Decycling Planar Graphs

by Matthew Sullivan

A thesis submitted to the
Department of Mathematics and Statistics
in partial fulfillment of the
requirements for the degree of
Honours Bachelor of Science

Department of Mathematics and Statistics
Memorial University

April 19 2014

St. John's Newfoundland

Abstract

A decycling set D of a graph G is a set of vertices from the vertex set of G whose removal leaves the remaining graph cycle free. Given a positive integer k and a graph G deciding whether there exists a decycling set $D \subset V(G)$ such that $|D| \leq k$ is an NP-complete decision problem. In this thesis, we will go through some history of the decycling set problem, and improve on a recent result.

Definitions

Definition. *A graph is an ordered pair $G = (V, E)$ comprising a set V of vertices together with a set E of edges.*

Definition. *A planar graph is a graph that can be drawn on the plane with the property that the edges of the graph only intersect at their endpoints.*

It is known that a graph G is planar if and only if it does not contain a $K_{3,3}$ or a K_5 as graph minors. This is known as Kuratowski's theorem which is found in [3] and will be used in this thesis.

Definition. *A decycling set $D \subseteq V(G)$ is a set of vertices of G such that $G - D$ is a forest. A decycling set $\bar{D} \subset V(G)$ is called minimum if for all decycling sets $D \subset V(G)$ we have*

$$|\bar{D}| \leq |D| \text{ and } \nabla(G) = |\bar{D}|,$$

where $\nabla(G)$ is called the decycling number of the graph G.

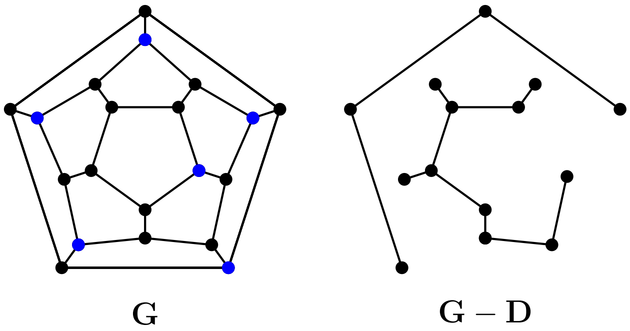


Figure 1: A decycling set D of G

Example. Let G be the following graph shown in Figure 1 and let the blue vertices be the set D.

$G - D$ is clearly a forest, therefore D is a decycling set.

Definition. *A cycle packing $\mathcal{C}$ is a collectioon of vertex disjoint cycles from the graph G. A cycle packing $\bar{\mathcal{C}}$ of G is called maximum if for all cycle packings $\mathcal{C}$ of G we have*

$$|\mathcal{C}| \leq |\bar{\mathcal{C}}| \text{ and } cp(G) = |\bar{\mathcal{C}}|,$$

where $cp(G)$ is called the cycle packing number of G.

Example. For the following graph G shown in Figure 2, let $\mathcal{C}$ be the set of blue cycles, then $\mathcal{C}$ is a cycle packing of G.

Clearly $cp(G) \leq \nabla(G)$ for all graphs G since we must remove at least one vertex from every disjoint cycle of a maximum cycle packing to decycle G.

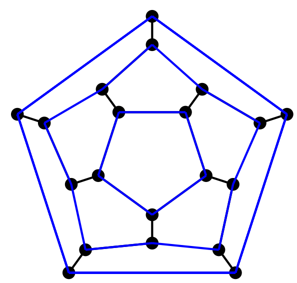


Figure 2: A cycle packing $\mathcal{C}$ of G

Decycling Planar Graphs

Problem. *Given a graph G and a positive integer k, does their exist a decycling set $D \subseteq V(G)$ such that $|D| \leq k$?*

This is an NP-complete decision problem. It is one of the first NP-complete problems shown by Richard Karp in [3]. Even for planar graphs the problem remains NP-complete. This problem is very important and has applications in deadlock prevention for operating systems. A process in an operating system will relinquish its resources only when it has completed its task. When every process requires a resource from another process to complete its task then no process will finish and this is called a deadlock. We would like to know what is the smallest number of processes required to stop to maximixe the number of completed processes, a clear connection to decycling sets.

Lemma 1. *Every rich planar graph has a face of length at most 5.[1]*

In the above lemma by Kloks, Lee and Liu, a rich planar graph is defined as a planar graph G with $\delta(G) \geq 3$, where $\delta(G) = \min\limits_{v \in V(G)} \{\deg_G(v)\}$. We will omit the proof of this lemma since we will prove something much stronger further into the paper. The proof can be found in [1] and is used to prove the following theorem.

Theorem 1. *For every planar graph G, $\nabla(G) \leq 5cp(G)$.[1]*

This is one of the first results bounding $\nabla(G)$ by some constant multiple of $cp(G)$ for planar graphs. The proof for this theorem can be found in [1] and is omitted in this thesis since we will prove a much stronger result.

Lemma 2. *Every 2-edge connected triangle free planar graph G with minimum degree at least three has either a C_4 containing a 3-vertex or a C_5 containing at least four 3-vertices.[2]*

First it is important to note that for any given positive integer k, a k-vertex is a vertex of degree k.

