Contents

1 Basic Riemannian Geometry

1.1 Differentiable Manifolds

Intuitively, manifolds describe topological spaces which locally look like subsets of $\mathbb{R}^n$. This means that a neighbourhood of any point in a manifold can be mapped in a continuos way to an open set in $\mathbb{R}^n$. The whole manifold can be thought of loosely as these sets 'sewn' together in a smooth way. Manifolds generalize the notion of a surface in $\mathbb{R}^n$ (indeed there are certain theorems which establish that one can always think of a manifold as a surface for sufficiently large n). The canonical example is the sphere S^2. This structure allows us to carry forward familiar concepts from calculus on $\mathbb{R}^n$ such as differentiation, integration, and geometry (lengths, angles, volumes). The topology (global properties) of a manifold can be quite different to $\mathbb{R}^n$.

SInce we are more interested in applications of differentiable manifolds to physics, we will not be overly concerned with rigour but focus on what we need.

First recall that an *open set* in $\mathbb{R}^n$ is a subset that can be written as union of *open balls*. An open ball $B_r(x_0)$ centred at the point x_0 of radius $r > 0$ consists of the points that lie within a ball Euclidean distance r from x_0:

$$B_r(x_0) = \{x \in \mathbb{R}^n : |x - x_0| = \left[(x^1 - x_0^1)^2 + \ldots + (x^n - x_0^n)^2\right]^{1/2} < r\} \tag{1}$$

This is natural extension of an open interval (a, b) in $\mathbb{R}$. The open balls form a basis for the topology of $\mathbb{R}^n$. They allow us to talk about continuity of functions, and in turn calculus.

Definition. *An n dimensional smooth manifold is a set M and a collection of subsets $\{O_\alpha\}$ of M such that*

1. $\cup_\alpha O_\alpha = M$ *(i.e. every point $p \in M$ belongs to at least one of these O_α).*

2. *For each α, there is a bijection (1-1 and onto) map $\phi_\alpha : O_\alpha \to U_\alpha$ where U_α is an open set of $\mathbb{R}^n$. The maps (O_α, ϕ_α) are called* coordinate systems *or* charts*. The collection of charts $(O_\alpha, \{\phi_\alpha\})$ is called an atlas.*

3. *(Transition maps) If $O_\alpha \cap O_\beta \neq \phi$ then the map $\phi_\beta \circ \phi_\alpha^{-1}$ from $\phi_\alpha(O_\alpha \cap O_\beta) \subset U_\alpha \to \phi_\beta(O_\alpha \cap O_\beta) \subset U_\beta$ has to be smooth (infinitely differentiable) as a map from $\mathbb{R}^n \to \mathbb{R}^n$. This condition means that the patches of M are glued together in a smooth way. The compositions $\phi_\beta \circ \phi_\alpha^{-1}$ are called transition functions, or 'coordinate transformations'.*

Given a point $p \in M$, the map $\phi_\alpha(p) \in (x^1(p), x^2(p), \ldots x^n(p)) \in \mathbb{R}^n$ defines a point in $\mathbb{R}^n$. We usually refer to $x^a(p)$ as the 'coordinates of p' . In concrete application, there is a natural chart that covers most of the spacetime manifold that we are interested in. However, there will sometimes be regions of a spacetime for which we must pass to another chart because our original chart fails. Such a situation occurs in the study of the event horizon of a black hole.

It is clear that M may admit many atlases, and we do not want all these different possibilities to define a different manifold. Thus we also require that the open cover $\{O_\alpha\}$ is *complete* or *maximal* in the sense that all charts compatible with requirements (2) and (3) are included. The construction

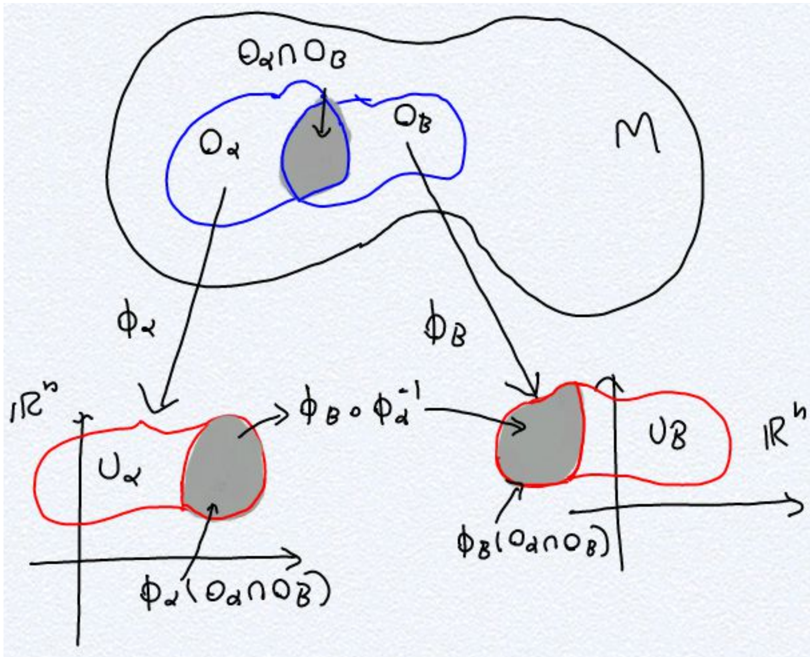

Figure 1: Two charts and their region of overlap. The map $\phi_\beta \circ \phi_\alpha^{-1}$ is smooth as a map from $\mathbb{R}^n \to \mathbb{R}^n$.

naturally defines a topology of M as follows: a set $A \subset M$ is an open set if $\phi_\alpha(A \cup O_\alpha)$ is an open set in $\mathbb{R}^n$ for all α. In this topology, the sets O_α are open and the maps ϕ_α are homeomorphisms.

The manifolds M we consider are topological spaces which are Haussdorff and have a countable basis (M can be covered by a countable number of charts) . The former condition is needed to show that the limit of a converging sequence is unique. The second is needed to define a *partition of unity*, which allows us to extend certain locally defined notions, like integration, to all of M. You are not responsible for these terms.

Example. Consider $\mathbb{R}^n$. It can be covered by a single coordinate chart, the familiar Cartesian one. In other words we can identify any point in $\mathbb{R}^n$ by its cartesian coordinates $x^a(p)$. Of course there are other charts - take $n = 2$ for simplicity. Then we can also use the polar coordinate chart (r, θ) with the transition functions

$$(x, y) = (r \cos\theta, r \sin\theta) \tag{2}$$

with inverse $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}(y/x)$. Now condition (3) demands that the transition functions above be smooth functions as maps from $\mathbb{R}^2 \to \mathbb{R}^2$. By looking at the expression for r defined above, it is clear that the function $r(x, y)$ fails to even be differentiable once at $(x, y,) = (0, 0)$. Hence the polar coordinate chart does not cover the origin, and $r > 0$ Also, the function $\theta(x, y)$ fails to be continuous as one moves in a circle around the origin (i.e. it jumps by 2π as the $x-$axis is crossed). So really polar coordinates only cover $\mathbb{R}^2$ with the semi infinite axis $(x, 0) : x \geq 0$ removed.

Example. A simple example is the circle S^1 which we can think of the locus of points with $x^2 + y^2 = 1$. We need two charts to cover it. If we tried to cover it with one chart, say (O, θ), suppose $\theta(p) = 0$ where p is the point at the farthest right on the circle. Now we move around the circle and return to p. The map has to be one to one, so we cannot continuously assign p a value. We could assign $\theta \in [0, 2\pi)$ but this is not an open interval of $\mathbb{R}$. The remedy is to define a chart (O_1, θ_1) with image $(0, 2\pi)$ and (O_2, θ_2) which maps to $(-\pi, \pi)$. The transition function on the overlap region $S^1 - \{(-1, 0) \cup (1, 0)\}$ would be $\theta_2 = \theta_1$

Example. We now consider an important and non-trivial example: the sphere S^2. We will take a unit sphere centred at the origin $x^2 + y^2 + z^2 = 1$. We will use two charts (O_1, ϕ_1) and (O_2, ϕ_2). O_1 consists of S^2 with the North Pole $(0, 0, 1)$ removed; and the O_2 is S^2 with the South Pole $(0, 0, -1)$ removed. We will use the stereographic projection to map O_1 to an open set $U_1 \subset \mathbb{R}^2$. Draw a line passing through $(0, 0, 1)$, intersecting the sphere at a point, that intersects the plane $z = -1$ at $(\xi^1, \xi^2, -1)$. In this way we associate any point on O_1 to a point $(\xi^1, \xi^2) \in \mathbb{R}^2$. Explicitly one can check this nap is given by

$$\phi_1(x, y, z) = (\xi^1, \xi^2) = \left(\frac{2x}{1-z}, \frac{2y}{1-z}\right) \tag{3}$$

To see this fix a point (x, y, z) on O_1 and a straight line $r(t) = (0, 0, 1 - t) + t(x, y, z)$. This intersects the $z = -1$ plane when $t = 2(1 - z)^{-1}$. Thus we read off (ξ^1, ξ^2) as given above. Note that $(\xi^1, \xi^2) = (0, 0)$ corresponds to the South Pole $(0, 0, -1)$ of S^2. Note that

$$(\xi^1)^2 + (\xi^2)^2 = 4\frac{1+z}{1-z} \to z = \frac{(\xi^1)^2 + (\xi^2)^2 - 4}{(\xi^1)^2 + (\xi^2)^2 + 4} \tag{4}$$

Similarly we can define a chart on O_2 by drawing a line starting from $(0, 0, -1)$ that intersects a

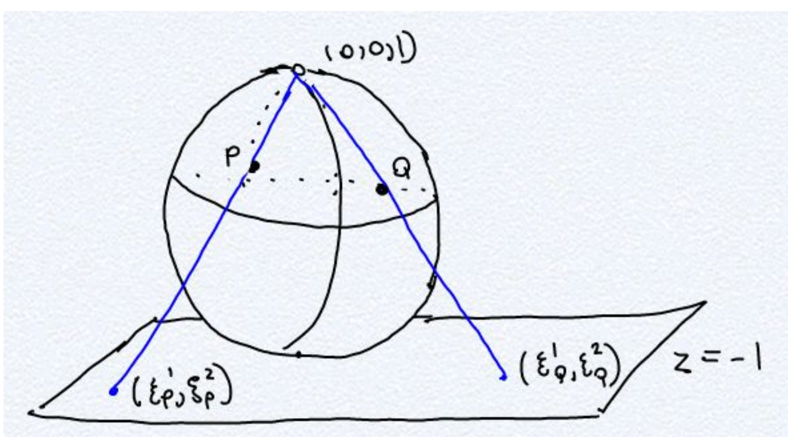

Figure 2: Stereographic projection of $S^2 - \{(0, 0, 1)\}$ to the plane with coordinates (ξ^1, ξ^2).

point (x, y, z) on O_2 and intersects the $z = +1$ plane at $(\psi^1, \psi^2) \in \mathbb{R}^2$, giving

$$\phi_2(x, y, z) = (\psi^1, \psi^2) = \left(\frac{2x}{1+z}, \frac{2y}{1+z}\right) \tag{5}$$

Clearly all the points on S^2 belong to one of these charts. We can also compute the transition function 'change of coordinates' defined on the overlap region $S^2 - \{(0,0,1) \cup (0,0,-1)\}$:

$$\phi_2 \circ \phi_1^{-1} = (\psi^1(\xi^a), \psi^2(\xi^b)) = \left(\frac{4\xi^1}{(\xi^1)^2 + (\xi^2)^2)}, \frac{4\xi^2}{(\xi^1)^2 + (\xi^2)^2)} \right) \tag{6}$$

Note that on the overlap region, the $(\xi^1)^2 + (\xi^2)^2 > 0$. Thus the map is smooth since it is the quotient of two smooth functions and the denominator never vanishes.

1.2 Functions on Manifolds

The most elementary type of function on a manifold is a scalar function. By this we mean a map $f : M \to \mathbb{R}$ that associates to each $p \in M$ a unique real number $f(p)$. Suppose (O_α, ϕ_α) is a chart with $\phi_\alpha : O_\alpha \to U_\alpha \subset \mathbb{R}^n$. Then the composition $f \circ \phi_\alpha^{-1} : U_\alpha \to \mathbb{R}$ is usual multivariable function of n variables.

Definition. *A function $f : M \to \mathbb{R}$ is smooth iff for any chart, the composition $f \circ \phi^{-1} : U \to \mathbb{R}$ is a smooth function.*

Remark. The space of smooth functions on M is often denoted $C^\infty(M)$. It is a commutative ring (roughly , we can multiply smooth functions f, g to obtain new smooth functions $fg(p) = f(p)g(p)$, although their quotients f/g may not be smooth, since g may vanish on M).

In practice, we are working in a particular chart or coordinate system, so we drop the composition notation and simply use the symbol '$f(x)$' or '$f(x^a)$' to refer to a function on the manifold. Of course care must be taken to ensure that a locally defined function is defined and smooth everywhere, not just in a particular chart. The functions we consider in this course will always be assumed to be smooth in the above sense.

Example. Consider the previous example of the sphere S^2 with the two charts. Define $f(x, y, z) = x$ where x is the Cartesian coordinate on the sphere $x^2 + y^2 + z^2$. This is a smooth function. In terms of the first chart, we see

$$f(\xi^1, \xi^2) \equiv f \circ \phi_1^{-1} = \frac{\xi^1(1 - z(\xi^1, \xi^2))}{2} \tag{7}$$

where $z(\xi^1, \xi^2)$ is the function given in (4). The function is clearly smooth on the part of S^2 covered by O_1 since $z(\xi^1, \xi^2)$ is smooth. On the overlap region $O_1 \cup O_2$, we can find $f \circ \phi_2^{-1}$ by using

$$f(\psi^1, \psi^2) = f \circ \phi_2^{-1} = f \circ \phi_1^{-1} \circ \phi_1 \circ \phi_2^{-1} \tag{8}$$

and the composition $\phi_1 \circ \phi_2^{-1}$ is the 'change of coordinates' from ψ^1 to ξ^1. Some algebra shows that the transformation is given by

$$(\xi^1, \xi^2) = \left(\frac{4\psi^1}{(\psi^1)^2 + (\psi^2)^2}, \frac{4\psi^2}{(\psi^1)^2 + (\psi^2)^2} \right) \tag{9}$$

Thus to get $f(\psi^1, \psi^2)$ simply substitute these expressions for (ξ^1, ξ^2) into the expression for $f(\xi^1, \xi^2)$. The result is also a smooth function, because the composition of smooth functions is again smooth. Of course the overlap region does not include the point $(0, 0, 1)$. To show f is smooth on the whole S^2, we need to show it is smooth there as well. A direct computation shows

$$f \circ \phi_2^{-1} = \frac{\psi^1(1 + z(\psi^1, \psi^2))}{2} \tag{10}$$

where

$$z(\psi^1, \psi^2) = \frac{4 - ((\psi^1)^2 + (\psi^2)^2)}{4 + ((\psi^1)^2 + (\psi^2)^2)} \tag{11}$$

Thus $f \circ \phi_2^{-1}$ is also smooth. We could also compute f is *any* other compatible chart in the atlas by using composition.

The above example shows that even for a simple function on M, care must be taken to show it is globally defined. For simplicity, most of the time we will assume tacitly that functions defined locally in a particular coordinate chart can be extended globally to all of M.

1.3 Vectors

Motivation

Intuitively we think of vectors as 'arrows' with a magnitude and direction. This is helpful in $\mathbb{R}^n$ but not useful in the context of manifolds. More algebraically, we think of vectors as a collection of elements belonging to a vector space. $\mathbb{R}^n$ is special because it is *also* a vector space; for this reason we can identify points in $\mathbb{R}^n$ with vectors (e.g. an arrow from the origin to the point) and then add them and multiply them by constants. In general, M will not be a vector space so we cannot do this.

The correct notion of a vector on a manifold comes form the study of surface S in $\mathbb{R}^n$. At any point $p \in S$ we can define the tangent plane T_pS to be the collection of vectors in $\mathbb{R}^n$ that are tangent to S at p. This is vector space (i.e. sums and scalar multiples of tangent vectors are themselves tangent vectors). Note that T_pS and T_qS are *different* vector space; it makes no sense to add vectors belonging to different vectors spaces together.

How does one actually construct a tangent vector to a surface? Clearly, if $\gamma(t)$ is a curve that lies on the surface, then its tangent vector $\gamma'(t)$ at p belongs to T_pS. By considering all the tangent vectors of possible curves lying on the surface in a neighbourhood of p we construct T_pS. This is motivation for the definition of vectors, and vector fields, on a manifold M.

Definition. *A smooth curve on a differentiable manifold is a smooth function $\gamma : I \to M$ where M is an open interval in $\mathbb{R}$. This means the composition $\phi_\alpha \circ \gamma$ is a smooth function $I \to \mathbb{R}^n$ for all charts ϕ_α.*

This notion will be familiar from vector calculus; for example the curve $\gamma(t) = (\cos t, \sin t, 0)$ describes a unit circle in the $z = 0$ plane centred at the origin. In mathematics it is customary to use t as the parameter on the curve; in GR we will sometimes want to reserve t for one of the spacetime coordinates. For now we will stick to the usual notation and use t.

Now consider a smooth function f on M. The composition $f \circ \gamma : I \to \mathbb{R}$ defines a smooth, 1-variable function, which we can also write as $f(\gamma(t))$. We can differentiate this at $t = 0$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[(f \circ \gamma)(t)\right]\bigg|_{t=0} \tag{12}$$

This is the instantaneous rate of change of f along γ at $t = 0$. Recall from elementary vector calculus, this is written as $\gamma'(0) \cdot \nabla f$. Another way of writing it is as a directional derivative; if $\mathbf{v} = \gamma'(0)$ is the tangent to the curve at $p = \gamma(0)$, we can define the directional derivative: $D_{\mathbf{v}} f = \mathbf{v} \cdot \nabla f$. This is a scalar. We could also interpret it as a linear map from functions to numbers: $\mathbf{v}(f) : f \to \mathbb{R} = \mathbf{v} \cdot \nabla f$. Note that in the second way of looking at it, we can throw away the original curve $\gamma(t)$ and just focus on its tangent vector $\mathbf{v}$ at $t = 0$. This is the motivation for the following definition:

Definition. *Suppose $\gamma(t) : I \to M$ is a smooth curve with $\gamma(0) = p \in M$. The* **tangent vector** *to γ at p is the linear map $X_p : f \to \mathbb{R}$ given by*

$$X_p(f) = \frac{d}{dt}\left[(f \circ \gamma)(t)\right]\bigg|_{t=0} \tag{13}$$

Note that X_p is linear in its arguments, $X_p(f + g) = X_p(f) + X_p(g)$, $X_p(cf) = cX_p(f)$ where c is a constant, and the product rule $X_p(fg) = g(p)X_p(f) + f(p)X_p(g)$ holds.

Let us suppose we are working in an explicit chart ϕ with coordinates $x^a = (x^1, x^2, \ldots x^n)$. The composition $\phi \circ \gamma = x^a(t)$ represents the curve in our coordinate system. Meanwhile the composition $f(x^1, x^2, \ldots x^n) = f \circ \phi^{-1}$ represents the function in this chart. Finally $f \circ \gamma = f \circ \phi^{-1} \circ \phi \circ \gamma = f(x^a(t))$ represents the function f pulled back to the curve. Thus in our coordinate chart, using the above definition and the Chain Rule,

$$X_p(f) = \frac{\mathrm{d}}{\mathrm{d}t}\left[(f \circ \gamma)(t)\right]\bigg|_{t=0} = \frac{\mathrm{d}x^a(t)}{\mathrm{d}t}\bigg|_{t=0}\left(\frac{\partial f(x^b)}{\partial x^a}\right)\bigg|_{\phi(p)} \tag{14}$$

where $\phi(p) = (x^1(0), x^2(0), \ldots x^n(0))$ are the coordinates of p in our chart and we are using the Einstein summation convention. In vector calculus language (using the usual dot product of Euclidean space) this reduces to $\gamma'(t) \cdot \nabla f$.

Proposition 1. *The set of all tangent vectors at p forms an $n-$dimensional vector space, referred to as the tangent space to M at p, denoted $T_p(M)$.*

Proof. See, e.g. John Stewart's *Advanced General Relativity* text. □

Given that T_pM is a vector space, an obvious question is: is there a natural basis for it (i.e. a set of n linearly independent vectors which span the space) such that we can write a general vector as a linear combination of these basis vectors? The expression (14) gives us a natural coordinate basis for T_pM associated to a given chart. To see this, write (14) as

$$X_p(f) = \frac{\mathrm{d}}{\mathrm{d}t}\left[(f\circ\gamma)(t)\right]\bigg|_{t=0} = \left[\frac{\mathrm{d}x^a(t)}{\mathrm{d}t}\bigg|_{t=0}\left(\frac{\partial}{\partial x^a}\right)\bigg|_{\phi(p)}\right] f(x^b) = X^a\frac{\partial}{\partial x^a}f \tag{15}$$

In the proof of Proposition 1 it is shown that the set of n coordinate derivative operators $\partial_a = \partial/\partial x^a$, $a = 1\ldots n$ associated to the chart ϕ form a basis for T_pM. We refer to $X^a = \mathrm{d}x^a(t)/\mathrm{d}t|_{t=0}$ as the *components* of X in this basis. In general, we express an arbitrary vector $V \in T_pM$ as

$$V = V^a\partial_a \tag{16}$$

Note that although we used curves $\gamma(t)$ to define T_pM, we do not need any longer to think of vectors as 'belonging' to a particular curve; they are geometric objects in their own right.

Remark. The coordinate basis $\partial_a = (\partial/\partial x^1, \partial/\partial x^2, \ldots \partial/\partial x^n)$ can be thought of the tangent vectors to 'coordinate curves' that pass through p of the form $\gamma^a(t) = (x_0^1, x_0^2, \ldots x_0^a + t, \ldots x_0^n)$ where $\gamma^a(0) = (x_0^1, x_0^2, \ldots x_0^n)$ are the coordinates of p in the chart. This furnishes just one basis for T_pM. Sometimes it is convenient to choose a non-coordinate basis e_a. In this course we will restrict ourselves to the standard coordinate bases.

In many situations, we may want to work on overlapping charts that cover a neighbourhood of $p \in M$, and so we want to be able to determine an expression for a vector V in one chart in terms of another. Explicitly, suppose ϕ_2 is one chart with coordinates y^a and basis vectors ∂_{y^a} and ϕ_1 is another chart with coordinates x^a and basis vectors ∂_{x^a} (e.g. $y^a = (r, \theta)$ could be polar coordinates on $x^a = (x, y)$ standard Cartesian coordinates). We want to determine the components of a vector V in the first basis in terms of the components of the other basis.

Proposition 2. *Let (O_1, ϕ_1) and (O_2, ϕ_2) be two overlapping charts with coordinates x^a, y^a respectively such that in the overlap region, the change of coordinates is given by $y^a = y^a(x^b)$. Suppose that $V = V^a\partial_{x^a} \in T_pM$ is a vector based at $p \in O_1 \cup O_2$. Then the components V'^a in the coordinate basis ∂_{y^a} are given by*

$$V'^a = \frac{\partial y^a}{\partial x^b}V^b \tag{17}$$

Remark. The $n\times n$ matrix $\partial y^a/\partial x^b$ is sometimes referred to as the *Jacobian* of the transformation; sometimes the term is reserved only for the determinant of that matrix. This matrix is invertible in a neighbourhood of the point $\phi_1(p)$ (and $\phi_2(p)$) because the functions $y^a(x^b)$ have an inverse; from the definition of a manifold we can invert these to write $x^a(y^b)$. The inverse is simply $\partial x^a/\partial y^b$.

Proof. Let f be some smooth function. In the chart O_1, $f = f(x^a)$. By definition

$$V(f) = V^a\frac{\partial}{\partial x^a}(f(x^a)) \tag{18}$$

Now in the chart O_2, $f = f(y^a)$. $V(f)$ is a scalar and is independent of the chart chosen to evaluate it. Using the Chain Rule (for multivariable functions) and that change of coordinates $y^a = y^a(x^b)$ (i.e. we know the coordinates y^a in terms of the x^a)

$$V(f) = V^b \frac{\partial}{\partial x^b} f = V^b \frac{\partial y^a}{\partial x^b} \frac{\partial}{\partial y^a} f \tag{19}$$

but this must be equal to $V'^a \partial_{y^a} f$, leading to the above formula for V'^a. □

Remark. In more elementary approaches a vector is defined to be a geometric object that satisfies the above transformation rule.

Remark. Note that basis vectors transform as

$$\frac{\partial}{\partial x^b} = \frac{\partial y^a}{\partial x^b} \frac{\partial}{\partial y^a} \tag{20}$$

Example. Let us take the simple case of $\mathbb{R}^2$. In the standard Cartesian chart $x^a = (x, y)$, the coordinate basis vectors are $\partial_{x^a} = (\partial/\partial x, \partial/\partial y)$. In the polar coordinates basis (r, θ) given by (2) the basis vectors are $(\partial/\partial r, \partial/\partial\theta)$. We can work out the relation between these basis vectors by using the Chain Rule:

$$\frac{\partial}{\partial r} = \left(\frac{\partial x}{\partial r}\right) \frac{\partial}{\partial x} + \left(\frac{\partial y}{\partial r}\right) \frac{\partial}{\partial y} = \frac{x}{(x^2+y^2)^{1/2}} \frac{\partial}{\partial x} + \frac{y}{(x^2+y^2)^{1/2}} \frac{\partial}{\partial y} \tag{21}$$

and

$$\frac{\partial}{\partial \theta} = \left(\frac{\partial x}{\partial \theta}\right) \frac{\partial}{\partial x} + \left(\frac{\partial y}{\partial \theta}\right) \frac{\partial}{\partial y} = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} \tag{22}$$

Equivalently, we can use the vector transformation law applied to the vector $V_1 = \partial/\partial r$ and $V_2 = \partial/\partial\theta$ which respectively have components $V_1^a = (1, 0)$ and $V_2^a = (0, 1)$ in the polar basis, to read off the components of V in the Cartesian basis. Conversely, one can check that

$$\frac{\partial}{\partial x} = \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r} \frac{\partial}{\partial \theta}, \qquad \frac{\partial}{\partial y} = \sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r} \frac{\partial}{\partial \theta} \tag{23}$$

1.4 Differential one-forms (covectors)

From vector calculus one recalls that the *normal* to a surface $f(x, y, z) =$ constant is given by $\vec{n} = \nabla f$. If a tangent vector $\vec{v}$ is tangent to this surface, $\vec{n}(\vec{v}) \equiv \vec{n} \cdot \vec{v} = 0$. As we see below, it is not natural in general to consider the gradient of a function at a point $p \in M$ as a vector belonging to T_pM, but rather a linear map on T_pM into the real numbers (i.e. it belongs to the dual vector space to T_pM).

