Perfect $T(G)$ Triple Systems when G is a Matching

by

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Abstract

A $T(G)$ triple is formed by taking a graph G and replacing every edge with a 3-cycle, where all of the new vertices are distinct from all others in G. An edge-disjoint decomposition of $3K_n$ into $T(G)$ triples is called a $T(G)$ triple system of order n. If we can decompose K_n into copies of a graph G, such that we can form a $T(G)$ triple from each graph in the decomposition and produce a partition of the edges of $3K_n$, then the resulting $T(G)$ triple system is called perfect.

We give necessary and sufficient conditions for the existence of perfect $T(G)$ triple systems when G is a matching with λ edges, which we denote by $\cup_\lambda P_2$. We then give cyclic perfect decompositions of $3K_n$ into $T(\cup_\lambda P_2)$ triples for all $n \equiv 1 \pmod{2\lambda}$ when λ is even (except for $n = 4\lambda + 1$ when $\lambda > 8$) as well as completely solve the case $\lambda = 3$.

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My father and my brothers have been an inspiration to me. Stephen and Joe have stuck with me through everything. My mother has always been there to advise and support me, and for that I will always be grateful.

Table of Contents

List of Figures

List of Tables

Chapter 1

Introduction

1.1 Graph Theory

A **graph** $G = (V, E)$ is a non-empty set V, whose elements are called **vertices**, and a set E of unordered pairs of distinct elements from V called **edges**. Given a graph G we will refer to its corresponding vertex and edge sets as $V(G)$ and $E(G)$ respectively. An example of a graph is given in Figure 1.1.

The **complete graph** on n vertices, denoted K_n, is the unique graph on n vertices with an edge between every pair of vertices. An illustration of the complete graph on 9 vertices is given in Figure 1.2.

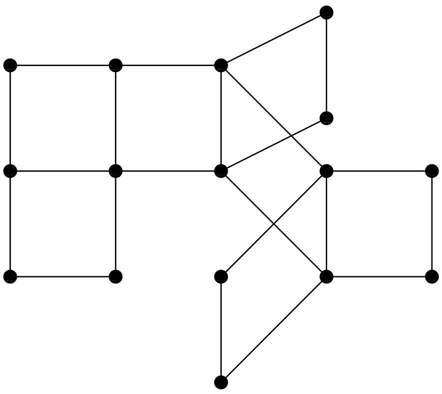

Figure 1.1: An example of a graph

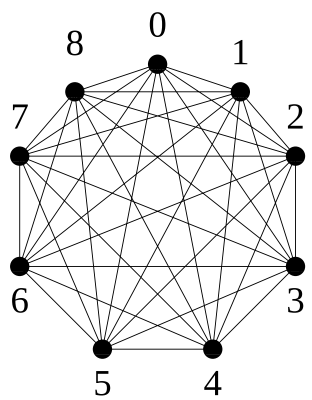

Figure 1.2: The complete graph on 9 vertices

The **complete bipartite graph** with bipartition sets M and N of sizes m and n respectively, denoted $K_{m,n}$, is the graph with all possible edges of the form $\{v_m, v_n\}$, where $v_m \in M$ and $v_n \in N$, but no edges of the form $\{v_m, w_m\}$ or $\{v_n, w_n\}$ where $v_m, w_m \in M$ and $v_n, w_n \in N$.

A **subgraph** of a graph $G = (V, E)$ is a graph $H = (V', E')$ with $V' \subseteq V$ and $E' \subseteq E$. A **decomposition** of a graph H is a family $\mathcal{F}$ of edge-disjoint subgraphs of H such that the union of all of the edges of the members of $\mathcal{F}$ is the edge set of H. A decomposition of K_6 into three subgraphs is given in Figure 1.3.

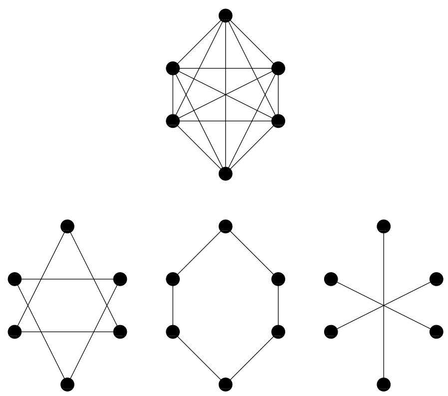

Figure 1.3: An example of a decomposition

Given a family of graphs $\mathcal{G}$, a **$\mathcal{G}$-decomposition** of H is a decomposition $\mathcal{F}$ such that every graph $F \in \mathcal{F}$ is isomorphic to some $G \in \mathcal{G}$. If $\mathcal{G} = \{G\}$ then we use the shorthand G-decomposition. A C_3-decomposition is given in Figure 1.4. G-decompositions are far more frequently the object of study than frivolous decompositions such as the one pictured in Figure 1.3.

A **multigraph** is a graph where we are permitted to have multiple edges between a single pair of vertices. The largest number of edges between any pair of vertices is called the **multiplicity** of the multigraph.

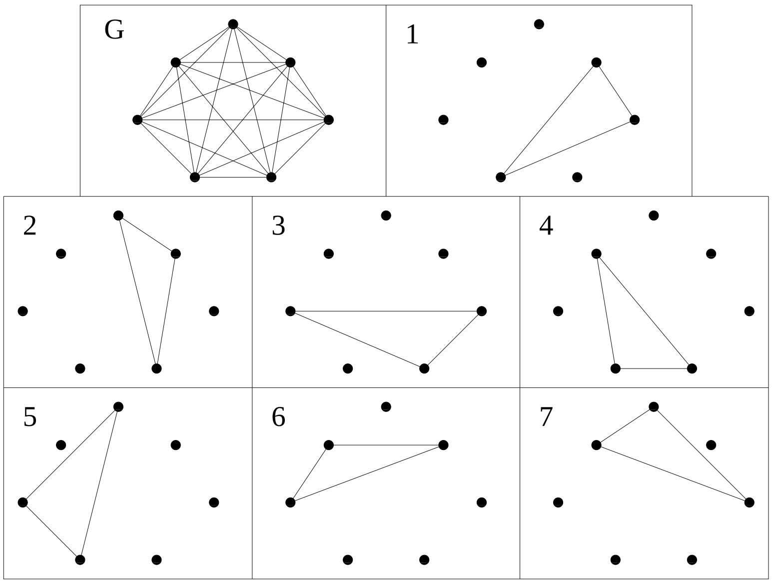

Figure 1.4: A C_3-decomposition of K_7.

We let cK_n be the complete multigraph where each edge has multiplicity c; that is, there are c edges between each pair of vertices. Figure 1.5 shows $3K_4$.

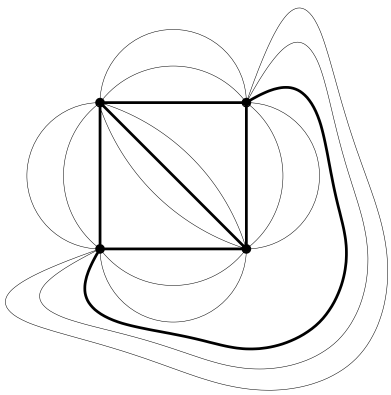

Figure 1.5: The multigraph $3K_4$

Given a graph $G = (V, E)$, a vertex $v \in V$ is **incident** with an edge $e \in E$ if and only if $e = \{v, v'\}$ for some $v' \in V$. The **degree** of a vertex v is the number of distinct edges incident with that vertex, and is usually denoted $\deg(v)$.

A **path** is a simple graph with all vertices of degree 2 except for two vertices of degree 1. We denote a path on n vertices by P_n. A path on 4 edges is shown in Figure 1.6. A graph is **connected** if for any two vertices u and v there is a path whose vertices of degree 1 are u and v.

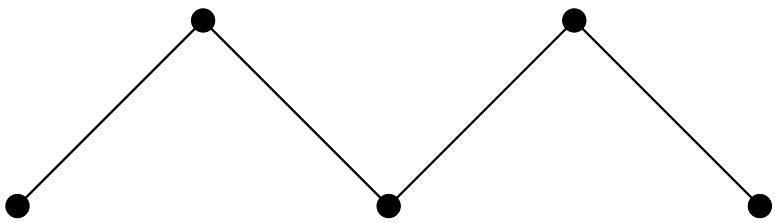

Figure 1.6: A path on 4 edges P_4

A connected graph where all vertices have degree 2 is called a **cycle**. A cycle on 6 vertices is shown in Figure 1.7. We denote a cycle on n edges (and vertices) by C_n. We will use the terms 3-cycle and triangle interchangeably for the graph C_3.

A **matching** in a graph G is a set of edges in a graph which have no vertices in common. Figure 1.8 gives an example of a matching as a subgraph of K_9. We denote the graph G which consists of λ disjoint copies of P_2 by $\cup_\lambda P_2$. This graph is sometimes referred to as a λ-matching.

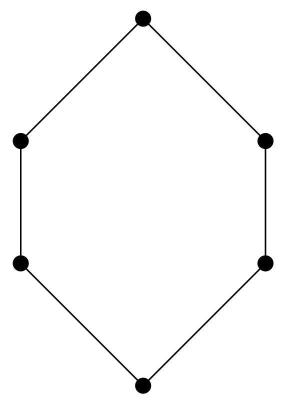

Figure 1.7: A cycle on 6 vertices C_6

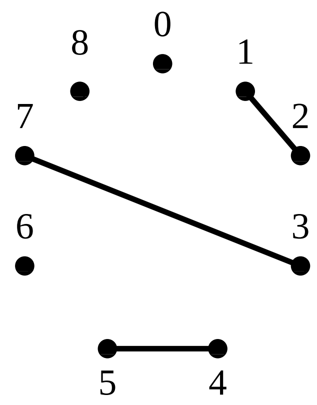

Figure 1.8: An example of a 3-matching

Given a complete graph on n vertices, we can assign an arbitrary and distinct label from the set $\{0, 1, 2, \ldots, n-1\}$ to each vertex. If this correspondence is given by the function $f\colon V(K_n) \rightarrow \{0, 1, 2, \ldots, n-1\}$, then we can refer to the length of the edge between vertices u and v by the absolute difference $\min\{|f(u) - f(v)|, |f(v) - f(u)|\}$

where the subtractions are computed modulo n. In any complete graph with an odd number of vertices (which we will be dealing with almost exclusively), the largest such distance between any two vertices is $\frac{n-1}{2}$ and by convention we will denote this quantity by m and write $n = 2m + 1$.

1.2 Design Theory

A **balanced incomplete block design** or **BIBD** is a pair (V, B) where V is a non-empty set of v elements often referred to as **points** and B is a family of b k-subsets of V, called **blocks**, such that every element of V occurs in r blocks and every pair of distinct elements from V occurs in exactly λ blocks.

A BIBD thus has five parameters, v, b, k, r, and λ. However, the following classic theorem, a proof of which can be found in any elementary design theory textbook such as [1], gives some relationships between them.

Theorem 1.1 *If v, b, k, r, and λ are the parameters of a BIBD then the following equalities hold:*

1. $r(k-1) = \lambda(v-1)$
2. $bk = rv$

Theorem 1.1 tells us that any three parameters of a BIBD determine the other two; we will thus refer to a BIBD(v, b, k, r, λ) more concisely as a BIBD(v, k, λ).

Example 1.2 If we let $V = \{0, 1, 2, 3, 4, 5\}$, then the unique BIBD(6, 3, 2) up to isomorphism is given by the set of blocks $B = \{\{0, 1, 2\}, \{0, 1, 3\}, \{0, 2, 4\}, \{0, 3, 5\}, \{0, 4, 5\}, \{1, 2, 5\}, \{1, 3, 4\}, \{1, 4, 5\}, \{2, 3, 4\}, \{2, 3, 5\}\}$.

A **parallel class** or **resolution class** in a design is a set of blocks which partition the point set V. A BIBD is **resolvable** if there exists a partition R of its block set B into parallel classes. R is called a **resolution** of the BIBD.

Example 1.3 Table 1.1 gives a resolvable BIBD(9, 3, 1) where each row is a parallel class.