Proof. Let G be a planar graph, We charge each vertex v of degree d_v by $4 - d_v$ and each face f of length ℓ_f by $4 - \ell_f$. The total charge on the graph is

$$\begin{aligned} & \sum_{v \in V(G)} (4 - d_v) + \sum_{f \in F(G)} (4 - \ell_f) \\ = \; & 4|V(G)| - 2|E(G)| + 4|f(G)| - 2|E(G)| \\ = \; & 4(|V(G)| - |E(G)| + |F(G)| = 8, \end{aligned}$$

by Euler's formula. Since the charge of the graph G is positive then there exist a vertex of degree 3. Discharge every 3-vertex equally to its incident faces. Since each 3-vertex in a 2-edge connected planar graph is exactly incident to three faces, we have that each 3-vertex gives a charge of $\frac{1}{3}$ to each of its three incident faces. Since the charge of every vertex is now non-positive and the total charge on the graph is positive then this implies some face f must have a positive charge after discharging. Thus, we have

$$4 - \ell_f + \frac{1}{3}k > 0$$

where k is the number of 3-vertices in the face f. The above inequality holds when $\ell_f \leq 5$ because $k \leq \ell_f$. Since there are no triangles we have $\ell_f \in \{4, 5\}$. When $\ell_f = 4$ we have $k \geq 1$ and when $\ell_f = 5$ we have $k \geq 4$, as desired. □

Theorem 2. *For every planar graph G, $\nabla(G) \leq 3cp(G)$.[2]*

Proof. The following algorithm takes a planar graph G as input and outputs a decycling set D and a cycle packing $\mathcal{C}$ with $|D| \leq 3|\mathcal{C}|$.
Start with $\mathcal{C} = D = \emptyset$ then,

A. Remove from G all vertices not lying on any cycle. If no vertex now exists, then output D and $\mathcal{C}$, otherwise continue to step B.

B. Repeatedly remove from the resulting graph 2-vertices that have non-adjacent neighbours and connect those two neighbours with an edge.

C. If there is a C_3 then place the three vertices of this C_3 into D and place the C_3 into $\mathcal{C}$. Remove all three vertices from the graph and go back to step A. Otherwise, continue to step D.

D. By Lemma 2, there is a C_4 containing a 3-vertex or a C_5 containing four 3-vertices. In the former case, place the C_4 into $\mathcal{C}$, place the three vertices that are not the 3-vertex into D, remove them from G and then go back to step A. In the latter case, place the C_5 into $\mathcal{C}$. There must be two non-adjacent 3-vertices in the C_5. Place the other three vertices into D, remove them from G, then go back to step A.

It is easy to see that given a planar graph G, any decycling set in G will be a decycling set of the remaining graph after steps A and B are applied. Once this algorithm terminates then D is a decycling set since it contains at least one vertex in every cycle. We must now show that $|D| \leq 3|\mathcal{C}|$. For every cycle placed in $\mathcal{C}$, three vertices were placed in D. All cycles in $\mathcal{C}$ are mutually disjoint since the vertices from each cycle are removed from the graph before finding another cycle to place in $\mathcal{C}$. Although step D leaves some 3-vertices, they turn into 1-vertices (since two of their neighbours were removed from the graph) and are removed in step A. We now have

$$\nabla(G) \leq |D| \leq 3|\mathcal{C}| \leq 3cp(G),$$

as desired. □

New Results

We have that $\nabla(G) \leq 3cp(G)$ for planar graphs but can we improve on this bound? Kloks, Lee and Liu conjectured the following in [1].

Conjecture 1. *For every planar graph G, $\nabla(G) \leq 2cp(G)$.*

They showed that the conjecture was true for the case when the graph G is outerplanar and could not verify that it was false for planar graphs. It turns out we can improve the result by Chen, Fu and Shih by proving for planar graphs if $cp(G) = 1$ then $\nabla(G) \leq 2$. If this is true then the algorithm stated above can be improved by removing two vertices instead of three when the graph has $cp(G) = 1$, thus giving us $\triangledown(G) < 3cp(G)$.

Lemma 3. *Let G be a graph with $cp(G) = 1$. If G has a C_3 with a 3-vertex or a C_4 with two non-adjacent 3-vertices then $\triangledown(G) = 2$.*

Proof. Let G be a graph as defined in the lemma. Since $cp(G) = 1$ then the intersection of every cycle with another in G is non-empty. This implies that the removal of any cycle leaves the graph cycle free.

Case 1: Assume there is a C_3 containing at least one 3-vertex in the graph G, then we know that $G - V(C_3)$ is a forest. Let x be a vertex of degree three in this C_3 and let v_1 and v_2 be the other two vertices in the cycle. We have $\deg_{G-\{v_1,v_2\}}(x) = 1$ since x is adjacent to v_1 and v_2 in G. Now we have $\triangledown(G-\{v_1, v_2\}) = \triangledown(G-\{v_1, v_2, x\})$ since x is not on any cyles in $G-\{v_1, v_2\}$. We are done by the following:

$$\triangledown(G - \{v_1, v_2\}) = \triangledown(G - \{v_1, v_2, x\}) = \triangledown(G - V(C_3)) = 0,$$

which says $G - \{v_1, v_2\}$ is a forest and $\triangledown(G) \leq 2$.