Definition. *The dual space V^* of a vector space V is the vector space of linear maps from V to $\mathbb{R}$.*

Proposition 3. *If dim $V = n$, then dim $V^* = n$. If $e_a, a = 1 \dots n$ is a basis for V, the* dual basis *of V^* θ^a is defined by $\theta^a(e_b) = \delta^a{}_b$.*

There is a natural (basis independent) isomorphism between V and $(V^*)^*$ so we identify them. Note the placement of the indices: basis vectors are labelled with an index 'downstairs' whereas the components of a vector v in a basis are denoted with an index 'upstairs', and vice versa for elements of the dual vector space. In general, if $v = v^a e_a \in V$ and $\omega = \omega_a \theta^a \in V^*$, then $\omega(v) = \omega_a \theta^a(v^b e_b) = \omega_a v^b \theta^a(e_b) = \omega_a v^b \delta^a{}_b = \omega_a v^a = \omega_1 v^1 + \omega_2 v^2 + \dots \omega_n v^n$.

Example. Consider an $n-$dimensional vector space whose elements are column vectors with constant entries. The dual vector space consists of row vectors. Each row vector provides a linear map from the column vectors to $\mathbb{R}$ (given by usual matrix multiplication of $1 \times n$ and $n \times 1$ matrices.

Definition. *Let T_pM be the tangent space at p. The dual space of this vector space is called the* cotangent space *at p and is denoted by T_p^*M. Elements of T_p^*M are called* co-vectors *or* one-forms*. If e_a is a basis for T_pM with associated dual basis θ^a, we expand a one-form $\omega \in T_p^*M$ in this basis and write $\omega = \omega_a \theta^a$, where ω_a are the components of ω.*

Remark. To determine the components ω_a of a one-form ω in a particular basis, simply act on the basis vectors e_a: $\omega_a = \omega(e_a)$.

An extremely important class of one-forms is provided by the gradient of a scalar function f.

Definition. *Let $f : M \to \mathbb{R}$ be a smooth function. The one-form $df \in T_p^*M$ defined by $(df)(X) = X(f)$, for any $X \in T_pM$, is called the* gradient *or* differential *of f at p.*

In other words, the action of $\mathrm{d}f$ on a tangent vector X is simply the directional derivative of f in the direction X.

Proposition 4. *Let (O, ϕ) be a chart on M with coordinates x^a and $p \in O$. The n one-forms $\{dx^a\}$ is the dual basis of T_p^*M associated to the coordinate basis vectors $\{\partial_{x^a}\}$ of T_pM.*

Proof. By definition of the differential,

$$\mathrm{d}x^a \left(\frac{\partial}{\partial x^b} \right) \bigg|_p = \frac{\partial x^a}{\partial x^b} \bigg|_p = \delta^a{}_b \tag{24}$$

□

In certain cases it is useful to work with non-coordinate bases, but in this course we will focus mostly on the coordinate basis one-forms and expand a general one-form as $\omega = \omega_a \mathrm{d}x^a$. Then if $X \in T_pM$ is a vector, $\omega(X) = \omega_a X^a \in \mathbb{R}$. The action of computing a scalar quantity by acting with a one form on a vector is referred to as *contraction*. Notice we indicate a contraction in the index notation by summing over *repeated upper and lower indices*.

What are the components in the coordinate basis of $\mathrm{d}f$? These are easy to compute: in a chart, $f \circ \phi^{-1}$ is simply $f = f(x^1, x^2 \ldots)$ and simply acting on the basis vectors ∂_{x^a} gives

$$\mathrm{d}f(\partial_{x^b}) = (\mathrm{d}f)_a \mathrm{d}x^a(\partial_{x^b}) = (\mathrm{d}f)_a \delta^a{}_b = (\mathrm{d}f)_b \tag{25}$$

but this must be equal to, by definition of the differential,

$$\mathrm{d}f(\partial_{x^b}) = \frac{\partial f}{\partial x^b} \tag{26}$$

hence $(\mathrm{d}f)_a = \partial_a f$, so the components of $\mathrm{d}f$ are simply the partial derivatives in each coordinate direction. This is why this one-form is called the gradient of f, which you are familiar with from calculus on $\mathbb{R}^n$. We will see below why gradients can be identified with vectors in Euclidean space.

Example. In the standard Cartesian chart of $\mathbb{R}^2$, $\{\mathrm{d}x, \mathrm{d}y\}$ are the basis one-forms whereas in polar coordinates , $\{\mathrm{d}r, \mathrm{d}\theta\}$ are the basis one-forms.

Finally, we turn to the transformation rules for components of one-forms. Suppose (O_1, ϕ_1) and (O_2, ϕ_2) are two charts with coordinates x^a and y^a respectively. The transition functions $\phi_2 \circ \phi_1^{-1}$ are explicitly given by $y^a = y^a(x^1, x^2, \ldots x^n) = y^a(x^b)$. It is easy to see how the basis one-forms $\mathrm{d}y^a$ transform: treating y^a as functions of x^a, we have by definition of the gradient,

$$\mathrm{d}y^a = \frac{\partial y^a}{\partial x^b}\mathrm{d}x^b \tag{27}$$

This gives the transformation rule for the basis one-forms. Note the placement of indices; the basis one-forms transform in the same way as the *components* of vectors. From this fact we can easily find the transformation rule for one-forms. If $\omega = \omega'_a \mathrm{d}y^a$ in (O_2, ϕ_2) and $\omega = \omega_a \mathrm{d}x^a$ in (O_1, ϕ_1) then on overlap regions,

$$\omega'_a \mathrm{d}y^a = \omega'_a \frac{\partial y^a}{\partial x^b}\mathrm{d}x^b = \omega_b \mathrm{d}x^b \to \omega_b = \omega'_a \frac{\partial y^a}{\partial x^b} \tag{28}$$

Alternatively we can write

$$\omega'_a = \frac{\partial x^b}{\partial y^a}\omega_b \tag{29}$$

upon inverting the above expression. Compare this to the transformation rule for vector components given in (17).

Example. Returning to polar coordinates, it is simply to see

$$\mathrm{d}x = \cos\theta \mathrm{d}r - r\sin\theta \mathrm{d}\theta \qquad \mathrm{d}y = \sin\theta \mathrm{d}r + r\cos\theta \mathrm{d}\theta \tag{30}$$

What is the $\mathrm{d}r(\partial/\partial x)$ and $\mathrm{d}r(\partial/\partial y)$? To do this, you can either use the above expression for $\mathrm{d}r$ in terms of $\mathrm{d}x, \mathrm{d}y$, or alternatively express the Cartesian basis vectors in terms of polar basis vectors.

1.5 Tensors

One-forms are linear maps $T_pM \to \mathbb{R}$ and vectors can be viewed as linear maps $T_p^*M \to \mathbb{R}$. More generally, we have

Definition. *A type* (r,s) tensor *at* $p \in M$ *is defined to be a multilinear map* $T_p^*M \times \ldots \times T_p^*M \times T_pM \times \ldots \times T_pM$ *where there are* r *factors of* T_p^*M *and* s *factors of* T_pM*. Here multilinear means the map is linear in each of its* $r+s$ *arguments.*

Remark. We can regard vectors as $(1,0)$ tensors, and one-forms as $(0,1)$ tensors.

Let T be a $(1,1)$ tensor. Its components in a coordinate basis are obtained by finding its action on the basis coordinate vectors and basis one-forms:

$$T^a_{\ b} = T(\mathrm{d}x^a, \partial_{x^b}) \tag{31}$$

We also slightly generalize the above definition by distinguishing tensors by the *order* of their arguments. The above example of a $(1,1)$ tensor is a multilinear map: $T_p^*M \times T_pM \to \mathbb{R}$. We distinguish this from a tensor $S : T_pM \times T_p^*M \to \mathbb{R}$ which would have components $S(\partial_{x^a}, \mathrm{d}x^b) = S_a^{\ b}$. We write these objects geometrically as

$$T = T^a_{\ b}\frac{\partial}{\partial x^a} \otimes \mathrm{d}x^b \qquad S = S_a^{\ b}\mathrm{d}x^a \otimes \frac{\partial}{\partial x^b} \tag{32}$$

As usual repeated indices are summed over. The symbol $\otimes$ represents the tensor product. In general all the equations we study will be tensor equations, relating tensors of the *same* type. It is then conventional to drop writing the explicit basis one forms and basis vectors, and simply just using tensorial indices. In other words, if X and V are two vectors that are equal, we simply write $X^a = V^a$ with the understanding that this equation, being tensorial, holds in all bases. This is useful when doing calculations and is called the *abstract index convention.*

Example. The most important example of a $(0,2)$ tensor is the *metric tensor* which we will study in great detail. This is a symmetric, multilinear map $g : T_pM \times T_pM \to \mathbb{R}$, so if $V, W \in T_pM$, then $g(V,W) = g(W,V)$ is a real number. In Euclidean geometry using Cartesian coordinates, its components g_{ab} form the identity matrix and we identity $g(V,V)$ as the square of the length of V. Singling out a special $(0,2)$ tensor in this way gives a manifold geometric structure.

1.6 Smooth tensor fields on Manifolds

The geometric objects defined so far are all based at a point $p \in M$. Of course, we are interested in studying field which vary from point to point on M.

Definition. *A vector field* X *is a map that associates any point* $p \in M$ *to a vector* $X_p \in T_pM$*. We say a vector field* X *is smooth if, given any smooth function* f*, the scalar function* $X(f) : M \to \mathbb{R}$ *defined by* $X(f) : p \to X_p(f)$ *is itself smooth.*

Remark. The basis vector fields $\{\partial_{x^a}\}$ associated to a chart (O, ϕ) are smooth vector fields in $O \subset M$. If we expand a vector field X in this basis,

$$X = X^a \left(\frac{\partial}{\partial x^a}\right) \tag{33}$$

it follows that X is smooth if and only if the component *functions* X^a are smooth functions.

Remark. The above definition applies in the obvious way to one-forms, and tensor fields, in general. For example, a one-form field ω maps any point $p \in M$ a unique one form $\omega_p \in T_p^* M$. Given a one-form field ω and a vector field X, consider the function $\omega(X) : M \to \mathbb{R}$ defined by $\omega(X) : p \to \omega_p(X_p)$. ω is smooth provided that this scalar function is smooth for any smooth vector field X. A tensor field is smooth if and only if its components in a coordinate chart are smooth functions.

We restrict attention hereafter to smooth tensor fields. Notice that the word 'field' is often dropped when talking about 'vectors', 'one-forms' etc. The expressions for the transformation rules for tensor components discussed above are the same for tensor fields.

1.7 Integral Curves

Given a vector field X and a fixed point $p \in M$, one can consider a curve $\gamma(t)$ passing through p (say $\gamma(0) = p$) such that its tangent vector at any point $\gamma(t)$ on the curve is given by $X(\gamma(t))$. If X represents the velocity vector field of a fluid, then these curves could represent the path of a particle moving along with the fluid.

Definition. *Let X be a smooth vector field on M and $p \in M$ An integral curve of X through p is a curve through p whose tangent at every point is X.*

In a local coordinate chart, the curve is described by a path $x^a(t)$ and without loss of generality, $x^a(0) = x_0$ are the coordinates of p. The integral curve satisfies the initial value problem

$$\frac{\mathrm{d}x^a(t)}{\mathrm{d}t} = X^a(x^b(t)), \qquad x^a(0) = x_0 \tag{34}$$

ODE theory (Picard-Lindelöf theorem) states at least for a small enough interval $t \in (t_1, t_2)$ a unique solution to this problem exists. As a trivial example, in the standard Cartesian coordinate chart of $\mathbb{R}^3$, the integral curve of the vector field $\partial/\partial x$ passing through the origin is simply $x^a(t) = (t, 0, 0)$.

1.8 The Metric Tensor

The central object of study in Riemannian geometry is the *metric tensor* g, which is a symmetric, non-degenerate $(0, 2)$ tensor. The metric tensor endows a smooth manifold M with a geometrical structure that allows for the calculations of lengths, angles, areas, and volumes of submanifolds.

From standard vector calculus on $(\mathbb{R}^3, \delta)$ given a parametrized curve $\mathbf{r}(t)$, $t \in [t_1, t_2]$, we compute the length as

$$L = \int_{t_1}^{t^2} [\mathbf{r}'(t) \cdot \mathbf{r}'(t)] \,\mathrm{d}t = \int_{t_1}^{t^2} [g_E(\mathbf{r}'(t), \mathbf{r}'(t))] \,\mathrm{d}t \tag{35}$$

where $\mathbf{r}'(t) = \mathrm{d}\,\mathbf{r}(t)/\mathrm{d}t$ is the tangent vector to the curve and $\cdot$ denotes the standard inner product with respect to the Euclidean metric g_E defined by $g_E(V, W) = V^1W^1 + V^2W^2 + V^3W^3$, or in index notation, $g_E(V, W) = \delta_{ij}V^iW^j$. In this sense, g_E can be thought of as a $(0, 2)$ tensor field mapping two vectors into $\mathbb{R}$, which we naturally call the inner product. Notice that this is different to the action of a one-form on a vector; for example on a smooth manifold M, $\mathrm{d}f : V \to \mathbb{R} = V(f)$ produces a number without the use of any inner product structure.

Definition. *A metric tensor g at $p \in M$ is a $(0, 2)$ tensor with the properties:*

1. *Symmetry. $g(X, Y) = g(Y, X)$ for all $X, Y \in T_pM$. Thus in any basis $\{e_a\}$ for T_pM, $g_{ab} = g(e_a, e_b) = g(e_b, e_a) = g_{ba}$.*

2. *Non-degeneracy: $g(X, Y) = 0$ for all $Y \in T_pM$ iff $X = 0$.*

Remark. In a coordinate basis of one-forms, note that

$$g = g_{ab}\mathrm{d}x^a \otimes \mathrm{d}x^b \tag{36}$$

where $g_{ab} = g(\partial_{x^a}, \partial_{x^b})$. It is conventional to write this in line element form:

$$\mathrm{d}s^2 = g_{ab}\mathrm{d}x^a\mathrm{d}x^b \tag{37}$$

This should be familiar to you from the way we express the line element form of the Euclidean metric $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$ on $\mathbb{R}^3$. Here $\mathrm{d}s^2$ represents the infinitesimal distance between two points with coordinates $x^a + \mathrm{d}x^a$ and x^a. In general relativity it is customary to express the metric in the line element form. It is easy to read off the components g_{ab} however from the line element: obviously, $g_{ab} = \delta_{ab}$ where δ is the identity 3×3 matrix in this example.

Remark. The 'length squared' of a vector X is given by the scalar $g(X, X)$. The notation $|X|^2$ or X^2 is often used (although care must be taken to ensure that you know X is a vector, not a scalar).

Remark. It is customary to use abstract index notation and refer to the metric tensor by its components as g_{ab}. Since it is non-degenerate, g_{ab} has an inverse denoted g^{ab}, which are the components of a $(2, 0)$ tensor g^{-1}. They are inverses in the sense of standard matrix multiplication,

$$g^{ab}g_{bc} = \delta^a_{\ c} \tag{38}$$

The components of g^{-1} are found by $g^{ab} = g^{-1}(\mathrm{d}x^a, \mathrm{d}x^b)$ in a coordinate basis; in practice one simply inverts the metric components g_{ab}.

As the metric tensor is symmetric, one can choose a basis for T_pM to diagonalize it. None of the eigenvalues can be zero, since it is non-degenerate (i.e $\det g \neq 0$). By rescaling these diagonal elements of g_{ab} can be scaled to be ± 1 (this corresponds to an orthonormal basis). The number of positive and negative elements is independent of the choice of basis as a consequence of a result of Sylvester. This is referred to as the *signature* of the metric.

A Riemannian metric, such as the standard Euclidean metric on $\mathbb{R}^n$ has signature $(+,+,+,\ldots+)$. This means g is *positive definite*: $g(X,X) \geq 0$, with equality if and only if $X = 0$. For the moment we will focus on Riemannian metrics. We have already seen that the Minkowski metric on $\mathbb{R}^{3,1}$ has signature $(-,+,+,+)$. Metrics with this signature (i.e. one 'minus' sign) are referred to as Lorentzian metrics. In general one refers to a metric with signature other than these types as a pseudo-Riemannian metric.

Definition. *A Riemannian (Lorentzian) manifold (M,g) is a smooth manifold M equipped with a Riemannian (Lorentzian) metric tensor field. Lorentzian manifolds are also referred to as spacetimes.*

Definition. *On a Riemannian manifold, the* norm *or* length *of a vector X is $|X| = (g(X,X))^{1/2}$ and the angle between vectors X, Y is determined by $\cos\theta = g(X,Y)/(|X||Y|)$.*

In the Lorentzian case, our experience with special relativity indicates that there are three cases to consider:

Definition. *On a Lorentzian manifold, a non-zero vector X is said to be* timelike *if $g(X,X) < 0$,* null *if $g(X,X) = 0$, and* spacelike *if $g(X,X) > 0$.*

Remark. In practice one just refers the scalar $|X|^2 = g(X,X)$ as 'X squared'.

Proposition 5. *Suppose in a chart (O_1, ϕ_1) with coordinates x^a the components of g are given by g_{ab} and in an overlapping chart (O_2, ϕ_2) with coordinates y^a, the components of g are denoted by g'_{ab}. These are related by*

$$g_{ab} = \frac{\partial y^c}{\partial x^a}\frac{\partial y^d}{\partial x^b} g'_{cd} \tag{39}$$

where $y^a = y^a(x^b)$ represents the transition functions $\phi_2 \circ \phi_1^{-1}$.

Proof. Exercise. □

When transforming a metric into a different coordinate system, it generally easiest to use the invariance of the line element:

$$\mathrm{d}s^2 = g_{ab}\mathrm{d}x^a\mathrm{d}x^b = g'_{ab}\mathrm{d}y^a\mathrm{d}y^b \tag{40}$$

and then simply calculate the differentials $\mathrm{d}y^a$ in terms of $\mathrm{d}x^a$.

Example. Euclidean space $(\mathbb{R}^2, g_E)$ has the metric

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 \tag{41}$$

in Cartesian coordinates. In polar coordinates, simply note $\mathrm{d}x = \cos\theta \mathrm{d}r - r\sin\theta \mathrm{d}\theta$ and $\mathrm{d}y = \sin\theta \mathrm{d}r + r\cos\theta \mathrm{d}\theta$. Squaring and adding these expressions gives

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2 \mathrm{d}\theta^2 \tag{42}$$

Of course we are not allowed to 'square' the one-forms $\mathrm{d}x$ and $\mathrm{d}y$; we are actually really taking their tensor product $\otimes$, using the fact that g is symmetric so $g_{xy}\mathrm{d}x \otimes \mathrm{d}y = g_{yx}\mathrm{d}y \otimes \mathrm{d}x$, etc. and then going back to the line element notation. In practice however, it is simplest to just formally follow this procedure.

Example. The so-called 'round' metric on S^2 (induced from the surface $x^2 + y^2 + z^2 = 1$ in $\mathbb{R}^3$) has the metric

$$\mathrm{d}s^2 = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2 \tag{43}$$

where $\theta \in (0, \pi)$ is the polar angle and $\phi \in (0, 2\pi)$ is the azimuthal angle running around the equator. We read off trivially that $g_{\theta\theta} = 1, g_{\phi\phi} = \sin^2\theta, g_{\theta\phi} = 0$. Notice the metric fails to be invertible at $\theta = 0, \pi$. This is a sign that the (θ, ϕ) chart has broken down at the poles. This is expected, because we know that we cannot cover S^2 with a single chart. However, it can be shown that the above metric can be smoothly extended to cover all of S^2 by choosing new charts around the poles.

Example. Minkowski spacetime is equipped with the familiar metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \tag{44}$$

in the standard coordinate chart. If we pass from Cartesian coordinates to spherical coordinates on $\mathbb{R}^3$ (exercise) we find

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right) \tag{45}$$

Here $r \in \mathbb{R}$ is the radial coordinate and surface of constant t and $r = R$ =constant is an S^2 with the metric $g_R = R^2 g_{S^2}$ where g_{S^2} is given above.

Example. A famous example of a metric we will study later is the Schwarzschild metric on $\mathbb{R}^{3,1} - 0$. This is a Lorentzian metric

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - \frac{2M}{r}} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right) \tag{46}$$

Here $r \in (2M, \infty)$, $t \in \mathbb{R}$, and θ, ϕ are coordinates covering S^2 as in the above example. Physically, this represents the spherically symmetric geometry produced by a point gravitational mass at the origin. The metric as $r \to \infty$ approaches the metric of Minkowski spacetime in spherical coordinates (see above). The surface $r = 2M$, where the above metric appears to be singular, corresponds to a black hole *event horizon*. To understand what is going on here, we will need to pass from the above chart with coordinates (t, r, θ, ϕ) to another one that covers the region *inside* the black hole.

Finally, the non-degeneracy of the metric tensor allows one to define a basis-independent, canonical isomorphism between vectors and one-forms.

Definition. *(Raising and lowering) Given a vector field X, consider the one-form defined by $\tilde{X} = g(X,\ )$. The expression on the right hand side means 'leave one argument of g free' and hence defines a one-form (i.e. map from $T_pM \to \mathbb{R}$). In components $\tilde{X}_a = g_{ab}X^b$. On the other hand, given a one-form $\tilde{X}$ we can define a vector field $\hat{X} = g^{-1}(\tilde{X},\ )$, or in components $\hat{X}^a = g^{ab}\tilde{X}_b$. It is easily seen that $\hat{X} = X$ (i.e. the maps are clearly inverses of each other). Explicitly, $\hat{X}^a = g^{ab}\tilde{X}_b = g^{ab}g_{bc}X^c = \delta^a{}_c X^c = X^a$ where we used the property $gg^{-1} = g^{-1}g = I$ where I is the identity matrix.*

Remark. The above shows that there is a canonical (natural) way to identity vectors and one-forms using the metric tensor. It is the convention to use *the same symbol* for a vector and one-form associated in this way and talk about 'raising' and 'lowering' their index. In other words if X^a represents a vector, then $X_a = g_{ab}X^b$ represents its associated one-form and vice versa. The metric tensor is used to raise and lower indices in this way of general tensors, i.e. if T is a $(0,3)$ tensor, then

$$T^a_{\ bc} = g^{ad}T_{dbc} \qquad T_{ab}{}^c = g^{cd}T_{abc} \tag{47}$$

defines different $(1,2)$ tensors obtained by raising the first index and the third index respectively.

Example. Now consider the Euclidean metric on $\mathbb{R}^3$. Given a function f, we can define the differential $\mathrm{d}f$, or equivalently in components, $\mathrm{d}f_a = \partial_a f$. Since the metric is just the identity matrix, raising an index does not actually change any of the components of $\mathrm{d}f$. The vector $(\mathrm{d}f)^a = g^{ab}(\mathrm{d}f)_b$ is really what one means by the 'gradient vector field'. However, both the one-form and vector have exactly the same components. Hence in vector calculus we do not distinguish them. However, you will surely have noticed that in spherical or cylindrical coordinates, there will be a difference when viewing $\mathrm{d}f$ as a vector as opposed to a vector. This is the reason why expressions for the curl and divergence are somewhat more complicated in these other coordinate systems for $\mathbb{R}^3$.

Example. Let us consider the metric on S^2 given above. Let $V = \partial/\partial\phi$, or in components $V^a = (0,1)$ (we order the basis as $x^1 = \theta, x^2 = \phi$). Then $V_a = g_{ab}V^b = g_{a\theta}V^\theta + g_{a\phi}V^\phi = g_{a\phi}$. Thus $V_a = (0, \sin^2\theta)$ since $g_{\phi\phi} = \sin^2\theta$. In other words the associated one-form is $V = V_a\mathrm{d}x^a = \sin^2\theta\mathrm{d}\phi$. Now let $\omega = \mathrm{d}\phi$ be a one-form. You should be able to verify that as a vector, $\omega = \csc^2\theta\partial_\phi$ since $g^{\phi\phi} = 1/\sin^2\theta$.

Example. Ih the Minkowski metric, the signature difference also introduces minus signs. For example, if $\omega = \mathrm{d}t$ is a one-form ($\omega_a = (1,0,0,0)$) then $\omega^a = (-1,0,0,0)$ or $\omega = -\partial/\partial t$. This is because $g^{tt} = -1$.

Remark. It is probably now clear that sometimes it will be unclear as to whether one is using a particular symbol to refer to a vector or one-form. In tensorial equations, all indices on both sides must 'match up' so that there is no confusion. However, some mathematics texts use the 'musical' notation of sharps and flats to refer to raised and lowered objects. So if ω is a 1-form, then $\omega^\#$ is the vector field $g^{-1}(\omega,\ )$ and so on.

1.9 Geodesics as Curves of Extremal Length

Let us now consider curves $\gamma : (a, b) \to M$ on (M, g). Such a curve with have a tangent vector T at each point along it.

Definition. *A curve in a Lorentzian manifold is called* timelike *if T is everywhere timelike, and analogously for null and spacelike curves.*

Remark. Of course some curves might change their character (i.e. the tangent changes from timelike to null). On a Riemannian manifold, all curves are spacelike in character since $g(T, T) > 0$.

In this section we will discuss a special set of curves which are critical points (i.e. extrema) of a certain functional defined on the set of all curves joining two fixed points.

For the moment, we will mostly pay attention to Riemannian manifolds and get to the Lorentzian case when we start discussing general relativity in detail. On a Riemannian manifold (M, g) we can define the length of a curve C. Let $\gamma : (a, b) \to M$ be be a parameterization of C with tangent vector X and start and endpoints $\gamma(a)$ and $\gamma(b)$ respectively. The length of the curve is

$$\text{Length} = \int_C \mathrm{d}s = \int_a^b \sqrt{g(X, X)}\mathrm{d}t \tag{48}$$

It can be shown that this definition is independent of the parametrization used to describe the curve. Given two fixed points $p, q \in M$, one can ask whether there exists a curve of extremal length amongst all possible smooth paths joining these points. Proving the existence of such a extremal curve (and further whether it is a minimizer of length) is a subtle question that requires global considerations. However, if we *assume* that an extremal curve exists, one can use techniques from the calculus of variations to determine this curve.

Variational Calculus and the Euler-Lagrange equations

Proposition 6. *Given a functional $S[x]$ on the class of admissible smooth function $x(t)$ with fixed boundary values $x(t_2) = x_2, x(t_1) = x_1$ of the form*

$$S[x] = \int_{t_1}^{t_2} L(x, \dot{x}, t)\, dt\,, \qquad \dot{x} \equiv \frac{dx}{dt} \tag{49}$$

the function $x(t)$ which extremize S, assuming it exists, must satisfy the Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \tag{50}$$

Proof. Given in lectures □

Remark. In classical mechanics problems, the function $L = L(x, \dot{x}, t)$ is called the Lagrangian function. However variational techniques apply more generally than only to mechanics problems. We will refer to it as the Lagrangian.

Proposition 7. *Suppose the Lagrangian in the above problem is independent of x. Then for solutions $x(t)$ of the Euler-Lagrange equations,*

$$\frac{\partial L}{\partial \dot{x}} = C \tag{51}$$

where C is a constant. This is referred to a conserved momentum, where $p = \partial L/\partial \dot{x}$ is the associated momentum.