{0, 1, 2}	{3, 4, 5}	{6, 7, 8}
{0, 3, 6}	{1, 4, 7}	{2, 5, 8}
{0, 4, 8}	{1, 5, 6}	{2, 3, 7}
{0, 5, 7}	{1, 3, 8}	{2, 4, 6}

Table 1.1: A resolvable BIBD(9, 3, 1)

Let $b_0 \in B$ be a block of a BIBD(v, k, λ) and let ϵ be the permutation $(0\ 1\ 2\ \ldots\ v - 1)$. Applying the permutation ϵ to the block $b_0 = \{a_0, a_1, \ldots, a_w\}$

of size w produces a block $b_1 = \{a_0 + 1, a_1 + 1, \ldots, a_w + 1\}$, where the additions are performed modulo v. Note that b_0 and b_1 are distinct when $k < v$, and that b_1 is not necessarily a block of the same BIBD. Suppose that applying ϵ to b_0 produces a different block b_1, applying ϵ to b_1 produces a block b_2 distinct from all previous blocks, and so on until applying ϵ to some block b_x produces the block b_0. We say that the set of distinct blocks $\{b_0, b_1, \ldots, b_x\}$ produced by applying the permutation ϵ to successive blocks is the **orbit of b_0 under ϵ**.

Let $B_0, B_1, \ldots, B_x$ be the disjoint orbits of the x distinct blocks $b_0, b_1, \ldots, b_x$ under the (not necessarily distinct) permutations $\epsilon_0, \epsilon_1, \ldots, \epsilon_x$ respectively. A BIBD whose block set is the disjoint union $B_0 \cup B_1 \cup \ldots \cup B_x$ of the orbits of some set $b_0, b_1, \ldots, b_x$ of disjoint blocks is called **cyclic**. The blocks $b_0, b_1, \ldots, b_x$ are called the **base blocks** of the design. We will be primarily concerned with taking the orbits of base blocks under the canonical permutation $(0\ 1\ 2\ \ldots\ v\ -\ 1)$. An example of a cyclic design constructed by this procedure using a single base block is given in Example 1.4.

Example 1.4 Consider the point set $V = \{0, 1, 2, 3\}$ and the block $A_0 = \{0, 1, 2\}$. By applying the permutation $\epsilon = (0\ 1\ 2\ 3)$ to A_0, we obtain a new block $A_1 = \{1, 2, 3\}$. Similarly, by applying ϵ to A_1, we obtain another distinct block $A_2 = \{2, 3, 0\}$; and finally by applying ϵ to A_2 we obtain a fourth block $A_3 = \{3, 0, 1\}$. By letting $B = \{A_0, A_1, A_2, A_3\}$, we obtain a BIBD(4, 3, 2). Note that applying ϵ to A_3 produces

A_0 again, so B is in fact the orbit of A_0 under ϵ.

As in Example 1.4, we will often have v distinct blocks in the orbit of b_0 under ϵ; we say that a base block $b_0 \in B'$ generates a 'short' orbit under ϵ if fewer than v distinct blocks occur as translations of b_0.

A **λ-fold triple system** or TS(v, λ) is a BIBD(v, 3, λ). A **3-fold triple system** is a BIBD(v, 3, 3).

Theorem 1.5 gives a necessary and sufficient condition for the existence of a λ-fold triple system, and is given in II.2.2 of [5].

Theorem 1.5 *A TS(v, λ) exists if and only if* $v \neq 2$ *and* $\lambda \equiv 0 \pmod{\gcd(v-2, 6)}$.

A **Steiner triple system on v points** or STS(v) is a BIBD(v, 3, 1). Necessary and sufficient conditions for their existence are given by Theorem 1.6, which is one of the earliest and most famous results in design theory. In fact, a resolvable Steiner triple system of order v is known as a **Kirkman triple system on v points**, in honour of T. P. Kirkman, who proved Theorem 1.6.

Theorem 1.6 *[8] An STS(v) exists if and only if* $v \equiv 1, 3 \pmod 6$.

Since we will be primarily interested in cyclic triple systems, Theorem 1.7 will also be relevant.

Theorem 1.7 *[14] A cyclic STS(v) exists if and only if* $v \equiv 1, 3 \pmod 6$ *and* $v \neq 9$.

An example of a cyclic Steiner triple system of order 7 is given in Example 1.8. We note that an STS(v) is equivalent to a C_3-decomposition of K_v, and that this example was in fact also illustrated in Figure 1.4.

Example 1.8 If we let $V = \{0, 1, 2, 3, 4, 5, 6\}$, then an STS(7) is given by the set of blocks $B = \{\{0,\ 1,\ 3\},\ \{1,\ 2,\ 4\},\ \{2,\ 3,\ 5\},\ \{3,\ 4,\ 6\},\ \{4,\ 5,\ 0\},\ \{5,\ 6,\ 1\},\ \{6,\ 0,\ 2\}\}$. Note that this design can be produced cyclically by taking the orbit of the block $A = \{0,\ 1,\ 3\}$ under the permutation $\epsilon = (0\ 1\ 2\ 3\ 4\ 5\ 6)$.

Let K and G be sets of positive integers and let λ be a positive integer. A **group divisible design** of index λ and order v, denoted as a (K, λ)-GDD, is a triple ($\mathcal{V}$, $\mathcal{G}$, $\mathcal{B}$) where $\mathcal{V}$ is a finite set of cardinality v, $\mathcal{G}$ is a partition of $\mathcal{V}$ into parts (called groups) whose sizes lie in G, and $\mathcal{B}$ is a family of subsets (called blocks) of $\mathcal{V}$, which satisfy the following properties.

1. If $B \in \mathcal{B}$ then $|B| \in K$.
2. Every pair of distinct elements of $\mathcal{V}$ occur in exactly λ blocks or one group, but not both.
3. $|\mathcal{G}| > 1$.

Furthermore, if $v = a_1g_1 + \ldots a_sg_s$ and there are a_i groups of size s_i for $i = 1, \ldots, s$ then a (K, λ)-GDD is of type $g_1^{a_1} \ldots g_s^{a_s}$. If $\lambda = 1$ and $K = \{k\}$, then we say that a (K, λ)-GDD is a k-GDD.

1.3 Description of Problem

In 2004, Küçükçifçi and Lindner introduced the concept of a **perfect hexagon triple system**, which is a partition of $3K_n$ into the graph shown in Figure 1.9, with the additional property that the shaded 'interior' triangles form a decomposition of K_n into 3-cycles, i.e. a Steiner triple system [9].

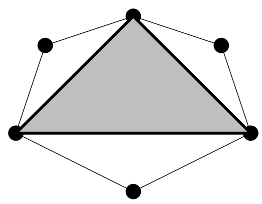

Figure 1.9: A hexagon triple

This definition has been generalized by several authors. First, Billington and Lindner defined a $T(G)$ **triple** to be the graph obtained by taking any subgraph G of K_n and forming the graph with a 3-cycle on each edge, where each third vertex is distinct from any other in G [3]. Examples of $T(G)$ triples where G is a cycle and a path are given in Figures 1.10 and 1.11 respectively. An edge-disjoint decomposition of $3K_n$ into $T(G)$ triples is called a $T(G)$ **triple system of order** n [3].

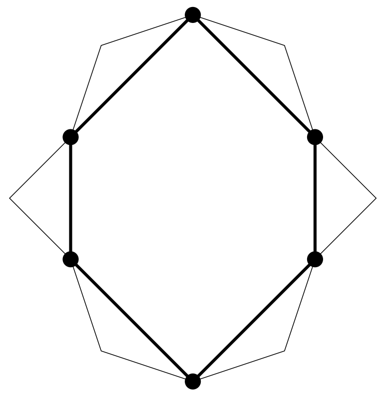

Figure 1.10: A $T(C_6)$ triple

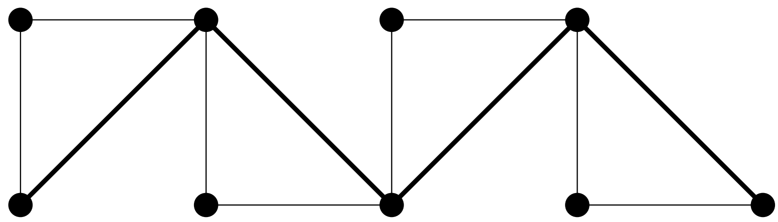

Figure 1.11: A $T(P_4)$ triple

Now suppose that K_n can be partitioned into copies of G. If a G-decomposition exists where we can form a $T(G)$ triple from each graph in the decomposition and produce a partition of $3K_n$, then Billington, Lindner, and Rosa referred to the resulting $T(G)$ triple system as **perfect** [4]. Equivalently, we define a perfect decomposition of $3K_n$ into $T(G)$ to be a G-decomposition $\mathcal{F}$ of K_n such that for each graph $G \in \mathcal{F}$ we can form a $T(G)$ triple and produce a $T(G)$-triple decomposition of $3K_n$. The set of positive integers n for which a decomposition of $3K_n$ into a perfect $T(G)$ triple system exists is called the **spectrum**.

Consider the following series of examples, which give the construction for perfect $T(C_4)$ triple systems from [4]. A $T(C_4)$ triple is given in Figure 1.12.

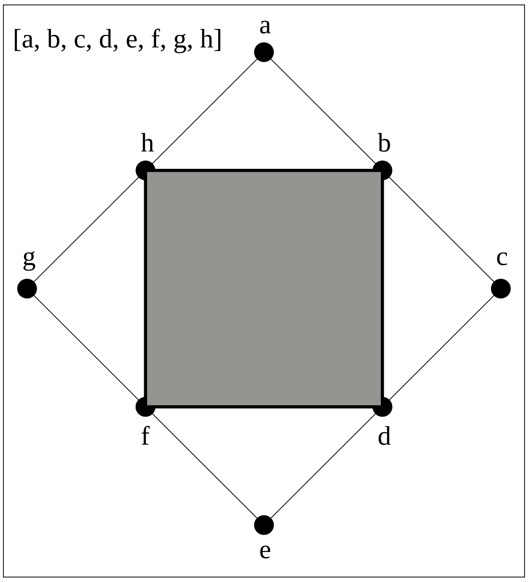

Figure 1.12: A $T(C_4)$ triple

Example 1.9 A perfect $T(C_4)$ triple system of order 9 is given cyclically by the block [1, 0, 5, 8, 6, 4, 3, 7] $\pmod 9$.

Example 1.10 A perfect $T(C_4)$ triple system of order 17 is given cyclically by the blocks [16, 0, 11, 8, 9, 3, 10, 2] and [8, 0, 5, 7, 14, 1, 16, 4] $\pmod{17}$.

Example 1.11 The graph $K_{4,4,4}$ is given in Figure 1.13. The graph $3K_{4,4,4}$ is obtained by taking 3 copies of each edge in $K_{4,4,4}$.

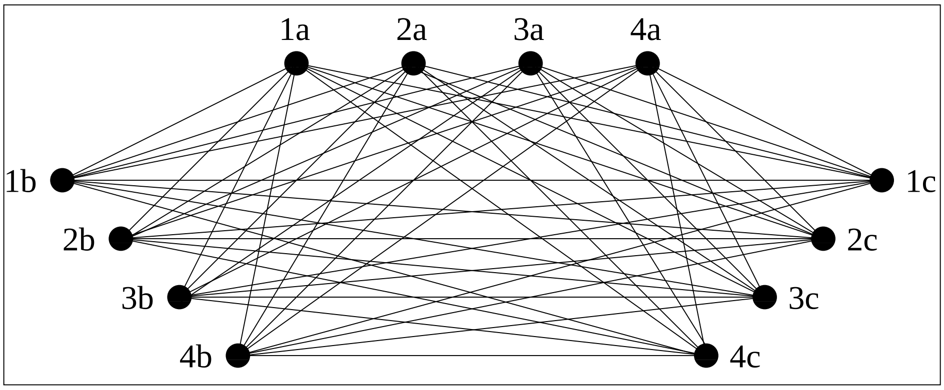

Figure 1.13: The graph $K_{4,4,4}$

A decomposition of $3K_{4,4,4}$ into $T(C_4)$ triples is given by the blocks $[1_c, 1_b, 4_c, 2_a, 2_c, 2_b, 3_c, 1_a]$, $[4_c, 3_b, 1_c, 2_a, 3_c, 4_b, 2_c, 1_a]$, $[2_c, 1_b, 3_c, 4_a, 1_c, 2_b, 4_c, 3_a]$, $[3_c, 3_b, 2_c, 4_a, 4_c, 4_b, 1_c, 3_a]$, together with 8 other blocks obtained by applying the permutation (a, b, c) to the subscripts of the previous 4 blocks.