Case 2: Assume there is a C_4 in the graph G comprised of two non-adjacent 3-vertices then we know that $G - V(C_4)$ is a forest. Let v_1 and v_3 be non-adjacent 3-vertices on this C_4 and let v_2 and v_4 be the other two vertices on this C_4. We know that $\deg_{G-\{v_2,v_4\}}(v_1) = \deg_{G-\{v_2,v_4\}}(v_3) = 1$ since v_2 and v_4 are each adjacent to both v_1 and v_3. By a similar argument as the previous case we get $G - \{v_1, v_3\}$ is a forest and $\triangledown(G) \leq 2$. ◻

Theorem 3. *Every cubic planar graph G with $cp(G) = 1$ has $\triangledown(G) \leq 2$.*

By Theorem 2 we know for every planar graph G with $cp(G) = 1$ we have $\triangledown(G) \leq 3$. So we will assume $\triangledown(G) = 3$ in our proof and find a contradiction.

Proof. Suppose we have a cubic planar graph G with $cp(G) = 1$. Assume $\nabla(G) = 3$, let $N = \{1, 2, 3\}$ and let $D = \{v_1, v_2, v_3\}$ be a minimum decycling set.

Case 1: Assume $\max\limits_{i,j\in N, i\neq j}\{|N_G(v_i) \cap N_G(v_j)|\} = 0$. Since $N_G(v_i) \cap N_G(v_j) = \emptyset, \forall i, j \in N, i \neq j$, then we have that each vertex in the graph $G - D$ is adjacent to at most one vertex of D in G, which implies $\delta(G-D) = 2$. Since every finite forest contains at least two vertices of degree one (otherwise known as leaves) we have that $G - D$ is not a forest, a contradiction with D being a decycling set.

Case 2: Assume $\max\limits_{i,j\in N, i\neq j}\{|N_G(v_i) \cap N_G(v_j)|\} = 1$.

Case 2.a: Assume $\exists! i, j \in N, i \neq j$, such that $|N_G(v_i) \cap N_G(v_j)| = 1$. Without loss of generality let $i = 1, j = 2$ and let $x \in N_G(v_1) \cap N_G(v_2)$. x is the only vertex in $G - D$ adjacent to two vertices of D in G. Every other vertex in $G - D$ is adjacent to at most one vertex of D in G. This implies that x is the only leaf in $G - D$. Since every forest has at least two leaves then $G - D$ is not a forest, a contradiction with D being a decycling set.

Case 2.b: Assume $\exists! i, j \in N, i \neq j$, such that $|N_G(v_i) \cap N_G(v_j)| = 0$. Without loss of generality let $i = 1, j = 3$, $N_G(v_1) \cap N_G(v_3) = \emptyset$, and let $x \in N_G(v_1) \cap N_G(v_2)$ and $y \in N_G(v_2) \cap N_G(v_3)$. Since x and y are the only leaves in $G-D$ then $G-D$ is exactly an $x-y$ path, call this path P. Define $Int(P) = V(P)\backslash\{x, y\}$, then every vertex in $Int(P)$ is a 2-vertex in $G - D$. This implies every vertex in $Int(P)$ is adjacent to exactly one vertex of D in G. We would like to know the possible cases for $|Int(P)|$. For every cubic graph we have

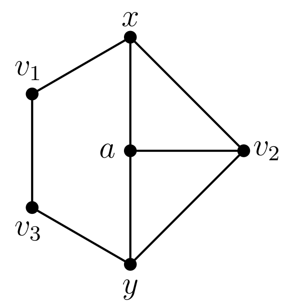
$$3n = \sum_{v\in V(G)} \deg_G(V) = 2|E(G)|$$

where $n = |V(G)|$. Clearly n is even and since

$$n = |V(G)| = |D| \cup |\{x, y\}| \cup |Int(P)|$$

then $|Int(P)|$ is odd.

Case 2.b.i: Assume $|Int(P)| = 1$. Let $a \in Int(P)$.



If a is adjacent to v_2 then v_1 can be adjacent to at most two vertices in G, i.e. x and v_3. But $deg_G(v_1) = 3$, therefore a is adjacent to v_1 or v_3. Without loss of generality assume a is adajcent to v_1 then $(v_1 x a)$ is a C_3 in the graph G. We have a C_3 with a 3-vertex in a graph G with $cp(G) = 1$ therefore by lemma 3 $\nabla(G) \leq 2$, a contradiction.

Case 2.b.ii: Assume $|Int(P)| = 3$. If v_1 or v_3 are adjacent to v_2 in G then one of $(v_1 x v_2)$ or $(v_3 y v_2)$ exists and is a C_3, a contradiction by lemma 3. This implies that v_1 and v_3 are not adjacent to v_2. We actually have that v_1 and v_3 are adjacent for if not then no vertex in D is adjacent to another and this would imply $|Int(P)| \geq 5$, which it is not. So now we have for every vertex in $Int(P)$ it must be adjacent to exactly one vertex in D and every vertex in D must be adjacent to exactly one vertex in $Int(P)$. It is either there is a vertex, say $x_1 \in Int(P)$, adjacent to x that is adjacent to v_1 or v_3 in G, or a vertex, say $y_1 \in Int(P)$, adjacent to y that is adjacent to v_1 or v_3 in G.