Proposition 8. *Consider the above variational problem for the action functional $S[x]$. Suppose L has no explicit dependence on t, i.e. $\partial L/\partial t = 0$. Then for solutions of the Euler-Lagrange equations, we have the conservation law*

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}}\dot{x} - L\right] = 0 \tag{52}$$

Proof. Direct computation and use the Euler-Lagrange equation. □

Remark. The associated conserved quantity in mechanical systems can be thought of as the Hamiltonian (energy) function, which is the Legendre transform of the Lagrangian L. For some Lagrangians the equation

$$\frac{\partial L}{\partial \dot{x}}\dot{x} - L = C \tag{53}$$

is a *first order* equation in $x(t)$ and hence easier to integrate directly than the Euler-Lagrange equations.

Proposition 9. *(Multiple functions) Suppose a functional depends on a set of functions $x^a(t), a = 1 \ldots n$ and their derivatives $\dot{x}^a$, that is*

$$S[x] = \int_{t_1}^{t_2} L(x^a, \dot{x}^a, t)\, dt\ . \tag{54}$$

with $x^a(t_1) = x^a_1, x^a(t_2) = x^a_2$ fixed. Then there are n Euler-Lagrange equations, one for each function:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^a}\right) = \frac{\partial L}{\partial x^a} \tag{55}$$

We will now apply this to our extremization problem for the length functional:

$$S[x^a] = \int_{t_1}^{t_2} L(x^a, \dot{x}^a)\ \mathrm{d}t\ , \qquad L = \sqrt{g_{ab}\dot{x}^a\dot{x}^b} \tag{56}$$

Note that this Lagrangian function has no explicit t dependence, i.e. $\partial L/\partial t = 0$. This simplifies a great deal of calculations.

Proposition 10. *The Euler-Lagrange equations following from varying the functional* (56) *are*

$$\ddot{x}^a + \Gamma^a_{bc}\dot{x}^a\dot{x}^b = 0 \tag{57}$$

and is known as the geodesic equation. Here

$$\Gamma^a_{bc} = \frac{g^{ad}}{2}\left(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc}\right) \tag{58}$$

are the Christoffel symbols *associated to the metric g and we are assuming that the parameter t on the extremal curve is proportional to the arclength, so that $g_{ab}\dot{x}^a\dot{x}^b = C > 0$.*

In practice for an explicit metric it is much easier to use a slightly different Lagrangian, which can be interpreted as giving rise to an 'energy' functional:

Proposition 11. *The geodesic equations of motion* (57) *can be derived from varying the functional*

$$S[x^a] = \int_{t^1}^{t^2} L(x^a, \dot{x}^a)\ dt\ , L = g_{ab}\dot{x}^a\dot{x}^b \tag{59}$$

For solutions, the quantity $g_{ab}\dot{x}^a\dot{x}^b$ is constant .

It is this form of the Lagrangian that we will use when discussing the motion of matter and light in general relativity.

Proof. Straightforward computation. To show $g_{ab}\dot{x}^a\dot{x}^b$ is a constant, note that the conserved quantity (52) reduces to $2L$ in this case. □

Example. Let us find solutions to the geodesic equations in $(\mathbb{R}^3, g_E)$ in Cartesian coordinates. The metric is

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \tag{60}$$

which means the associated Lagrangian we will consider is simply

$$L = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 \tag{61}$$

Now solve the 3 Euler-Lagrange equations associated to L. Note that $\partial L/\partial x^i = 0$ for $i = 1, 2, 3$. So we immediately read off

$$\frac{\partial L}{\partial \dot{x}^i} = 2\dot{x}^i = 2c_i \tag{62}$$

where c_i are constants (possibly different for each i.). This is trivial to integrate: $x^i(t) = c_i t + d_i$ where d_i are also constants. It is clear that these curves are simply straight line, as we expect.

Example. A more complicated example is S^2. Using the round metric on S^2 defined previously, we start with

$$L = \dot{\theta}^2 + \sin^2\theta\dot{\phi}^2 \tag{63}$$

We then can find the Euler-Lagrange equations. Firstly notice that the ϕ-equation is simple. Using

$$\frac{\partial L}{\partial \phi} = 0 \rightarrow \frac{\partial L}{\partial \dot{\phi}} = 2\sin^2\theta\dot{\phi} = 2J \tag{64}$$

for some constant J (the factor of '2' is simply put there for convenience). We could now directly solve the θ Euler-Lagrange equation. This is second-order, however, so we will now use a more directly and useful approach. We know that on solutions L itself is a (positive) constant; therefore, we must have

$$C^2 = \dot{\theta}^2 + \frac{J^2}{\sin^2\theta} \tag{65}$$

where we have eliminated $\dot{\phi}$ in favour of J. Without loss of generality we can choose our parameter t to be an arc-length parameter, so that $C = 1$ (i.e. the tangent vector has unit length). Suppose first $J = 0$. Then $\dot{\phi} = 0$ so $\phi = \phi_0$ a constant. Then $\dot{\theta}^2 = \pm 1$ so that $\theta = \pm t + t_0$. Hence our geodesic is given by the coordinate curve $x^a(t) = (t + t_0, \phi_0)$. This curve represents a meridian (a line of longitude) on S^2 (note that it is a great circle). Next, suppose $J \neq 0$. We must solve two (non-linear) equations, and a nice way to do this is to eliminate the t variable, by writing $\theta(\phi(t))$ and using the Chain rule to give

$$\left(\frac{\mathrm{d}\theta}{\mathrm{d}\phi}\right)^2 = \frac{\dot{\theta}^2}{\dot{\phi}'^2} = \frac{\sin^2\theta}{J^2}\left(\sin^2\theta - J^2\right) \tag{66}$$

This can be immediately integrated to give

$$\phi - \phi_0 = \pm\int \frac{J\mathrm{d}\theta}{\sin\theta\sqrt{\sin\theta - J^2}} \tag{67}$$

We will take the upper sign without loss of generality. This can be solved by setting $u = \cot\theta$ and $\mathrm{d}u = -\csc^2\theta\mathrm{d}\theta$ to give

$$\phi - \phi_0 \;=\; \int \frac{-J\mathrm{d}u}{\sqrt{1 - J^2\csc^2\theta}} \tag{68}$$

since $\sin\theta\csc\theta = 1$. But $1 + \cot^2\theta = 1 + u^2 = \csc^2\theta$ so

$$\phi - \phi_0 \;=\; \int \frac{-J\mathrm{d}u}{\sqrt{1 - J^2(1 + u^2)}} = \int \frac{-\mathrm{d}u}{\sqrt{\alpha^2 - u^2}} \qquad \text{where } \alpha = \frac{\sqrt{1 - J^2}}{J} \tag{69}$$

$$= \arccos(u/\alpha) \tag{70}$$

So the end result is that the geodesics satisfy

$$\cot\theta = \alpha\cos(\phi - \phi_0) = \alpha(\cos\phi\cos\phi_0 + \sin\phi\sin\phi_0) \tag{71}$$

and this can be rewritten

$$\cos\theta = \alpha(\cos\phi\cos\phi_0 + \sin\phi\sin\phi_0)\sin\theta \Leftrightarrow z = ax + by \tag{72}$$

for constants a, b satisfying $a^2 + b^2 = \alpha^2$ and we have used the map from spherical coordinates (see Assignment 2) to write our curve in terms of Cartesian (x, y, z) coordinates in $\mathbb{R}^3$. Recall of course we are on S^2, so $x^2 + y^2 + z^2 = 1$ also holds. So one can easily see that the geodesics are precisely the intersection of S^2 with planes that pass through the origin $(0, 0, 0)$, i.e. the centre of the sphere - these are, of course, the great circles. Note that these geodesics depend on two arbitrary constants (α, ϕ_0))(or alternatively (a, b, α) with one constraint between them) which determine the initial position and velocity.

Timelike and Null geodesics

In general relativity, freely-falling test particles - that is, particles that are not acted on by any external forces - travel in the Lorentzian manifold (M, g) along the the 'straightest lines', or geodesics. We have presented a variational approach to determine geodesics at least locally as critical curves of the energy functional with $L = g_{ab}\dot{x}^a\dot{x}^b$. This is *not* a postulate of the theory, but actually a consequence of the Einstein field equations. In fact the Einstein equations imply that a sufficiently small body (that does not have self-gravity that backreacts against the surrounding spacetime) will move along a geodesic path in the background spacetime. (The 'test' in 'test particles' refers to the fact we can use these bodies to determine the gravitational field, much like we determine the electromagnetic field by considering the motion of test charges).

We can summarize this as follows. We can treat timelike, null, and spacelike geodesics on a similar footing by working with the Euler-Lagrange equations arising from

$$L = g_{ab}\dot{x}^a\dot{x}^b \tag{73}$$

We have shown above that for the energy functional S associated to this Lagrangian, the quantity L itself is a constant along geodesics (we say it is 'conserved' or a 'constant of the motion'). Thus in a given coordinate system, the trajectories of particles are given by[a] $x^a(\lambda)$ where

$$\ddot{x}^a + \Gamma^a_{bc}\dot{x}^a\dot{x}^b = 0\ , \qquad g_{ab}\dot{x}^a\dot{x}^b = \epsilon\ , \qquad \dot{x}^a \equiv \frac{\mathrm{d}x^a}{\mathrm{d}\lambda} \tag{74}$$

where $\epsilon = 0, -C^2, +C^2$ for null, timelike, and spacelike geodesics respectively. It is usually the case that for timelike and spacelike curves we choose our parameter λ on the geodesic to be the proper time or arc-length respectively; in this case $C = 1$. Here we recall that the proper time is defined to be the parameter on the curve with the property

$$g_{ab}\frac{\mathrm{d}x^a}{\mathrm{d}\tau}\frac{\mathrm{d}x^b}{\mathrm{d}\tau} = -1 \tag{75}$$

Much as spacelike geodesics are seen to minimize the distance between two points, timelike geodesics can be thought of as 'maximizing' the proper time between two events in spacetime.

[a]from now on we reserve the variable t for one of the spacetime coordinates, and use λ as a parameter along a curve

Example. It is a simple exercise to show that the geodesics in Minkowski spacetime are simply straight lines: $x^a(\lambda) = u^a\lambda + c^a$ where $x^a(0) = c^a = (c^0, c^1, c^2, c^3)$ represents the particles initial position and u^a is a constant four-vector. For a timelike curve, if the the parameter λ is such that $g(u, u) = u \cdot u = -1$, then $\lambda = \tau$; for a null path, $g(u, u) = 0$; and for a spacelike geodesics, if the parameter λ is such that $g(u, u) = 1$ then we identify λ with the arc-length parameter.

2 Connections and Curvature

2.1 Connections

We need to be able to differentiate vector fields, one-forms, and tensor fields in general in order to describe physics. For scalar fields we have defined the gradient of a scalar field, the one-form $\mathrm{d}f$. For tensor fields the situation is more complicated. Essentially the problem is that differentiation involves computing the *difference* of the values of a tensor field at two different points on M, but these belong to different vector spaces. What one needs is the additional structure of a *connection.* Intuitively this gives a precise way to compare two tensors fields at different points and then decide if they are 'the same' or if not, 'how much' they differ.

In dealing with derivatives of vector fields in $\mathbb{R}^3$, you may be familiar with certain *combinations* of partial derivatives, e.g. $\nabla \cdot V$ and $\nabla \times V$. These could be thought of as subsets of a general matrix of partial derivatives $\nabla_i V^j$ (so, for example, the divergence of V would be the trace of this matrix). This is the motivation for the chart-independent definition.

Definition. *A covariant derivative ∇ on a smooth manifold M is a map that sends the pair of smooth vector field X, Y to a new smooth vector field $\nabla_X Y$ such that, if f, g are functions,*

1. $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$
2. $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$ *(linearity)*
3. $\nabla_X(fY) = f\nabla_X Y + (\nabla_X f)Y$ *(product rule)*

and $\nabla_X f \equiv X(f)$, i.e. the usual directional derivative.

Remark. We say $\nabla_X Y$ is the 'covariant derivative of Y along X'. Intuitively, it is the first-order change in Y if one moves infinitesimally along a curve whose tangent is X.

Remark. for *fixed* Y, the map $\nabla Y : X \to \nabla_X Y$ is a linear map from T_pM to itself. So one can think of ∇Y as a $(1, 1)$ tensor (i.e. given a vector X and a one-form ω, it produces the real number $\omega(\nabla_X Y)$.

Definition. *The covariant derivative of the vector field Y is the* $(1, 1)$ *tensor field ∇Y. In components this is written $(\nabla Y)^a{}_b$, but is generally shortened to $\nabla_b Y^a$.*

Remark. It is very important when writing $\nabla_b Y^a$ not to think of the covariant derivative as acting on each component function Y^a; rather, it acts on the total vector field Y. The notation $(\nabla Y)^a{}_b$ makes this more explicit, but for convenience one usually adopts the former notation.

Remark. The covariant derivative of a function f is written $\nabla_a f$. We already know $\nabla f : X \to \nabla_X f = X(f)$. Thus $\nabla f = \mathrm{d}f$. In components, $\nabla_a f = \partial_a f$ so it reduces to the usual partial derivative.

Now suppose we are working in a particular coordinate chart. We would like to find an expression for the covariant derivative. In this basis our two vector fields $X = X^a \partial_a$ and $Y = Y^a \partial_a$ say, where $\partial_a = \partial / \partial x^a$ is our coordinate basis for vector fields. By definition

$$\begin{aligned} \nabla_X Y &= \nabla_X (Y^a \partial_a) = \nabla_X (Y^a) \partial_a + Y^a \nabla_X \partial_a \qquad \text{(product rule)} && (76) \\ &= X(Y^a) \partial_a + Y^a \nabla_{X^b \partial_b} \partial_a && (77) \\ &= X^b \partial_b (Y^a) \partial_a + Y^a X^b \nabla_{\partial_b} \partial_a && (78) \end{aligned}$$

At this stage we need to determine what are the covariant derivatives of the basis vector fields along the basis vector fields. By definition this is another smooth vector field. We write this as

$$\nabla_{\partial_b} \partial_a = \Gamma^c_{ab} \partial_c \tag{79}$$

Since the left hand side is a vector field, the right side is writing it as a linear combination of the basis vector fields, and the coefficients in this expansion are the Γ^c_{ba}, which are known as the *connection components* in the coordinate basis. The connection coefficients are **not** the components of a $(1, 2)$ tensor - rather they are basis dependent. For those familiar with electromagnetism, they are analogous to the gauge field potential A, and we know Maxwell's equations are invariant under 'gauge transformations' - this is why A is not physically meaningful, at least classically. The physical fields are the electric and magnetic fields, which are obtained by taking derivatives of A. The analogous quantity here is precisely the curvature tensor, which we will get to shortly.

At this stage you might wonder how we actually *determine* what the connection is. This is actually a choice, amounting to what it means to be 'parallel'. For a Riemannian manifold there is a natural, unique choice for the connection. With this choice, the connection coefficients are precisely the Christoffel symbols we have seen in our discussion of geodesics. Continuing onwards,

$$\begin{aligned} \nabla_X Y &= X^b \partial_b (Y^a) \partial_a + Y^a X^b \Gamma^c_{ab} \partial_c && (80) \\ &= X^b (\partial_b Y^a + Y^c \Gamma^a_{cb}) \partial_a && (81) \end{aligned}$$

(note the dummy indices have been rearranged). So $\nabla_X Y$ is the vector field with components

$$(\nabla_X Y)^a = X^b (\partial_b Y^a + Y^c \Gamma^a_{cb}) \tag{82}$$

and it is for fixed Y, a linear map on X and the quantity in the brackets is intrinsic to Y (i.e. does not depend on X). So the components of the covariant derivative are

$$(\nabla Y)^a{}_b = \nabla_b Y^a = \partial_b Y^a + Y^c \Gamma^a_{cb} \tag{83}$$

Note that the first term is what one would expect - i.e. the derivative of the components of Y, whereas the second term comes from the basis the basis vector fields themselves can be changing as we move from point to point on a chart. In a Cartesian coordinate system, the basis vectors i, j, k say are constant and so these terms are zero, which is why you did not encounter them before.

We can extend this to define covariant derivatives of general tensor fields, using the Leibniz property. We will work in components for simplicity. Consider a one-form ω with components ω_a. Then $\omega_a Y^a$ is a scalar. Hence

$$\nabla_b(\omega_a Y^a) = Y^a \partial_b \omega_a + \omega_a \partial_b Y^a \tag{84}$$

just using the fact the covariant derivative reduces to the partial derivative when acting on the function $\omega_a Y^a$. On the other hand,

$$\partial_b Y^a = \nabla_b Y^a - \Gamma^c_{ab} Y_c \tag{85}$$

Now if we demand that the covariant derivative satisfy the product rule, we must have

$$\nabla(\omega(Y)) = (\nabla\omega)(Y) + \omega(\nabla Y) \tag{86}$$

or equivalently, in the index notation

$$\nabla_b(\omega_a Y^a) = (\nabla_b \omega_a) Y^a + \omega_a (\nabla_b Y^a) \tag{87}$$

We have already computed the left hand side above in (84). Equating gives

$$\begin{aligned}
(\nabla_b \omega_a) Y^a &= Y^a \partial_b \omega_a + \omega_a \partial_b Y^a - \omega_a(\partial_b Y^a + \Gamma^a_{cb} Y^c) && (88)\\
&= Y^a \partial_b \omega_a - \omega_a \Gamma^a_{cb} Y^c && (89)\\
&= (\partial_b \omega_a - \omega_c \Gamma^c_{ab}) Y^a && (90)
\end{aligned}$$

Hence we can conclude

$$\nabla_b \omega_a = \partial_b \omega_a - \omega_c \Gamma^c_{ab} \tag{91}$$

The extension to a general tensor field $T^{ab\ldots}{}_{cd\ldots}$ is obvious; there is a term corresponding to the partial derivative, and then additional terms, with the proper index structure, with a $+$ sign for each contravariant 'upstairs' index and a $-$ sign for each covariant 'downstairs' index. For example

$$\nabla_c g_{ab} = \partial_c g_{ab} - \Gamma^d_{ac} g_{db} - \Gamma^d_{bc} g_{ad} \qquad \nabla_c T^a{}_b = \partial_c T^a{}_b + \Gamma^a_{dc} T^d{}_b - \Gamma^d_{bc} T^a{}_d \tag{92}$$

Note that the covariant derivative of an (r, s) tensor field is a $r, s+1$ tensor field; the latter has an extra vector argument in which is inserted the direction along which the covariant derivative is to be taken.

The Levi-Civita connection

Definition. *A connection is torsion free if $\nabla_a \nabla_b f = \nabla_b \nabla_a f$. This is equivalent to $\Gamma^a_{bc} = \Gamma^a_{cb}$ in a coordinate basis.*

This is a natural property in the sense that it is a coordinate-independent way of saying covariant derivatives commute on functions. Note that this is not automatic, as it is for partial derivatives. To see, this calculate

$$\nabla_a \nabla_b f = \partial_a(\nabla_b f) - \Gamma^c_{ba}(\nabla_c f) = \partial_a \partial_b f - \Gamma^c_{ba} \partial_c f \tag{93}$$

and $\nabla_b \nabla_a f$ can easily seen to be simply the above expression with the roles of a, b exchanged. The torsion-free condition implies that the connection coefficients be symmetric in its lower indices.

On a (pseudo-)Riemannian manifold (M, g) the metric singles out a preferred connection.

Theorem 1. *Given the manifold (M, g), there exists a unique torsion-free connection ∇ with the property that the metric is covariantly constant: $\nabla g = 0$ (i.e. $\nabla_a g_{bc} = 0$) and is called the Levi-Civita connection.*

The proof of this statement is straightforward and can be found in any decent GR or Riemannian geometry textbook. It is in fact known as the Fundamental Theorem of Riemannian geometry. In a coordinate basis, one can show that the Levi-Civita connection coefficients are

$$\Gamma^a{}_{bc} = \frac{g^{ad}}{2}(\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc}) \tag{94}$$

which is precisely the Christoffel symbols defined previously in our discussion of geodesics as extremal curves of the distance functional (this is not a coincidence). Using the Levi-Civita connection, it is a good exercise in index manipulation to explicitly verify that $\nabla_a g_{bc} = 0$.

What would we wish to impose that g be covariantly constant? It *does not* imply that g_{ab} are actually constant functions - indeed the theorem above says one can choose ∇ so that $\nabla g = 0$ for an arbitrary metric. To understand we need to understand the notion of parallel transport.

Parallel Transport

Let us begin with a simpler idea. Given a function f and a curve $x^a(\lambda)$ with tangent vector X^a, we would define the rate of change of f along the curve as

$$\frac{\mathrm{d}f(x^a(\lambda))}{\mathrm{d}\lambda} = X^a \partial_a f = X^a \nabla_a f \tag{95}$$

We would then conclude that f is *constant* along the curve if and only if $X(f) = X^a \nabla_a f = 0$. We want a similar definition for what it means for a tensor field to be 'constant' along the curve. Thinking of vectors as arrows, one could imagine that a vector is 'constant' along a curve it does not change direction or its length (note these latter ideas require a metric, but the following definition just requires a connection).

Definition. *Let X be the tangent vector to a curve. The tensor field T is parallelly transported along the curve if $\nabla_X T = 0$.*

One could look at this definition as an initial value problem. That is, fix p on the curve and specify the tensor T at p. Then the condition that T is parallelly transported on the curve will uniquely determine T everywhere else on the curve. As an example suppose we are given a fixed vector V_0 at the point p, and a curve $x^a(\lambda)$ with tangent X^a. The requirement $\nabla_X V = 0$ gives

$$0 = X^b(\partial_b V^a + \Gamma^a_{cb} V^c) \tag{96}$$

But we of course identify $X^b\partial_b$ to be $\mathrm{d}/\mathrm{d}\lambda$, i.e the directional derivative along the curve. So we can write this as

$$0 = \frac{\mathrm{d}V^a}{\mathrm{d}t} + X^b \Gamma^a_{cb} V^c \tag{97}$$

and treat everything as a function of λ as we are restricting to the curve. Then this is simply a first-order (linaer) ODE with initial value condition $V^a(0) = V^a_0$. Existence and uniqueness of a solution $V^a(\lambda)$ of this equation is guaranteed on some open set $\lambda \in (-\lambda_0, \lambda_0)$.

Thus we now have a scheme for comparing vector fields at different points on the curve. Let p, q be two distinct points on a fixed curve γ and suppose we have a tensor field T. Evaluate $T(p)$ and $T(q)$. We cannot compare these two tensors. However, we can now construct a new tensor $\hat{T}$ at q by parallelly transporting $T(p)$ to q, by solving the above initial value problem as just discussed. We can now compute the difference $\hat{T}(q) - T(q)$ to see whether T has changed. Notice that in general *this procedure depends on the curve chosen* to transport the tensor from p to q. In general, parallel transport is path dependent. The mathematical device that measures this path-dependence is the **Riemann curvature tensor**.

For the Levi-Civita connection, lengths and angles between parallelly transported vectors are preserved.

Proposition 12. *Let V, W be two vectors that are parallel transported along a curve with tangent X, i.e. $\nabla_X V = \nabla_X W = 0$. Then $g(V, W)$ is a constant (in particular $g(V, V)$ is constant).*

Proof. Let X be the tangent vector to the curve. Note $g(V, W)$ is a function. Explicit calculation gives

$$\nabla_X g(V, W) = (\nabla_X g)(V, W) + g(\nabla_X V, W) + g(V, \nabla_X W) = 0 \tag{98}$$

where we used the fact $\nabla_X g = 0$. $\square$

This is why the Levi-Civita connection is natural: it preserves angles and lengths of vectors that by definition are 'parallel' as they move along a curve . It is a good exercise again to use indices to verify the above statement. That is, compute

$$X^a \nabla_a (g_{cd} V^c W^d) = X^a [V^c W^d \nabla_a g_{cd} + g_{cd} V^c \nabla_a W^d + g_{cd} W^d \nabla_a V^c] = 0 \tag{99}$$

Geodesics, again

Let us now return to the notion of a geodesic. We obtained a characterization of geodesics as critical curves of an energy functional defined on curves with tangent vector $\dot{x}^a$. There are certain disadvantages to this description, as clearly it a priori assumes an extremal curve exists between two points and further standard ODE theory does not guarantee a unique solution for large enough parameter values. However the notion of parallel transport gives us a cleaner characterization of geodesics as ‘straightest lines’.

Definition. *Let M be a manifold with connection ∇. An affinely parameterized geodesic is an integral curve of a vector field X that satisfies $\nabla_X X = 0$.*

In light of the above discussion of parallel transport, this is saying that a geodesic is a curve whose tangent is parallelly transported ‘along itself’. This captures the notion of a straight line on a general manifold. In $\mathbb{R}^3$ we would say a straight line does not change direction; its tangent vector does not change as it moves along the curve. Let us see how this comes about more concretely.

Suppose we are a given a geodesic curve $x^a(\lambda)$. The tangent vector to the curve is $X^a = \dot{x}^a$ and is defined only along the curve; but we can extend it to a neighbourhood of the curve, so that X^a becomes a vector field, and the curve is an integral curve of this vector field. Now note that

$$\ddot{x}^a = \frac{\mathrm{d}}{\mathrm{d}\lambda} X^a(\lambda) = \dot{x}^b \partial_b X^a = X^b \partial_b X^a \tag{100}$$

where we used the Chain rule (alternatively think of $\mathrm{d}/\mathrm{d}\lambda = X^a \partial_a$ as the directional derivative along the curve). Thus the geodesic equation becomes

$$0 = \ddot{x}^a + \Gamma^a_{bc} \dot{x}^b \dot{x}^c = X^b \partial_b X^a + X^b X^c \Gamma^a_{bc} = X^b \left(\partial_b X^a + \Gamma^a_{bc} X^c \right) = X^b \nabla_b X^a = \nabla_X X \tag{101}$$

Thus the vector field X tangent to a geodesic is parallelly transported along the geodesic. This is a local characterization of a geodesic and is more useful in many contexts (although practically the Euler-Lagrange equations give the easiest way to find geodesics in explicit situations).

Now what does it mean to be ‘affinely parametreized’? Note that since $\nabla_X X = 0$, it follows $\nabla_X g(X, X) = 0$, i.e. the length of X is preserved - the curve moves on a constant velocity in this parameterization. More generally, to be a straightest possible path, one need only assume that the rate of change of X^a is in the direction of X itself, i.e.

$$\nabla_X X = f X \tag{102}$$

where f is some function. Such a parameterization of a geodesic is called ‘non-affine’. It is an easy exercise to show that by changing parameters to say $\tilde{\lambda}$ appropriately, one can always find an affine parameter for a geodesic (to see this write $\lambda = \lambda(\tilde{\lambda})$ and find the differential equation satisfied by this function in order to make the geodesic affintely parameterized in terms of $\tilde{\lambda}$). Thus unless there is some particular natural parameterization for a curve, we will simply assume all our geodesics are affintely parameterized. Note that in the variational approach, this condition follows from the fact $L = g_{ab} \dot{x}^a \dot{x}^b$ is constant.