Example 1.12 Necessary condition for the existence of a perfect $T(C_4)$ triple system are that $n \equiv 1 \pmod 8$ and $n \geq 8$.

Perfect $T(C_4)$ triple systems of orders 9 and 17 were given in Examples 1.9 and 1.10. For $n > 17$, set $n = 8s + 1$ with $s > 2$ and take $V(K_n) = \{\infty\} \cup \{(i, j) \mid 1 \leq i \leq 2s, 1 \leq j \leq 4\}$. We form the blocks F of a perfect $T(C_4)$ triple system by the following four steps.

1. If $s \equiv 0, 1 \pmod 3$, then we take a perfect $T(C_4)$ triple system of order 9, as in Example 1.9, place it on the vertex set $\{\infty\} \cup \{(2i-1, j), (2i, j) \mid 1 \leq j \leq 4\}$ for $1 \leq i \leq s$, and then place the resulting blocks in F.

2. If $s \equiv 2 \pmod 3$, then we take a perfect $T(C_4)$ triple system of order 17, as in Example 1.10, place it on the vertex set $\{(i, j) \mid 1 \leq i \leq 4, 1 \leq j \leq 4\}$, place the resulting blocks in F, and then repeat 1. for $3 \leq s$.

3. If $s \equiv 0, 1 \pmod 3$, then we take a 3-GDD of type 2^s on $\{1, 2, \ldots, 2s\}$ with groups $\{2i-1, 2i\}$ for $1 \leq i \leq s$. Such a GDD is known to exist [5]. For each block xyz in the GDD, take the blocks of a decomposition of $3K_{4,4,4}$ into $T(C_4)$ triples as in Example 1.11 with vertex set $\{(x, j) \mid 1 \leq j \leq 4\} \cup \{(y, j) \mid 1 \leq j \leq 4\} \cup \{(z, j) \mid 1 \leq j \leq 4\}$ and place these blocks in F.

4. If $s \equiv 2 \pmod 3$ then we take a 3-GDD of type $4^1 2^{s-2}$ on $\{1, 2, \ldots, 2s\}$ with groups $\{1, 2, 3, 4\}$ and $\{(2i-1, 2i)\}$, for $3 \leq i \leq s$, and then repeat 3. for each block xyz of the GDD. Once again, such a GDD is known to exist [5].

The resulting system $(V(K_n), F)$ is a perfect $T(C_4)$ triple system of order n.

Clearly, the hexagon triple in Figure 1.9 can be described as a $T(K_3)$ triple. Küçükçifçi and Lindner determined that the spectrum for perfect hexagon triple systems is $n \equiv 1, 3 \pmod 6$ [9]. For $n \equiv 1, 9 \pmod{12}$ and $n \neq 9$, they took a decomposition of $2K_n$ into C_6 and subdivided each 6-cycle into two hexagon triples to form a perfect $T(K_3)$ triple system. They then proceeded to construct cyclic perfect hexagon triple systems for all orders $n \equiv 7 \pmod{12}$. To handle the final case, $n \equiv 3 \pmod{12}$, they considered 2 subcases. When $\frac{n}{3} \equiv 1 \pmod 6$, they used GDDs together with derangements (permutations which do not leave any elements fixed) in a manner similar to Example 1.12; otherwise they used Kirkman triple systems and GDDs, again in a way similar to Example 1.12.

The constructions of Billington, Lindner, and Rosa for perfect $T(C_4)$ triple systems were given in Example 1.12. They also determined the spectra for perfect $T(G)$ triple systems when G is the 'kite' graph in Figure 1.14 using the same techniques [4]. They also determined the spectra for perfect decompositions of $3\lambda K_n$ into $T(C_4)$ and $T(G)$ triples when G is a kite.

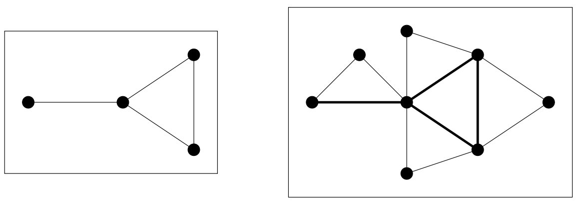

Figure 1.14: A 'kite' G on the left and the corresponding $T(G)$ triple on the right

Lindner and Rosa determined the spectrum for perfect $T(K_4)$ triple systems, which they referred to as perfect dexagon triple systems [11]. They began by showing that any BIBD(v, 4, 2) can be expanded to a $T(K_4)$ triple system which is not perfect. They then exploited the structure of several GDDs to construct perfect dexagon triple systems of all orders $n \equiv 1 \pmod{12}$ except for several small cases they handled separately, using techniques similar to Example 1.12.

Billington and Lindner then determined the spectra for all other graphs on 4 or fewer vertices [3]. In particular, their cyclic constructions for perfect $T(P_2)$ and $T(P_2 \cup P_2)$ triple systems will be given in Chapter 3. They also used cyclic constructions to build perfect $T(P_3)$ triple systems, and perfect $T(P_4)$ and $T(K_{1,3})$ triple systems when $n \equiv 1 \pmod 6$. To construct perfect $T(K_4 - e)$ triple systems, where e is any edge of K_4, they used GDDs as in Example 1.12. They also used the same techniques to construct perfect $T(P_4)$ and $T(K_{1,3})$ triple systems when $n \equiv 3 \pmod 6$.

The spectra for perfect $T(G)$ triple systems have thus been determined for all graphs G on 4 or fewer vertices. However, the spectra for graphs on larger numbers of vertices is still an open problem.

We consider the problem of determining the spectrum for perfect $T(G)$ triple systems when G is matching with λ edges, where λ is an arbitrary positive integer. Note that this λ is not related to the one introduced as a parameter of a BIBD introduced earlier. We will begin by describing this problem as both a graph theory and a combinatorial design problem.

In essence, we are studying the following graph decomposition and colouring problem. Take n vertices, and draw two red edges and one blue edge between each pair of vertices. Decompose the blue copy of K_n into λ-matchings and then construct an edge-disjoint decomposition of $3K_n$ into sets of λ-disjoint 3-cycles, by adding two incident red edges to each edge of the blue matching. Given λ, the question is to determine the spectra of n for which such a decomposition exists, and describe what the decompositions look like.

We will indicate which edges of $3K_n$ are coloured blue in our graphs by drawing them in bold.

The analogous design problem is unusual for two particular reasons. As with many designs, we begin with a set of v points with non-negative integer labels. Furthermore, we wish to arrange them into triples such that each pair occurs exactly thrice — if these were the only conditions, the problem would in fact be that of finding 3-fold triple systems. However, we introduce two additional constraints — the triples must be further arranged into families of a constant number of point-disjoint triples, and there must be a way of selecting one pair from each triple in the design to form a BIBD(v, 2, 1). We illustrate in Example 1.13 that in some cases there exist specific examples of known designs which satisfy these conditions, but we are not aware that this problem has been previously studied in general.

Given that there are many open problems in this area, it is quite reasonable to ask why we chose to look at matchings. In fact, Billington and Lindner suggested looking at paths or cycles, since the problems of decomposing K_n into copies of P_m or C_m have been extensively studied [2], [15], [18], [19].

The $T(G)$ triples when G is a cycle or a path, which were given in Figures 1.10 and 1.11 respectively, consist of 3-cycles which share some vertices. The $T(G)$ triples when G is a matching are sets of disjoint triangles, which are more closely related to other decompositions which have been studied previously. Thus, we concluded that studying perfect $T(\cup_\lambda P_2)$ triple systems would be both a more generalized and tractable problem than that of perfect $T(P_m)$ or $T(C_m)$ triple systems.

1.4 Examples

Example 1.13 The blocks for a perfect $T(P_2 \cup P_2 \cup P_2)$ triple system with $n = 9$ are given in Table 1.2. For the first three rows, the first pair in each triple constitutes the blue edge. In the remaining nine rows, the first and third elements form the blue edge for the first two triples, while the last two elements form the blue edge in the last triple. One reason that this example is illustrative is because the last nine rows have repeated triples, but with different blue edges each time. In addition, note that the first row of triangles generates a short orbit.

(**4**, **5**, 0)	(**1**, **2**, 6)	(**7**, **8**, 3)
(**5**, **6**, 1)	(**2**, **3**, 7)	(**8**, **0**, 4)
(**6**, **7**, 2)	(**3**, **4**, 8)	(**0**, **1**, 5)
(**0**, 1, **3**)	(**2**, 4, **7**)	(5, **6**, **8**)
(**1**, 2, **4**)	(**3**, 5, **8**)	(6, **7**, **0**)
(**2**, 3, **5**)	(**4**, 6, **0**)	(7, **8**, **1**)
(**3**, 4, **6**)	(**5**, 7, **1**)	(8, **0**, **2**)
(**4**, 5, **7**)	(**6**, 8, **2**)	(0, **1**, **3**)
(**5**, 6, **8**)	(**7**, 0, **3**)	(1, **2**, **4**)
(**6**, 7, **0**)	(**8**, 1, **4**)	(2, **3**, **5**)
(**7**, 8, **1**)	(**0**, 2, **5**)	(3, **4**, **6**)
(**8**, 0, **2**)	(**1**, 3, **6**)	(4, **5**, **7**)

Table 1.2: A perfect $T(P_2 \cup P_2 \cup P_2)$ triple system of order 9

The base triangles are given in Figure 1.15, while the full perfect $T(\cup_3 P_2)$ triple system is given in Figure 1.16. Note furthermore that by taking the rows of Table 1.2 or the sets of triangles in Figure 1.16 as parallel classes, this is a resolvable BIBD(9, 3, 3).

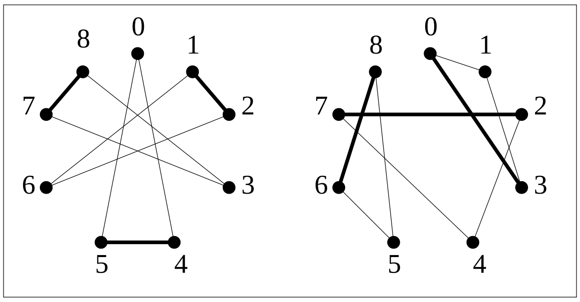

Figure 1.15: The base triangles for a perfect $T(\cup_3 P_2)$ triple system with $n = 9$

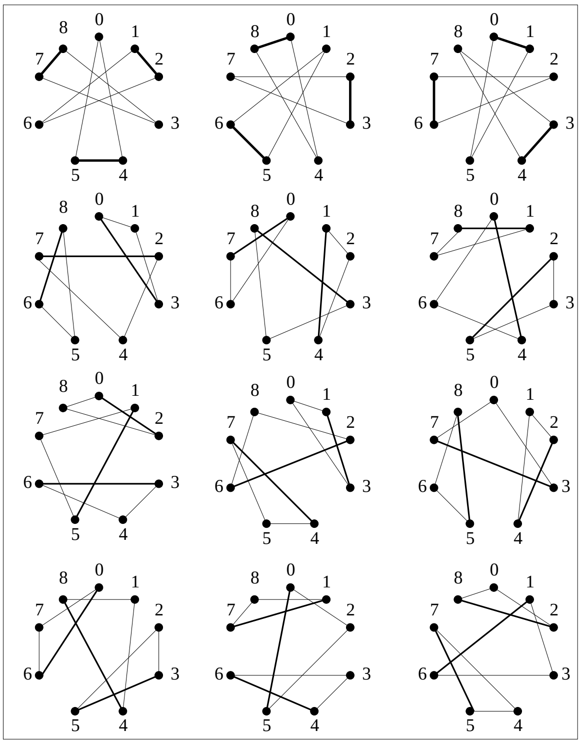

Figure 1.16: A perfect decomposition of $3K_9$ into $T(\cup_3 P_2)$ triples

Example 1.14 The base blocks for a perfect $T(\cup_3 P_2)$ triple system with $n = 13$ are given in Figure 1.17.

