Morley’s Theorem: The three intersections of the trisectors of the angles of a triangle, lying near the three sides respectively, form an equilateral triangle.

**Proof:** The result will be obvious when we compute the length of $FE$ in terms of the angles $\alpha, \beta, \gamma$ and the circumradius $R$. Using the sine law on triangle $ABF$, the fact that $\alpha + \beta + \gamma = \pi$, and $2R = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ we have

$$AF = \frac{c \sin \beta}{\sin(\alpha + \beta)} = \frac{2R \sin \beta \sin 3\gamma}{\sin(\frac{\pi}{3} - \gamma)} = \frac{2R \sin \beta \sin \gamma(3 \cos^2 \gamma - \sin^2 \gamma)}{\sin(\frac{\pi}{3} - \gamma)} = \frac{8R \sin \beta \sin \gamma \sin(\frac{\pi}{3} + \gamma) \sin(\frac{\pi}{3} - \gamma)}{\sin(\frac{\pi}{3} - \gamma)} = 8R \sin \beta \sin \gamma \sin(\frac{\pi}{3} + \gamma).$$

Similarly, $AE = 8R \sin \beta \sin \gamma \sin(\frac{\pi}{3} + \beta)$. Then from the cosine law we have,

$$FE^2 = AE^2 + AF^2 - 2AE \cdot AF \cos \alpha = 64R^2 \sin^2 \beta \sin^2 \gamma \left[ \sin^2(\frac{\pi}{3} + \gamma) + \sin^2(\frac{\pi}{3} + \beta) - 2 \sin(\frac{\pi}{3} + \gamma) \sin(\frac{\pi}{3} + \beta) \cos \alpha \right].$$

What is truly amazing is that the expression within square brackets is equal to $\sin^2 \alpha$. This can be computed by showing that $\sin^2 \gamma^* + \sin^2 \beta^* - 2 \sin \gamma^* \sin \beta^* \cos \alpha = \sin^2 \alpha$, where $\beta^* = \frac{\pi}{3} + \beta$ and $\gamma^* = \frac{\pi}{3} + \gamma$ and hence $\alpha + \beta^* + \gamma^* = \pi$. This follows with minimal effort using the substitution $\sin \gamma^* = \sin(\alpha + \beta^*)$. (Or, note that the cosine law holds if the side lengths are replaced by the sines of the corresponding angles.) Hence

$$FE^2 = 64R^2 \sin^2 \beta \sin^2 \gamma \sin^2 \alpha.$$

From the symmetry, it follows that triangle $DEF$ is equilateral. \qed