Computation of Γ^a_{bc}

In the computation of covariant derivatives, for a given metric g the easiest way to calculate the Christoffel symbols is to read them off directly from the Euler-Lagrange equations. That is, we start with a given metric and write the geodesic equations in the general form (57) and then by looking at the terms quadratic in the $\dot{x}^a$ it is easy to determine the non-zero connection coefficients Γ^a_{bc}.

Example. Compute the non-vanishing Christoffel symbols associated to the unit sphere metric (43) and the Schwarzschild exterior metric (46).

The Christoffel symbols in a given basis are the coefficients of the expansion $\nabla_{e_a} e_b$ in terms of basis vectors e_c. They are, as mentioned above, not the components of a tensor. Let us suppose we have two overlapping coordinate charts with coordinates x^a, y^a. We would like to know the relation between the Γ^a_{bc} in each chart. For shorthand we will write $\partial_a = \partial/\partial x^a$ and $\partial'_a = \partial/\partial y^a$ to represent coordinate basis vectors in each chart; they are related by $\partial_a = J^b{}_a \partial'_b$ and $J^b{}_a = \partial y^b/\partial x^a$. We then have by definition

$$\nabla_{\partial_a} \partial_b = \Gamma^c_{ba} \partial_c \tag{103}$$

but if we expand out the left hand side in terms of the ∂'_a, we get

$$\begin{aligned} \nabla_{\partial_a} \partial_b &= \nabla_{J^d{}_a \partial'_d} (J^e{}_b \partial'_e) && (104)\\ &= J^d{}_a \nabla_{\partial'_d} (J^e{}_b \partial'_e) = J^d{}_a \nabla_{\partial'_d} (J^e{}_b) \partial'_e + J^d{}_a J^e{}_b \nabla_{\partial'_d} (\partial'_e) && (105)\\ &= J^d{}_a \partial'_d (J^e{}_b) \partial'_e + J^d{}_a J^e{}_b \Gamma'^f_{ed} \partial'_f && (106) \end{aligned}$$

where Γ'^f_{ed} are the Christoffel symbols associated to the y^a coordinate basis. This allows us to read off

$$\Gamma^c_{ba} = J^d{}_a (J^{-1})_e{}^c \partial'_d (J^e{}_b) + J^d{}_a J^e{}_b (J^{-1})_f{}^c \Gamma'^f_{ed} \tag{107}$$

and J^{-1} is the inverse to J, i.e. $(J^{-1})_f{}^c = \partial x^c/\partial y^f$. The second term in the transformation above is what one would expect for the components of a tensorial object; however the first term is *not* linear in Γ'^a_{bc} and hence not tensorial. In fact one can define a *connection* as a geometric object that transforms in this way. Those of you familiar with electromagnetism may compare this with how the gauge field A transforms under a gauge transformation $A \to A + \mathrm{d}f$. The upshot of this is that even if $\Gamma^a_{bc} = 0$ in one coordinate chart, they may not be zero in another chart.

Example. Calculate the connection coefficients in a cylindrical coordinate system in $\mathbb{R}^3$.

Commutators

Before moving to curvature, there is one other operation it is worth defining (see Assignment). Let X and Y be smooth vector fields and f a smooth function. The quantity $Y(f)$ is itself a function, and so we can act on it with X to form yet another function $X(Y(f))$. However we cannot define a vector field $XY : f \to XY(f)$ in this way, because $XY(fg)) = X(fY(g) + gY(f)) = fX(Y(g)) + gX(Y(f)) + X(f)Y(g) + X(g)Y(f)$ so the Leibniz rule is not satisfied (the last two terms ruin this). However we can define a new vector field by

Definition. *The commutator of two vector field X and Y is the vector field denoted $[X, Y]$ defined by*

$$[X, Y](f) = X(Y(f)) - Y(X(f)) \tag{108}$$

for any smooth function f.

Remark. The commutator of two vector fields is also known as a Lie bracket. In this context, one often writes $L_X Y = [X, Y]$ where $L_X Y$ is the 'Lie derivative of Y along X'. This gives a notion of derivative on a manifold which is very basic (i.e. does not require any structure such as a connection or metric).

In a coordinate basis, the components of $[X, Y]$ are

$$[X, Y]^a = X^b \partial_b Y^a - Y^b \partial_b X^a \tag{109}$$

If X, Y are coordinate basis vectors, then it follows that $[X, Y] = 0$. One of the geometrical meanings of commutativity (zero bracket) of vector fields relies on the *flows* generated by these two vector fields. WIthout going into the precise details, two vector fields are commutative if (and only if) there is no difference starting at one point p, traveling a distance t_1 over a curve with tangent X and then a time t_2 along the curve with tangent Y, or, instead, traveling first t_2 along a curve with tangent Y and then t_1 along a curve with tangent X. One ends up at the same point if and only if $[X, Y] = 0$.

2.2 Curvature on a Riemannian Manifold

2.2.1 Preliminary comments

We would like a precise way to discuss how a particular geometry described by a metric tensor g differs from Euclidean (or Minkowski) space. This question is subtle because we have a freedom in the choice of chart (diffeormorphsim invariance) to describe the geometry in a neighbourhood of a point. Even in Euclidean space, one can choose a spherical chart in which the metric components are not constant and indeed the Chrisfoffel symbols do not vanish. Hence these cannot give an invariant measure of a 'curvature' of a given metric g . This suggests that these second derivatives contain intrinsic data about the metric g which is chart-independent.

Suppose we choose a point $p \in (M, g)$. There is a natural coordinate system that can always be introduced in a neighbourhood of p by an appropriate coordinate transformation. In this chart, called *normal coordinates*, one can show that the metric takes the form

$$g_{ab} = \delta_{ab} + s_{ab}(x^a(q) - x^a(p))(x^b(q) - x^b(p)) + \ldots \tag{110}$$

where $x^a(p)$ are the coordinates of p in this chart, and $x^a(q)$ are the coordinates of a second point q. This can be intuitively understood as a Taylor series of the metric about the point p. We call these 'normal' coordinates because to 2nd order, $g_{ab} = \text{diag}(1, 1, \ldots 1)$. Observe that the *first derivatives* of g_{ab} evaluated at p in this chart vanish; this can always be achieved by an appropriate choice. Hence in this chart, $\Gamma^a_{bc} = 0$ at the point p. A simple counting argument show, however, that one *cannot arrange for the second derivatives of g_{ab} to vanish at p.* For details on this point consult the text of B. Schutz.

Let us think of this in terms of physics. We know that even in a gravitational field in a sufficiently small region around a point in spacetime, one can introduce 'freely falling' observers for which the usual laws of special relativity hold: think of the reference frame of someone jumping off a building or in a falling elevator. This is the physical statement of the above statement that one can always introduce a normal coordinate chart in which the metric is flat to 2nd order. Note that in this chart, the geodesic equation is simply

$$\ddot{x^a} = -\Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0 \tag{111}$$

in a neighbourhood of p since the Christoffel symbols vanish. Hence freely-falling objects will move on straight lines in this chart, which of course is what we expect in an inertial frame. The normal coordinate chart is also therefore called a 'locally inertial coordinate chart' in the general relativity context, although it's just a statement in Riemannian geometry.

2.2.2 The Riemann tensor

We have seen parallel transport with respect to a connection allows us to define what it means for a vector field to change as we move from point to point on M. As explained above, starting from a fixed vector V at p, the vector one obtains by transporting V to a point q depends on the curve chosen. Consider a closed path formed by a parallelogram whose sides are geodesics with tangents X and Y. We know $\nabla_X X = \nabla_Y Y = 0$. We now parallel transport a vector field V around the closed curve. By definition $\nabla_X V = \nabla_Y V = 0$. It follows that the angle V makes with X and Y while being transported along the curves is constant, i.e. $g(X, V)$ is constant when transporting V along the integral geodesic curve of X and similarly for $g(Y, V)$. Doing this on the plane one sees that V will return to itself after completing a loop. However on S^2 the vector V will have *rotated* relative to its initial direction, even though its length cannot change, by definition of parallel transport. Hence we have a orthogonal transformation $V \to O(V)$ as one moves around the path. In components, $V^a \to O^a_b V^b$ where O is some orthogonal matrix. Now on a general manifold, change in rotation will depend on the tangent vectors X and Y (i.e. parallel transport is path-dependent). As we take this loop to zero size, we find that the change ΔV^a in V^a can be written $\Delta V^a = R^a_{bcd} V^b X^c Y^d$. The object on the right hand side is a $(1, 3)$ tensor called the Riemann curvature tensor. We will define this below.

Definition. *Let X, Y, Z be vector fields and M be a manifold with connection ∇. The Riemann curvature tensor ${R^a}_{bcd}$ is the $(1, 3)$ tensor defined by*

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \tag{112}$$

and in components we write $R(X, Y)Z)^a = R^a_{bcd} Z^b X^c Y^d$. For fixed X, Y, we think of this as a linear transformation on Z, i.e. $Z^a \to {\mathcal{R}^a}_b Z^b$ where ${\mathcal{R}^a}_b = {R^a}_{bcd} X^c Y^d$.

This chart-independent definition is a bit abstract but is useful because it demonstrates this really defines a $(1, 3)$ tensor .Recall this simply means that $R(X, Y)Z$ must be linear in all its arguments *even with functions*. Clearly $R(X,Y)Z = -R(Y,X)Z$ so this needs to only be checked for X and Z, say. Linearity means in particular that $R(fX,Y)Z = fR(X,Y)Z$ and $R(X,Y)(fZ) = fR(X,Y)Z$. Let us try the first one:

$$
\begin{aligned}
R(fX,Y)Z &= \nabla_{fX}\nabla_Y Z - \nabla_Y\nabla_{fX}Z - \nabla_{[fX,Y]}Z && (113)\\
&= f\nabla_X\nabla_Y Z - \nabla_Y(f\nabla_X Z) - \nabla_{f[X,Y]-Y(f)X}Z && (114)\\
&= f\nabla_X\nabla_Y Z - f\nabla_Y\nabla_X Z - Y(f)\nabla_X Z - \nabla_{f[X,Y]}Z + \nabla_{Y(f)X}Z && (115)\\
&= f\nabla_X\nabla_Y Z - f\nabla_Y\nabla_X Z - Y(f)\nabla_X Z - f\nabla_{[X,Y]}Z + Y(f)\nabla_X Z && (116)\\
&= fR(X,Y)Z && (117)
\end{aligned}
$$

The next one is a bit more complicated:

$$
\begin{aligned}
R(X,Y)(fZ) &= \nabla_X\nabla_Y(fZ) - \nabla_Y\nabla_X(fZ) - \nabla_{[X,Y]}(fZ)\\
&= \nabla_X(f\nabla_Y Z + Y(f)Z) - \nabla_Y(f\nabla_X Z + X(f)Z) - f\nabla_{[X,Y]}Z - [X,Y](f)Z\\
&= f\nabla_X\nabla_Y Z + X(f)\nabla_Y Z + Y(f)\nabla_X Z + X(Y(f))Z - f\nabla_Y\nabla_X Z - Y(f)\nabla_X Z\\
&- X(f)\nabla_Y Z - Y(X(f))Z - f\nabla_{[X,Y]}Z - [X,Y](f)Z\\
&= fR(X,Y)Z
\end{aligned}
$$

This is enough to demonstrate that the Riemann curvature tensor is indeed a tensor.

In a given coordinate basis we can find the components ${R^a}_{bcd}$ by setting $X = \partial_c, Y = \partial_d, Z = \partial_b$. Note that the commutator $[\partial_c, \partial_d] = 0$ automatically. Thus

$$
\begin{aligned}
R(e_c, e_d)e_b &= \nabla_c\nabla_d(\partial_b) - \nabla_d\nabla_c(\partial_b) && (118)\\
&= \nabla_c(\Gamma^a_{bd}\partial_a) - \nabla_d(\Gamma^a_{bc}\partial_a) && (119)\\
&= \partial_c(\Gamma^a_{bd})\partial_a + \Gamma^a_{bd}\Gamma^e_{ac}\partial_e - \partial_d(\Gamma^a_{bc})\partial_a - \Gamma^a_{bc}\Gamma^e_{ad}\partial_e && (120)\\
&= (\partial_c(\Gamma^a_{bd}) + \Gamma^e_{bd}\Gamma^a_{ec} - \partial_d(\Gamma^a_{bc}) - \Gamma^e_{bc}\Gamma^a_{ed})\partial_a && (121)
\end{aligned}
$$

Thus we have got the components:

$$
{R^a}_{bcd} = \partial_c(\Gamma^a_{bd}) - \partial_d(\Gamma^a_{bc}) + \Gamma^e_{bd}\Gamma^a_{ec} - \Gamma^e_{bc}\Gamma^a_{ed} \tag{122}
$$

Schematically, $R \sim \partial\Gamma + \Gamma\Gamma$. From the form above it is obvious we have ${R^a}_{bcd} = -{R^a}_{bdc}$, i.e. antisymmetry in the final two indices.

Proposition 13. *For any torsion-free connection (including the Levi-CIvita connection) the Ricci identity*

$$
\nabla_c\nabla_d Z^a - \nabla_d\nabla_c Z^a = {R^a}_{bcd}Z^b \tag{123}
$$

holds.

Proof. Expand the left hand side of the identity in terms of the Christoffel symbols and use the symmetry properties. Note that all derivatives of Z must cancel out as the right hand side only depends on Z. □

2.2.3 Curvature and parallel transport

Here we will just sketch the argument. We choose two vector fields X, Y that are linearly independent and commute, so that $[X, Y] = 0$. One can always define a coordinate chart in a small enough neighbourhood so that $x^a = (s, t, \ldots)$ where $X = \partial_s$ and $Y = \partial_t$ are coordinate basis vectors. Let $p \in M$ be a point with coordinates $(0, 0, \ldots 0)$ and points q, r, u have coordinates $(\Delta s, 0, 0, \ldots), (\Delta s, \Delta t, \ldots), (0, \Delta t, 0 \ldots)$ where $\Delta s, \Delta t$ are small. The points p and q can be connected along a $s-$coordinate curve (tangent X) and q and r can be connected by a $t-$ coordinate curve (tangent Y), resulting in a quadrilateral. Now fix a vector Z at T_pM and parallel transport it via pqr to a new vector $Z_r \in T_rM$. Then transport it via pur to obtain a new vector $Z'_r \in T_rM$. We want to compute $\Delta Z^a_r = Z'^a_r - Z^a_r$. A careful calculating, using the fact that $\nabla_X Z = 0$ and $\nabla_Y Z = 0$ along the appropriate curves, gives

$$R^a{}_{bcd} Z^b X^c Y^d)_r = \lim_{\Delta t, \Delta s \to 0} \frac{\Delta Z^a_r}{\Delta s \Delta t} \tag{124}$$

and hence the Riemann tensor measures the path-dependence of parallel transport.

Example. Consider Minkowski spacetime or Euclidean space. In standard Cartesian coordinates we already know that $\Gamma^a_{bc} = 0$ and hence the Riemann tensor is identically zero in this chart. Since it is zero in one chart, it will be zero in any other chart (unlike the Γ^a_{bc}, which are not tensor components). Hence the Riemann tensor vanishes and we say Minkowski spacetime (Euclidean space) is *flat*. If a manifold (M, g) has the property that the Riemann tensor vanishes at each point we say it is *locally flat*.

Remark. In a general Riemannian manifold (M, g), the Riemann tensor will be non vanishing. Intuitively, the Riemann tensor is a **precise measure of how the geometry of (M, g) differs from Euclidean (or Lorentzian) space $(\mathbb{R}^n, g_E)$.**

2.2.4 Symmetries of the Riemann tensor

In n dimensions the Riemann tensor appears to have n^4 independent components. However, due to its various symmetry properties, this number is reduced.

Proposition 14. *The Riemann tensor possesses the following symmetry properties:*

1. $R^a{}_{bcd} = -R^a{}_{bdc}$ *(by definition)*
2. $R_{abcd} = -R_{bacd}$ *where we lower the first index using* g_{ab}
3. $R^a{}_{[bcd]} = 0$ *Here the* $[\ldots]$ *notation stands for antisymmetrization.*
4. $R_{abcd} = R_{cdab}$
5. $\nabla_{[e} R^a{}_{|b|cd]} = 0$ *(Bianchi identity)*

Remark. The algebraic symmetries imply that the Riemann tensor has

$$\frac{n^2(n^2-1)}{12} \tag{125}$$

independent components. This gives 20 in $n=4$, and 1 in $n=2$. Hence in two dimensions, the curvature of a Riemannian manifold is determined by a single scalar function. This quantity is known as the Gauss curvature, often denoted by K. However for $n>2$ the curvature is a tensorial in nature.

Let us consider these symmetries for a moment. The first four are algebraic while the last is differential. First of all, the process of symmetrization is denoted by round brackets, and involves summing over all even (an even number of pairs of indices are exchanged) and odd permutations (an odd number of paris of indices are exchanged) of the indices and dividing by the total number of permutations, i.e.

$$T_{(ab)} = \frac{1}{2!}(T_{ab}+T_{ba}) \tag{126}$$

The result is a tensor which is automatically symmetric. Antisymmetrization, on the other hand, involves summing over all permutations, but now odd permutations are assigned a $-$ sign:

$$T_{[ab]} = \frac{1}{2!}(T_{ab}-T_{ba}) \tag{127}$$

THere will be $n!$ terms in either case, where $n=r+s$ for a type (r,s) tensor. For the Riemann tensor, the we have

$$0 = R^a{}_{[bcd]} = \frac{1}{3!}\left(R^a{}_{bcd}+R^a{}_{cdb}+R^a{}_{dbc}-R^a{}_{cbd}-R^a{}_{dcb}-R^a{}_{bdc}\right) \tag{128}$$

In the notation used for the Bianchi identity above, the $|b|$ notation means that this index is *not* to be antisymmetrized. Thus the Bianchi identity can be written

$$0 = \nabla_{[e}R^a{}_{|b|cd]} = \frac{1}{3!}\left(\nabla_e R^a{}_{bcd}+\nabla_c R^a{}_{bde}+\nabla_d R^a{}_{bec}-\nabla_c R^a{}_{bed}-\nabla_d R^a{}_{bce}-\nabla_e R^a{}_{bdc}\right) \tag{129}$$

The proof of these symmetry properties is most conveniently done by working in the normal coordinate chart introduced above, because $\Gamma^a_{bc}=0$. Once they are proved in a particular chart, since $R^a{}_{bcd}$ is a tensor, these properties will hold in *any* chart. In normal coordinates, the second derivatives of g_{ab} do not vanish, so we have

$$\partial_d\Gamma^a_{bc} = \frac{1}{2}g^{ae}(\partial_d\partial_b g_{ec}+\partial_d\partial_c g_{eb}-\partial_d\partial_e g_{bc}) \tag{130}$$

and (check) the Riemann tensor is

$$R^a{}_{bcd} = \frac{1}{2}g^{ae}(\partial_b\partial_c g_{ed}-\partial_d\partial_b g_{ec}+\partial_e\partial_d g_{bc}-\partial_e\partial_c g_{bd}) \tag{131}$$

One can then lower the index to find R_{abcd} in the normal coordinate chart. In this chart, the symmetry properties (2)-(5) can be verified

Note that covariant derivatives are simply partial derivatives in this chart, so the Bianchi identity is fairly simple to prove. At p,

$$\nabla_e R^a_{bcd} = \partial_e R^a_{bcd} \tag{132}$$

since in normal coordinates, the Christoffel symbols vanish. Schematically, $\partial R = \partial^2\Gamma - \Gamma\partial\Gamma$ and the second term vanishes in normal coordinates. Thus we get

$$\nabla_e R^a_{bcd} = \partial_e\partial_c\Gamma^a_{bd} - \partial_e\partial_d\Gamma^a_{bc} \tag{133}$$

Antisymmetrizing then gives the Bianchi identity at p:

$$R^a{}_{b[cd;e]} = 0 \tag{134}$$

in this basis at the point p . Here we are using the 'semi-colon' notation for the covariant derivative, which is often useful: $V^a_{;b} \equiv \nabla_b V^a$ (the comma is often used for partial derivatives, so $V^a_{,b} \equiv \partial_b V^a$). The Riemann tensor is complicated object. Einstein's equations are in fact formulated in terms of its 'trace' , known as the Ricci tensor:

Definition. *The Ricci curvature tensor is the symmetric* $(0, 2)$ *tensor defined by*

$$R_{ab} = R^c{}_{acb} = -R^c{}_{abc} \tag{135}$$

Remark. Note that the Ricci tensor is symmetric. This is because $R_{ab} = R^c{}_{acb} = g^{cd}R_{dacb} = g^{cd}R_{cbda} = R^d{}_{bda} = R_{ba}$ where we used the symmetry property $R_{dacb} = R_{cbda}$.

Remark. Note that one contracts the first and third indices of the Riemann tensor. There is no metric tensor used to raise or lower an index.

Finally, one can also define the *scalar* curvature:

Definition. *The scalar curvature of the Riemannian manifold* (M, g) *is the scalar function defined by*

$$R = g^{ab}R_{ab} = g^{ab}R^c{}_{acb} \tag{136}$$

The scalar curvature is often referred to as the 'Ricci scalar'. Neither the Ricci scalar or the Ricci tensor contain all the information about the curvature, but may be thought of as certain 'averages' of the curvature (in the sense that the trace of a matrix is a scalar associated to the matrix).

Example. Let us consider an explicit example. Consider the canonical round metric on S^2

$$\mathrm{d}s^2 = \mathrm{d}\theta^2 + \sin^2\mathrm{d}\phi^2 \tag{137}$$

We already have computed the non-vanishing components of the Levi-Civita connection: $\Gamma^1_{22} = -\sin\theta\cos\theta, \Gamma^2_{12} = \Gamma^2_{21} = \cot\theta$ where $x^i = (\theta, \phi)$. We already know that in two dimensions $n = 2$ there will only be a *single* independent component of the curvature from which all other components can be deduced. First note that $R^a{}_{b11} = R^a{}_{b22} = 0$ for any choice of a, b automatically, so we need only consider $R^a{}_{b12} = -R^a{}_{b21}$. Further we also read off that $g^{11} = 1, g^{22} = (\sin^2\theta)^{-1}, g^{12} = g^{21} = 0$. Thus

$$R^1{}_{112} = g^{1a}R_{a112} = g^{11}R_{1112} = 0, \qquad R^2{}_{212} = g^{2a}R_{a212} = g^{22}R_{2212} = 0 \tag{138}$$

since $R_{abcd} = -R_{bacd}$. This leaves $R^1{}_{212}, R^2{}_{112}$. But of course

$$R^2{}_{112} = g^{2a}R_{a112} = g^{22}R_{2112} = -g^{22}R_{1212} = -g^{22}g_{1b}R^b{}_{212} = -g^{22}g_{11}R^1{}_{212} \tag{139}$$

So the only component we need is R^1_{212}. By definition

$$R^1{}_{212} = \partial_1\Gamma^1_{22} - \partial_2\Gamma^1_{12} + \Gamma^a_{22}\Gamma^1_{a1} - \Gamma^a_{21}\Gamma^1_{a2} \tag{140}$$

Now we know all quantities are independent of ϕ, so the second term must vanish. Next $\partial_1\Gamma^1_{22} = \sin^2\theta - \cos^2\theta$. Also $\Gamma^1_{a1} = 0$ for any a. Finally the last term

$$-\Gamma^a_{21}\Gamma^1_{a2} = -\Gamma^2_{21}\Gamma^1_{22} = \cos^2\theta \tag{141}$$

Putting this together $R^1{}_{212} = \sin^2\theta$. This is the single independent component of the Riemann tensor. By the above we also have $R^2{}_{112} = -g^{22}g_{11}\sin^2\theta = -1$. Now let us compute the Ricci tensor. We have

$$\begin{aligned} R_{11} &= R^c{}_{1c1} = R^1{}_{111} + R^2{}_{121} = -R^2{}_{112} = 1 && (142)\\ R_{12} &= R_{21} = R^c{}_{1c2} = R^1{}_{112} + R^2{}_{122} = 0 && (143)\\ R_{22} &= R^c{}_{2c2} = R^1{}_{212} + R^2{}_{222} = \sin^2\theta && (144) \end{aligned}$$

We have thus shown that for the unit sphere metric,

$$R_{ab} = g_{ab} \tag{145}$$

This is a tensor equation and hence holds in any basis. A Riemannian manifold with the property that the Ricci tensor is proportional (by a constant) to the metric tensor is called an *Einstein manifold*. This is a very rare property. This form of the Ricci tensor makes it very easily to compute the scalar curvature

$$R = g^{ab}R_{ab} = g^{ab}g_{ab} = 2 \tag{146}$$

Those of you familiar with differential geometry will recognize that the scalar curvature in two dimensions is related to the Gaussian curvature K by $R = 2K$.

2.3 Geodesic Deviation

The Riemann tensor is the mathematical object that describes how physical manifestation of gravity on particles: gravity tends to cause freely falling particles to converge. Mathematically, this means that geodesics on a Riemannian manifold that begin initially parallel should start to converge. In Euclidean space, the Riemann tensor vanishes and initially parallel geodesics (i.e. straight lines) remain parallel. This would not be true on a sphere.

We begin with the notion of a *smooth one-parameter family* of geodesics $\gamma : I \times I' \to M$ where I, I' are open intervals of $\mathbb{R}$, which we denote by $\gamma(s,t)$. For fixed $s = s_0$, $\gamma(s_0, t)$ is a geodesic with affine parameter t. As we vary s we move to different geodesics in the family of curves. We require that the map $(s,t) \to \gamma(s,t)$ is smooth and one to one with a smooth inverse. The family of geodesics form a 2d surface in M and one can think of (s,t) as coordinates labelling points on this surface.

Let T be the tangent vector field to the geodesics (i.e. tangent to the curves of constant s) and S be the vector fields tangent to curves of constant t. If we have a coordinate chart, x^a, then points on the family of geodesics are specified by $x^a(s,t) = x^a(\gamma(s,t))$, and $T = \partial x^a(s,t)/\partial t$ and $S = \partial x^a(s,t)/\partial s$. Think of the coordinates of two nearby geodesics at fixed t:

$$x^a(s+\epsilon, t) = x^a(s,t) + \epsilon S^a(s,t) + O(\epsilon^2) \tag{147}$$

One can think of ϵS^a as pointing from one geodesic with labels (s,t) to the infinitesimally nearby one with $(s+\epsilon, t)$. Thus S^a is referred to as the *deviation* vector.

Since s, t naturally parametrize the surface Σ in M corresponding to the family of geodesics, it seems natural to choose coordinates $x^a = (s, t, \ldots)$ where $\ldots$ refer to the $n-2$ coordinates along directions that do not lie tangent to Σ. Then $T = \partial_t, S = \partial_s$. As coordinate basis vectors, S, T commute, i.e. $[S,T] = 0$. Recall this means that the coordinate curves 'close up' in the sense one arrives at the same point if one travels δs units along S and then δt units along T or vice versa.