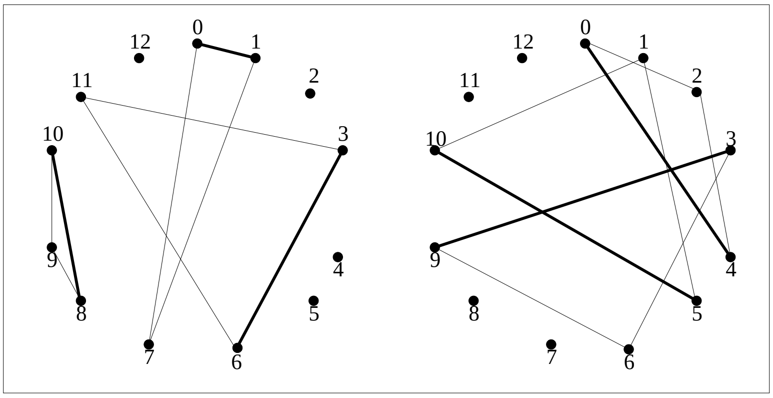

Figure 1.17: The base triangles for a perfect $T(\cup_3 P_2)$ triple system with $n = 13$

Example 1.15 The base blocks for a $T(\cup_3 P_2)$ triple system with $n = 15$ are given in Figure 1.18. Note that that the first set of triangles generate a short orbit.

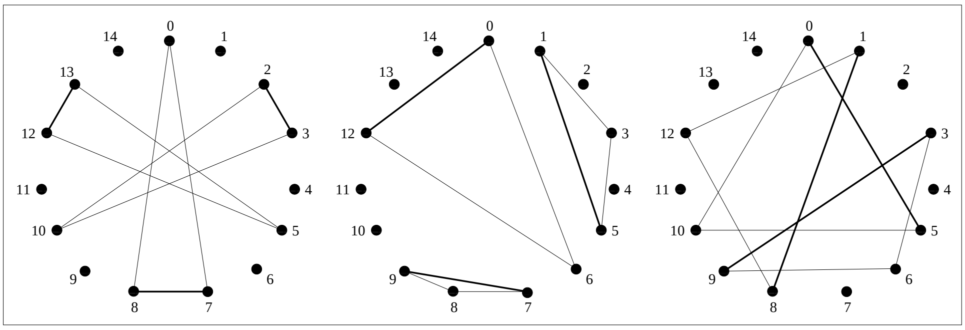

Figure 1.18: The base triangles for a perfect $T(\cup_3 P_2)$ triple system with $n = 15$

1.5 Generalized Skolem Sequences

A Skolem-type sequence is a sequence $(S_i)_{i=1}^{2n}$ of $2n$ integers from the set $\{1, 2, \ldots, n\}$ such that each integer occurs exactly twice, and if S_j and S_k are the two occurrences of the integer m then $|j - k| = m$. They are named in honour of Thoralf Skolem, who solved the problem of partitioning the integers from 1 to 2n into the differences from 1 to n in 1957 [17]. These sequences have connections to Steiner triple systems [1] and applications to graph labeling [13]. Skolem sequences can also be generalized so that each integer occurs more than twice [16].

A **generalized Skolem sequence of order** t **and multiplicity** s is a sequence GS =$(a_1^i, a_2^i, \ldots, a_{ts}^i)$, $i \in \{1, 2, \ldots, s\}$ of ts integers from $\{1, 2, \ldots, t\}$ that satisfies the following two properties.

1. For every $k \in \{1, 2, \ldots, t\}$ and for every $i \in \{1, 2, \ldots, s\}$, there exist s elements in GS, $\{a_{j_1}^i, a_{j_2}^i, \ldots, a_{j_s}^i\}$ such that $a_{j_1}^i = a_{j_2}^i = \cdots = a_{j_s}^i = k$.

2. If $a_{j_u}^i = a_{j_{(u+1)}}^i$ then $j_{(u+1)} - j_u = k$ for $1 \leq u \leq s - 1$.

A **generalized extended Skolem sequence of order** t **and multiplicity** s **with** d **zeros** is a sequence GES =$(a_1^i, a_2^i, \ldots, a_{ts+d}^i)$, $i \in \{1, 2, \ldots, s\}$ of ts integers from $\{1, 2, \ldots, t\}$, $d \geq 1$ that satisfies the following three properties.

1. For every $k \in \{1, 2, \ldots, t\}$ and for every $i \in \{1, 2, \ldots, s\}$, there exist s elements in GES, $\{a^i_{j_1}, a^i_{j_2}, \ldots, a^i_{j_s}\}$ such that $a^i_{j_1} = a^i_{j_2} = \cdots = a^i_{j_s} = k$.

2. If $a^i_{j_u} = a^i_{j_{(u+1)}}$ then $j_{(u+1)} - j_u = k$ for $1 \leq u \leq s - 1$.

3. There are exactly d zeros in the sequence (where d is the minimum number of zeros that can exist in the sequence).

Example 1.16 The following sequence is a generalized extended Skolem sequence of order 8 and multiplicity 3.

$$1\ 1\ 1\ 2\ 4\ 2\ 7\ 2\ 4\ 8\ 6\ 0\ 4\ 7\ 5\ 3\ 6\ 8\ 3\ 5\ 7\ 3\ 6\ 0\ 5\ 8$$

There has also been some study of odd- and even- Langford sequences, which can be considered as Skolem sequences using only odd or only even integers, possibly with some differences missing [12]. We will apply the following generalizations of these sequences to the problem of constructing perfect $T(\cup_\lambda P_2)$ triple systems in Chapter 4.

If t is even then a **generalized extended even Skolem sequence of order t and multiplicity s with d zeros** is a sequence $GEES = (a^i_1, a^i_2, \ldots, a^i_{\frac{s(t+1)}{2}+d})$, $i \in \{1, 2, \ldots, s\}$ of $\dfrac{s(t+1)}{2}$ integers from $\{2, 4, 6, \ldots, t-2, t\}$, $d \geq 1$ that satisfies the following three properties.

1. For every $k \in \{2, 4, 6, \ldots, t-2, t\}$ and for every $i \in \{1, 2, \ldots, s\}$, there exist s elements in $GEES$, $\{a^i_{j_1}, a^i_{j_2}, \ldots, a^i_{j_s}\}$ such that $a^i_{j_1} = a^i_{j_2} = \cdots = a^i_{j_s} = k$.

2. If $a^i_{j_u} = a^i_{j_{(u+1)}}$ then $j_{(u+1)} - j_u = k$ for $1 \leq u \leq s-1$.

3. There are exactly d zeros in the sequence (where d is the minimum number of zeros that can exist in the sequence).

Example 1.17 The following sequence is a generalized extended even Skolem sequence of order 8 and multiplicity 3.

$$4\ 0\ 8\ 0\ 4\ 6\ 0\ 0\ 4\ 0\ 8\ 6\ 2\ 0\ 2\ 0\ 2\ 6\ 8$$

If t is odd then a **generalized extended odd Skolem sequence of order t and multiplicity s with d zeros** is a sequence $GEOS = (a^i_1, a^i_2, \ldots, a^i_{\frac{ts}{2}+d})$, $i \in \{1, 2, \ldots, s\}$ of $\frac{ts}{2}$ integers from $\{1, 3, 5, \ldots, t-2, t\}$, $d \geq 1$ that satisfies the following three properties.

1. For every $k \in \{1, 3, 5, \ldots, t-2, t\}$ and for every $i \in \{1, 2, \ldots, s\}$, there exist s elements in $GEOS$, $\{a^i_{j_1}, a^i_{j_2}, \ldots, a^i_{j_s}\}$ such that $a^i_{j_1} = a^i_{j_2} = \cdots = a^i_{j_s} = k$.

2. If $a^i_{j_u} = a^i_{j_{(u+1)}}$ then $j_{(u+1)} - j_u = k$ for $1 \leq u \leq s-1$.

3. There are exactly d zeros in the sequence (where d is the minimum number of zeros that can exist in the sequence).

Example 1.18 The following sequence is a generalized extended odd Skolem sequence of order 7 and multiplicity 3.

3 7 5 3 0 0 3 5 7 1 1 1 5 0 0 7

Chapter 2

A theorem of Lamken and Wilson and the existence of perfect $T(\cup_\lambda P_2)$ triple systems

In this chapter, we will introduce terminology in order to describe a theorem of Lamken and Wilson on the existence of graph decompositions. We will then use that theorem to prove a result about the existence of perfect $T(\cup_\lambda P_2)$ triple systems.

2.1 Definitions and a theorem of Wilson and Lamken

This next set of definitions are modified from [10] and lead up to Theorem 2.1. We denote the integers and the natural numbers by **Z** and **N** respectively. Similarly, we denote the set of n-vectors with elements in **N** by $\mathbf{N}^{\,n}$.

A **directed graph** or **digraph** $G = (V,\, A)$ is a set V of vertices and a set A of ordered pairs from V called **arcs**. Note that graphs can be considered as a special case of directed graphs where for every arc $(u,\, v) \in A$ there is an arc $(v,\, u) \in A$. Using this correspondence, all of the following results about directed graphs hold trivially for simple graphs.

We consider only [directed] graphs with finite numbers of [arcs] edges and vertices.

An r**-edge-coloured** graph [digraph] has every edge [arc] associated with exactly one of a set of r colours. Such a graph is **simple** if it has no loops (arcs of the form (u, u) for some vertex u of G). Simple digraphs have the additional condition that given any pair of vertices x, y there is only one of the arcs $(x,\, y)$ or $(y,\, x)$. Given a digraph, the opposite of an arc $(x,\, y)$ is the arc $(y,\, x)$.

We denote the complete digraph on n vertices with arcs of each of r colours between every ordered pair of vertices by $K_n^{(r)}$. This digraph has $rn(n-1)$ arcs and is simple if and only if $r = 1$. Note that in the case of the (undirected) complete graph with edges of r colours between every pair of vertices, there are $\frac{rn(n-1)}{2}$ edges but it is again simple if and only if $r = 1$.

Given a vertex x of an r-edge-coloured digraph G the **degree-vector** of x, $\tau(x)$ is the $2r$-vector

$$\tau(x) = (\text{in}_1(x), \text{out}_1(x), \text{in}_2(x), \text{out}_2(x), \ldots, \text{in}_r(x), \text{out}_r(x))$$

where $\text{in}_j(x)$ and $\text{out}_j(x)$ denote the number of arcs of colour j at the vertex x that are directed inward and outward respectively. Note that this is just a generalization of the concept of degree which encompasses both direction and colour.

In the case of the simple digraph $K_n^{(r)}$, then we trivially have that for each vertex v in $V(K_n^{(r)})$, $\tau(x) = (n-1, n-1, \ldots, n-1)$. If we then want to talk about a decomposition $\mathcal{G}$ of $K_n^{(r)}$, then for each vertex v in $V(K_n^{(r)})$ we must have that $\sum_{G \in \mathcal{G}} (\text{in}_1(x), \text{out}_1(x), \text{in}_2(x), \text{out}_2(x), \ldots, \text{in}_r(x), \text{out}_r(x)) = (n-1, n-1, \ldots, n-1)$. We will therefore desire some notation to describe linear combinations of $\tau(x)$.

If we denote the vertices of a graph G by $x_1, x_2, \ldots, x_n$ then we can find constants $c_1, c_2, \ldots, c_n$ so that $\sum_{i=1}^{n} c_i \tau(x_i)$ =(t_0, t_0, …, t_0) and $t_0 \in \mathbb{Z}$ is minimal. Since this minimal constant t_0 is a parameter of the graph G, we define $\gamma(G) = t_0$.

We can now state Theorem 2.1, which is a result about the existence of G-decompositions of $K_n^{(r)}$.

Theorem 2.1 *[10] Let G be a simple r-edge-coloured digraph with m edges of each of r different colours. There exists a constant $n_0 = n_0(G)$ such that the complete r-edge-coloured digraph $K_n^{(r)}$ admits a G-decomposition for all integers $n \geq n_0$ that satisfy the following conditions:*

$$n(n-1) \equiv 0 \pmod{m} \tag{2.1}$$

$$n-1 \equiv 0 \pmod{\gamma(G)}. \tag{2.2}$$

2.2 The existence of perfect $T(\cup_\lambda P_2)$ triple systems

We can derive a result about the existence of perfect $T(\cup_\lambda P_2)$ triple systems using Theorem 2.1, but first we derive necessary conditions using counting arguments which more clearly reveal the reasons for these constraints.

Proposition 2.2 *Necessary conditions for the existence of a perfect $T(\cup_\lambda P_2)$ triple system of order n are that $n(n-1) \equiv 0 \pmod{2\lambda}$, $n \equiv 1 \pmod{2}$, and $n \geq 3\lambda$.*

Proof: To have λ distinct triangles we must have at least 3λ distinct vertices.

Since every vertex in the decomposition into triangles is adjacent to exactly two vertices, every vertex must have even degree and n must be odd.

