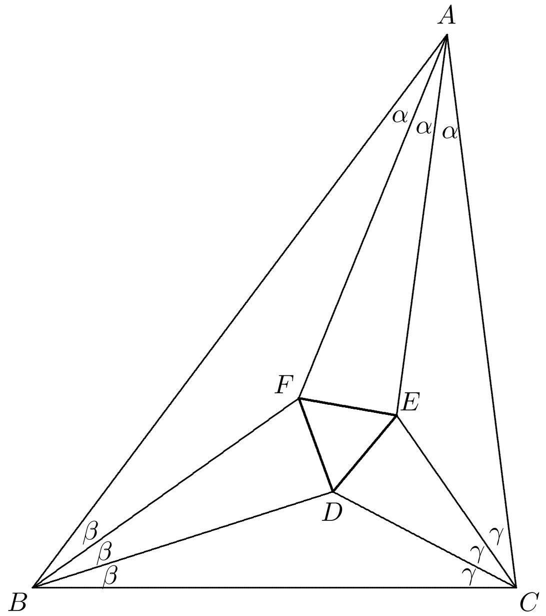
Morley's Theorem: *The three intersections of the trisectors of the angles of a triangle, lying near the three sides respectively, form an equilateral triangle.*



Proof: The result will be obvious when we compute the length of FE in terms of the angles α, β, γ and the circumradius R. Using the sine law on triangle ABF, the fact that $\alpha+\beta+\gamma=\frac{\pi}{3}$, and $2R=\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$ we have

$$\begin{aligned} AF &= \frac{c\sin\beta}{\sin(\alpha+\beta)} \\ &= \frac{2R\sin\beta\sin 3\gamma}{\sin(\frac{\pi}{3}-\gamma)} \\ &= \frac{2R\sin\beta\sin\gamma(3\cos^2\gamma-\sin^2\gamma)}{\sin(\frac{\pi}{3}-\gamma)} \\ &= \frac{8R\sin\beta\sin\gamma\sin(\frac{\pi}{3}+\gamma)\sin(\frac{\pi}{3}-\gamma)}{\sin(\frac{\pi}{3}-\gamma)} \\ &= 8R\sin\beta\sin\gamma\sin(\frac{\pi}{3}+\gamma). \end{aligned}$$

Similarly, $AE=8R\sin\beta\sin\gamma\sin(\frac{\pi}{3}+\beta)$. Then from the cosine law we have,

$$\begin{aligned} FE^2 &= AE^2+AF^2-2AE\cdot AF\cos\alpha \\ &= 64R^2\sin^2\beta\sin^2\gamma\left[\sin^2(\frac{\pi}{3}+\gamma)+\sin^2(\frac{\pi}{3}+\beta)-2\sin(\frac{\pi}{3}+\gamma)\sin(\frac{\pi}{3}+\beta)\cos\alpha\right]. \end{aligned}$$

What is truly amazing is that the expression within square brackets is equal to $\sin^2\alpha$. This can be computed by showing that $\sin^2\gamma^*+\sin^2\beta^*-2\sin\gamma^*\sin\beta^*\cos\alpha=\sin^2\alpha$, where $\beta^*=\frac{\pi}{3}+\beta$ and $\gamma^*=\frac{\pi}{3}+\gamma$ and hence $\alpha+\beta^*+\gamma^*=\pi$. This follows with minimal effort using the substitution $\sin\gamma^*=\sin(\alpha+\beta^*)$. (Or, note that the cosine law holds if the side lengths are replaced by the sines of the corresponding angles.) Hence

$$FE^2=64R^2\sin^2\beta\sin^2\gamma\sin^2\alpha.$$

From the symmetry, it follows that triangle DEF is equilateral. $\square$