We are interested in how fast nearby geodesics are 'spreading' or 'converging' . They are affintely parametrized geodesics, and so $\nabla_T T = 0$ by definition. On the other hand the quantity $\nabla_T \nabla_T S$ is measures the relative 'acceleration' between nearby geodesics, as one travels 'up' the family (i.e. forward along the geodesics). This can be calculated as follows:

$$\nabla_T \nabla_T S = \nabla_T \nabla_S T \tag{148}$$

because $[S,T] = \nabla_S T - \nabla_T S = 0$ (the fact that one can write the commutator in terms of covariant derivatives holds for any torsion-free connection, and in particular for the Levi-Civita connection). Using the definition of the Riemann tensor,

$$R(T,S)T = \nabla_T \nabla_S T - \nabla_S \nabla_T T - \nabla_{[T,S]} T \tag{149}$$

we then have

$$\nabla_T \nabla_T S = R(T,S)T + \nabla_S \nabla_T T \tag{150}$$

But since T tangent to affinely parametrized geodesics, $\nabla_T T = 0$, so the second term vanishes. So we end up with the simple expression for the (covariant) relative acceleration of nearby geodesics in the family:

$$\nabla_T \nabla_T S = R(T,S)T \tag{151}$$

In the index notation this is

$$T^c\nabla_c(T^d\nabla_d)S^a = R^a{}_{bcd}T^bT^cS^d \tag{152}$$

Note that if we contract both sides with T_a we must get zero since $T^aR_{abcd}T^bT^cS^d = 0$; the deviation vector does not accelerate in the direction tangent to the geodesics (i.e. the geodesics are not 'speeding up' relative to each other).

This is actually how one would measure the curvature. Start at a point p and select an arbitrary initial direction T for the geodesics. By the theory of ODEs we can always find in a small enough neighbourhood a family of geodesics (this corresponds physically to dropping two nearby balls and seeing what happens to them). THen measure the relative acceleration of these two balls. This would then determine $R^a{}_{(bc)d}$ (note that T appears symmetrically on both sides). Then the full Riemann tensor can be obtained via the non-obvious identity

$$R^a{}_{bcd} = \frac{2}{3}(R^a{}_{(bc)d} - R^a_{(bd)c}) \tag{153}$$

Example. Let us take a very simple example. Consider S^2 with the canonical metric. We know that the great circles are geodesics. So take $T = \partial/\partial\theta$ (these are tangent to great circles that are 'longitude' lines running from the North to the South pole). We take S to be ∂_ϕ, i.e. it tangent to the curves of constant θ. The quantity

$$R^a{}_{bcd}T^bT^cS^d = R^a{}_{\theta\theta\phi} \tag{154}$$

We have calculated the Riemann tensor for this case and we know $R^\theta{}_{\theta\theta\phi} = 0$ and $R^\phi{}_{\theta\theta\phi} = -1$ (we used the notation $R^2{}_{112}$ for this component in the example). Thus

$$(\nabla_T\nabla_TS)^\theta = 0, \qquad (\nabla_T\nabla_TS)^\phi = -1 \tag{155}$$

Thus as expected, the deviation vector has negative acceleration in the ϕ direction, i.e. the geodesics are getting closer together.

3 The Einstein Field Equations

Before we turn to the gravitational field equations, we first briefly discuss the Newtonian case.

3.1 Physics in Curved spacetimes

Here we wish to gain some intuition for dealing with particle motion in general relativity. Recall that the motivation for studying curved Lorentzian manifolds was to model the paths of freely-falling test particles in a gravitational field as 'straight line' (geodesics) in a curved geometry. We have not yet reached the stage where we can discuss how exactly the geometry is determined - the Einstein field equations - but let us assume we are considering a situation with a weak gravitational field. In this case the Newtonian potential Φ, which satisfies the Poisson equation

$$\nabla^2\Phi = 4\pi\rho \tag{156}$$

where ρ is the energy density, fully determines the metric. Far from a source of mass M, we will choose units where $\Phi \to -M/r$. We will assume the geometry is approximately determined by

$$\mathrm{d}s^2 = -(1+2\Phi)\mathrm{d}t^2 + (1-2\Phi)(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2) \tag{157}$$

where (t, x, y, z) are usual Cartesian coordinates taking real values. The field being 'weak' requires that the spacetime closely resembles Minkowski spacetime, so $|\Phi| << 1$ and the above metric is really only valid to linear order in Φ. This is known as the 'weak-field' approximation to general relativity, and would model the geometry far from a massive object, just as the Sun.

Let us consider the motion of particles of unit mass (timelike geodesics). We will use affinely parameterized curves, i.e.

$$g_{ab}\dot{x}^a\dot{x}^b = -1 \tag{158}$$

In this case, the parameter on the curve is the proper time, since if $u^a = \dot{x}^a$ then $u \cdot u = -1$. We also recall from our discussion of special relativity that the four-momentum $p^a = mu^a$ and the component p^0 had the interpretation of energy of the particle in the reference frame associated with the above coordinate chart, and p^i, $i = 1, 2, 3$ were the components of the 3-momentum. Starting from the Lagrangian

$$L = -(1+2\Phi)\dot{t}^2 + (1-2\Phi)(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \tag{159}$$

we can derive the geodesic equation for $t(\tau)$:

$$(1+2\Phi)\ddot{t} + \partial_t\Phi\dot{t}^2 + 2\partial_i\Phi\dot{x}^i\dot{t} - \partial_t\Phi(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = 0 \tag{160}$$

Now if the particle is moving at non-relativistic speeds, $\dot{t}^2 >> \dot{x}^2 + \dot{y}^2 + \dot{z}^2$ and $\dot{t} >> \dot{x}^i$. Using this approximation, and the fact $(1+2\Phi)^{-1} \approx 1 - 2\Phi$, to lowest order in Φ we are left with

$$\ddot{t} = -\partial_t\Phi\dot{t}^2 \tag{161}$$

but since $\dot{t}^2 = +1$ (because $u \cdot u = -1$) we have, after multiplying both sides of the equation by the rest mass m,

$$\frac{\mathrm{d}p^0}{\mathrm{d}\tau} = -m\partial_t\Phi \tag{162}$$

This states that if Φ is time-independent, then the particle's energy is conserved in this frame, as one would expect. Now if one considers the geodesic equation along spatial directions, one gets

$$(1-2\Phi)\ddot{x}^i - 2(\partial_j\Phi\dot{x}^j)\dot{x}^i - 2\partial_t\Phi\dot{t}\dot{x}^i + \partial_i\Phi\dot{t}^2 + \partial_i\Phi(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = 0 \tag{163}$$

By the above approximation, assuming $\dot{t} >> \dot{x}^j$ for all j, and expanding to lowest order in Φ, and $\dot{t}^2 \approx 1$, gives

$$\ddot{x}^i = -\partial_i\Phi \tag{164}$$

or upon multiplying by m,

$$\frac{\mathrm{d}p^i}{\mathrm{d}\tau} = -m\partial_i\Phi \tag{165}$$

This is simply Newton's law of motion $F = \mathrm{d}p/\mathrm{d}\tau$, with $F = -m\nabla\Phi$ the force associated to the gravitational potential Φ. Thus, in the approximation scheme of low velocities and weak gravitational fields, we can naturally recover Newtonian gravity from an assumption that the spacetime metric takes the form (157).

The phenomena of *gravitational redshift* can also be deduced from our weak-field metric (157). The weak equivalence principle states that if two test bodies initially have the same position and velocity then they will follow exactly the same trajectory in a gravitational field (this is not true of other forces: in an electromagnetic field, bodies with different charge to mass ratio will follow different trajectories.) This suggested to Einstein that the trajectories of test bodies in a gravitational field are determined by the structure of spacetime alone and hence gravity should be described geometrically.

Consider the proper time τ between two infinitesimally nearby events in the geometry described by (157) :

$$\mathrm{d}\tau^2 = -\mathrm{d}s^2 = (1+2\Phi)\mathrm{d}t^2 - (1-2\Phi)(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2) \tag{166}$$

and Φ is independent of t (time-independent gravitational field). Suppose A is an observer with spatial position $x^i_A = (x_A, y_A, z_A)$ and B has spatial position x^i_B. At time t_A A sends B a light signal, and then sends a second light signal at $t_A + \Delta t$. B will receive the first signal at some time t_B. We want to determine when the 2nd light signal is received at B. WIthout caring exactly how the light travels in this geometry, what is clear is that both light signals travel in the same way, since the geometry is time-independent. So B must receive the 2nd signal at $t_B + \Delta t$. The proper time interval between the signals sent by A is given by

$$\Delta\tau_A^2 = \left(1 + 2\Phi(x^i_A)\right)\Delta t^2 \tag{167}$$

where we have used the fact that $\Delta x^i = 0$ since the two light signals are sent from the same spatial position, i.e. x^i_A. Using a Taylor expansion this means

$$\Delta\tau_A \approx (1 + \Phi(x^i_A))\Delta t \tag{168}$$

By the same logic the proper time difference between signal received by B is

$$\Delta\tau_B \approx (1 + \Phi(x^i_B))\Delta t \tag{169}$$

where again the signals are received in the same spatial position x^i_B so $\Delta x^i = 0$. Eliminating t gives

$$\Delta\tau_B \approx (1 + \Phi(x^i_B))(1 + \Phi(x^i_A))^{-1}\Delta\tau_A \approx (1 + \Phi(x^i_B) - \Phi(x^i_A))\Delta\tau_A \tag{170}$$

Now if we assume A stands at the bottom of a building on Earth and B stands at the top of the building, and we have chosen our potential to be negative and vanish at infinity, then $\Phi(x^i_B) > \Phi(x^i_A)$ (for example take $\Phi = -GM/r$ and $z_B > z_A, x_A = y_A = x_B = y_B = 0$). The end result is that the proper time between the light signals received by B is greater than that between the light signals originally sent by A. In other words time is running 'faster' when the gravitational field is weaker, or equally time is running slower in a stronger gravitational field. This effect has actually been experimentally measured. If we think of waves of light, this would mean the period between successive wavecrests is longer as received by B than a sent from A; and since $\tau = \lambda/c = \lambda$ where λ is the wavelength, the end result is that $\lambda_B > \lambda_A$; in this sense, the light has been *redshifted* since it has a longer wavelength when received than when it was emitted. Alternatively, signals sent from B to A would be *blueshifted*, i.e. the wavelength is measured to be shorter when received than when emitted. For a black hole, one actually has $\Phi = -2M/r$ and as we will see, the gravitational field is so strong near the horizon that if A sends a light signal to B , the time between signals will become infinite; B will never get the second signal.

3.2 The Einstein Tensor

We now motivate the field equations governing the gravitational field, obtained independently by Einstein and Hilbert by a variational formulation in 1915. Ultimately there is no 'derivation' of these equations: their validity rests on its ability to describe physical observations. Nonetheless from the point of view of mathematics, they are a very natural set of equations to study on a (pseudo-)Riemannian manifold and are actually related to the classification problem of 2-manifolds. We begin with a definition:

Definition. *The Einstein tensor G_{ab} is the symmetric $(0,2)$ tensor defined by*

$$G_{ab} \equiv R_{ab} - \frac{1}{2} g_{ab} R \tag{171}$$

Proposition 15. *The Einstein tensor is divergenceless, that is $\nabla^a G_{ab} = 0$.*

Proof. This follows from the Bianchi identity (134). If we explicitly write it out the antisymmetrization, this implies

$$\nabla_e R^a{}_{bcd} + \nabla_c R^a{}_{bde} + \nabla_d R^a{}_{bec} - \nabla_c R^a{}_{bed} - \nabla_d R^a{}_{bce} - \nabla_e R^a{}_{bdc} = 0 \tag{172}$$

Noting the antisymmetry on the last two indices, this can be simplified to

$$\nabla_e R^a{}_{bcd} + \nabla_c R^a{}_{bde} + \nabla_d R^a{}_{bec} = 0 \tag{173}$$

Now contract the indices a and c (this commutes with differentiation). This gives

$$\nabla_e R_{bd} + \nabla_a R^a{}_{bde} - \nabla_d R_{be} = 0 \tag{174}$$

Now multiply by g^{eb} (since the metric is covariantly constant, it can pass through the derivatives):

$$\nabla_e R^e{}_d + \nabla_a R^a{}_d - \nabla_d R = 0 \tag{175}$$

using $\nabla_a g^{eb} R^a{}_{bde} = \nabla^a g^{eb} R_{baed} = \nabla^a R_{ad}$. Relabelling the dummy indices gives

$$2\nabla_e R^e{}_d - \nabla_d R = 0 \Rightarrow \nabla^e \left(R_{ed} - \frac{1}{2} g_{ed} R \right) = 0 \tag{176}$$

and thus $\nabla^e G_{eb} = 0$. Here $\nabla_d = g_{ed} \nabla^e$. $\square$

The significance of this symmetric rank-2 tensor field that has identically vanishing divergence becomes apparent when one considers matter fields.

3.3 Energy-momentum Tensor

In general relativity, matter in spacetime (e.g. a star, a planet) is described by a tensor field called the *energy-momentum tensor*. In special relativity, for a particle with four-velocity u, the four-momentum is $p = mu$. The energy of the particle as measured by an observer with four-velocity v is then

$$E = -\eta_{ab} v^a p^b \tag{177}$$

Note that in an inertial framet where $v^a = (1, 0, 0, 0)$, this would just give $E = -\eta_{00} p^0$ so the energy would just be p^0. Moving the general relativity, essentially the equivalence principle asserts that we should replace η_{ab} with g_{ab} in the general case, so define

$$E = -g_{ab} v^a p^b \tag{178}$$

and $p_a p^b = g_{ab} p^a p^b = -m^2$ is the rest mass. Notice that in this equation, p and v must belong to the same tangent space $T_x M$; an observer at $x \in M$ cannot measure the energy of a particle at a different point in M.

For a *continuous* distribution of matter and energy, we need a more general geometric to describe its properties. Here we think of the distribution macroscopically, rather than as a large number of individual particles. Energy and momentum is then associated to this bulk object, which can most conveniently thought of as a fluid. The stress tensor is denoted T^{ab} and roughly defined as the flux of momentum p^a across a surface of constant x^b. In an inertial frame in which this fluid is at rest, T^{00} would correspond to the energy density ρ, T^{0i} to momentum density in the spatial directions x^i, and T^{ij} are referred to as 'stress', representing the forces beteen neighbouring elements of the fluid. In particular off diagonal terms in T^{ij} would be shearing terms (e.g. think of viscosity) . On the other hand diagonal terms like T^{ii} (no summation) are 'pressures' p^i.

The most commonly used fluid to model physical systems is a *perfect fluid*. The energy-momentum tensor takes the form

$$T_{ab} = (\rho + P) U_a U_b + P g_{ab} \tag{179}$$

where ρ, P are the energy density and pressure respectively and the fluid is imagined to be moving with four-velocity vector field U^a satisfying $U \cdot U = -1$ (do not think of U^a as the tangent to a single curve, as in the case of a particle, but rather as a vector field at every point in spacetime, pointing in the direction that the fluid is moving). For simplicity, let us consider the situation in special relativity in an inertial coordinate system, and assume the fluid's velocity field is $U^a = (1, 0, 0, 0)$, so it is at rest in the frame. Then a simple calculation shows (in terms of $T^{ab} = (\rho + P) U^a U^b + P g^{ab}$)

$$T^{00} = (\rho + P) + (-P) = \rho \qquad T^{0i} = 0 \qquad T^{ij} = P\delta^{ij} \tag{180}$$

where δ^{ij} are the components of the identity matrix in three dimensions. Note that the energy density $\rho = T_{ab} U^a U^b$ in general (not just the special frame used above).

The fluid is not arbitrarily specified, but is required to satisfy the conservation equation

$$\nabla_a T^{ab} = 0 \tag{181}$$

which is equivalent to

$$U^a \nabla_a \rho + (\rho + P)\nabla^a U_a = 0 \qquad (P + \rho)u^a \nabla_a U_b + (g_{ab} + U_a U_b)\nabla^a P = 0 \tag{182}$$

To get some physical insight into these equations, return to special relativity with our perfect fluid with $U^a = (1, \vec{v})$ (so we give our fluid a non-relativistic 3-velocity $\vec{v}$ and $\rho >> P$). Then the first of the above equations is

$$\partial_t \rho + \vec{v} \cdot \rho + (\rho + P)\mathrm{div}\vec{v} = 0 \Rightarrow \partial_t \rho + \mathrm{div}(\rho \vec{v}) \approx 0 \tag{183}$$

where we have used ignored the P term and div is the usual divergence operator in $\mathbb{R}^3$. This is easily seen to be the usual Newtonian equation for conservation of mass. The second equation above can be shown, in the non-relativistic limit where $\partial_t P << |\partial_i P|$, to be simply Euler's equation for a fluid.

We will not discuss the specific form of T_{ab} in these lectures, as we are often interested in the vacuum ($T_{ab} = 0$) case. We do mention that important physical fields, such as Maxwell fields, Yang-Mills fields, and scalar fields, as well as fluids, can be described in terms of a energy-momentum tensor satisfying the above conservation equation (181). This gives rise to the following

Postulate. Energy, momentum, and stresses of physical matter can be described by an $(0, 2)$ symmetric tensor field T_{ab} that is covariantly conserved, $\nabla^a T_{ab} = 0$.

We conclude by noting that in general relativity, certain physical requirements are imposed on T_{ab} to describe realistic matter. These include the weak energy condition, the dominant energy condition, and the strong energy condition. These are more advanced topics, and we will briefly discuss these after introducing the field equations.

3.4 Gravitational Field Equations

In Newtonian theory, the gravitational field is a conservative field with potential Φ satisfying a Poisson equation

$$\vec{\nabla}^2 \Phi = 4\pi G \rho \tag{184}$$

where we have reinstated the gravitational constant G and $\vec{\nabla}^2$ represents the standard Laplace operator in $(\mathbb{R}^3, g_E)$. We seek a tensorial equation of motion, and under general grounds we wish this to be second-order in derivatives of g_{ab}, as in most classical field theories in physics. As discussed above, it is natural to interpret $T_{ab} v^a v^b$ as the energy density of a fluid as observed by a timelike observer with four-velocity v^a. The left hand side of the above equation should then be related to (second) derivatives of the metric tensor g_{ab}. Indeed we have discussed that in the weak-field limit, an 'effective' metric is given by

$$\mathrm{d}s^2 = -(1 + 2\Phi)\mathrm{d}t^2 + (1 - 2\Phi)(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2) \tag{185}$$

where (t, x, y, z) are Cartesian coordinates taking real values and $\Phi \ll 1$. We have already calculated the Christoffel symbols associated to this metric (in the low velocity, time-independent limit) and from here one can continue to calculate the Riemann tensor. One gets

$$\Gamma^0_{00} = \partial_t \Phi, \qquad \Gamma^i_{00} = \partial_i \Phi \tag{186}$$

and it is straightforward to derive

$$R_{00} = \vec{\nabla}^2 \Phi \tag{187}$$

This would then suggest, in light of Poisson's equation, the tensorial equation

$$R_{ab} = 4\pi G T_{ab} \tag{188}$$

Indeed Einstein considered this possibility for tensorial (frame-independent, in physical terms) dynamical equations for the gravitational field. The problem arises from the fact that, as we have stated, T_{ab} must satisfy the conservation law (181). From what we have discussed, the aptly named Einstein tensor G_{ab} has identically vanishing divergence. This immediately leads to

$$G_{ab} = \kappa T_{ab} \tag{189}$$

for some coupling constant κ. Note that both sides are symmetric by construction. In order to match with Poisson's equation, note that it can be shown that the scalar curvature associated to the weak-field metric (157) is simply

$$R = g^{ab} R_{ab} = 2\vec{\nabla}^2 \Phi \tag{190}$$

From this it follows $G_{00} = R_{00} - \frac{1}{2} g_{00} R = 2\vec{\nabla}^2 \Phi$ where we used $g_{00} = -(1+2\Phi)$ and kept terms only to linear order in Φ. Then in order to get Poisson's equation from (189), we must take $\kappa = 8\pi G$. Thus we arrive at

$$G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R = 8\pi G T_{ab} \tag{191}$$

These are Einstein's field equations for general relativity. It is motivated by mathematical consistency and matching with the Newtonian gravitational field equations in the appropriate limits. Remarkably though it has been tested *outside* these limits and still been found to be utterly accurate. Note usually we will set $G = 1$.

We can rewrite the Einstein equation in an alternative form as follows. Take the trace of both sides (i.e. contract with g^{ab}) to get

$$R - \frac{1}{2}(4)R = -R = 8\pi G g^{ab} T_{ab} =: 8\pi G T \tag{192}$$

where T is defined to be the trace of the energy-momentum tensor. Hence $R = -8\pi G T$ and we can rewrite (191) as

$$R_{ab} = 8\pi G T_{ab} + \frac{1}{2} g_{ab} R = 8\pi G \left(T_{ab} - \frac{1}{2} g_{ab} T \right) \tag{193}$$

This form is actually arguably more useful in specific cases, because for a given T_{ab}, one does not need the scalar curvature. In particular if we are in the *vacuum* so that $T_{ab} = 0$ we get the simple equation

$$R_{ab} = 0 \qquad \text{(Einstein's equations in the vacuum)} \tag{194}$$

These form a set of purely geometric PDEs on (M, g). Empty spacetime is described by these equations. Note though that these are **local** equations. So just because there is no matter at a particular neighbourhood of spacetime, there is no reason why the only solution of (194) is Minkowski spacetime; indeed we could be in the region outside a star or a black hole, in which case there will still be gravitational field present created by this source.

The cosmological constant

Since the metric tensor is itself convariant constant $\nabla g = 0$, it is automatically divergenceless, so $\nabla^a g_{ab} = 0$. Furthermore the metric tensor is by definition symmetric. Thus the geometric requirements that led to Einstein's equations are unaffected if we add an additional term to (191):

$$R_{ab} - \frac{1}{2}g_{ab}R + \Lambda g_{ab} = 8\pi G T_{ab} \tag{195}$$

where $\Lambda \in \mathbb{R}$ is a constant known as the *cosmological constant*. Moving it to the right hand side of the equation, one can imagine it to be a sort of constant energy density, as opposed to a purely geometric term. Note that for $T_{ab} = 0$, the sign of Λ is chosen so that $R = 4\Lambda$, or equivalently

$$R_{ab} = \Lambda g_{ab} \tag{196}$$

which is just the condition for an Einstein metric. Hence $\Lambda > 0$ is has positive curvature and $\Lambda < 0$ would have negative curvature. It appears, based on observations, that there is in fact a small and positive cosmological constant. The maximally symmetric solution of (196) with $\Lambda > 0$ is *de Sitter spacetime*. On the other hand, for theoretical reasons motivated by developments in string theory, it is generally believed that one can define a consistent theory of quantum gravity on curved spcaetimes which asymptotically approach the maximally symmetric solution of (196) with $\Lambda < 0$, known as *Anti-de Sitter spacetime*.

3.4.1 The Nature of Einstein's equations

Let us consider (191) in a concrete setting. We have a set of coordinates, say $x^a = (t, x^i)$ (they need not be Cartesian in general). Our goal is to solve these PDEs for the components of the metric tensor in this coordinate basis, $g_{ab} = g_{ab}(t, x^i)$. There are 10 independent components, since one can think of g_{ab} as a non-degenerate symmetric 4×4 matrix. The Einstein equations thus form a set of non-linear, 2nd order coupled PDEs for the functions g_{ab}. They are 2nd order because the Ricci scalar and scalar curvature contain derivatives up to second order of the metric; they are non linear clearly, because there are terms of the schematic form $\Gamma \cdot \Gamma$ appearing in the curvature; and they are coupled in these sense one needs to solve for all components simultaneously in general - one does not get 10 independent equations for each component g_{ab} that can be solved separately from the last.

Finally, the most important feature of the Einstein equations are that they are *dynamical*. The functions g_{ab} are in general time dependent. From the PDE standpoint the field equations are **hyperbolic**, like the wave equation, i.e. they are non-linear tensorial versions of

$$\left(-\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2\right) f = 0$$

which is the scalar wave equation. The solutions in general represent phenomena propagating in time. A good example is gravitational waves, which represent signals travelling at the speed of light. An important subclass of solutions are the time-independent sector, which represent equilibrium situations. This is analogous to time-independent solutions of the above scalar wave equation, which are simply solutions of Laplace's equation. The most elementary example of such a solution is discussed in length next.

4 The Schwarzschild Geometry and Black Holes

4.1 Basic Properties

We now turn to studying arguably the best known non-trivial *vacuum* solution of Einstein's equations (194)

$$\mathrm{d}s^2 = -\left(1-\frac{2M}{r}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1-\frac{2M}{r}} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right) \tag{197}$$

where $M \geq 0$ is constant which will have the interpretation of mass. The coordinates used are as *Schwarzschild coordinates* and have the ranges

$$t \in \mathbb{R}, r > 2M, \theta \in (0,\pi), \phi \in [0, 2\pi) \tag{198}$$

The metric has coordinate singularities as $\theta = 0, \pi$ but these are the usual ones coming from the fact one cannot cover S^2 with a single coordinate chart, so they are not serious. More worrying is a possible singularity at $r = r_s = 2M$. The constant r_s is known as the Schwarzschild radius.

The solution was obtained in 1916 by Karl Schwarzschild (translate the name literally from German), shortly after the publication of the field equations in 1915. As we will discuss, it represents the gravitational field outside of a spherically symmetric massive object. For example, the Sun is spherically symmetric and one would expect that the gravitational field produced by its presence would accordingly be spherically symmetric. It is important to re-emphasize that the geometry here represents spacetime *outside* an isolated mass, in the absence of matter. That is, it satisfies

$$R_{ab} = 0 \tag{199}$$

The above solution cannot be used to describe the *inside* of a star or planet, for example, where $T_{ab} \neq 0$. Indeed for this situation one would need a so-called 'interior' solution, and we could model the inside of the star by a perfect fluid or more complicated energy-matter distribution. In general, $r_s = 2M$ is an extremely small number for given M (for the Earth, it is of order of centimetres, and for the Sun it is around 3km). Hence we expect that the Schwarzschild solution will not be valid for such a small value of r, because one would no longer be in empty space but rather 'inside' . Thus the apparent singularity at $r = r_s$ can safely be ignored. If, however, the massive object collapses to a size so small that it lies inside its own Schwarzschild radius (i.e. the Sun collapses to a size less than 3km without losing any of its mass) then we must investigate what happens in a neighbourhood of $r = 2M$. This then leads to the concept of a **black hole**.

