Applied Mathematics 2130

Lab 2006W–4B

"Reasoners Resemble Spiders"

An iterative process (such as a sequence) where the "next" approximation to a solution is found in terms of the previous approximation(s), is a useful method for finding solutions to certain classes of problems. Newton's Method, found in most Calculus texts, is an example. The iterative process can take the form $x_{n+1} = f(x_n)$ (for instance, $x_{n+1} = \frac{x_n^2+2}{2x_n}$), or $x_{n+2} = f(x_{n+1}) + g(x_n)$ (for example, $x_{n+2} = 2x_{n+1} - x_n$). With any iterative process, it is necessary to have a starting value, say x_0. Sometimes this is given, or it may be freely chosen from a certain interval.

In implementing an iteration, one must have a criterion under which the process halts. Specifically, one would like to stop at a 'good' approximation to a true solution. The difficulty is that one does not know in advance the value of a true solution. Thus deciding when to stop is a problem. Two approaches are usually taken. One is based on the fact that as x_n approaches a true solution, the value of $|x_n - x_{n+1}|$ must get small. The other uses the fact that near a root of a function $f(x)$, $|f(x_n)|$ must get small. Both of these two facts may be used as part of a stopping criterion.

As part of this project we are to choose a selection of values for the real number λ in the interval $(0, 1]$ for the function

$$f(x) = \lambda \sin(\pi x)$$

and investigate the iteration

$$x_{n+1} = f(x_n).$$

To fully understand what is happening, it may be helpful to explore with several different values of λ, using a program that can generate the corresponding iterated sequence $x_0, x_1, \ldots$ One question that we might consider while exploring the effects of varying values for λ is whether the corresponding sequence converges, and if so, to what? If not, then can the behaviour of the sequence be described?

A closely related area of interest concerns the iterated function $f^{(n)}(x) = f(f^{(n-1)}(x))$ on the interval $[0, 1]$. For instance, by comparing the graphs of $f^{(2)}(x) = f(f(x))$, $f^{(3)}(x) = f(f(f(x)))$, $f^{(4)}(x) = f(f(f(f(x))))$, etc. with that of $f(x)$, can we draw any conclusions about the function $f^{(n)}(x)$ on the interval $[0, 1]$?

Following the *Suggested Report Format*, prepare a report that addresses the issues mentioned above. Include suitable illustrative graphics supporting your conclusions on the behaviour of this iterative process. As always, supporting mathematical reasoning is essential.

NOTE: Before beginning this lab, it is recommended that you review the article in *Scientific American* (November 1981 issue, pp 22–43) by Douglas Hofstadter (the author of *Gödel, Escher, Bach*).