rests.

In summary, the purpose of this book is to provide mentors with the mathematical information they need to ensure the children in their care master arithmetic. There are five aspects to this:

1. applying the A+ Table to identify topics critical to a child's success on a grade-by-grade basis;

2. providing the procedural knowledge on which computations are based so mentors are able to help children master the computations of arithmetic;

3. providing theoretical knowledge so that mentors can explain the **why** of computations;

4. providing detailed information as to what mentors should expect a child to know on completion of each grade levels;

5. suggesting specific activities that help and encourage children engaged in the learning process so as to ensure their child achieves appropriate levels of mastery.

3.3.2 Tracking Tests

Testing has something of an evil name, certainly among students. The fact is, there is only one way to determine whether a child knows something and that is to ask them.

In the opinion of the author, the only legitimate purpose of testing in mathematics is to:

> **determine whether a student has learned the material that has been taught.**

This may seem obvious, but often times test questions seem to have another purpose; for example, finding out if a student is clever. The results of tests should be used only to inform parents and teachers whether a child has assimilated the mathematical knowledge required to proceed.

Here is what we recommend in respect to testing. Each year your child should have a notebook devoted solely to tests in arithmetic. As each new test is added to the collection, you and your child should go over the test question by question. Going over all questions provides an opportunity to hand out praise. Going over questions that are wrong provides the opportunity to find out if your child knows why the answer was wrong and whether there are issues in respect to the question that need to be addressed. We stress: **going over a test is not an adversarial process**. It is about helping your child learn what is needed.

There may be rare situations in which neither you nor your child knows what is wrong and what the correct solution is. In that case, ask the teacher for the correct solution with an explanation.

Finally, this notebook will provide pointers as to what sorts of additional practice your child needs to master a topic. The next section shows where you can get help with this.

3.4 Math Websites for Mentors and Teachers

There are any number of websites that have been developed to support K-6 math and offer material suitable for children focused on A+ topics. Some sites offer worksheets that are available for download at no cost. We provide a sample list of such sites with descriptive material taken from the site. We also provide a list of subscription sites.

The website

http://www.achievethecore.org/parent-community-common-core/parent-resources/

has more information on the Common Core movement, including detailed information on expectations.

The website

http://www.pta.org/advocacy/content.cfm?ItemNumber=3552

contains numerous items on the Common Core movement including a number of math videos on specific topics, for example, *coherence*. Parental guides from PTA can be found at http://pta.org/parents/content.cfm?ItemNumber=2910

The website

http://everydaymath.uchicago.edu/parents/

has information for parents on curriculum topics at all grade levels. Of particular interest to parents is the material on alternative algorithms for performing standard computations. Short videos demonstrate the use of these methods, for example, the partial sums method, for performing computations. It also provides access to on-line learning games such as **Bunny Count** and **Connect the Dots**. We will refer to this site as EDM.

3.4.1 Free Websites Containing Worksheets

The website

http://www.math-aids.com/

contains the following content description:

> The website contains over 72 different math topics with over 847 unique worksheets. These math worksheets may be customized to fit your needs and may be printed immediately or saved for later use. These math worksheets are randomly created by our math worksheets, so you have an endless supply of quality math worksheets at your disposal. These high quality math worksheets are delivered in a PDF format and include the answer keys. Our math worksheets are free to download, easy to use, and very flexible. These math worksheets are a great resource for Kindergarten through 12th grade. A detailed description is provided in each math worksheets section.

The home page at this site has a list of topics on the left. Clicking on a topic produces a worksheet page that begins with a description of the types of worksheet. The *Kindergarten* button produces worksheets on a variety of entry-level topics. Across the top is a list of buttons, one of which is a site map which is very useful. There is also a button leading to links to other resources, most of which are commercial. We will refer to this site as M-A.

The website

http://www.superkids.com/aweb/tools/math/

has an interactive engine that will produce worksheets on a fairly complete list of computational topics from arithmetic. It also covers pre-algebra and exponents. Access is free. The following is from the site description:

> Have you ever wondered where to find math drill worksheets? Make your own here at SuperKids for free! Simply select the type of problem, the maximum and minimum numbers to be used in the problems, then click on the button! A worksheet will be created to your specifications, ready to be printed for use. We will refer to this site as SKids.

The website

http://www.softschools.com/

provides math worksheets and interactive games on an extensive list of topics. There is a lot of interactive stuff on counting, so it's good for Pre-K. The following is from their site description

> SoftSchools.com provides free math worksheets, free math games, grammar quizzes and free phonics worksheets and games. Worksheets and games are organized by grades and topics. These printable math and phonics worksheets are auto generated. There are many counting games at Pre-K level. We will refer to this site as SS.

The website

http://www.kidzone.ws/

has **free** printable worksheets. Unfortunately, the items covered are limited in scope to counting, the basic operations, and word problems related to these computations. The grade levels covered are Pre-K to Grade 5. The site does not have material on fractions.

The website

http://www.pbs.org/parents/education/math/

has lots of general education stuff. There are on-line games for children and their parents. It is probably more valuable to new parents and parents of the very young.

3.4.2 Subscription Websites Containing Worksheets

The website

http://www.adaptedmind.com/Math-Worksheets.html?type=hs

has worksheets organized by grade level. The list of topics is extensive and parents will have to pick and choose. There is useful material on place value in topics for Grades 1 and 2. Not free: $10/mo.

The website

http://edhelper.com/math.htm

offers extensive list of worksheets and activities covering K-12 at a cost of $19.99 per/year for a limited subscription and $39.99 for a complete subscription.

The website

http://themathworksheetsite.com/

has worksheets available on all aspects of arithmetic. A subscription is $25.

The website

http://ca.ixl.com/

contains practice problems that are interactive. It is comprehensive and geared to the Canadian curriculum. For this reason parents will need to be choosey about which activities they ask their child to undertake. As long as parents focus on curriculum elements from the CCSS-M, there's lots of great interactive material to provide practice for your child. Unfortunately, access is not free, but twenty free worksheets can be had from http://www.math-drills.com/ which appears to be a subsidiary of IXL. Other subsidiary sites are

http://www.mathsisfun.com/worksheets/
http://www.dadsworksheets.com/
http://www.mathworksheetsland.com/