Furthermore the number of edges in each graph of the decomposition must divide the total number of edges, so $\frac{3n(n-1)}{2} \equiv 0 \pmod{3\lambda}$. ■

Although $n(n-1) \equiv 0 \pmod{2\lambda}$ may have many solutions for a fixed λ, we remark that $n \equiv 1 \pmod{2\lambda}$ is always a solution, and we will show in Corollary 4.9 that if λ is a power of 2 then it is the only odd solution.

We can in fact get a stronger result than Proposition 2.2 as a consequence of Theorem 2.1 by associating every arc with an opposite arc of the same colour, then corresponding each of these pairs of opposite arcs of the same colour with an undirected edge of that colour, and setting $r = 3$. Furthermore, we consider the perfect $T(\cup_\lambda P_2)$ triple system problem, by making each undirected edge in each 3-cycle a different colour, then associating two of the colours with each other.

Corollary 2.3 *There exists a constant $n_0 = n_0(\cup_\lambda P_2)$ such that perfect $T(\cup_\lambda P_2)$ triple systems of order n exist for all $n \geq n_0$ where n also satisfies $n(n-1) \equiv 0 \pmod{2\lambda}$ and $n \equiv 1 \pmod{2}$.*

Proof: Comparing Theorem 2.1 with the statement of Corollary 2.3, we claim that $m = 2\lambda$ and $\gamma(\cup_\lambda P_2) = 2$.

Since each of the λ 3-cycles has 2 edges of each colour, it should be clear that $m = 2\lambda$.

We wish to establish that $\gamma(\cup_\lambda P_2) = 2$. Suppose that $v_0, v_1, \ldots, v_{3\lambda-1}$ are the vertices of $\cup_\lambda P_2$. We need to show that 2 is the minimal constant t_0 such that $\sum_{i=1}^{3\lambda} c_{i-1}\tau(v_{i-1}) = (t_0, t_0, \ldots, t_0)$, for any constants $c_0, c_1, \ldots, c_{3\lambda-1}$.

Because we are dealing exclusively with undirected multigraphs,

$$\forall v \in V(\cup_\lambda P_2),\ \text{in}_i(v) = \text{out}_i(v) = \deg_i(v) \text{ for } i = 1{,}2{,}3.$$

Using this fact we define a bijection $\mathbf{N^6} \to \mathbf{N^3}$ given by

$$\tau(v) = (\text{in}_1(v), \text{out}_1(v), \text{in}_2(v), \text{out}_2(v), \text{in}_3(v), \text{out}_3(v))$$

$$\to \tau'(v) = (\deg_1(v), \deg_2(v), \deg_3(v)).$$

Denote the vertices of $\cup_\lambda P_2$ by v_i, $i = 0, 1, \ldots, 3\lambda - 1$ (where v_0, v_1, v_2 are in the same triangle).

It is easy to show that $\gamma(\cup_\lambda P_2) \leq 2$. Since every vertex $v \in V(\cup_\lambda P_2)$ is a vertex of a 3-cycle, it has exactly two neighbours and $\tau(v) = 2$. There are many possible combinations of constants $c_0, c_1, \ldots, c_{3\lambda-1}$ which give $\sum_{i=1}^{3\lambda} c_i\tau'(v_i)$; for example consider

$$\sum_{i=1}^{3\lambda} c_i\tau'(v_i) = (2, 2, 2) \text{ where } c_i = \begin{cases} 1, & \text{if } i = 1, 2, 3 \\ 0, & \text{otherwise.} \end{cases}$$

We prove that $\gamma(\cup_\lambda P_2) \geq 2$ by way of contradiction.

Suppose that $\gamma(\cup_\lambda P_2) = 1$ so that

$$\begin{aligned} c_1 + c_2 + c_4 + c_5 + \ldots + c_{3\lambda-2} + c_{3\lambda-1} &= 1 \\ c_1 + c_3 + c_4 + c_6 + \ldots + c_{3\lambda-2} + c_{3\lambda} &= 1 \\ c_1 + c_2 + c_3 + c_4 + c_5 + \ldots + c_{3\lambda-1} + c_{3\lambda} &= 1 \end{aligned}$$

since

$$\left(\begin{array}{cccccccccc|c} 1 & 1 & 0 & 1 & 1 & 0 & \ldots & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & \ldots & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & \ldots & 0 & 1 & 1 & 1 \end{array}\right)$$

(where the unaugmented part has 3λ columns) row reduces to

$$\left(\begin{array}{cccccccccc|c} 1 & 0 & 0 & 1 & 0 & 0 & \ldots & 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 & 1 & 0 & \ldots & 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 & 1 & \ldots & 0 & 0 & 1 & \frac{1}{2} \end{array}\right)$$

which does not have an integral solution, so $\gamma(\cup_\lambda P_2) > 1$. ■

Comparing Proposition 2.2 with Corollary 2.3, we would conjecture that $n_0 = 3\lambda$.

In Chapter 3, we give constructions for cyclic perfect $T(\cup_\lambda P_2)$ triple systems for small values of λ and all corresponding values of n. In Chapter 4 we construct cyclic perfect $T(\cup_\lambda P_2)$ triple systems of orders $n \equiv 1 \pmod{2\lambda}$ when λ is even, allowing us to completely solve the conjecture for an infinite family of graphs. This provides support for our conjecture and allows us to look at the structure of these decompositions.

Chapter 3

Constructing cyclic perfect $T(\cup_\lambda P_2)$ triple systems for small λ

In this chapter we construct perfect $T(\cup_\lambda P_2)$ triple systems when $\lambda = 1$, 2 and 3 for all values of n which satisfy Proposition 2.2. The results and constructions for $\lambda = 1$ and $\lambda = 2$ come from [3]; the constructions for $\lambda = 3$ are new results.

3.1 Constructions of Billington and Lindner

These constructions enabled completion of the spectra of perfect $T(G)$ triple systems where $G \subseteq K_4$. Since these constructions were designed to handle a particular number of vertices and we are interested in determining the spectra of perfect $T(G)$ triple systems where G has an arbitrary number of vertices, we will need to make some significant adjustments to cover other cases.

3.1.1 Constructing cyclic perfect $T(P_2)$ triple systems

Theorem 3.1 *A cylic perfect $T(P_2)$ triple system of order n exists if and only if $n \geq 3$ is odd.*

Proof: Since $\lambda = 1$, Proposition 2.2 says that if a perfect $T(P_2)$ triple systems of order n exists then $n(n-1) \equiv 0 \pmod 2$, $n \equiv 1 \pmod 2$, and $n \geq 3$. Therefore $n \equiv 1 \pmod 2$.

The construction is given in Table 3.1, where $1 \leq i \leq \dfrac{n-1}{2}$. This construction is illustrated in Figure 3.1, where $n = 2m + 1$. The column on the right gives the edge lengths of the triangles while the column on the left gives distinct vertices of base triangles with those edge lengths. The blue edges have the edge length of the third entry of the triple in the column on the left and are between the first and second vertices in the column on the right. An edge of each possible edge length occurs directly as an edge of the form $\{0, i\}$, for each i from 1 to $\dfrac{n-1}{2}$. Every other edge of the blue copy of K_n is obtained by taking the orbits of these edges under $\epsilon = (0\ 1\ \ldots\ n-1)$. Since there are exactly n edges in the orbits of each of these edges, there are exactly $\dfrac{n(n-1)}{2}$ blue edges, which is exactly the number of edges in K_n. Now consider the red edges given in Table 3.1. Each of the edge lengths from 1 to $\dfrac{n-1}{2}$ occur exactly once as edge between i and $2i$ since each such edge has length i. The red edges of the form $\{0, 2i\}$ have length $2i$. Taking $1 \leq i \leq \left\lfloor \dfrac{n-1}{4} \right\rfloor$ gives all of the even lengths between and possibly including 2 and m. If $\left\lfloor \dfrac{n-1}{4} \right\rfloor \leq i \leq \dfrac{n-1}{2}$ then $2i$ modulo $2m+1$ is odd and we produce all possible odd lengths. Furthermore, the table directly gives $\dfrac{3(n-1)}{2}$ edges. Each triangle in the table has exactly n triangles in its orbit under $\epsilon = (0\ 1\ \ldots\ n-1)$, so the triangles in the table together with their orbits under ϵ give all $\dfrac{3n(n-1)}{2}$ edges of $3K_n$. ■

Triangle with edge lengths	Vertices
$(2i,\, i,\, i)$	$(0,\, i,\, 2i)$

Table 3.1: Construction for perfect $T(P_2)$ triple sytems

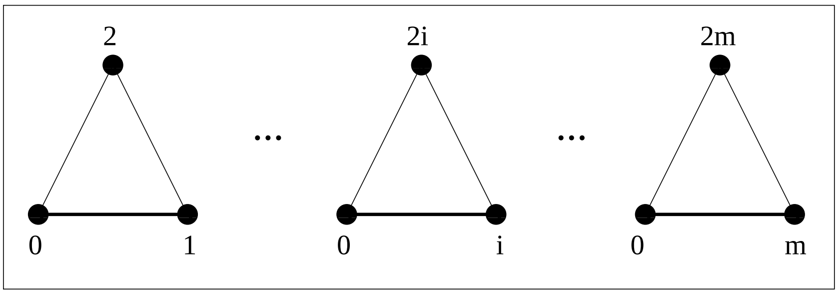

Figure 3.1: The base triangles for perfect $T(P_2)$ triple systems

3.1.2 Constructing cyclic perfect $T(P_2 \cup P_2)$ triple sytems

Theorem 3.2 *A cylic perfect $T(P_2 \cup P_2)$ triple system of order n exists if and only if $n \equiv 1 \pmod 4$ and $n \geq 6$.*

Proof: Since $\lambda = 2$, Proposition 2.2 says that if a perfect $T(P_2)$ triple systems of order n exists then $n(n-1) \equiv 0 \pmod 4$, $n \equiv 1 \pmod 2$, and $n \geq 6$. Since we must have $n \equiv 1 \pmod 2$, it must be true that $n \equiv 1, 3 \pmod 4$. However, if $n \equiv 3 \pmod 4$ then $n(n-1) \equiv 2 \pmod 4 \not\equiv 0 \pmod 4$. We conclude that we must have $n \equiv 1 \pmod 4$.

The main construction is given in Table 3.2 where $1 < i < \frac{n-1}{4}$ and the double lines separate the general construction from those for the differences where $i = 1$ and $i = \frac{n-1}{4}$. This construction works for $n > 9$. The left hand column gives the edge lengths of the base triangles while the column on the right gives distinct vertices with corresponding edge lengths. The base triangles are illustrated in Figure 3.2, where $n = 2m + 1$. The blue edges again have the edge length of the third entry of the triple in the column on the left and are between the first and second vertices. All odd edge lengths occur exactly once as edges of the form $\{0, 2i-1\}$ except for 1 which occurs as the edge $\{0, 1\}$ and $\dfrac{n-3}{2}$ which occurs as the edge $\left\{0, \dfrac{n-3}{2}\right\}$. All of the even blue edge lengths occur exactly once as edges of the form $\{1, 2i+1\}$ except for

2 and $\dfrac{n-1}{2}$ which occur as the edges $\{3,5\}$ and $\left\{2, \dfrac{n+3}{2}\right\}$ respectively. All other edges from the blue copy of K_n occur in orbits of these edges under the permutation $\epsilon = (0\ 1\ \dots\ n-1)$.

Now consider the red edges. One edge of each odd length occurs as an edge of the form $\{2i-1, 4i-2\}$, which has length $2i-1$ except for 1 which occurs in the edge $\{1,2\}$ of the triple (0, 1, 2) and $\dfrac{n-3}{2}$ which occurs in the edge $\left\{\dfrac{n-3}{2}, n-3\right\}$ while one edge of each even length occurs as an edge of the form $\{2i+1, 4i+1\}$ which has length $2i$ except for 2 and $\dfrac{n-1}{2}$ which occur as the edges $\{5,7\}$ and $\{1, 2m+2\}$ respectively.

Another red edge of each length congruent to 2 modulo 4 occurs as an edge of the form $\{0, 4i-2\}$, $1 < i \leq \left\lfloor \dfrac{n-3}{8} \right\rfloor$ except for 2 which occurs in the edge $\{0,2\}$. The remaining even red edges, which have lengths congruent to 0 modulo 4, occur as edges of the form $\{0, 4i+1\}$, $1 < i \leq \left\lfloor \dfrac{n-1}{8} \right\rfloor$, except for 4 which occurs in the edge $\{3,7\}$.