We have stated the above solution is *spherically symmetric* . What does this mean? Let us consider a scalar PDE, such as Laplace's equation in $\mathbb{R}^3$

$$\vec{\nabla}^2\Phi = 0 \tag{200}$$

The *fundamental solution*, or what is often called the 'Green's function' for this equation is

$$U = -\frac{1}{4\pi r} \tag{201}$$

where r is the usual radial coordinate in spherical coordinates on $\mathbb{R}^3$. This represents a point charge at the origin, and away from $r = 0$, $\vec{\nabla}^2 U = 0$. (In physics language one would write $\vec{\nabla}^2 U \propto \delta(r)$ where $\delta(r)$ represents the Dirac-delta 'function'). It is obvious that this solution is spherically symmetric in that its value at a point (x, y, z) depends only on the distance from the origin $r = \sqrt{x^2 + y^2 + z^2}$.

For a vector field or tensor field in general, the concept of a symmetry is more complicated than simply requiring that its components in some chart are independent of a particular coordinate. The correct coordinate-invariant notion of an infinitesimal symmetry of the metic tensor g_{ab} is given by the notion of a Killing vector field.

Definition. *A Killing vector field ξ satisfies the Killing equations*

$$\nabla_a \xi_b + \nabla_b \xi_a = 0 \tag{202}$$

A Killing vector field is an infinitesimal generator of a symmetry transformation. Equivalently this equation can be written

$$\xi^a \partial_a g_{bc} + \partial_b \xi^a g_{ac} + \partial_c \xi^a g_{ba} = 0 \tag{203}$$

Killing fields are examined in Assignment 4. The integral curves of this vector field (i.e. the curves whose tangent is ξ) are curves along which g_{ab} does not change, referred to a 'isometries'.

For a spacetime to be spherically symmetric, we require that it has the same isometry group as the canonical metric on S^2,

$$\mathrm{d}s^2 = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2 \tag{204}$$

As you show in Assignment 4, this space has 3 Killing vector fields K_1, K_2, K_3 which satisfy certain commutation relations. It is very easy to see in this coordinate system that $K_1 = \partial/\partial\phi$ is Killing from (203). This is because in coordinates, $K_1 = (0, 1)$. Hence all its partial derivatives are zero, and further each component g_{ab} is independent of ϕ. Hence the left hand side of (203) is easily seen to vanish. On the other hand, the Killing fields K_2, K_3 take a more complicated form. We say that the coordinates ϕ is 'adapted' to the Killing field K_1. It is *not always possible* to find coordinates adapted to every Killing field in spacetime. For example, in Minkowski spacetime in the usual Cartesian coordinates, the Killing fields $\partial_t, \partial_x, \partial_y, \partial_z$ are associated to the usual coordinates (t, x, y, z) but these coordinates are not adapted to the other symmetry transformations related to Loretnz boosts and rotations. For another example, spherical coordinates on $\mathbb{R}^3$ are adapted to rotations around the z-axis (the coordinate ϕ and associated Killing field ∂_ϕ) but not to the translations along the x, y and z directions, or rotations about the y- and $x-$axes.

Symmetries of the Schwarzschild Solution

It can be easily checked that (197) is invariant under the same Killing fields K_1, K_2, K_3 and is hence spherically symmetric. These symmetries form the mathematical structure of a *Lie group*, the rotational group $SO(3)$. Informally, this is because is because (197) 'contains' the 2d spherical metric as a building block but otherwise does not have any other dependence on the angles (θ, ϕ). The surfaces of constant t, r are spheres S^2 of radius r, and they have area $4\pi r^2$.

The Schwarzschild solution also has another symmetry. It is obvious that the vector field $\xi = \partial_t$ is also a Killing vector field. To see this simply note that in this coordinate system $\xi = (1, 0, 0, 0$. Hence all its partial derivatives vanish. Further the metric components g_{ab} are obviously $t-$independent. Thus Killing's equations (203) are satisfied. Note that

$$g(\xi, \xi) = g_{ab}\xi^a\xi^b = g_{tt} = -\left(1 - \frac{2M}{r}\right) \tag{205}$$

and hence for $r > 2M$, ξ is a timelike vector field. Intuitively we think of the geometry (197) as time-independent, or physically describing a gravitational field that is in equilibrium.

Definition. *A spacetime is said to be* stationary *if it possesses a timelike Killing vector field.*

Thus the Schwarzschild solution is stationary, at least provided we restrict to $r > r_s = 2M$. In fact it is also a *static* spacetime. This is a stronger property and best stated in terms of differential forms. We will simply say that staticity is equivalent to the covariant condition

$$\xi_{[a}\nabla_b\xi_{c]} = 0 \tag{206}$$

which is the condition that ξ is orthogonal to hypersurfaces. For our purposes, all that matters is that (197) does not contain any 'cross-terms' of the form g_{ti} where $i = (r, \theta, \phi)$. This is why it is 'static' . There are non-static generalizations of the Schwarzschild geometry, most notably the Kerr solution, for which this property does not hold. The Kerr geometry represents the stationary, non-static gravitational field outside a *rotating* star or a rotating black hole.

It is a remarkable property that assuming the vacuum Einstein equations $R_{ab} = 0$ and spherical symmetry is sufficient to *guarantee* the spacetime is static (!).

Theorem 2. *(Birkhoff) The unique spherically symmetric solution of the vacuum Einstein equations is the Schwarzschild solution, which in particular is static.*

This shows that the gravitational field outside a spherical body is independent of time, regardless of whether the body creating the field is dynamical (changing in time). But as long as the process is spherically symmetric and one considers the gravitational field in the region of spacetime 'outside' , one will be in the Schwarzschild geometry.

Asymptotic flatness

Let us now consider an important property of the Schwarzschild solution. If $M = 0$ the solution is simply

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{207}$$

which is just Minkowski spacetime expressed in terms of spherical coordinates. For large $r \gg 2M$ if we pass to Cartesian coordinates it can easily be seen that

$$g_{ab} = \eta_{ab} + O(r^{-1})h_{ab} \qquad r \to \infty \tag{208}$$

where h_{ab} is rank-2 symmetric tensor. Hence the metric approaches the Minkowskian metric on $\mathbb{R}^{13}$ with a specific fall-off rate. This is what it means for the spacetime to be *asymptotically flat*. The proper definition of asymptotic flatness requires more care (and can have weaker decay rates), but essentially it means outside a compact region constant time slices of the manifold look like $\mathbb{R}^3$ minus a ball. Iin fact, for large r if we use a Taylor expansion,

$$g_{rr} = \left(1 - \frac{2M}{r}\right)^{-1} \approx \left(1 + \frac{2M}{r}\right) \tag{209}$$

as $r \gg 2M$. Thus we see that

$$\mathrm{d}s^2 \approx -\left(1 + \frac{2M}{r}\right)\mathrm{d}t^2 + \left(1 + \frac{2M}{r}\right)\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 \tag{210}$$

where $\mathrm{d}\Omega^2$ represents a shorthand notation for the S^2 metric. So this means for large r,

$$\mathrm{d}s^2 \sim \mathrm{d}^2_{\text{Mink}} - \frac{2M}{r}\mathrm{d}t^2 + \frac{2M}{r}\mathrm{d}r^2 \tag{211}$$

where $\mathrm{d}s^2_{\text{Mink}}$ is the Minkowski metric written in spherical coordinates as above. At large distances, the spacetime is indeed flat, plus small corrections.

To see this another way, if we change coordinates to

$$r = R\sqrt{1 + \frac{2M}{R}} \tag{212}$$

and ignore terms of order M^2/R^2 we get (note $r \approx R + \ldots$)

$$\mathrm{d}s^2 \approx -\left(1 - \frac{2M}{R}\right)\mathrm{d}t^2 + \left(1 + \frac{2M}{R}\right)(\mathrm{d}R^2 + R^2\mathrm{d}\Omega^2) \tag{213}$$

If we now pass to usual Cartesian coordinates (x, y, z) then obviously

$$\mathrm{d}R^2 + R^2\mathrm{d}\Omega^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \tag{214}$$

Hence the metric takes precisely the weak-field form (185) with the Newtonian potential

$$\Phi = -\frac{M}{r} \tag{215}$$

Therefore we conclude the Schwarzschild solution describes the geometry of a body mass M. We will therefore take $M > 0$. A more advanced analysis reveals that for asymptotically flat spaces one can define a mass independent of coordinates (i.e. in an invariant way) and the famous Positive Mass Theorem of General Relativity, proved by Schoen and Yau, proves that that the mass $M \geq 0$ with equality if and only if the spacetime is Minkowski.

4.2 Geodesics in Schwarzschild spcaetime

We now consider timelike and null geodesics in this geometry. Physically this represents the paths of planets and light rays in the spherically symmetric field produced by the Sun, assuming of course that the planets' masses are so small that they can thought of as 'test' particles that do not affect the gravitational field. These trajectories differ slightly from the Newtonian predictions, and these differences have been confirmed by experiment. The two main novel predictions are the precession of Mercury, and the bending of light by the Sun.

We start with the Lagrangian

$$L = -\left(1 - \frac{2M}{r}\right)\dot{t}^2 + \frac{\dot{r}^2}{1 - \frac{2M}{r}} + r^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) \tag{216}$$

Recall *along affinely parametrized geodesics* the quantity L is actually a constant, with a sign which determines whether the geodesic is timelike, spacelike, or null. Along solutions we will set $L = -\epsilon$ where $\epsilon = 1, 0$ for timelike and null geodesics respectively.

Restriction to the Equatorial Plane

We first make a useful observation. Since

$$\frac{\partial L}{\partial \theta} = 2r^2 \sin\theta\cos\theta\dot{\phi}^2 \qquad \frac{\partial L}{\partial \dot{\theta}} = 2r^2\dot{\theta} \tag{217}$$

the Euler-Lagrange equation for $\theta(\lambda)$ is

$$\ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \sin\theta\cos\theta\dot{\phi}^2 = 0 \tag{218}$$

Consider the function $\theta(\lambda) = \frac{\pi}{2}$. One can check it solves the above geodesic equation (since $\ddot{\theta}(\lambda) = \dot{\theta}(\lambda) = 0$ and $\cos\theta(\lambda) = 0$. The standard uniqueness theorems for 2nd order ODEs guarantee that this is the unique solution which satisfies the initial condition $\theta(0) = \pi/2$. Thus, if orient our axes so that the particle motion starts in the $\theta = \pi/2$ plane then we see that the motion will stay in that plane forever. Thus justifies why we can just consider geodesics moving in the plane $\theta = \pi/2$. From now on we will assume we have oriented our axes so that the particle is restricted to this plane. Effectively this removes θ as a coordinate.

Geodesic equation

Set $\theta = \pi/2$ in the Lagrangian. Immediately we see that t, ϕ are so called 'ignorable coordinates' because they immediately give rise to constants of motion. To wit,

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial L}{\partial \dot{t}}\right) = \frac{\partial L}{\partial t} = 0 \Rightarrow \frac{\partial L}{\partial \dot{t}} = -2E \tag{219}$$

for some constant E. This gives

$$\dot{t} = \frac{E}{1 - \frac{2M}{r}} \tag{220}$$

Likewise

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{\partial L}{\partial \phi} = 0 \Rightarrow \frac{\partial L}{\partial \dot{\phi}} = 2h \tag{221}$$

for some constant h. This gives

$$\dot{\phi} = \frac{h}{r^2} \tag{222}$$

Here E and h will have the physical interpretations as the energy and angular momentum (per unit mass) of the particle as measured by an observer at rest at spatial infinity. To see this, suppose an observer is at rest at infinity, so that their 4-velocity is $v^a = (1, 0, 0, 0)$. Then $E_{\mathrm{obs}} = -g_{ab}v^a p^a$ where $p^a = \dot{x}^a$ for a unit mass particle. Note that the observation can be made only at infinity, because we can only compute the scalar product of two vectors at the same spacetime point. It is easy to see that $E_{\mathrm{obs}} = E$. That is, E is the 'energy' of the particle if it were to start from rest at $r \to \infty$ as measured by our static observer.

It is a straightforward exercise to show that condition $L = -\epsilon$ gives (see Assignment)

$$\dot{r}^2 = E^2 - V(r) \qquad V(r) = \left(1 - \frac{2M}{r}\right)\left(\epsilon + \frac{h^2}{r^2}\right) \tag{223}$$

We do not need to explicitly solve the Euler-Lagrange equation for $r(\lambda)$, which is second order. However it is still worth computing this in order to determine the Christoffel symbols efficiently.

Those familiar with classical mechanics will recognize this equation to be the one-dimensional trajectory of a particle with energy $E^2/2$ moving within an effective potential field $V(r)/2$. Explicitly

$$V(r) = \epsilon - \frac{2M\epsilon}{r} + \frac{h^2}{r^2} - \frac{2Mh^2}{r^3} \tag{224}$$

The new general relativistic term is the final one, which dominates for smaller values of r; it is this term which causes Mercury, which travels closest to the Sun, to behave slightly differently to what Newtonian gravity predicts.

Finally note that we must have $\dot{r}^2 \geq 0$ with 'turning points' when $\dot{r} = 0$. Thus for a given E, the only allowed values of r allowed are those for which $E^2 \geq V(r)$, with a turning point at $r = r_0$ (i.e. $\dot{r}$ switches sign) at $E^2 = V(r_0)$. For the moment we consider the cases where $r > 2M$. The analysis of trajectories reduces to determining the qualitative behaviour of $V(r)$. Generally we expect both free trajectories (e.g. a comet approaches the Sun, then flies past and escapes to the far-field region) and bound orbits (e.g. a planet in an elliptic orbit). This will depend crucially on the magnitude of h. Many features of the trajectories are explored in the appropriate Assignment, but there will be some overlap here.

Null Geodesics

Let $\epsilon = 0$ for null geodesics. The potential is

$$V(r) = \left(1 - \frac{2M}{r}\right)\frac{h^2}{r^2} \tag{225}$$

What does the potential look like? If $h = 0$ then $V(r) = 0$; the particle motion is just

$$\dot{r}^2 = E^2 \tag{226}$$

Note that we cannot have $E = 0$. Since $h = 0$, $\phi = \phi_0$ is a constant. If $E = 0$ then $t = t_0$ is a constant as well. But then the null condition $g_{ab}\dot{x}^a\dot{x}^b = g_{rr}\dot{r}^2 \geq 0$ with equality if and only if $\dot{r} = 0$. But if this were true, then $\dot{x}^a = 0$ identically. Hence we have $E \neq 0$, and thus fixing $E > 0$,

$$\dot{r} = \pm E \qquad \text{Radial Motion of Null geodesics} \tag{227}$$

This easily integrates to $r = \pm E\lambda$. Physically this corresponds to a light ray moving radially away or towards the central object depending on the sign chosen; it does not change direction.

Now suppose we have $h \neq 0$. The the light ray has angular momentum 'around' the central object (i.e. it is not moving directly towards or away from it). The potential $V(r)$ has a critical point $V'(r_*) = 0$ when

$$V'(r) = -\frac{2h^2(r - 3M)}{r^4} \tag{228}$$

vanishes, i.e. $r_* = 3M$. It is easy to check that $V''(r) = 6h^2(r - 4M)/r^5$. Thus $V''(r_*) > 0$ so this critical point is a *maximum*. We can graph this below. Thus the trajectories fall into two cases.

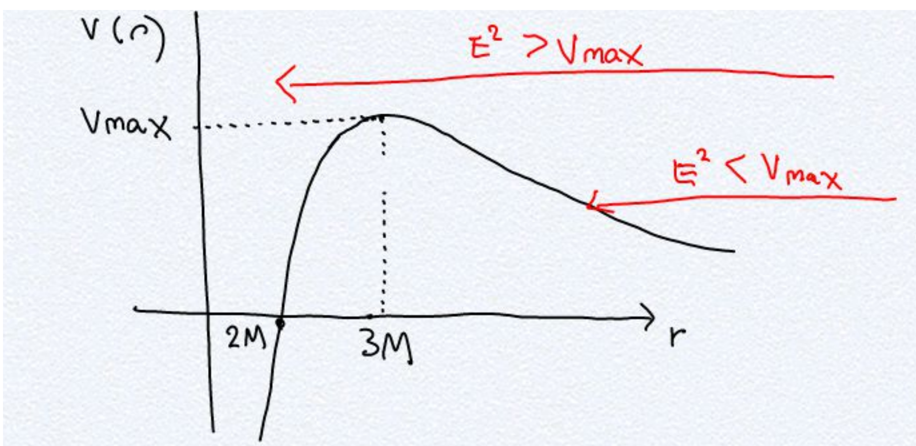

Figure 3: Effective potential for null geodesic motion for $h \neq 0$

Define

$$V_{max} = V(r_*) = \frac{h^2}{27M^2} \tag{229}$$

Let us assume the light ray is coming from large r. If $E^2 > V_max$ then the light ray has enough energy to overcome the potential 'barrier' and it can travel to arbitrarily small r (at least until the Schwarzschild geometry is no longer a valid description of spacetime, so for example the light ray is absorbed by the surface of a star). If $E^2 < V_{max}$ then an incoming light ray from large r will approach the object and then get no closer than the point $r_o > r_*$ where $E^2 = V(r_0)$ and then turn around and 'escape' .

On the other hand if the light ray started at some $r < r_*$ and is travelling 'away' then it can only escape if $E^2 > V_{max}$; otherwise it will travel out to some $r_0 < r_max$ and then turn back and spiral to smaller r (in this sense it is 'trapped'). Finally, if the light ray has arranged itself so that

$$E^2 = V(r_*) = \frac{h^2}{27M^2} \tag{230}$$

then $\dot{r} = 0$ for all λ. This is the circular orbit at $r = 3M$. Note that the light ray is orbiting in a circle with $\dot{\phi} = h/9M^2$. The orbit is clearly *unstable* as discussed in the Assignment. This is because a small perturbation of the orbit to either larger or small r (i.e. increase or decrease E) will cause it to 'roll down the potential' and either escape or be trapped in the potential.

Deflection of Light

Define the impact parameter

$$b \equiv \frac{h}{E} \tag{231}$$

The condition that an incoming null ray overcomes the potential barrier and be captured is therefore $b^2 < 27M^2$. What does b represent? It can be thought of roughly as the distance of 'closest approach' to the massive object. We will explain this below, and then discuss an important prediction of GR, the gravitational bending of light.

Suppose we are in flat space, so that $M = 0$. Again fix motion onto the equatorial plane. Then the constant

$$b^2 = \frac{h^2}{E^2} = \frac{r^4\dot{\phi}^2}{\dot{t}^2} \tag{232}$$

Since b is a constant we can calculate it anywhere we want. We will do so at large r. On a light

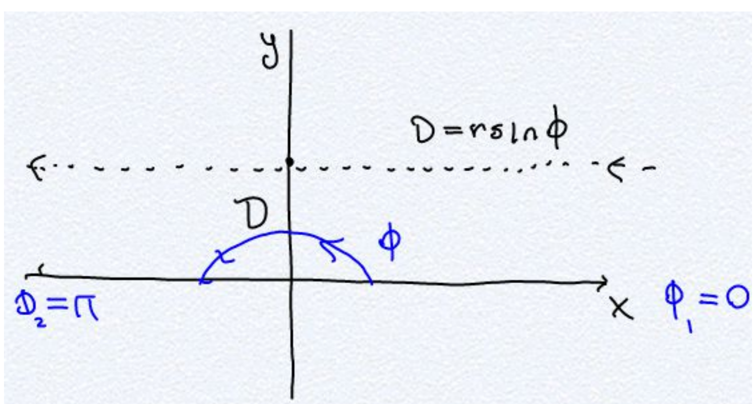

Figure 4: Impact parameter b

ray, $g_{ab}\dot{x}^a\dot{x}^b = 0$ so that $\dot{t}^2 = \dot{r}^2 + r^2\dot{\phi}^2$. Now in flat space, we know the straight lines are geodesics. So in the usual Cartesian coordinates on the equatorial plane $z = 0$, we have

$$x = r\cos\phi \qquad y = r\sin\theta \tag{233}$$

The path $y = D$ is a geodesic, which means $r \sin\phi = D$. When r is very large, $\sin\phi = D/r$ is very small, and hence $\sin\phi \approx \phi$. Thus $\phi = D/r$ for large r , and thus

$$\dot{\phi} = -\frac{D\dot{r}}{r^2} \tag{234}$$

Meanwhile the null condition for large r reduces to

$$\dot{r}^2 = E^2 \tag{235}$$

and hence $\dot{r}^2/\dot{t}^2 = 1$. So putting this altogether implies that, evaluating b for large r,

$$b^2 = \frac{r^4\dot{\phi}^2}{\dot{t}^2} = D^2 \tag{236}$$

This explains why b can be interpreted as the distance of 'closest' approach to the origin. We have done this calculation in flat space with $M = 0$, but since the Schwarzschild geometry is well approximated by the flat geometry at large r, we can keep this interpretation.

Now imagine in flat space the path of null ray that travels on the line $y = b =$ constant. In terms of the angle ϕ, it goes from $\phi_1 = 0$ (large positive x) to $\phi_2 = \pi$ (large negative x). Hence the total amount of angle travelled is $\phi_2 - \phi_1 = \pi$. For $M > 0$ it can be shown that the angle travelled is actually larger than π - this difference is the *deflection angle*.

To calculate it, we will do a change of variables. Let $u = 1/r$. Then $r \to \infty$ corresponds to $u = 0$. We have $\dot{u} = -\dot{r}/r^2$. We can then eliminate the parameter λ in favour of ϕ to obtain

$$u'^2 = \left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2 = \frac{\dot{u}^2}{\dot{\phi}^2} = \frac{\dot{r}^2}{r^4\dot{\phi}^2} = \frac{1}{b^2} - (1 - 2Mu)\, u^2 \tag{237}$$

where we used our expression for $\dot{r}$ for null geodesics above and the definition of b. Differentiating gives the simple equation

$$u'' + u = 3Mu^2 \tag{238}$$

When $M = 0$, this just has the simple solution $u_0 = \sin\phi/b$ for some constant b (setting an integration constant to zero so that $u = 0$ corresponds to $\phi_1 = 0$ and $\phi_2 = \pi$). This of course is just the solution we already knew, $b = r\sin\phi$ where b is the impact parameter , which is a straight line $y = b$ in Cartesian coordinates on the plane. The idea is then to seek a approximate solution for small M of the form

$$u = u_1 + u_0 \tag{239}$$

and neglect terms of order u_1^2 and Mu_1. This is equivalent to simply replacing the u^2 on the RHS of (238) with u_0^2. One can verify easily that the following is a solution:

$$u = \frac{\sin\phi}{b} + \frac{M(1 - \cos\phi)^2}{b^2} \tag{240}$$

where we have chosen the integration constants to $\phi_1 = 0$ corresponds to $u = 0$. Now we want to find ϕ_2 defined by $u(\phi_2) = 0$. We expect this to occur near $\phi = \pi$ as in flat space. So let $\phi_2 = \pi + \Delta\phi$ where $\Delta\phi$ is small. We get

$$0 = u(\pi + \Delta\phi) = \frac{\sin(\pi + \Delta\phi)}{b} + \frac{M(1 - \cos(\pi + \Delta\phi))^2}{b^2} \tag{241}$$

Now use the double angle formulae

$$\begin{aligned}
\sin(\pi + \Delta\phi) &= \sin\pi\cos\Delta\phi + \cos\pi\sin\Delta\phi = -\sin\Delta\phi \approx -\Delta\phi + O(\Delta\phi^3) & (242)\\
\cos(\pi + \Delta\phi) &= \cos\pi\cos\Delta\phi - \sin\pi\sin\Delta\phi = -\cos\Delta\phi = -1 + O(\Delta\phi^2) & (243)
\end{aligned}$$

Substituting this into (241) we get

$$-\Delta\phi + \frac{4M}{b^2} = 0 \tag{244}$$

which gives the *deflection angle* $\Delta\phi = \phi_2 - \phi_1 - \pi$ relative to a straight line trajectory:

$$\Delta\phi = \frac{4MG}{b^2c^2} \tag{245}$$

where we have reinstated the gravitational constant G and the speed of light c to indicate how small this deflection is. The derivation assumes that M/b is very small, which is indeed true, so this formula would not be accurate in a very strong gravitational field. For a light ray that just grazes the surface of the Sun, the deflection angle is approximately $\Delta\phi \approx 1.75''$ (arcseconds). To observe it, consider a distant star as a point source emitting a light ray 'from infinity' that reaches Earth. In the night sky, it will have a certain position. The effect of the Sun can be measured doing a solar eclipse (otherwise the Sun's light is too bright to observe the deflection) and has the effect of causing the location of the star have moved relative to its original position. The predicted amount has been confirmed to high accuracy by experiment. There are additional general-relativistic effects which can be tested, most notably the prediction of the precession of the orbits of Mercury. We will not study this here, but the calculation involves an analysis of perturbations of bound orbits of the $\mathrm{d}u/\mathrm{d}\phi$ geodesic equation derived above.

Timelike Geodesics

We now consider timelike geodesics. The motion is one-dimensional and governed by the first order ODE for $r(\lambda)$ (223)

$$\dot{r}^2 = E^2 - V(r) \qquad V(r) = \left(1 - \frac{2M}{r}\right)\left(1 + \frac{h^2}{r^2}\right) \tag{246}$$

Properties of solutions are investigated in the relevant Assignment, but we will summarize the key results here. Note that as $r \to \infty$, $V(r) \to 1$ and that $V(2M) = 0$.

The first case to consider is a particle with no angular momentum $h = 0$. Then the motion is purely radial ($\dot{\phi} = 0$). $V(r) \leq 1$ for all $r \geq 2M$ and is a monotonically increasing function of r.

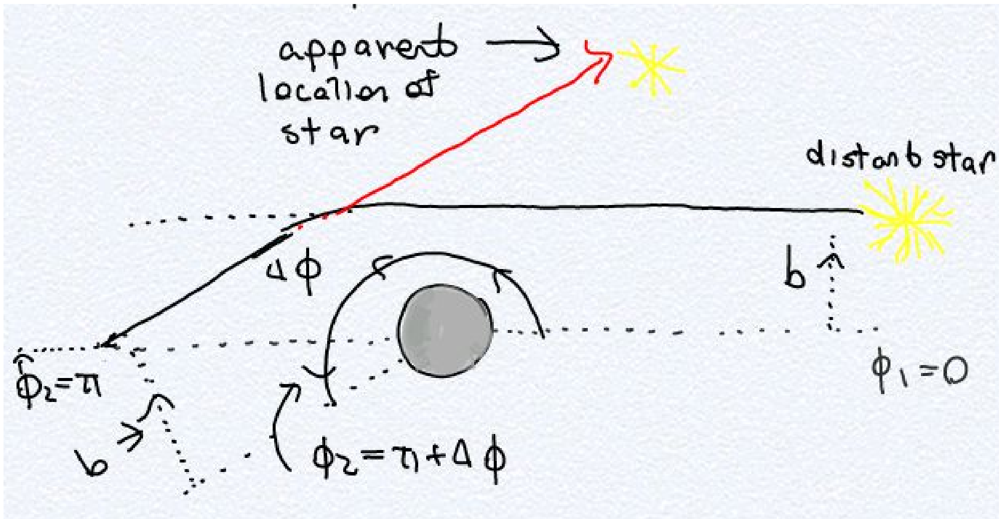

Figure 5: Deflection of light from a star by the Sun

An ingoing particle will fall into the region $r = 2M$. Notice that we must have $E^2 \geq V(r)$ at all points on the trajectory so $E^2 \geq 1$. A similar property holds for outgoing radial geodesics. Note (obviously) there are no curves which are 'fixed' in space ,i.e. $r(\lambda) = r_0$ is constant. To see this, if $\dot{r} = 0$ for all λ, then it would follow $\ddot{r} = 0$. By the Chain rule this implies $V'(r_*) = 0$. But for $h = 0$ the potential has no critical points, so this cannot occur.