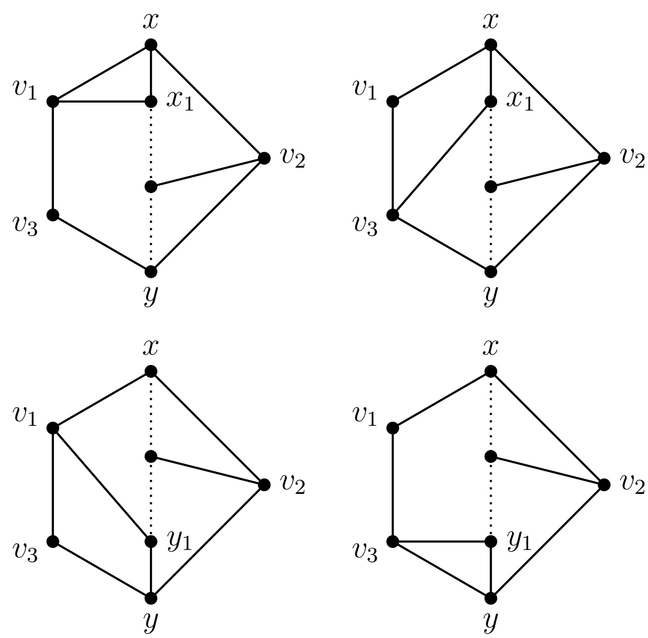
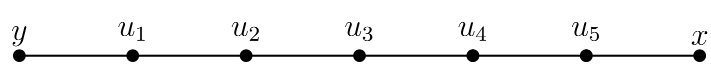


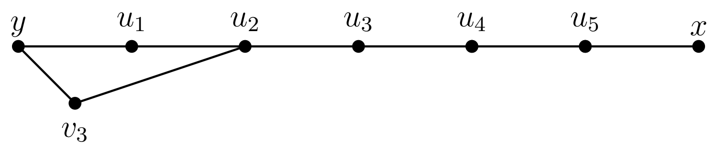
Figure 3: The four possible subgraphs

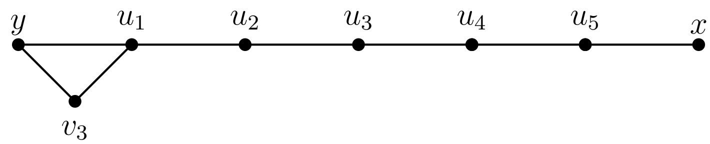
One of the graphs in Figure 3 must be a subgraph of G by our argument. Since each of these graphs contains a C_3 or a C_4 and G is cubic then we can use lemma 3 to show that $\nabla(G) \leq 2$, a contradiction.

Case 2.b.iii: Assume $|Int(P)| = 5$. Since each vertex in $Int(P)$ is adjacent to exactly one vertex from D (because they are of degree two in $G - D$) then none of of the vertices in D are adjacent to any other vertex in D (by a simple counting argument and using the fact G is cubic). This tells us that v_3 is adjacent to two vertices in $Int(P)$. Label the vertices in $Int(P)$, u_1 to u_5 as follows.



If v_3 is adjacent to u_1 or u_2 then we can find a C_3 or a C_4 that satisfie the conditions of lemma 3, a contradiction.

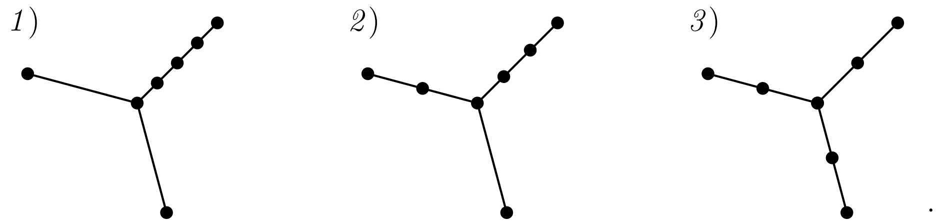




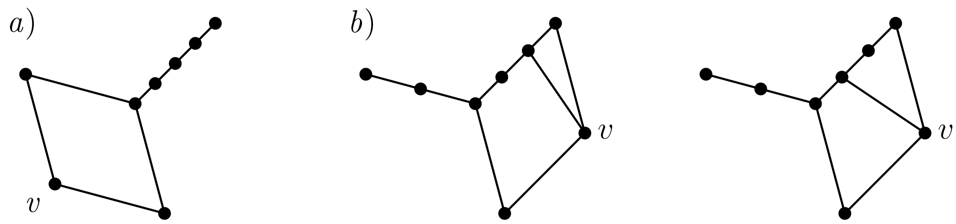
Therefore v_3 must be adjacent to two of u_3, u_4, u_5 and then we can find a C_3 or a C_4 that satisfie the conditions of lemma 3 to get a contradiction to the assumption that $\nabla(G) = 3$.

Case 2.b.iv: Assume $|Int(P)| > 5$, then v_2 can be adjacent to at most one vertex from $Int(P)$, v_1 and v_3 can each be adjacent to at most two vertices from $Int(P)$ and every vertex from $Int(P)$ is adjacent to exactly one vertex in D. Since $|Int(P)| > 5$ then this is clearly not possible.