For the edges of the form $\{0, 4i-2\}$, when $\left\lfloor \dfrac{n+3}{8} \right\rfloor < i < \dfrac{n-1}{4}$ we have that $4i-2 > \dfrac{n-1}{2}$. We thus consider such edges to have length $n-4i+2$, which is less than or equal to $\frac{n-1}{2}$, and congruent to 3 modulo 4 since $n \equiv 1 \pmod 4$. This gives another copy of all red edge with lengths congruent to 3 modulo 4 except for 3 which occurs in the edge $\{0, n-3\}$. Similarly, the edges of the form $\{1, 4i+1\}$ with

$\left\lfloor \frac{n-1}{8} \right\rfloor < i < \frac{n-1}{4}$ have length $n - 4i$, which gives the remaining red edges with lengths congruent to 1 modulo 4 except for 1 which occurs in the edge $\{1, 2\}$ of the triple $\left\{1, 2, \frac{n+1}{2}\right\}$.

Furthermore, Table 3.2 directly gives $2\left(\frac{3(n-1)}{4}\right)$ edges. Each triangle in the table has exactly n triangles in its orbit under ϵ, so the triangles in the table together with their orbits under ϵ give all $\frac{3n(n-1)}{2}$ edges of $3K_n$.

Triangle with edge lengths	Vertices
(2, 1, 1)	(0, 1, 2)
(4, 2, 2)	(3, 5, 7)
$(4i-2,\ 2i-1,\ 2i-1)$	$(0,\ 2i-1,\ 4i-2)$
$(4i,\ 2i,\ 2i)$	$(1,\ 2i+1,\ 4i+1)$
$(3, \frac{n-3}{2}, \frac{n-3}{2})$	$(0, \frac{n-3}{2}, n-3)$
$(1, \frac{n-1}{2}, \frac{n-1}{2})$	$(2, \frac{n+1}{2}, 1)$

Table 3.2: Construction for perfect $T(\cup_2 P_2)$ triple systems

The construction for a perfect $T(\cup_2 P_2)$ triple system of order 9 is given in Table 3.3, where the double line separates the two sets of two vertex disjoint base triangles; these base triangles are illustrated in Figure 3.3. ■

Triangle with edge lengths	Vertices
(1, 2, 3)	(0, 1, 3)
(2, 3, 1)	(2, 4, 5)
(3, 2, 4)	(2, 5, 0)
(4, 4, 1)	(3, 7, 8)

Table 3.3: Construction for a perfect $T(\cup_2 P_2)$ triple system of order 9

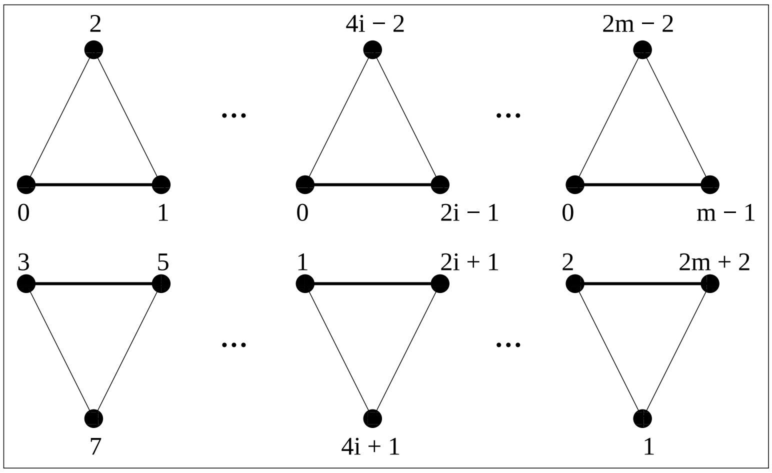

Figure 3.2: The base triangles for perfect $T(\cup_2 P_2)$ triple systems

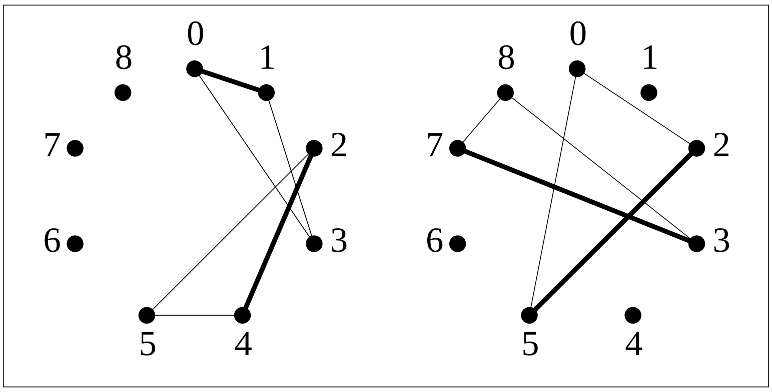

Figure 3.3: The base triangles for a perfect $T(\cup_2 P_2)$ triple system of order 9

3.2 Constructing cyclic perfect $T(P_2 \cup P_2 \cup P_2)$ triple systems

According to Proposition 2.2 the necessary conditions for the existence of a perfect $T(\cup_3 P_2)$ triple system are $n \equiv 1, 3 \pmod 6$ and $n \geq 9$. In this section we construct perfect $T(\cup_3 P_2)$ triple systems of all such orders n. Since n is odd, we use the notation introduced in Chapter 1 by letting $n = 2m + 1$.

We construct the desired decompositions as triple systems using design theory techniques. We give a set of base triangles, which together with their orbits under the permutation $\epsilon = (01 \ldots n)$, gives a perfect $T(\cup_3 P_2)$ triple system of order n. Note that although all of our constructions are cyclic, there may exist perfect $T(\cup_3 P_2)$ triple systems which are not.

The constructions are given in tables, with the first column indicating the edge lengths given by the vertices in the second column. The first length in the column on the left is the length of the blue edge, which we call our indexing edge length. If this length is odd then it is between the first two vertices in the column on the right; otherwise the length is even and it is between the first and third vertices in the second column.

We will consider two cases, when $n \equiv 1 \pmod 6$ and when $n \equiv 3 \pmod 6$. First, consider $n \equiv 1 \pmod 6$. Let $n = 6k + 1$ for some positive integer k. Then there are $3k$ distinct edge lengths that must be considered in a cyclic construction. In particular, since we wish to partition $3K_n$, each of these lengths must occur 3 times in our construction.

We will form triples of edge lengths of the form $(d_1,\ d_2,\ d_3)$ in the following manner. If the indexing difference d_1 is odd, then we set $d_2 = d_3 = \dfrac{n - d_1}{2}$. We call the triangles formed by such edges of differences d_1, d_2, d_3 'outer' triangles. If the indexing difference d_1 is even, then we set $d_2 = d_3 = \dfrac{d_1}{2}$. We call the triangles formed by these kinds of differences 'inner' triangles. We call the first difference in these triples, d_1, the **indexing difference**. If we consider the set of triples formed by letting d_1 vary from 1 to $3k$, we will see that in fact every length occurs exactly three times, and in particular only once as the first coordinate of the triple.

We will partition these triples into subsets of three triples such that each of these subsets will have indexing differences either of the form $6i+1$, $6i+2$, and $6i+3$, or of the form $6i+4$, $6i+5$, and $6i+6$. (Slightly different constructions will be necessary in both cases.) From these subsets of triples, we will form triangles with those edge lengths.

We do something very similar in the case $n \equiv 3 \pmod 6$, but there is a slight difference. Letting $n = 6k+3$, we see that this time, there are $3k+1$ distinct edge lengths. We will consider the triple $(1, 3k+1, 3k+1)$ as a special case, and again form triples as in the previous case, but this time we will form subsets by using the indexing differences $6i+2$, $6i+3$, and $6i+4$, and also the differences $6i+5$, $6i+6$, and $6i+7$. Then, as before, we will construct triangles with these edge lengths.

Lemma 3.3 *If $n \equiv 1 \pmod 6$ and $n \geq 13$ then there exists a cyclic perfect $T(\cup_3 P_2)$ triple system of order n.*

Proof: In this case we note that $m \equiv 0 \pmod 3$. Therefore we can arrange each of the m blue edge lengths into triples.

First we consider the case where the indexing edge length is congruent to 1, 2, or 3 modulo 6.

By restricting ourselves to the case $n \geq 13$ we know that $m \geq 6$ so that there exist blue edges with lengths in these congruence classes. Therefore, we can write the blue edge lengths as $6i+1$, $6i+2$, and $6i+3$ where $1 \leq i \leq \dfrac{n-13}{12}$ if $n \equiv 1 \pmod{12}$ and $1 \leq i \leq \dfrac{n-7}{12}$ when $n \equiv 7 \pmod{12}$. Since the largest possible difference in $3K_n$ is m, we deduce that $m \geq 6i+3$ and thus $n \geq 12i+7$.

The general construction for triangles with blue edges of lengths $6i+1$, $6i+2$, $6i+3$ is given in Table 3.4.

Triangle with edge lengths	Vertices
$(6i+1,\ m-3i,\ m-3i)$	$(0,\ 6i+1,\ m+3i+1)$
$(6i+2,\ 3i+1,\ 3i+1)$	$(1,\ 3i+2,\ 6i+3)$
$(6i+3,\ m-3i-1,\ m-3i-1)$	$(2,\ 6i+5,\ m+3i+4)$

Table 3.4: Construction for blue edges of lengths $6i+1$, $6i+2$, and $6i+3$ when $n \equiv 1 \pmod 6$

Since $n \geq 12i+7$ all of the positions of the vertices are distinct integers for $i \geq 1$. If $i = 0$ we use the triangles cyclically generated by the construction in Table 3.5.

Triangle with edge lengths	Vertices
$(1,\ m,\ m)$	$(0,\ 1,\ m+1)$
$(2,\ 1,\ 1)$	$(6,\ 7,\ 8)$
$(3,\ m-1,\ m-1)$	$(2,\ 5,\ m+4)$

Table 3.5: Constructions for blue edges of lengths 1, 2, and 3 when $n \equiv 1 \pmod 6$

The positions in Table 3.5 are distinct whenever $m + 1 \geq 9$ and thus for $n \geq 19$ using the fact that $n \equiv 1 \pmod 6$. The only missing case for $n \geq 9$ is $n = 13$. The generating triangles for this case are given in Table A.1 in Appendix A.

Taking the orbits of the appropriate triangles under the permutation $\epsilon = (0\ 1\ \ldots\ n\ -\ 1)$ gives all blue edges with lengths congruent to 1, 2, or 3 modulo 6.

Next we consider the case where the indexing edge length is congruent to 4, 5, or 0 modulo 6.

Once again, by restricting ourselves to the case $n \geq 13$ we know that there exist blue edge lengths in these congruence classes. This time we write the blue edge lengths as $6i + 4$, $6i + 5$, and $6i + 6$ where $1 \leq j \leq \dfrac{n-13}{12}$ if $n \equiv 1 \pmod{12}$ and $1 \leq i \leq \dfrac{n-19}{12}$ if $n \equiv 7 \pmod{12}$. Since $m \geq 6i + 6$, we conclude that $n \geq 12i + 13$. The general construction for these base triangles is given in Table 3.6.

As with the previous base triangles the positions are distinct for $i \geq 1$. If $i = 0$ use the constructions from Table 3.7.