For $h \neq 0$ the situation is more interesting since the potential $V(r)$ has turning points. Since (check)

$$V'(r) = \frac{1}{r^4}(2Mr^2 - 2h^2 r + 6Mh^2) \tag{247}$$

the potential has critical points when

$$r = r_{\pm} \equiv \frac{h^2 \pm h\sqrt{h^2 - 12M^2}}{2M} \tag{248}$$

and without loss of generality we will assume $h > 0$. There are no turning points if $h^2 < 12M^2$ - the particle's angular momentum is not sufficient to prevent it from falling towards $r = 2M$. This case is qualitatively similar to the $h = 0$ (see Figure). If $h^2 > 12M^2$ then bound orbits are possible, and indeed even *circular* orbits can occur.

One can check (exercise) that $V''(r_+) > 0$ and $V''(r_-) < 0$. This implies r_+ is a minimum and r_- is a maximum (see Figure 4.2). As discussed in the Assignment one can show

$$3M < r_- < 6M < r_+ \tag{249}$$

Note the strict inequalities. The degenerate case $h^2 = 12M^2$ is actually a maximum. In this case $r_- = r_+ = 6M$ and one finds $V''(6M) = 0$ as well. However $V'''(6M) > 0$, indicating that in this case one also has a minimum. One can then easily see how bound orbits are possible. If E^2 of the particle is such that it lies within a potential 'well' than particle will 'bounce' back and forth

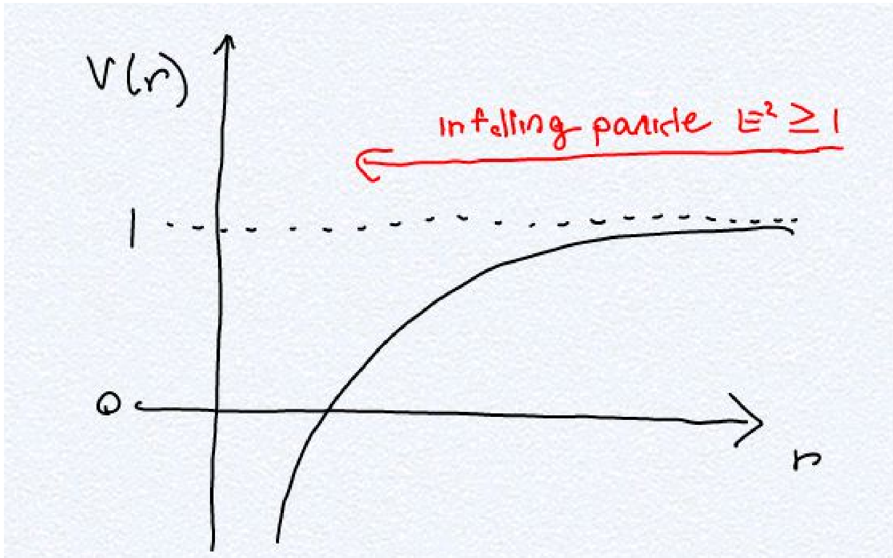

Figure 6: Radial potential $V(r)$ for $h^2 < 12M^2$.

between the two turning points (this would correspond approximately to an elliptic orbit). If E^2 is sufficiently large ($E^2 > V(r_-)$) than an infalling particle will spiral into the region $r \to 2M$. Of course in astrophysical situations $r = 2M$ is far inside a star, for example, so the particle would strike the surface of the star far before this.

Circular orbits $r = r_c$ is constant is possible if and only if $\dot{r} = \ddot{r} = 0$ for all λ. The requirement $\dot{r} = 0$ implies $E^2 = V(r_c)$ for these special cases. The requirement $\ddot{r} = 0$ then implies (see Assignment) that $V'(r_c) = 0$, i.e. they occur at critical points of the potential. As is obvious from the diagram, a circular orbit is *stable* if the critical point is a minimum (small perturbations will remain bounded for all λ) whereas the orbit is *unstable* if the critical point is a maximum (a small perturbation will case the particle to 'roll' down the potential). Thus stable circular orbits occur if $h^2 > 12M^2$ and $r = r_+$, and there is an unstable orbit at $r = r_-$. The degenerate case when $h^2 = 12M^2$ has a stable circular orbit at $r = 6M$, often referred to as the *innermost stable circular orbit*.

In the case of non-circular bound orbits oscillating around a minimum at $r = r_+$, one can still define a point of closest approach, called the perihelion of an orbit. In Newtonian theory as is well known, these orbits are ellipses and the perihelion occurs at the same value of r on every circuit; the change $\Delta\phi$ between consecutive perihelions is just 2π. A calculation using the timelike geodesic equation predicts to precession of the perihelion, so $\Delta\phi > 2\pi$, due to the extra $1/r^3$ in the effective potential $V(r)$. For Mercury, which is closest to the Sun, this effect is most prominent. GR predicts a precession of 42.98 seconds of arc per century; taking all other effects into account, such as the Earth's own precession of its rotation axis, the attraction of Venus, etc.) the observed rate is 42.98 ± 0.04 arc seconds per century. GR appears to be remarkably accurate.

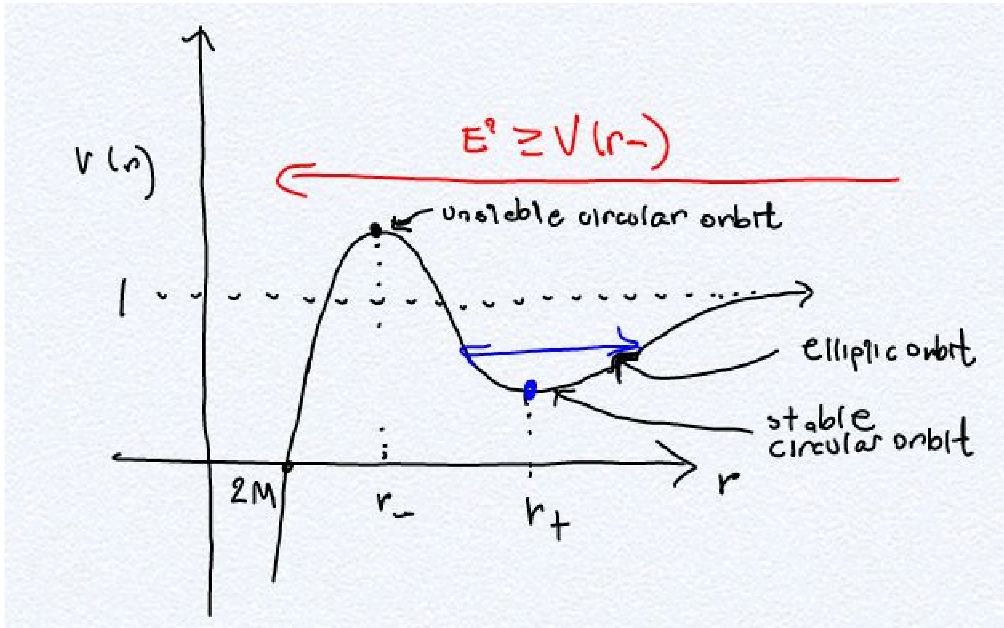

Figure 7: For $h^2 > 12M^2$ there are two critical points. A bound orbit is shown in blue along with circular orbits at the critical points.

4.3 The Schwarzschild Black Hole

We will now consider what happens if the Schwarzschild geometry holds for all $r > 0$, and in particular up to $r = 2M$. Such a situation could arise as the end point of a spherically symmetric gravitational collapse, in which a star of mass M contracts to a size smaller than $r = 2M$. We will see this describes a *black hole*.

To begin note that the Schwarzschild geometry is clearly singular as $r \to 2M$. We could ask whether this is a serious pathology or only a failure of the (t, r, θ, ϕ) coordinate chart. Consider a timelike geodesic that is falling radially ($h = 0$) towards $r = 2M$. From our equations

$$\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 = \frac{\dot{r}^2}{\dot{t}^2} = \frac{E^2 - (1 - \frac{2M}{r})}{\frac{E^2}{(1-2M/r)^2}} \tag{250}$$

This implies that the total coordinate time taken to reach $r = 2M$, starting from some finite $r = R > 2M$ is

$$\Delta t = \int_R^{2M} \left((1 - 2M/r)\sqrt{1 - E^{-2}(1 - 2M/r)}\right)^{-1} \mathrm{d}r \tag{251}$$

This integral actually diverges. Hence it takes an infinite amount of coordinate time for the freely falling particle to fall to $r = 2M$. The coordinate time is that proper time of an observer at rest at infinity. This is what is meant by saying such an observer will never see an object actually reach the Schwarzschild radius. For radial null particles, one gets

$$\Delta t = \int_R^{2M} \left(1 - \frac{2M}{r}\right)^{-1} \mathrm{d}r \tag{252}$$

which can easily be seen to diverge as well.

On the other hand consider the situation from the point of view of an observer actually freely falling. The relevant parameter here is the observer's proper time τ. By a similar integral,

$$\Delta\tau = \int_R^{2M} \dot{r}\, \mathrm{d}r = \int_R^{2M} \left(E^2 - (1 - \frac{2M}{r}) \right)^{-1/2} \mathrm{d}r \tag{253}$$

This integral is actually finite. This suggests that to an observer *nothing significant* occurs as one reaches $r = 2M$, and indeed one can pass to regions of $r < 2M$.

Another simple check that nothing happens on the 2-sphere $r = 2M$ is to compute a scalar curvature invariant (since it is scalar it takes the same value in all coordinate charts). A well known result shows that the so-called Kretchmann scalar

$$R_{abcd}R^{abcd} = \frac{48M^2}{r^6} \tag{254}$$

which is clearly finite at $r = 2M$. This is a necessary although not sufficient condition to ensure regularity there. Of course, it is clearly divergent at $r = 0$, suggesting something seriously goes wrong there - this is the *curvature singularity* . The point $r = 0$ does not in fact belong to the manifold as the metric is not defined there (recall we are dealing with smooth Lorentzian manifolds).

Eddington-Finkelstein coordinates

By rescaling our affine parameter for radial null geodesics we can arrange $E = 1$. The geodesic equations are of course

$$\frac{\mathrm{d}r}{\mathrm{d}\lambda} = \pm 1 \,, \qquad \frac{\mathrm{d}t}{\mathrm{d}\lambda} = \left(1 - \frac{2M}{r}\right)^{-1} \tag{255}$$

The + sign refers to outgoing geodesics (i.e. as λ increases, r increases). Then

$$\frac{\mathrm{d}t}{\mathrm{d}r} = \pm \left(1 - \frac{2M}{r}\right)^{-1} \tag{256}$$

We now introduce a new coordinate r_* defined by the integral

$$\frac{\mathrm{d}r_*}{\mathrm{d}r} = \left(1 - \frac{2M}{r}\right)^{-1} \Rightarrow r_* = r + 2M \log|(2M)^{-1}r - 1| \tag{257}$$

The advantage of this is that

$$\frac{\mathrm{d}t}{\mathrm{d}r_*} = \pm 1 \qquad t \mp r_* = c \tag{258}$$

where the + (lower) sign corresponds to the ingoing geodesics. Thus along ingoing null geodesics $t + r_*$ is a constant - that is, each such geodesics is labelled by this constant. This suggests we define a coordinate labelling the ingoing geodesics,

$$v = t + r_* \tag{259}$$

Now rewrite the metric in terms of the (v, r, θ, ϕ) chart. Note that although it seems like we are using the same r, ∂_r is not the same. We get

$$\mathrm{d}t = \mathrm{d}v - \frac{\partial r_*}{\partial r}\mathrm{d}r = \mathrm{d}v - \left(1 - \frac{2M}{r}\right)^{-1} \mathrm{d}r \tag{260}$$

which leads to the line element in Eddington-Finkelstein coordinates:

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right) \mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}r + r^2\mathrm{d}\Omega_2^2 \tag{261}$$

In this chart the metric tensor components g_{ab} are invertible at $r = 2M$ and indeed the metric is *smooth* for all $r > 0$, including in particular $r = 2M$. . Note that now $\partial/\partial r$ is a null vector field whereas

$$g(\partial_v, \partial_v) = g_{vv} = -(1 - \frac{2M}{r}) \leq 0 \tag{262}$$

is timelike for $r > 2M$ and vanishes at $r = 2M$. This change of a timelike vector field to a null vector field is characteristic of an *event horizon*.

What is significant about the surface $r = 2M$? Consider a future-pointing timelike curve Future pointing simply means that if λ is a parameter on the curve, then $\dot{v} > 0$. In particular if we choose the proper time as the parameter along the (not necessarily geodesic) curve, then we must have

$$-1 = -\left(1 - \frac{2M}{r}\right) \dot{v}^2 + 2\dot{v}\dot{r} + r^2(\dot{\theta}^2 + \dot{\phi}^2) \tag{263}$$

We can rearrange this to get

$$2\dot{v}\dot{r} = -1 + \left(1 - \frac{2M}{r}\right) \dot{v}^2 - r^2(\dot{\theta}^2 + \dot{\phi}^2) \tag{264}$$

Now if $r > 2M$ we can arrange for the RHS to be positive. Thus since $\dot{v} > 0$, we can have $\dot{r} > 0$ - that is the timelike curve can be moving *away* from $r = 2M$. But if $r < 2M$, then the RHS is strictly negative. Thus we must have $\dot{r} < 0$; a timelike curve *must fall towards smaller* r. Note this applies not just to freely falling geodesics but any timelike curve, such as the worldline of a astronaut with a rocket that allows her to travel arbitrarily close to the speed of light. In this sense a timelike curve is trapped and the surface $r = 2M$ is a one-way membrane - a matter particle can enter but it cannot leave.

What about light rays? In the Eddington-Finkelstein coordinate system, we have by definition the curves with $\dot{v} = \dot{\theta} = \dot{\phi} = 0$ are ingoing radial null geodesics. But consider the *outgoing* null geodesics. These have $t - r_* = v - 2r_* =$ constant. Thus

$$v = 2r_* + \text{constant} \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}r} = \frac{2}{1 - \frac{2M}{r}} \tag{265}$$

Alternatively we can write in terms of an affine parameter,

$$\dot{v} = \frac{2\dot{r}}{1 - \frac{2M}{r}} \tag{266}$$

Consider again a future-pointing outgoing null geodesics so that $\dot{v} > 0$. The RHS for $r > 2M$ is also positive provided $\dot{r} > 0$. Thus outside the surface $r = 2M$ the outgoing null geodesics are indeed moving larger r unsurprisingly. But for $r < 2M$ we must have $\dot{r} < 0$. Hence *both the ingoing and outgoing future directed radial null geodesics* have decreasing r. This is why we say that the spacetime region $r < 2M$ is 'trapped'. Light cannot escape this region. The intermediate case $r = 2M$ occurs for light rays that lie exactly tangent to the surface $r = 2M$ for all affine parameter values. This gives rise to the statement that a black hole is a region from which 'nothing, not even light, can escape'. It can be shown this property holds for any null curve, not just radial or geodesic motion. Because no signal can leave the region $r \leq 2M$ to an outside observer, this region is effectively 'black' . Indeed the defining characteristic of a black hole spacetime is that it is an asymptotically flat spacetime which contains a region from which no signal can escape to the asymptotic region.

The Event horizon

Let $f(r) = 1 - 2M/r$. The EF metric can be written

$$\mathrm{d}s^2 = -f^2\mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}r + r^2\mathrm{d}\Omega_2^2 \tag{267}$$

Proposition 16. *The vector field $K = \partial/\partial v$ is a Killing field.*

Proof. Clearly since the metric components g_{ab} are independent of the coordinate v. K is a Killing vector field. □

Direct calculation using the above coordinate transformation

$$\frac{\partial}{\partial v} = \frac{\partial t}{\partial v}\frac{\partial}{\partial t} + \frac{\partial r}{\partial v}\frac{\partial}{\partial r} = \frac{\partial}{\partial t} \tag{268}$$

where we are writing $t = t(v, r'), r = r(v, r') = r'$ (it is conventional to use the same coordinate symbol r for r', although they are actually *different*). Note that $\partial r/\partial v = 0$.

Proposition 17. *The surface $r = 2M$ is a null hypersurface whose normal is the Killing field K.*

Proof. First of all the surface $r = 2M$ has normal $n = \mathrm{d}r$. A hypersurface is said to be null if its normal is null. To compute $\mathrm{d}r \cdot \mathrm{d}r$ we need the inverse metric g^{ab}:

$$g^{vv} = 0 \qquad g^{vr} = 1 \qquad g^{rr} = f(r) \tag{269}$$

Thus $g^{ab}n_a n_b = g^{rr} = 1 - 2M/r$. When $r = 2M$ clearly this vanishes. Hence the surface is null. To show that its normal is Killing, raise the index of the normal to turn it into a vector:

$$n^a = g^{ab}n_b = g^{ar} = (1, f, 0, 0) \tag{270}$$

Now on the surface $r = 2M$, $f = 0$ and hence

$$n^a = (1, 0, 0, 0) \qquad r = 2M \tag{271}$$

or equivalently $n = K$. □

Remark. A null hypersurface whose normal is a Killing vector field is called a *Killing horizon*. Hawking's rigidity theorem states that the event horizon of a stationary (analytic) black hole is a Killing horizon.

Remark. Note that this truly is a *hypesruface* (i.e. it is three-dimensional). The coordinates on this hypersurface are (v, θ, ϕ). Seen as a Lorenzian manifold in its own right, it has signature $(0, +, +)$ with the '0' corresponding to the null direction $\partial/\partial v$.

Note that a constant v cross-section of the S^2 with radius $r = 2M$ has area

$$A = 4\pi(2M)^2 = 16\pi M^2 \tag{272}$$

This is often referred to as the 'area of black hole'.

Next we turn to another important physical and geometrical quantity that characterizes a stationary black hole: its surface gravity κ. Roughly this measures the 'local acceleration due to gravity' at the event horizon as measured by an observer at rest at infinity. As a comparison, on the Earth this quantity would be 9.8 m$/s^2$. We will give a geometric definition below.

Proposition 18. *The Killing field K is geodesic on the event horizon.*

Proof. Recall that a non-affintely parameterized geodesic $\gamma(\lambda)$ has a tangent vector $T = \dot{\gamma}$ that satisfies

$$T^a \nabla_a T^b = F(\lambda) T^b \tag{273}$$

for some function F. This means that the tangent vector is not changing direction (i.e. it is staying 'straight') although its magnitude is allowed to vary. Let us compute the left-hand side for the choice $T^a = K^a = (1, 0, 0, 0)$ in the Eddington-Finkelstein coordinates. We get

$$K^a \nabla_a K^b = K^v \nabla_v K^b = \partial_v K^b + \Gamma^b_{va} K^a = \partial_v K^b + \Gamma^b_{vv} K^v = \Gamma^b_{vv} \tag{274}$$

where we used $K^v = 1$ with the other components zero. By direct computation

$$\Gamma^b_{vv} = \frac{1}{2} g^{bc}(2\partial_v g_{cv} - \partial_c g_{vv}) = \frac{1}{2} g^{br} \partial_r f(r) \tag{275}$$

Now we are interested in what happens on the event horizon $r = 2M$. We know that $g^{br} = 0$ if $b \neq v$ on the event horizon. If $b = v$ we get

$$\Gamma^v_{vv} = \frac{1}{2} \left(\frac{2M}{r^2} \right) \Bigg|_{r=2M} = \frac{1}{4M} \tag{276}$$

Putting this altogether we get

$$K^a \nabla_a K^b = \left(\frac{1}{4M}, 0, 0, 0 \right) = \frac{1}{4M} K^b = \kappa K^b \tag{277}$$

where we have defined the *surface gravity*

$$\kappa \equiv \frac{1}{4M} \tag{278}$$

Thus as claimed K is a non-affintely parametrized geodesic on the surface $r = 2M$. □

K is the path of a static observer in the Schwarzschild geometry (i.e. one at rest in space). Such a path is of course not a geodesic, because some force is required to keep it 'in place' and not falling towards the massive object. One can think of the quantity $K^a \nabla_a K^b$ as an 'acceleration' or unit force required to keep an observer with tangent K at rest. Thus κ can be thought of as the force required to hold up an observer at rest just outside the event horizon. Note that it is proportional to $1/M$ - more massive black holes actually have lower surface gravity. A theorem of Bardeen, Carter, and Hawking proves that κ is a constant for any stationary black hole, not just Schwarzschild.

First Law of Black Hole mechanics

Our solution is completely characterized by one free parameter, M. We have already seen $A = 16\pi M^2$. An elementary exercise shows

$$\mathrm{d}M = \frac{\mathrm{d}A}{32\pi M} = \frac{\kappa}{2\pi}\frac{\mathrm{d}A}{4} \tag{279}$$

SUch a formula holds quite generally for stationary black holes and is known as the first law of black hole mechanics. It formally resembles the first law of thermodynamics

$$\mathrm{d}E = T\mathrm{d}S \tag{280}$$

where T is the temperature and S is the entropy of the system, provided we make the ad hoc identifications

$$M \to E\ , \qquad \kappa \to 2\pi T\ , \qquad A \to 4S \tag{281}$$

The first of these is plausible as we can think of M as a rest mass energy of our massive object. The other two are classically nonsensical, because a blackbody at temperature T should radiate particles over a distribution of wavelengths given by Planck's formula. Moreover one would expect for a black hole the entropy, or total amount of information in the system, is infinite, corresponding to all the different kinds of ways an object of mass M could have formed. Remarkably Hawking showed that using ideas from quantum field theory in curved spacetime, that black holes in fact do radiate at the above temperature and have an associated entropy S. Thus the analogy is not merely a formal one. The other laws of thermodynamics also can proven to have mathematical analogies in the theory of black holes: he zeroth law that the temperature is a constant in a system at equilibrium corresponds to κ being constant on the event horizon; and the entropy increase law $\mathrm{d}S \geq 0$ has an equivalent mathematical statement as the fact that $\mathrm{d}A \geq 0$ for physical processes.

Kruskal Extension

Eddington-Finkelstein coordinates cover a region of coordinates for $r > 0$, and in particular extend to regions 'to the future' of $r = 2M$. We could also the choosing coordinates adapted to the *outgoing* null radial geodesics. Recall these satisfy $t - r_* =$ constant. We can define a coordinate that labels each of these geodesics:

$$u = t - r_* \tag{282}$$

in terms of which the exterior Schwarzschild metric becomes

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r + r^2\mathrm{d}\Omega^2 \tag{283}$$

Note that $g_{ur} < 0$. As before the metric and its inverse are smooth functions for all $r > 0$ and in particular at $r = 2M$, and we can extend to regions $r < 2M$. An analogous analysis of timelike curves and null radial geodesics show that now $r = 2M$ has a very different behaviour. Future directed outgoing geodesics have u constant, with $\mathrm{d}r/\mathrm{d}\lambda = 1$ by (255). Thus $\dot{r} > 0$ everywhere, so these curves are *always* outgoing, regardless of r. On the other hand the ingoing null geodesics are defined by the equation

$$c = t + r_* = u + 2r_* \tag{284}$$

and thus

$$\frac{\mathrm{d}u}{\mathrm{d}r_*} = -2 \Rightarrow \dot{u} = -\frac{2\dot{r}}{1 - \frac{2M}{r}} \tag{285}$$

For $r > 2M$, if $\dot{u} > 0$ (future directed) then $\dot{r} < 0$. But for $r < 2M$ future directed ingoing null geodesics have $\dot{r} > 0$. The situation is the same as in our previous study, but in reverse: both ingoing and outgoing null curves have $\dot{r} > 0$ for $r < 2M$. This shows that the region $r < 2M$ in the (u, r, θ, ϕ) chart is different to the $r < 2M$ region in the (v, r, θ, ϕ) chart. Indeed no light signal can be sent from a point with $r > 2M$ to a point with $r < 2M$. A similar analysis as above shows that any timelike curves starting with $r < 2M$ must reach through $r = 2M$ in finite proper time and leave to the region $r > 2M$. Thus the $r < 2M$ region in the outgoing coordinate system is referred to as a *white hole*. They are, in a sense, time reversed versions of black holes. We expect that black holes are stable in the sense small perturbations of the metric stay small in a suitable sense and the spacetime remains a black hole. On the other hand a white hole, being the time reverse of this situation, would be unstable, consistent with the fact no one has observed one.

We have shown that the original (t, r, θ, ϕ) chart breaks down at $r = 2M$ and can be extended in two different ways . One should think of the Schwarzschild manifold as admitting different coordinate patches that cover different regions. An obvious question is whether there are more regions which are not covered by the ingoing and outgoing Eddington-Finkelstein charts. The answer is yes. There is a maximal extension of the original Schwarzschild solution, obtained by Kruskal and Szekeres who introduced a new chart which cannot be extended further.

Let us begin by first introducing a 'double null' chart (u, v, θ, ϕ) (in the following we will not always explicitly write down the spherical parts of the metric). That is, we write the metric in terms of $v = t + r_*, u = t - r_*$. It is easy to check

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}u\mathrm{d}v + r^2\mathrm{d}\Omega^2 \tag{286}$$

Here $r = r(u, v)$ is defined implicitly through the relation

$$r_* = r + 2M\log\left|\frac{r}{2M} - 1\right| = \frac{v - u}{2} \tag{287}$$

Now introduce the Kruskal-Szekes coordinates (U, V, θ, ϕ)

$$U = -e^{-u/4M}, \qquad V = e^{v/4M} \tag{288}$$

where, since u, v are real, $U < 0$ and $V > 0$. Note that

$$UV = -e^{r/2M}\left(\frac{r}{2M} - 1\right) = F(r) \tag{289}$$

This gives an implicit relation for r in terms of U, V, although we cannot explicitly solve for $r(U, V)$. The inverse function theorem guarantees that a unique solution for $r(U, V)$ exists for $r > 0$ ($F(r)$ is smooth and $F'(r) < 0$ for all $r > 0$ so the function is monotonically decreasing). Meanwhile by taking the quotient of U, V,

$$\frac{V}{U} = -e^{\frac{t}{2M}} \tag{290}$$

This equation determines t uniquely, because the right hand is obviously monotonic (i.e. the derivative of the right hand side does not change sign). So we have shown the coordinate transformation is invertible. In terms of the $(U, V, \theta\phi)$ chart one finds

$$\mathrm{d}s^2 = -\frac{32M^3 e^{-r/2M}}{r}\mathrm{d}U\mathrm{d}V + r^2\mathrm{d}\Omega^2 \tag{291}$$

where we emphasize $r = r(U, V)$ is implicitly determined by (289).