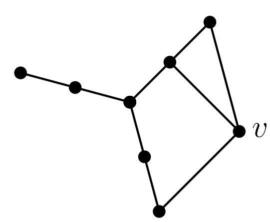
Case 2.c: Assume $\forall i, j \in \{1, 2, 3\}, i \neq j$, $|N_G(v_i) \cap N_G(v_j)| = 1$. We know $G-D$ is a forest, but we can say something stronger about it. Since every tree has at least two leaves and $G-D$ has three leaves we can say $G-D$ is a tree. Since $G-F$ is a tree with three leaves then it must also contain a 3-vertex and possibly some 2-vertices. None of the vertices of D can be adjacent to any other vertex in D or we could find a C_3 that satisfies the conditions of lemma 3 to find a contradiction, therefore each vertex in D must be adjacent to one 2-vertex and two leaves from $G-D$ in G. This tells us that there are at most three 2-vertices in $G-D$. $G-D$ must be one of the following forests:



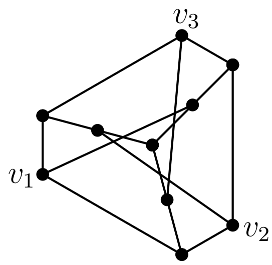
Let us try to find a contradiction with the above cases by looking at the following graphs.



If $G - D$ is forest *1*), then $\exists v \in D$ such that *a*) is a subgraph of G that clearly contains a C_4. In this case G is a cubic graph containing a C_4, by lemma 3 we can find a contradiction. If $G - D$ is forest *2*), then $\exists v \in D$ such that one of the graphs in *b*) is a subgraph of G. Both graphs contain a C_3 or a C_4. Yet again, since G is cubic containing a C_3 or a C_4 then by lemma 3 we can find a contradiction.

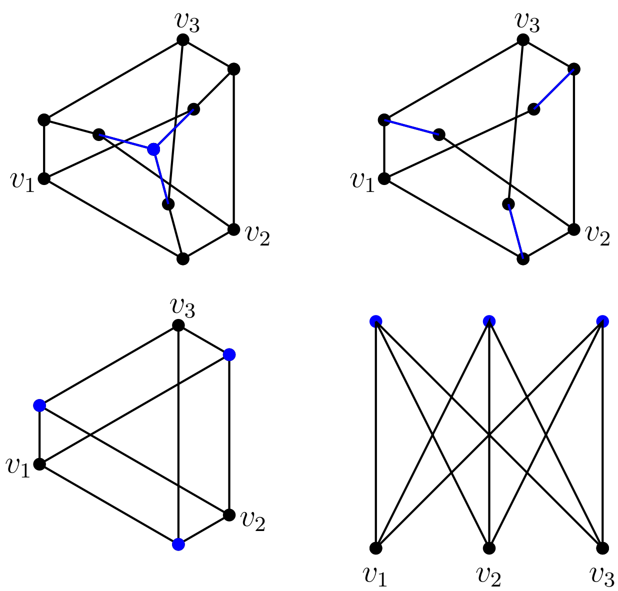


If $G - D$ is forest *3*) and the above subgraph occurs in G where $v \in D$ then we can find a C_3 and a contradiction. Therefore G must take the form of the following graph:



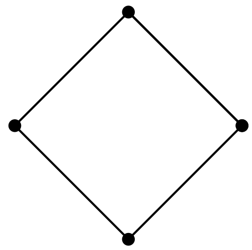
.

G has many crossing edges so let us try to find a $K_{3,3}$ as a graph minor to form a contradiction with G being planar. Under the following deletions and contractions we find a $K_{3,3}$ as a graph minor of G.



We have now contradicted case 2.c and we can now consider case 3.

Case 3: Assume $\max_{i,j \in N, i \neq j} \{|N_G(v_i) \cap N_G(v_j)|\} \geq 2$. Without loss of generality let $|N_G(v_1) \cap N_G(v_2)| = 2$. Then clearly G contains a C_4 and using lemma 3 $\nabla(G) \leq 2$ contradicting $\nabla(G) = 3$.



Since cases 1, 2, and 3 do not happen then we have finally made a contradiction with the original assumption that $\nabla(G) = 3$, therefore $\nabla(G) = 2$, as desired. □

Now the question becomes: can we extend this even further to general planar graphs with $\delta(G) \geq 3$? The answer is yes we can.

Theorem 4. *Every planar graph G with $\delta(G) \geq 3$ and $cp(G) = 1$ has $\nabla(G) \leq 2$.*

If we analyze the proof of Theorem 3 for cubic graphs then case 1 still holds for graphs with $\delta(G) \geq 3$ therefore we need only prove for $\max_{i,j \in N, i \neq j} \{|N_G(v_i) \cap N_G(v_j)|\} \geq 1$.

Proof. Suppose we have a planar graph G with $\delta(G) \geq 3$, and $cp(G) = 1$. Assume $\nabla(G) = 3$, let $N = \{1, 2, 3\}$ and let $D = \{v_1, v_2, v_3\}$ a minimum decycling set.