The positions of the vertices in Table 3.7 are distinct if $m + 8 \geq 11$; this holds in all cases since $m \geq 4$. Again taking the orbits of the appropriate triangles under $\epsilon = (0\ 1\ \ldots\ n - 1)$ gives all blue edges with lengths congruent to 4, 5, or 0 modulo 6, so together with the previous construction we have the desired system. ■

Triangle with edge lengths	Vertices
$(6i+4,\ 3i+2,\ 3i+2)$	$(0,\ 3i+2,\ 6i+4)$
$(6i+5,\ m-3i-2,\ m-3i-2)$	$(2,\ 6i+7,\ m+3i+5)$
$(6i+6,\ 3i+3,\ 3i+3)$	$(3,\ 3i+6,\ 6i+9)$

Table 3.6: Construction for blue edges of lengths $6i+4$, $6i+5$, and $6i+6$ when $n \equiv 1 \pmod 6$

Triangle with edge lengths	Vertices
(4, 2, 2)	(0, 2, 4)
$(5,\ m-2,\ m-2)$	$(5,\ 10,\ m+8)$
(6, 3, 3)	(3, 6, 9)

Table 3.7: Construction for blue edges of lengths 4, 5, and 6 when $n \equiv 1 \pmod 6$

Lemma 3.4 *If $n \equiv 3 \pmod 6$ and $n \geq 9$ then there exists a cylic perfect $T(\cup_3 P_2)$ triple system of order n.*

Proof: Since $n \equiv 3 \pmod 6$, the largest minimum positive edge length is $m = \dfrac{n-1}{2} \equiv 1 \pmod 3$. We conclude that after splitting the blue edge lengths into triples we will still have one blue edge length left over. First, since $n \equiv 3 \pmod 6$, write $n = 2m+1 = 6t+3$ so that $m = 3t+1$. We then generate n edges of length 1 and $2n$ edges of length m using the base triangles with vertices at $(0, m, m+1)$, $(t, t+1, m+t+1)$, and $(2t+1, m+2t+1, m+2t+2)$ by applying the permutation $(0\ 1\ \ldots\ n-1)$ exactly $\dfrac{n}{3}$ times since these base triangles produce a short orbit. Note that these positions are distinct for $m \geq 4$, since $m = 3t+1$. The blue edges are between the first two vertices so that we produce exactly n blue edges of length 1.

We consider the case when the blue edge lengths are congruent to 2, 3, and 4 modulo 6.

We restrict ourselves to $n \geq 9$ so there exist blue edge lengths in these classes. This time we write the blue edge lengths as $6i+2$, $6i+3$, and $6i+4$ where $1 \leq i \leq \dfrac{n-15}{12}$ if $n \equiv 3 \pmod{12}$ and $1 \leq i \leq \dfrac{n-9}{12}$ if $n \equiv 9 \pmod{12}$. Certainly $m \geq 6j+4$ so $n \geq 12j+9$. The most general construction is given in Table 3.8.

The positions in Table 3.8 are again distinct for $i \geq 1$. For $i = 0$ the construction is given in Table 3.9.

Triangle with edge lengths	Vertices
$(6i+2,\, 3i+1,\, 3i+1)$	$(4,\, 3i+5,\, 6i+6)$
$(6i+3,\, m-3i-1,\, m-3i-1)$	$(0,\, 6i+3,\, m+3i+2)$
$(6i+4,\, 3i+2,\, 3i+2)$	$(1,\, 3i+3,\, 6i+5)$

Table 3.8: Construction for blue edges of lengths $6i+2$, $6i+3$, and $6i+4$ when $n \equiv 3 \pmod 6$

Triangle with edge lengths	Vertices
(2, 1, 1)	(1, 2, 3)
$(3,\, m-1,\, m-1)$	$(0,\, m-1,\, 2m-2)$
(4, 2, 2)	(4, 6, 8)

Table 3.9: Constructions for blue edge of lengths 2, 3, and 4 when $n \equiv 3 \pmod 6$

The positions in Table 3.9 are distinct for $m \geq 10$; the remaining cases are given in Tables A.2, A.3, and A.4 respectively, all of which are in Appendix A. Taking the orbits of the relevant triangles under the permutation $\epsilon = (0\ 1\ \ldots\ n-1)$ gives all of the blue edges with lengths congruent to 2, 3, or 4 modulo 6.

Next we consider when the blue edge lengths are congruent to 5, 0, and 1 modulo 6, when the length is bigger than 1.

If $n = 9$ then the only blue edges in these congruence classes are those of length 1 which we have already constructed. We thus restrict ourselves to looking at $n \geq 15$. We write the blue edge lengths as $6i + 5$, $6i + 6$, and $6i + 7$ where $1 \leq i \leq \dfrac{n-15}{12}$ if $n \equiv 3 \pmod{12}$ and $1 \leq i \leq \dfrac{n-21}{12}$ if $n \equiv 9 \pmod{12}$. In this case $m \geq 6i + 7$ so $n \geq 12i + 15$. The general construction for this case is given in Table 3.10.

Triangle with edge lengths	Vertices
$(6i+5,\ m-3i-2,\ m-3i-2)$	$(0,\ 6i+5,\ m+3i+3)$
$(6i+6,\ 3i+3,\ 3i+3)$	$(1,\ 3i+4,\ 6i+7)$
$(6i+7,\ m-3i-3,\ m-3i-3)$	$(2,\ 6i+9,\ m+3i+6)$

Table 3.10: Construction for blue edges of lengths $6i + 5$, $6i + 6$, and $6i + 7$ when $n \equiv 3 \pmod 6$

The positions in Table 3.10 are distinct for $i \geq 1$; for $i = 0$ use Table 3.11.

Triangle with edge lengths	Vertices
$(5,\ m-2,\ m-2)$	$(0,\ 5,\ m+3)$
$(6,\ 3,\ 3)$	$(1,\ 4,\ 7)$
$(7,\ m-3,\ m-3)$	$(2,\ 9,\ m+6)$

Table 3.11: Construction for blue edges of lengths 5, 6, and 7 when $n \equiv 3 \pmod 6$

These positions are distinct for $m+3 \geq 10$ and hence for $m \geq 7$. For $n = 9$ use Table A.2 and for $n = 15$ use Table A.5, both of which are in Appendix A.

Again taking the orbits of the relevant triangles under ϵ gives all of the blue edges whose edge lengths are congruent to 5, 0, or 1 modulo 6 when they are greater than 1; together with the construction for other edge length congruencies this gives all desired blue edges. ■

Note that part of the construction for $n = 9$ was illustrated in Figure 1.15.

The combined results of Lemmas 3.3 and 3.4 constitute a proof for the following theorem.

Theorem 3.5 *A cyclic perfect $T(\cup_3 P_2)$ triple system of order n exists if and only if $n \equiv 1, 3 \pmod 6$ and $n \geq 9$.*

Chapter 4

Constructing cylic perfect $T(\cup_\lambda P_2)$ triple systems for $\lambda = 2\omega$ and $n \equiv 1 \pmod{2\lambda}$

The main problem we wish to consider is perfectly decomposing $3K_n$ into $T(\cup_\lambda P_2)$ triples when $\lambda \equiv 0 \pmod 2$ and $n \equiv 1 \pmod{2\lambda}$. Although Proposition 2.2 potentially gives many values of n for which such a decomposition exists, we consider only $n \equiv 1 \pmod{2\lambda}$ for two important reasons: $n \equiv 1 \pmod{2\lambda}$ is a solution to $n(n-1) \equiv 0 \pmod{2\lambda}$ and $n \equiv 1 \pmod 2$ for any value of λ and it implies that $\frac{n-1}{2} \equiv 0 \pmod{\lambda}$. This second fact greatly simplifies our approach to the problem, since it means we can group the $\frac{n-1}{2}$ distinct blue edge lengths into sets of λ. In particular, we will divide the blue edge lengths into the sets $\{1, 2, \ldots, \lambda\}$, $\{\lambda+1, \lambda+2, \ldots, 2\lambda\}$, ..., $\left\{\frac{n-2\lambda+1}{2}, \ldots, \frac{n-1}{2}\right\}$. Note furthermore that together $\lambda = 2\omega$ and $n \equiv 1 \pmod{2\lambda}$ imply that $n \equiv 1 \pmod{4\omega}$.

The number of edges in the matching is determined $\lambda = 2\omega$. As always, let $n = 2m+1$. If n is odd, then the largest edge length is $\frac{n-1}{2}$, or m. All of the edges from $3K_n$ thus have lengths from the set $\{1, 2, \ldots, m\}$. If the edge $\{u, v\}$ in $E(3K_n)$ has length d, then we can obtain $n-1$ other edges of length d by taking the orbit of the edge $\{u, v\}$ under the permutation $(0\ 1\ \ldots\ n-1)$ to obtain the edges $\{u+1, v+1\}$, $\{u+2, v+2\}$, ..., $\{u+n-1, v+n-1\}$ modulo n respectively. It is therefore possible to take one edge of each length from 1 to m and cyclically generate K_n.

We start with one edge of each length $1, 2, \ldots, m$, which will be the blue edge length, and then set up triangles (d_1, d_2, d_3) in the same manner as in Section 3.3. Clearly, there will be at least $\frac{n-1}{2\lambda}$ distinct base triangles.

We denote by $L_{\lambda,m}(\rho\lambda+1, (\rho+1)\lambda)$ the set of edge lengths associated with the indexing edge lengths $\rho\lambda+1, \rho\lambda+2, \ldots, (\rho+1)\lambda$, where $0 \leq \rho \leq \frac{n-1}{2\lambda}$. For example, $L_{8,64}(17, 24)$ refers to the edge lengths given in Table 4.1, where the first edge length in each triple is for a blue edge.

Our goal becomes to find triples of distinct vertices such that the edge lengths

Triangle with edge lengths
(17, 61, 61)
(18, 9, 9)
(19, 60, 60)
(20, 10, 10)
(21, 59, 59)
(22, 11, 11)
(23, 58, 58)
(24, 12, 12)

Table 4.1: The triples of edge lengths referenced by $L_{8,64}(17, 24)$

between them are given by $L_{\lambda,m}(\rho\lambda+1,(\rho+1)\lambda)$, for $0 \leq \rho \leq \dfrac{n-4\omega-1}{4\omega}$.

4.1 Constructions for $\rho \geq 2$

Lemma 4.1 *If $\lambda \geq 4$ is even, $n \equiv 1 \pmod{2\lambda}$, and $\rho \geq 2$ then there exist λ base triangles with edges of the lengths in $L_{\lambda,m}(\lambda\rho+1, \lambda(\rho+1))$ such that the triangles are vertex disjoint.*

PROOF: The general constructions are given in Table 4.2, where $0 \leq i \leq \omega - 1$ and $2 \leq \rho \leq \dfrac{n-4\omega-1}{4\omega}$.

Triangle with edge lengths	Vertices
$(2\omega\rho+2i+1, m-\omega\rho-i, m-\omega\rho-i)$	$(\omega+i+1,\ 2\omega\rho+\omega+3i+2,\ m+\omega\rho+\omega+2i+2)$
$(2\omega\rho+2i+2, \omega\rho+i+1, \omega\rho+i+1)$	$(i,\ \omega\rho+2i+1,\ 2\omega\rho+3i+2)$

Table 4.2: Primary Construction for $\rho \geq 2$ and $\omega \equiv 1, 2 \pmod 3$

The bulk of the proof is to show that the vertices are distinct for any values of i when ρ is fixed. Rather than showing that positions are not equivalent modulo n, we use the following 6 conditions, the first of which shows that all of the positions are least nonnegative residues modulo n. The remaining conditions show that each set of positions is strictly smaller than those with the next largest coefficient. Note that these bounds are stronger than absolutely necessary, but they suffice for our purposes.

The two sets of positions with the smallest coefficients are different since $i \leq \omega - 1 < \omega + 1 \leq w + i + 1$.

The set of positions written as $\omega + i + 1$ are different from those identified as $\omega\rho + 2i + 1$ since $\omega + i + 1 \leq 2\omega < \omega\rho + 1 \leq \omega\rho + 2i + 1$ whenever $\rho \geq 2$.

The next biggest sets of positions are distinct since $\omega\rho + 2i + 1 \leq \omega\rho + 2\omega - 1 < 2\omega\rho + 2 \leq 2\omega\rho + 3i + 2$ whenever $\rho \geq 2$. Note that also $\omega\rho + 2i + 1 \leq \omega\rho + 2\omega - 1 < 2\omega\rho + \omega + 2 \leq 2\omega\rho + \omega + 3i + 2$.

For the largest positions we have that $2\omega\rho + \omega + 3i + 2 \leq 2\omega\rho + 4\omega - 1 < m + \omega\rho + \omega + 2 \leq m + \omega\rho + \omega + 2i + 2$ as well as $2\omega\rho + \omega + 3i + 2 \leq 2\omega\rho + 4\omega - 1 < m + \omega\rho + \omega + 2 \leq m + \omega\rho + \omega + 2i + 2$ whenever $\rho \geq 1$.

The family of positions with the largest coefficient is $m + \omega\rho + \omega + 2i + 2$. It can be be placed in at worst the largest vertex, $2m$, since $m + \omega\rho + \omega + 2i + 2 \leq m + \omega\rho + 3\omega + 2 \leq 2m$ whenever $\rho \geq 1$. Since we have already shown that all of the other positions are smaller, this means that all of the positions are at most n.