The original 'Schwarzschild' coordinates cover the region $r > 2M$, and from the expressions in terms of u, v we know $U < 0, V > 0$ here. THe form of the metric given by (291) makes it clear that there is a definite singularity at $r = 0$, whereas nothing particular happens at $r = 2M$. Indeed the surface $r = 2M$ corresponds to *either* $U = 0$ or $V = 0$ and in the U, V coordinates, the metric is clearly regular as we move into this region. Thus let us define $r(U, V)$ by (289) even for $U \geq 0, V \leq 0$. The metric and its inverse are smooth here and can extend our chart to cover regions with $U > 0, V < 0$.

We first see the event horizon $r = 2M$ corresponds to two surfaces, one with $U = 0$ and one with $V = 0$ which intersect at $(U, V) = (0, 0)$. Meanwhile the curvature singularity at $r = 0$ gives $UV = 1$. In the $U - V$ plane this is a hyperbola with two branches. Finally, note that the vector fields

$$\frac{\partial}{\partial U}, \qquad \frac{\partial}{\partial V} \tag{292}$$

are both null. These vector fields are tangent to radial null geodesics of constant V and constant U respectively (i.e. a curve $x^a(\lambda) = (U_0, \lambda)$ will have tangent vector $\partial/\partial V$). We can plot the entire Schwarzschild solution on the $U - V$ plane, oriented so lines of constant U and V are straight lines running at 45 degrees to the horizontal, reflecting the fact these are the trajectories of light rays in the spacetime. The entire spacetime is represented by a *Kruskal diagram*. Here imagine 'time' is increasing in the vertical direction from bottom to top. Curves with $r =$constant correspond to $UV =$constant (hyperbolae) and $r = 2M$ is the limiting case where the hyperbolae degenerate and form two lines crossing at 45 degrees. This represents the event horizon. Meanwhile $t =$constant implies V/U is constant, which are simply straight lines.

The Kruskal diagram is naturally split into four regions. Region I is the part of spacetime covered by the original Schwarzschild coordinates $U < 0, V > 0$, the exterior to the black and white holes. Region II has $U, V > 0$, and covers the interior of the 'future' event horizon. Regions II and I are covered by ingoing Eddington-Finkelstein coordinates (v, r, θ, ϕ). Regions III $(U, V < 0)$ and I are covered by the outgoing Eddington-Finkelstein coordinates (u, r, θ, ϕ). As expected, the 'future' event horizon' corresponding to the black hole is not the same as the 'past event horizon' corresponding to the white hiole, which has $U < 0, V < 0$.

The rather interesting part is Region IV. This has $U > 0, V < 0$ and $r > 2M$; hence one could describe Region IV back in terms of our original exterior Schwarzschild coordinates. In fact Regions I and IV are *isometric* (i.e. they have the same metric) and the transformation between is given by $(U, V) \to (-U, -V)$. It represents another asymptotically flat region. Causal curves cannot travel from Region I to Region IV and vice versa directly, so the two regions are truly separated. In order to go between them one would have to travel on a spacelike curve moving horizontally in the Kruskal diagram (i.e. 'move faster than light'). There are more exotic solutions with unphysical matter content that do allow timelike curves to pass between two such regions, which are referred to as 'wormholes' . One could imagine, of course, that two physicists (who don't believe in mathematics) in each separate region could each travel on timelike curves into Region II and then meet up. However external observers would never be able to observe this meeting, as it occurs within the future event horizon. And in any case the physicists would have to end up at the singularity in finite proper time (see Assignment) and presumably be annihilated.

One final point concerns the spacetime singularity $r = 0$. This is a spacelike surface since the normal $\mathrm{d}r$ is actually *timelike* for $r < 2M$, as is easily checked. Thus it is incorrect to refer to the Schwarzschild singularity as a 'place' - it more like a moment in 'time' that a causal curve inside $r < 2M$ can no more avoid than the reader can avoid an exam at the end of term. There are other black hole solutions we will not study such as the Reissner-Nordstrom solution, which have timelike singularities which can be avoided even for observers inside the event horizon. They can then pass into new universes.

Regions IV and III are generally considered unphysical. We expect that gravitational collpase will create a black hole that settles into a Schwarzschild solution at late times; in such a situation, the entire region to the left of the timelike curve representing the surface of the star should be ignored since the vacuum Einstein's equations do not hold. Thus one is left essentially with Regions I and II.

Alternatively one can introduce Cartesian-like coordinates

$$T = \frac{1}{2}(V + U) \qquad X = \frac{1}{2}(V - U) \tag{293}$$

in which case (291) takes the form

$$\mathrm{d}s^2 = \frac{32M^3 e^{-r/2M}}{r}\left(-\mathrm{d}T^2 + \mathrm{d}X^2\right) + \mathrm{d}\Omega^2 \tag{294}$$

In this system, the null lines take the familiar form $T = \pm X$.

Finally, we conclude with some comments on the general situation for gravitational collapse. It may appear as if the presence of an event horizon and singularities are a mere artefact of the rather

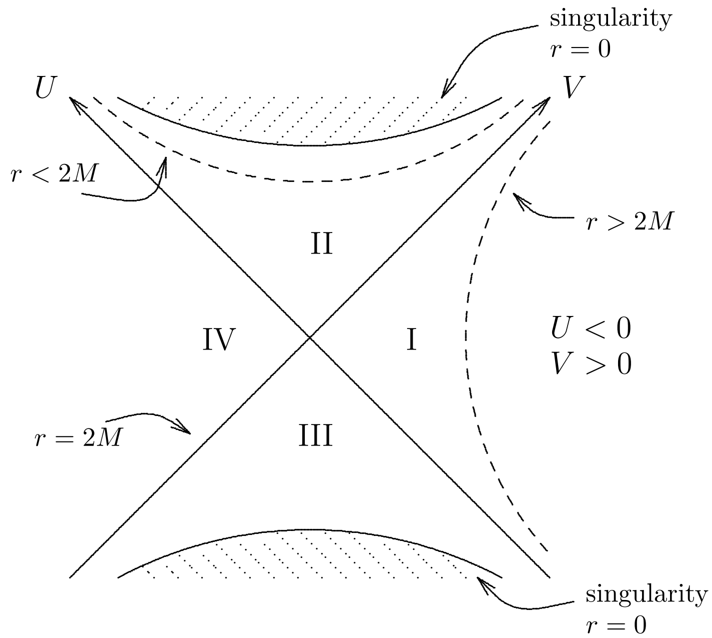

Figure 8: Kruskal diagram (from P.K. Townsend, Part III Black Holes lectures)

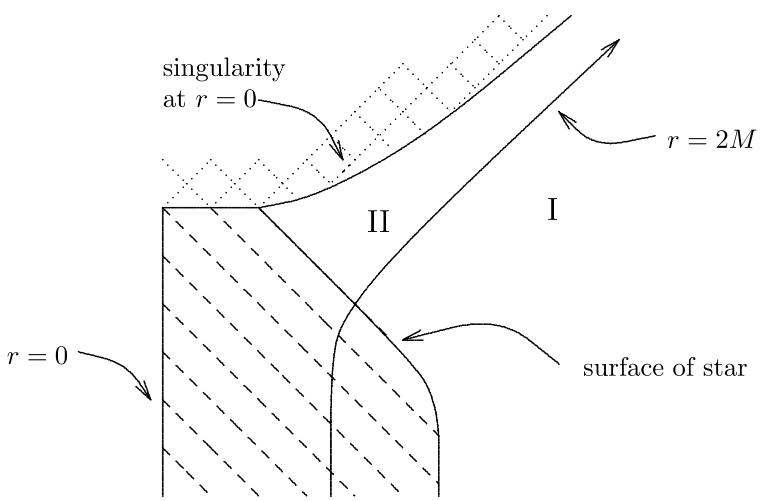

Figure 9: Gravitational collapse (from P K Townsend, Part III black holes lectures)

artificial assumption of spherical symmetry. However quite generally the singularity theorem of Penrose shows that gravitational collapse does generically lead to the formation of a singularity. A important open problem in general relativity is the proof of Penrose's cosmic censorship conjecture, which asserts that a singularity must be hidden behind an event horizon (and so is not observable).

5 Basic Cosmology

Let us now briefly consider how general relativity describes the large-scale behaviour of the Universe. Our basic observational facts are that (i) the Universe contains galaxies, radiation (light), as well as so-called dark matter; (ii) the Universe is expanding, in that the distance between nearby galaxies is increasing in time; (iii) at large scales the Universe is spatially homogenous. The is sometimes called the 'Copernican Principle' as it means there are no privileged points in the Universe. At appears true at scales of 10^8 light years . Mathematically this means that the spacetime (M, g) admits a group of isometries whose orbits are 3d spacelike surfaces. Roughtly, this means that at a fixed moment in time, all points on the resulting spacelike hypersurface are equivalent. Finally (iv) the Universe is isotropic (looks the same in all directions). The strongest evidence for this is the uniformity in all directions of the cosmic microwave background.

Of course the Universe is not temporally homogeneous (i.e. not all times are the same, as there is expansion). To make these ideas precise one needs some familiarity with groups and their action on manifolds. However we can still get an intuitive understanding.

Consider the two-sphere S^2 equipped with its natural round metric. Clearly there are no preferred directions, as one could arbitrarily choose what to label as the 'North pole' and set up a chart accordingly. More precisely, one could act with the group $SO(3)$ of rotations to transform any given point p to another points q without changing the geometry of the sphere. Similarly it is isotropic, as clearly all directions from a given point look the same. On the other hand the infinite cylinder $S^1 \times \mathbb{R}$ is not isotropic; clearly, one direction runs along the axis of the cylinder, whereas the other goes around the S^1. It is however homogeneous; one could use symmetry transformations to transport a given point to another with any given height and location on the S^1. The space $\mathbb{R}^2$, like S^2, is homogeneous and isotropic, as is the hyperbolic space $\mathbb{H}^2$, which we have briefly seen in the ASsignments. The simplest models of the universe assume that the three-dimensional spatial hypersurfaces of constant time are homogeneous and isotropic. This implies that these are (in a precise sense) *maximally symmetric*, so they admit the largest number of linearly independent Killing vector fields allowed in dimension 3, which is 6 (in general, it is $n(n+1)/2$ in n dimensions). Ii is a theorem that a maximally symmetric Riemannian manifold must locally admit a constant-curvature metric of the following form:

$$\mathrm{d}s_3^2 = h_{ij}\mathrm{d}x^i\mathrm{d}x^j = \frac{\mathrm{d}r^2}{1-kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2) \tag{295}$$

and here $k = 1, 0, -1$ corresponding to the canonical metric on S^3, Euclidean space $\mathbb{R}^3$, and hyperbolic space $\mathbb{H}^3$. These 3-manifolds will represent the geometry on constant 'time' slices of our model of the Universe. We will often use h_{ij} to refer to this three-dimensional metric.

There will be a preferred set of observers which travel on curves that are normal to the surfaces of spatial homogeneity. They are, loosely speaking 'at rest' and are referred to as *comoving* observers. Their four-velocity should be normal to spatial surfaces, and one can introduce coordinates such that the spacetime metric takes the form

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2\mathrm{d}s_3^2 \tag{296}$$

Note that the constant time surfaces have normal $n = \mathrm{d}t$; the comoving observers travel with 4-velocity u^a given by $u^a = g^{ab} n_b = (1, 0, 0, 0)$. Note $u \cdot u = -1$.

The function $a(t)$ is referred to as the *scale factor* and a cosmological spacetime with metirc (296) is called a Friedmann-Lemaitre-Robertson-Walker universe (shortened often to FRW or FLRW). The scale factor encodes how the universe is expanding or shrinking, and is determined by Einstein's equations as we shall see below.

Definition. *A FRLW Universe is called flat, closed, or open if* $k = 0, 1, -1$ *respectively.*

Hubble's law

Suppose we have two galaxies separated by a fixed spatial distance d calculated using the metric h_{ij}. The proper distance between the two points is a function of t; i.e. $D = a(t)d$. Thus the rate of change of the distance is

$$v = \frac{\mathrm{d}D}{\mathrm{d}t} = \dot{a}d = \frac{\dot{a}}{a} D = HD \tag{297}$$

which is known as Hubble's law; the relative velocity between two galaxies, say, should be proportional to the distance. Here $H = \dot{a}/a$, and its current value H_0 measured now is called Hubble's constant. We observe that nearby galaxies are actually moving apart so $H > 0$, or $\dot{a} > 0$.

Friedmann's Equations

We now want to use the Einstein field equations to determine the behaviour of $a(t)$. It is an exercise to show that the Einstein tensor G_{ab} for (296) is simply

$$G_{00} = 3\left(\frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right) \qquad G_{ij} = -(2a\ddot{a} + \dot{a}^2 + k)h_{ij} \tag{298}$$

On the other hand, we must determine the stress-energy tensor T_{ab}. We will assume at large scales galaxies are comoving, and distributed uniformly in space, and we treat all the matter and radiation as a perfect fluid:

$$T_{ab} = (\rho + P)u_a u_b + P g_{ab} = \rho u_a u_b + P h_{ab} \tag{299}$$

where $g_{ab} = h_{ab} - u_a u_b$ and recall $u^a = (1, 0, 0, 0)$ so that $T_{00} = \rho$ is the mass density. Here ρ, P are assumed to be functions of t only. The two cases we are interested in *dust* for which $P = 0$, and *radiation* for which $P = \rho/3$. Finally, suppose we consider Einstein's equations with $T_{ab} = 0$ but a non-vanishing cosmological constant Λ (see (196))

$$G_{ab} = -\Lambda g_{ab} \tag{300}$$

One sees the RHS of this equations takes the form of a perfect fluid with $P = -\rho$ and $\rho = \Lambda/8\pi$.

We can write all these in a unified way by in terms of the *equation of state*

$$P = w\rho \tag{301}$$

with appropriate constant w .

The conservation equation $\nabla_a T^{ab} = 0$ implies

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0 \tag{302}$$

which is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}(a^3\rho) = -P\frac{\mathrm{d}a^3}{\mathrm{d}t} \tag{303}$$

This makes physical sense: one can think of $a^3\rho$ as the energy density of a co-moving piece of volume, so the LHS looks like $\mathrm{d}E$, and $-P\dot{a^3}$ is like a $-P\mathrm{d}V$ term (work done by the fluid as it expands). Thus the rate of change of energy is equal to amount of work done by the fluid. This can be integrated using the equation of state to obtain

$$\rho(t) = \rho_0 \left(\frac{a_0}{a(t)}\right)^{3(1+w)} \tag{304}$$

where $\rho_0 = \rho(t_0)$ represents the density at the current time and $a_0 = a(t_0)$ is the current scale factor.

For pure dust, $w = 0$ and we get $\rho(t) \propto a(t)^{-3}$ which indeed makes sense; the density goes as one over the volume, and is decreasing as the Universe expands. For $w = 1/3$ (radiation) we get $\rho(t) \propto a(t)^{-4}$. Radiation spreads out 'quicker' and hence the energy density decreases more rapidly with the scale factor. The cosmological constant case has the property that $\rho(t)$ is *constant* during the expansion. We can roughly think that there is a background energy density that fills up spacetime.

Einstein's equations lead to just two non-trivial equations:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi\rho}{3} - \frac{k}{a^2}\,, \qquad \frac{\ddot{a}}{a} = -\frac{4\pi}{3}(\rho + 3P) \tag{305}$$

The first of these is a conservation of energy equation, and is usually referred to as the Friedmann equation. We will firstly discuss some qualitative properties of these two equations, and then look at simple solutions for a Universe dominated by radiation and then dust.

1. First, note that assuming $\rho > 0, P \geq 0$, that $\ddot{a} < 0$ for all t. This implies $\dot{a}$ is either always increasing or decreasing, with possibly a single moment at which $\dot{a} = 0$ when the behaviour 'flips'. In any case, given that $\dot{a} > 0$, but $\ddot{a} < 0$, this implies that $\dot{a}$ was larger in the past - the expansion was *faster* at earlier times in the Universe. The predicted expansion of the Universe is a striking prediction of GR, which has been confirmed . Historically, Einstein actually added a positive Λ term in order to *eliminate* what he felt was an incorrect prediction of the theory; in fact he was right in the first place.

2. Let us assume the expansion is a constant, so $\dot{a} = c_1 > 0$ for all time. Let us fix the current time to be $t = 0$. Then $a(t) = c_1 t + a(0)$ where $a(0)$ is the current scale factor. It would

therefore follow that at $t = -H_0^{-1} = -a(0)/\dot{a}$ (i.e. H_0^{-1} units of time in the past, where H_0 is the current Hubble parameter)) we must have had $a = 0$ and the distance between all spatially separated points was zero. In fact we know that since $\ddot{a} < 0$, the expansion must have been *faster* in the past, as stated above, and thus the moment of $a(t) = 0$ occurred even closer to our present time. Note that the density of matter would also be infinite at this time: one can check the FRLW metric has a curvature singularity at $a(t) = 0$. This singular state of the Universe is referred to as the Big Bang.

Critical Density

What is the value of k, which characterizes the geometry (and topology) of our maximally symmetric spatial hypersurfaces? If we rewrite the Friedmann equation in terms of $H = \dot{a}/a$, it reads

$$-H^2 + \frac{8\pi\rho}{3} = \frac{k}{a^2} \tag{306}$$

$H \neq 0$ and we can divide by it to obtain

$$\Omega - 1 = \frac{k}{H^2 a^2} \tag{307}$$

where

$$\Omega \equiv \frac{8\pi\rho}{3H^2} = \frac{\rho}{\rho_{crit}}\ , \qquad \rho_{crit} = \frac{3H^2}{8\pi} \tag{308}$$

The sign of k is thus determined by whether $\Omega > 1, \Omega = 1, \Omega < 1$. Astrophysicists can measure H currently and furthermore we can detect by observation the current matter density ρ (it appear to be mostly concentrated in the density of galaxies). Current observations indicate that the curvature that we can measure, $K = k/a^2$ is very close to zero (the Universe is nearly spatially flat). Thus $\Omega 1 \pm 0.03$. It can be shown, however, that under reasonable assumptions that in order to be this close presently, at very early times Ω must have been very finely tuned to unity (the difference being around 10^{-68}). This is troubling as it seems strange that things could be so perfectly arranged; this is known as the *flatness problem*.

Accelerated Universe and dark energy

Current observations suggest that the Universe is in fact *accelerating*, in contrast to the expected $\ddot{a} < 0$ behaviour, assuming $\rho, P \geq 0$ in (305). This could be a fundamental problem with general relativity for describing the Universe at large scales, although it seems this is an extreme conclusion. Such a behaviour can in fact be modelled by assuming that there is a component of matter-energy with $P < 0$. A simple way to describe this is with a equation of state with $w = -1$, which effectively implies there is a positive Λ term in Einstein's field equations. A fluid with $P = -\rho$ is not so exotic; positive pressure means one needs to exert work to compress it; negative pressure implies that the fluid actually *gains* energy as it expands, and so it is like the fluid has a 'tension' , like an elastic string. Of course, as explained above $\rho = \rho_0$ is constant for $w = -1$. Physically, this implies there is a constant energy density in the universe, referred to as the 'dark energy' . Notice that this energy is on top of all the other radiation and matter components of ρ.

We now investigate two simple cases for which simple solutions of (305) are known.

Flat Universes

Here we consider $k = 0$, which physically seems most relevant. It proves simplest to consider a 'matter dominated' and 'radiation dominated' Universe separately - one imagines that in the early Universe, most of the energy was concentrated in radiation, and matter had not begun to form. After a certain time (referred to as 'recombination') hydrogen and helium begin to form, and so begins a matter dominated phase.

Let us assume $k = 0$. Using our expression for $\rho(t)$ gives

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}\rho_0 \left(\frac{a_0}{a}\right)^{3(1+w)} \tag{309}$$

Assuming $w > -1$ this can be integrated to give

$$a(t) = a_0 \left(\frac{t}{t_0}\right)^{\frac{2}{3(1+w)}} \tag{310}$$

We have shifted an integration constant so that $a(t) = 0$ when $t = 0$. For dust ($w = 0$) we have $a(t) \propto t^{2/3}$ whereas for radiation, $a(t) \propto t^{1/2}$. We see in either case, the Universe is expanding monotonically. Tracing backwards in time, we find that $a(t) \to 0$ as $t \to 0^+$. Scalar curvature invariants (such as the Ricci scalar curvature R) can be shown to diverge as $a(t) \to 0$ and moreover, the density ρ is also diverging as $t \to 0^+$. This is the Big Bang singularity.

Closed Universe

Let us assume $k = 1$. Then the Friedmann equation implies that at some t, $\dot{a} = 0$. Given that $\ddot{a} < 0$, at least for physically reasonable matter, there will only be one such turning point, and from here onwards $\dot{a} < 0$ and the Universe will begin to contract towards a final state with $a(t_f) = 0$ at some finite $t = t_f$. This is called a 'Big Crunch' singularity. To obtain explicit solutions, again use the fact that for dust, $8\pi\rho a^3(t)/3 = C$ is a constant, while for radiation, $8\pi\rho a^4(t)/3 = C'$ is constant. The resulting Friedmann equation gives

$$\dot{a}^2 - \frac{C}{a} + 1 = 0 \qquad \text{or} \qquad \dot{a}^2 - \frac{C'}{a^2} + 1 = 0 \tag{311}$$

for dust and radiation respectively. These can be integrated, giving

$$a(\eta) = \frac{C}{2}(1 - \cos\eta)\,, \qquad t(\eta) = \frac{C}{2}(1 - \sin\eta) \quad \text{(dust)} \tag{312}$$

$$a(t) = \sqrt{C'}\left[1 - (1 - t/\sqrt{C'})^2\right]^{1/2} \quad \text{(radiation)} \tag{313}$$

For the dust case, note that when $\eta = 2\pi$ (corresponding to $t = C/2$) the scale factor $a(t)$ vanishes, implying the aforementioned Big Crunch. In the pure radiation case as well, when $t = 2\sqrt{C'}$, we have $a(t) = 0$.

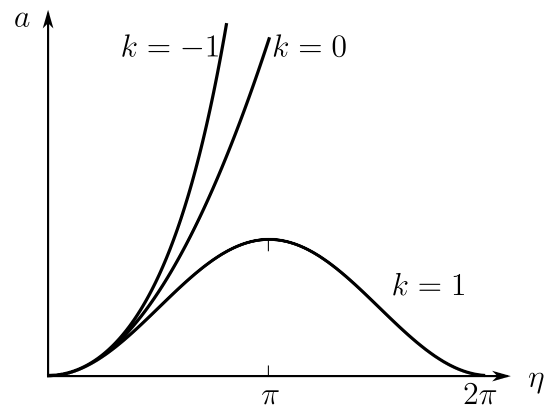

Figure 10: Scale factor in the matter-dominated case (from H Reall's Part III lectures on General Relativity.

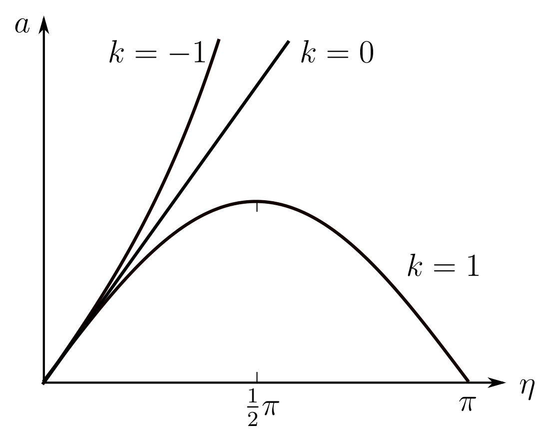

Figure 11: Scale factor in the radiation-dominated case (from H Reall's Part III lectures on General Relativity.

Einstein Static Universe and de Sitter spacetime

As mentioned earlier, Einstein introduced the cosmological constant in order to engineer a static Universe. To investigate the first class of spacetimes, we require $\dot{a} = \ddot{a} = 0$. Consider a stress energy tensor corresponding to pure dust *and* a positive cosmological constant term. This can be modelled by assuming there are *two* perfect fluids, i.e. $T_{ab} = T_{ab}^{dust} + T_{ab}^{\Lambda}$ where the former corresponds to pure dust with density ρ_m with pressure $P_m = 0$, and the second corresponds to a vacuum energy which, as stated earlier, corresponds to $P_\Lambda = -\rho_\Lambda$ and $P_\Lambda = \Lambda/8\pi$. The Friedmann equations are the same for our two-fluid model, but with $\rho \to \rho_m + \rho_\Lambda$ and $P \to P_m + P_\Lambda = P_\Lambda$.

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi(\rho_m + \rho_\Lambda)}{3} - \frac{k}{a^2}\,, \qquad \frac{\ddot{a}}{a} = -\frac{4\pi}{3}(\rho_m + \rho_\Lambda + 3P_\Lambda) = -\frac{4\pi}{3}(\rho_m - 2\rho_\Lambda) \tag{314}$$

We must have $\ddot{a} = 0$ for a constant scale factor, so $\rho_m = 2\rho_\Lambda$. Then we must have

$$k = 8\pi\rho_\Lambda a^2 > 0 \tag{315}$$

where $\rho_\Lambda = \Lambda/8\pi$. as discussed previously when we model a cosmological constant as a perfect fluid. Such a Universe is necessarily closed (spatial hypersurfaces have S^3 topology) and globally the Einstein static Universe is $\mathbb{R} \times S^3$. It is static since ∂_t is a everywhere non-vanishing timelike Killing vector field.

Finally, we can also model pure de Sitter spacetime as a FRW cosmology. As discussed, at large times, for an expanding Universe, one expects that the matter and radiation density to fall off quickly as the scalar factor grows. Thus one would expect, even if the constant energy density ρ_Λ corresponding to a cosmological constant is small, it will eventually dominate the dynamics - such an epoch is called 'vacuum dominated'. The de Sitter spacetime is actually maximally symmetric in four spacetime dimensions, not just spatially in three dimensions, so it has 10 Killing vector fields. However it does not have any globally defined timelike Killing fields and is *not* static. To find the scale factor, we note we have already shown that for $w = -1$, ρ is a constant; indeed it is simply

$$-P = \rho = \frac{\Lambda}{8\pi} \tag{316}$$

The only non-trivial equation to solve is

$$\dot{a}^2 = \frac{8\pi\rho a^2}{3} - k \tag{317}$$

Different choices of k correspond to different spatial foliations of de Sitter; we will choose flat slicings with $k = 0$ so spatial geometries are flat. The solution is

$$a(t) = e^{\sqrt{\frac{\Lambda}{3}}t} \tag{318}$$

where we have set $a(0) = 1$, and thus, defining the cosmological length scale

$$L \equiv \sqrt{\frac{3}{\Lambda}}\,, \tag{319}$$

gives

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + e^{\frac{2t}{L}} \left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \right) \tag{320}$$

Such a Universe is expanding exponentially, and as $t \to -\infty$, there is a Big Bang.