MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

FINAL EXAM **Pure Mathematics 3370** FALL 1998

Marks

[4] 1. Euclid defined perfect numbers and discovered a formula for even perfect numbers. Euler, 2000 years later, proved that this formula gave *all* the even perfect numbers. State clearly one of these results and prove it.

[2] 2. (a) Show how Maple can be used to generate the Fibonacci sequence.

[3] (b) Let $\{F_n\}_{n=1}^{\infty}$ be the Fibonacci sequence. Prove that $F_n \leq \alpha^{n-1}$, where α is the positive root of $x^2 - x - 1 = 0$.

[3] 3. Find all the incongruent solutions of $790x \equiv 30 \pmod{2000}$.

[3] 4. (a) Let m_1 and m_2 be positive integers with $(m_1, m_2) = 1$. Prove that the congruences $x \equiv a_1 \pmod{m_1}$ and $x \equiv a_2 \pmod{m_2}$ have a common solution which is unique modulo $m_1 m_2$.

[3] (b) Hence, state and finish proving the Chinese Remainder Theorem.

[2] (c) Find the common solution of the congruences $x \equiv 4 \pmod{100}$ and $x \equiv 13 \pmod{27}$.

[2] 5. (a) Give a formula that gives all the primitive Pythagorean triples.

[3] (b) State Fermat's Last Theorem. By whom and when was Fermat's Last Theorem proved.

[2] 6. (a) Find $\phi(4141)$.

[5] (b) Define a primitive root modulo n. Are there any primitive roots modulo 4141? (Hint: Consider $a^{\phi(4141)/2}$ modulo 4141, where $(a, 4141) = 1$.)

[4] (c) Let E be the following function from $\mathbf{Z}_{4141}^*$ to $\mathbf{Z}_{4141}^*$ defined by $E(\overline{x}) = \overline{y}$, where $y \equiv x^{51} \pmod{4141}$. Find the inverse of the function E. That is, find a number d such that $D(\overline{y}) = \overline{z}$, where $z \equiv y^d \pmod{4141}$, and such that $E \circ D$ is the identity function.

[4] 7. (a) Factor into Gaussian primes $\alpha = 264 - 168i$.

[4] (b) Prove that rational primes $p \equiv 3 \pmod 4$ are Gaussian primes.

[4] (c) Show that the ring $\mathbf{Z}[\sqrt{10}]$ is not a unique factorization domain by considering the factorization $2 \cdot 5 = (\sqrt{10})^2$.

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