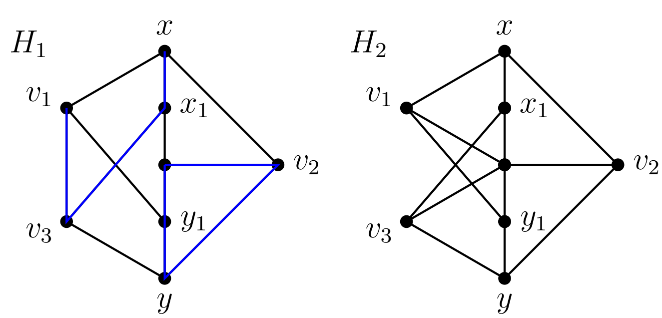
Assume $\max_{i,j \in N, i \neq j} \{|N_G(v_i) \cap N_G(v_j)|\} = 1$, then we have the following cases.

Case 1: Assume $\exists! i, j \in \{1, 2, 3\}, i \neq j$, such that $|N_G(v_i) \cap N_G(v_j)| = 1$. The proof for case 2.a in Theorem 3 holds for this case.

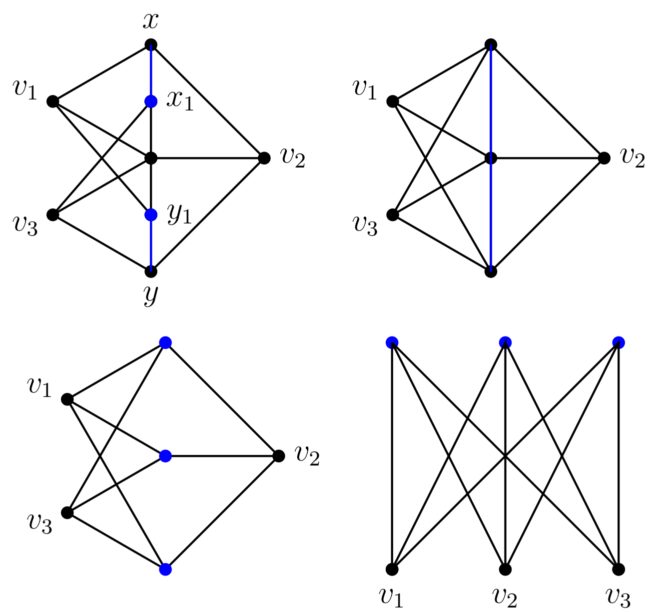
Case 2: Assume $\exists! i, j \in \{1, 2, 3\}, i \neq j$, such that $|N_G(v_i) \cap N_G(v_j)| = 0$. Without loss of generality let $N_G(v_1) \cap N_G(v_3) = \emptyset$ and define x, y, P and

$Int(P)$ to be the same as in Theorem 3 case 2.b. We know that $\deg_G(v) = 3$, for $v \in \{x, y\}$ since if not then $G - D$ would have less then two leaves and would not be a forest. If $|Int(P)| \leq 1$ then there exists a C_3 containing x or y, which leads to a contradiction . We can now say $|Int(P)| \geq 2$ and assume G is not a cubic graph since we have covered that case in Theorem 3. If v_1 or v_3 are adjacent to v_2 then we clearly have a C_3 containing x or y respectively and by lemma 3 we could find a contradiction. We know that v_1 and v_3 are each not adjacent to v_2.

Case 2.a: Assume $\deg_G(v) = 3$, $\forall v \in D$. If $|Int(P)| = 2$, then v_2 must be adjacent to a vertex in $Int(P)$ and we can find a C_3 containing x or y. If $|Int(P)| = 3$ then let x_1 be adjacent to x on P. Clearly, x_1 is not adjacent to v_1 or v_2, or there would be a C_3 in the graph containing x. We must have that x_1 is adjacent to v_3. Similarly, if we define y_1 to be adjacent to y on P then v_1 is adjacent to y_1, and v_2 and v_3 are not. Since we have the property $\delta(G) \geq 3$ then one of the following graphs is a subgraph of G.



If graph H_1 is a subgraph of G then we can clearly find two disjoint cycles. Therefore H_2 must be a subgraph of G. By the following deletions and contractions we can find a $K_{3,3}$ as a graph minor of G.



This contradicts the planarity of G therefore $|Int(P)| \neq 3$. If $|Int(P)| = 4$, and v_1 and v_3 are adjacent in G, than there exists a k-vertex in D with $k \geq 4$ since every vertex in $Int(P)$ is adjacent to at least one vertex in D, than the sum of the degrees of D must be at least ten, a contradiction with the vertices of D having degree three. Therefore v_1 and v_3 are not adjacent. Define x_1 and y_1 to be the same as in case 2.b.ii. Since the vertices of D all have degree three then one of the following graphs must be a subgraph of G:
Clearly the two graphs in Figure 4 graph have two disjoint cycles, contadicting $cp(G) = 1$, and therefore we have a contradiction with $|Int(P)| = 4$. If $|Int(P)| = 5$, then the graph is cubic since every vertex in $Int(P)$ is adjacent to at least one vertex in D and the vertices of D have degree three. We have already solved solved the problem for cubic graphs in Theorem 3. If $|Int(P)| \geq 5$, then a vertex in D must have degree higher than three, a contradiction. We have now contradicted that every vertex in D is of degree

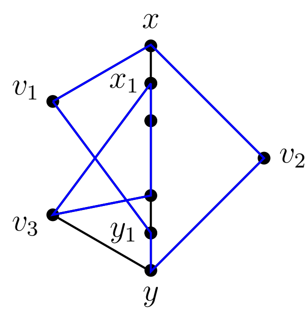


Figure 4: Subgraphs of G

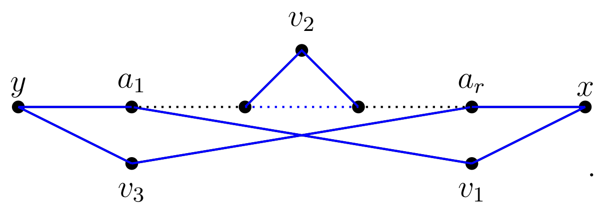
three.

Case 2.b: We can conclude that there exists a k-vertex in D where $k \geq 4$. Since v_1 and v_3 are not adjacent to v_2 then we have for some $v \in D$, v is adjacent to two vertices of $Int(P)$ in G. Label the vertices of $Int(P)$ as follows:

,

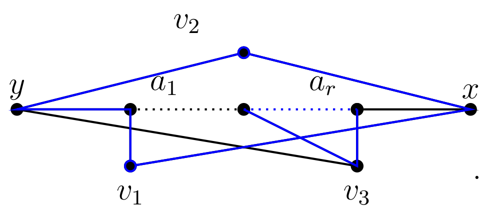
where $r = |Int(P)|$.

Case 2.b.i: Assume v_2 is adjacent to at least two vertices in $Int(P)$. If v_2 is adjacent to a_1 or a_r then there is a C_3 containing x or y. By lemma 3 we have a contradiciton. So v_2 is not adjacent to a_1 or a_r. If v_1 is adjacent to a_r then we have a C_3 containing x in G, therefore we must have v_3 is adjacent to a_r. Similarly, we get that v_1 is adjacent to a_1. Therefore, the following graph is a subgraph of G:



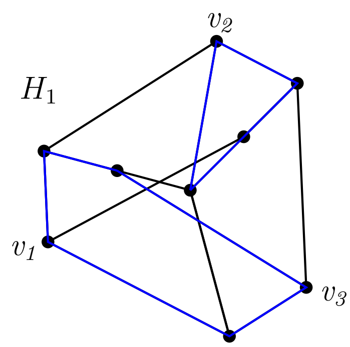
Clearly this is a contradiction with $cp(G) = 1$.

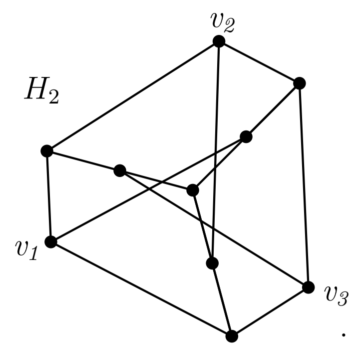
Case 2.b.ii: Without loss of generality assume v_1 is adjacent to at least two vertices in $Int(P)$. Then by a similar argument v_1 is not adjacent to a_r and by a similar argument as case 2.b.i we find that the following graph is a subgraph of G:



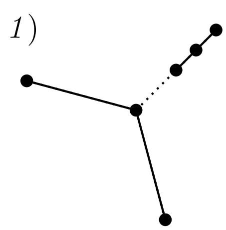
Since this graph has two disjoint cycles than this is a clear contradiciton with $cp(G) = 1$, and therefore no vertex in D can have degree higher than three. No vertex in D or $V(P)$ can have degree higher than three and the graph is not cubic by Theorem 3 therefore we have contradicted $\delta(G) \geq 3$.

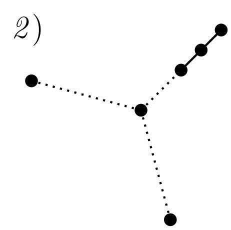
Case 3: Assume $\forall i, j \in \{1, 2, 3\}, i \neq j$, $|N_G(v_i) \cap N_G(v_j)| = 1$. Similar to case 2.c in Theorem 3, $G - D$ must be a tree with three leaves, one 3-vertex and possibly some 2-vertices. If $G - D$ contains zero or one 2-vertex then we can easily find a C_3 in G containing a 3-vertex; therefore $G - D$ must contain at least two 2-vertices. If $G - D$ has at most one 2-vertex on every branch then it must have one of the following graphs as a subgraph:



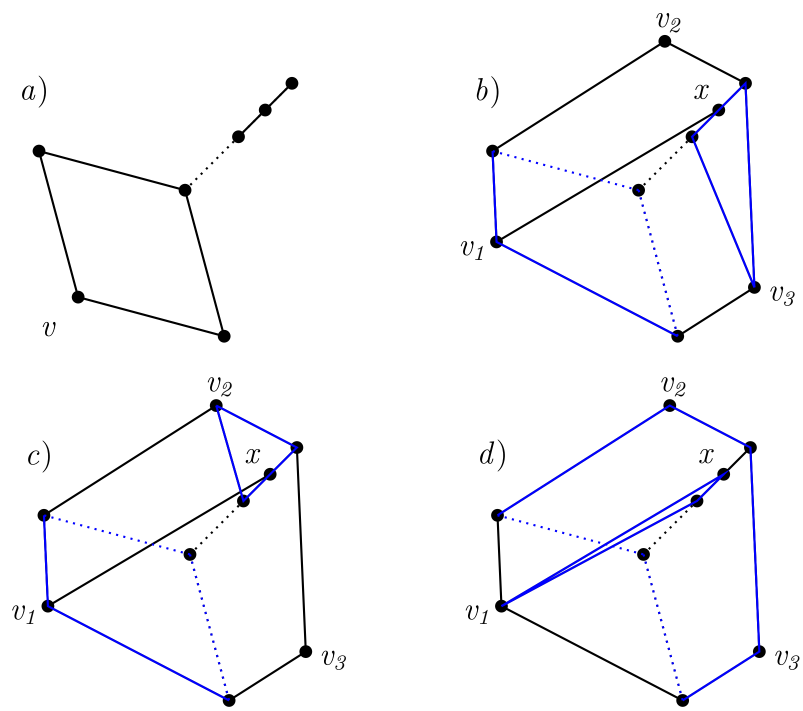


If this is not the case then we could easily find a C_3 with a 3-vertex which leads to a contradiction. Clearly if H_1 is a subgraph of G then we have two disjoint cyles, a contradiction. If H_2 is a subgraph of G then we have previously shown in case 2.c of Theorem 3 H_2 contains a $K_{3,3}$ as a graph minor. This implis G contains a $K_{3,3}$ as a graph minor, a contradiction. Therefore there exists a branch with two 2-vertices on it. Now the remaining possible cases for $G - D$ are the following graphs, where the solid lines are edges and the dotted lines are paths of length at least one:





Let us try to find a contradiction with the above cases by considering the following graphs.



If $G - D$ is tree *1*) then *a*) is a subgraph of G and it clearly contains a C_4. This C_4 has two non-adjacent 3-vertices in G since the leaves of $G - D$ have degree three in G, therefore we have a contradiction with $\nabla(G) = 3$ by lemma 3.

$G - D$ must be tree *2*). Now we have that either *b*), *c*), or *d*) (unlabelled) is a subgraph of G. Let x be defined as the vertex adjacent to the leaf in $G - D$ that is adjacent to v_2 and v_3 in G then one might ask why v_1 and x are adjacent? This is because if v_2 or v_3 were adjacent to x then it would be easy to find a C_3 with a 3-vertex. Since x must be adjacent to a vertex in D then it must be v_1. Since each one implies that G has at least two disjoint cycles then we have a contradiction with $cp(G) = 1$. We have now contradicted $|N_G(v_i) \cap N_G(v_j)| = 1, \forall i, j \in \{1, 2, 3\}, i \neq j$.

If $\max_{i,j \in N, i \neq j} \{|N_G(v_i) \cap N_G(v_j)|\} \geq 2$ then there are at most three 3-vertices in

$$\bigcup_{i,j \in N, i \neq j} N_G(v_i) \cap N_G(v_j).$$

If there were four or more 3-vertices then we could find a C_4 with two non-adjacent 3-vertices by a pigeonhole principle argument. Since there are at most three 3-vertices then $G-D$ takes the form of an x-y path for some x and y or a tree with three leaves, one 3-vertex and zero or more 2-vertices. Then we can apply arguments similar to the case of $\max_{i,j \in N, i \neq j} \{|N_G(v_i) \cap N_G(v_j)|\} = 1$ to find a contradiction. By our remark that cases 1 from Theorem 3 holds in this proof, then we have shown that every planar graph G with $\delta(G) \geq 3$ and $cp(G) = 1$ has $\nabla(G) \leq 2$. □

We can now adjust the algorithm by Chen,Fu and Shih to state when the planar graph G has $cp(G) = 1$, then place two vertices in D and one cycle in $\mathcal{C}$. This will strengthen their result and give us $\nabla(G) < 3cp(G)$ for planar graphs.

Conclusion

For planar graphs G, in the past ten years there have been significant improvements on bounding $\nabla(G)$ by a constant multiple of $cp(G)$. First we had $\nabla(G) \leq 5cp(G)$ by Kloks, Lee and Liu, than ten years later we had $\nabla(G) \leq 3cp(G)$ by Chen,Fu and Shih, and now we have $\nabla(G) < 3cp(G)$. It is possible that proving the conjecture $\nabla(G) < 2cp(G)$ is true for cubic planar graphs may lead to a general proof for planar graphs since this was the case for $cp(G) = 1$.

Acknowledgements

I would like to say thank you to:

Dr. David Pike, Professor at Memorial University, for helping tackle this problem with me, helping with the editing process along the way, and for being my supervisor for this thesis.

Dr. Danny Dyer, Professor at Memorial University, for helping me create graphs in LaTeX.

References

[1] T. Kloks,C.M. Lee, J. Liu, New Algorithms for k-Face Cover, k-Feedback Vertex Set, and k-Disjoint Cycles on Plane and Planar Graphs, in: *Proceedings of the 28th International Workshop on Graph-Theoretic Concepts in Computer Science, 2002, Vol.2573, pg.282-295.*

[2] H. Chen, H. Fu, C. Shih, Feedback Vertex Set on Planar Graphs, in: Taiwanese Journal of Mathematics, Vol.16, No.6, pg.2077-2082.

[3] R. M. Karp, Reducibility among comnibatorial problems, Complexity of Computer Computations, 1972, pg.85-103.

[4] N. L. Biggs, E. K. Lloyd, and R. J. Wilson, Graph Theory, 1976, pg.1736-1936