Since we have already shown that all of the vertices are least nonnegative residues, we need only show that $2\omega\rho + \omega + 3i + 2 \not\equiv 2\omega\rho + 3j + 2 \pmod 3$ which holds whenever $\omega \equiv 1, 2 \pmod 3$. In the case $\omega \equiv 0 \pmod 3$ we use the modified construction given in Table 4.3; note that all of the constraints for the construction in Table 4.2 are strong enough for this one as well.

Triangle with edge lengths	Vertices
$(2\omega\rho+2i+1, m-\omega\rho-i, m-\omega\rho-i)$	$(\omega+i, 2\omega\rho+\omega+3i+1, m+\omega\rho+\omega+2i+1)$
$(2\omega\rho+2i+2, \omega\rho+i+1, \omega\rho+i+1)$	$(i, \omega\rho+2i+1, 2\omega\rho+3i+2)$

Table 4.3: Primary construction modified for $\rho \geq 2$ and $\omega \equiv 0 \pmod 3$

We conclude that for the positions of a given set of base triangles (that is, a fixed value of ρ), we have disjoint 3-cycles. It should be clear that the edges from this construction correspond to $L_{\lambda,m}(\rho\lambda+1,(\rho+1)\lambda)$. ■

Returning to the constraints from above, the case $\omega = 1$ was solved by Billington and Lindner; therefore to complete the proof we need only show the cases $\rho = 0$ and $\rho = 1$.

4.2 Constructions for $\rho = 1$

Our first construction for $\rho = 1$ involves splitting the blue edges of even length, placing the small ones in the middle and the large ones at the end while keeping all of the blue edges of odd length together.

Triangle with edge lengths	Vertices
(17, 56, 56)	(0, 17, 73)
(19, 55, 55)	(1, 20, 75)
(21, 54, 54)	(2, 23, 77)
(23, 53, 53)	(3, 26, 79)
(25, 52, 52)	(4, 29, 81)
(27, 51, 51)	(5, 32, 83)
(29, 50, 50)	(6, 35, 85)
(31, 49, 49)	(7, 38, 87)
(18, 9, 9)	(39, 48, 57)
(20, 10, 10)	(40, 50, 60)
(22, 11, 11)	(41, 52, 63)
(24, 12, 12)	(42, 54, 66)
(26, 13, 13)	(88, 101, 114)
(28, 14, 14)	(89, 103, 117)
(30, 15, 15)	(90, 105, 120)
(32, 16, 16)	(91, 107, 123)

Table 4.4: Example of the first construction for $\rho = 1$ with $n = 129$ and $\lambda = 16$

Lemma 4.2 *If $\lambda \geq 4$ is even, $n \equiv 1 \pmod{2\lambda}$, and $\rho = 1$ then for $n > 15\omega + 1$ there exist λ base triangles with edges of the lengths in $L_{\lambda,m}(\lambda+1, 2\lambda)$ such that the triangles are vertex disjoint.*

Proof: The construction here is given in Table 4.5 where $0 \leq i \leq \omega - 1$, $0 \leq j \leq \left\lfloor \frac{\omega}{2} \right\rfloor - 1$, and $0 \leq k \leq \omega - \left\lfloor \frac{\omega}{2} \right\rfloor - 1$.

Triangle with edge lengths	Vertices
$(2\omega+2i+1,\, m-\omega-i,\, m-\omega-i)$	$(i,\, 2\omega+3i+1,\, m+\omega+2i+1)$
$(2\omega+2j+2,\, \omega+j+1,\, \omega+j+1)$	$(5\omega+j-1,\, 6\omega+2j,\, 7\omega+3j+1)$
$\left(2\omega+2\left\lfloor \frac{\omega}{2} \right\rfloor+2k+2,\, \omega+\left\lfloor \frac{\omega}{2} \right\rfloor+k+1,\, \omega+\left\lfloor \frac{\omega}{2} \right\rfloor+k+1\right)$	$\left(m+3\omega+k,\, m+4\omega+\left\lfloor \frac{\omega}{2} \right\rfloor+2k+1,\, m+5\omega+2\left\lfloor \frac{\omega}{2} \right\rfloor+3k+2\right)$

Table 4.5: Construction for $\rho = 1$

We prove that the triangles given in Table 4.5 are vertex disjoint by arguing that each set of positions (with variable i, j, k) are distinct from all others by establishing a chain of inequalities. We compare the largest of one set of positions with the smallest of those with the next biggest coefficient.

- $0 \leq i \leq \omega - 1 < 2\omega + 1 \leq 2\omega + 3i + 1$
- $2\omega + 3i + 1 \leq 5\omega - 2 < 5\omega - 1 \leq 5\omega + j - 1$

- $5\omega + j - 1 \leq 5\omega + \left\lfloor \frac{\omega}{2} \right\rfloor - 2 \leq \frac{11\omega}{2} - 2 < 6\omega \leq 6\omega + 2j$
- $6\omega + 2j \leq 6\omega + 2\left\lfloor \frac{\omega}{2} \right\rfloor \leq 7\omega < 7\omega + 1 \leq 7\omega + 3j + 1$
- $7\omega + 3j + 1 \leq 7\omega + 3\left\lfloor \frac{\omega}{2} \right\rfloor + 1 \leq \frac{17\omega}{2} + 1 < m + \omega + 1 \leq m + \omega + 2i + 1$ as long as $n > 15\omega + 1$
- $m + \omega + 2i + 1 \leq m + 3\omega - 1 < m + 3\omega \leq m + 3\omega + k$
- $m + 3\omega + k \leq m + 4\omega - \left\lfloor \frac{\omega}{2} \right\rfloor - 1 < m + 4\omega + \left\lfloor \frac{\omega}{2} \right\rfloor + 1 \leq m + 4\omega + \left\lfloor \frac{\omega}{2} \right\rfloor + 2k + 1$
- $m + 4\omega + \left\lfloor \frac{\omega}{2} \right\rfloor + 2k + 1 \leq m + \frac{11\omega}{2} - 1 < m + 6\omega + 2 \leq m + 5\omega + 2\left\lfloor \frac{\omega}{2} \right\rfloor + 3k + 2$
- $m + 5\omega \left\lfloor \frac{\omega}{2} \right\rfloor + 3k + 2 \leq m + 8\omega - \left\lfloor \frac{\omega}{2} \right\rfloor - 3 \leq m + \frac{15\omega}{2} - 3 \leq 2m$ provided that $n \geq 15\omega - 5$

Based on these observations, we conclude that for $n > 15\omega + 1$ the construction in Table 4.5 gives vertex-disjoint triangles.■

Hence for $\rho = 1$ we only need to consider $6\omega \leq n \leq 15\omega + 1$. However, once again noting that $\lambda = 2\omega$ and $n \equiv 1 \pmod{2\lambda} \equiv 1 \pmod{4\omega}$, we can further restrict ourselves to the cases $n = 12\omega + 1$ and $n = 8\omega + 1$.

It turns out that the case $n = 8\omega + 1$ is the most difficult since it provides the least amount of space for placing the triangles. We believe that perfect $T(\cup_\lambda P_2)$ triple system exist for such orders, as we have completely solved the cases $\lambda = 4, 6, 8$ in

Appendix B and have some general constructions for the case $n = 8\omega + 1$ and $\rho = 1$ in Appendix C.

4.2.1 Constructions for $\rho = 1$ and $n = 12\omega + 1$

Consider the construction for $\omega = 4$, $n = 12\omega + 1$, and $n = 49$ given in Table 4.6. The idea is to separate the edge lengths based on the congruence of the blue edge modulo 4. The differences in the same class as $2\omega + 1$ are placed in the earliest positions. The other odd congruence class is placed in the next available positions so that the largest positions from this class and the previous are not congruent modulo 3. One even congruence class is placed at the end and over the first set of positions; the other is placed after all of the other second positions, so that their third positions are in the unused congruence class modulo 3. In order for this construction to work in general, we need to determine how many spaces to leave between each congruence class.

Lemma 4.3 *If λ is even, $n \equiv 1 \pmod{2\lambda}$, and $\rho = 1$ then for $n = 12\omega + 1$ there exist λ base triangles with edges of the lengths in $L_{\lambda,m}(\lambda + 1, 2\lambda)$ such that the triangles are vertex disjoint.*

Proof: If $\omega \equiv 1, 5 \pmod 6$, we use the construction given in Table 4.7, where $0 \leq i \leq \dfrac{\omega - 1}{2}$ and $0 \leq j \leq \dfrac{\omega - 3}{2}$.

Triangle with edge lengths	Vertices
(9, 20, 20)	(0, 20, 40)
(10, 5, 5)	(26, 31, 36)
(11, 19, 19)	(3, 22, 41)
(12, 6, 6)	(48, 5, 11)
(13, 18, 18)	(1, 19, 37)
(14, 7, 7)	(25, 32, 39)
(15, 17, 17)	(4, 21, 38)
(16, 8, 8)	(47, 6, 14)

Table 4.6: Construction for $n = 49$, $\omega = 4$ and $\rho = 1$

Triangle with edge lengths	Vertices
$(2\omega + 4i + 1,\ 5\omega - 2i,\ 5\omega - 2i)$	$(i,\ 5\omega - i,\ 10\omega - 3i)$
$(2\omega + 4i + 2,\ \omega + 2i + 1,\ \omega + 2i + 1)$	$(6\omega - i + 1,\ 7\omega + i + 2,\ 8\omega + 3i + 3)$
$(2\omega + 4j + 3,\ 5\omega - 2j - 1,\ 5\omega - 2j - 1)$	$\left(\frac{\omega+1}{2} + j,\ 5\omega + \frac{\omega-1}{2} - j,\ 10\omega + \frac{\omega-3}{2} - 3j\right)$
$(2\omega + 4j + 4,\ \omega + 2j + 2,\ \omega + 2j + 2)$	$(12\omega - j - 1,\ \omega + j,\ 2\omega + 3j + 2)$

Table 4.7: Construction for $n = 12\omega + 1$, $\omega \equiv 1, 5 \pmod 6$, $\rho = 1$

We prove that the triangles given in Table 4.7 are vertex disjoint by arguing that each set of positions (with variable i, j, k) are distinct from all others by establishing a chain of inequalities. We compare the largest of one set of positions with the smallest of those with the next biggest coefficient. This case has an extra subtlety. Several of the positions have the same coefficient, so we must prove that they are not congruent modulo 3. Furthermore, in order to maintain the chain of inequalities which shows that all of the positions are less than or equal to $2m$, we need to show that the set of positions with the next smallest coefficient is small than all of them, and that they are smaller than the set of positions with the next largest coefficient.

- $0 \leq i \leq \frac{\omega-1}{2} < \frac{\omega+1}{2} \leq \frac{\omega+1}{2} + j \leq \omega - 1 < \omega \leq \omega + j$
- $\omega + j \leq \frac{3\omega-3}{2} < 2\omega + 2 \leq 2\omega + 3j + 2 \leq \frac{7\omega-9}{2} < \frac{9\omega-1}{2} < 5\omega - i$
- $5\omega - i \leq 5\omega < 5\omega + 1 \leq 5\omega + \frac{\omega-1}{2} - j$
- $5\omega + \frac{\omega-1}{2} - j \leq \frac{11\omega-1}{2} < \frac{11\omega+3}{2} \leq 6\omega - i + 1 \leq 6\omega + 1$
- $6\omega - i + 1 \leq 6\omega + 1 < 7\omega + 2 \leq 7\omega + i + 2 \leq \frac{15\omega+3}{2} < 8\omega + 3 \leq 8\omega + 3i + 3$
- 1. $7\omega + i + 2 \leq \frac{15\omega+3}{2} < 8\omega + 3 \leq 8\omega + 3i + 3$
  2. $7\omega + i + 2 \leq \frac{15\omega+3}{2} < \frac{17\omega-7}{2} \leq 10\omega - 3i$
  3. $7\omega + i + 2 \leq \frac{15\omega+3}{2} < 9\omega - 3 \leq 10\omega + \frac{\omega-3}{2} - 3j$

- 1. $8\omega + 3 + 3i \not\equiv 10\omega - 3i \pmod 3$

  2. $8\omega + 3 + 3i \not\equiv 10\omega + \frac{\omega - 3}{2} - 3j \pmod 3$

  3. $10\omega - 3i - 4 \not\equiv 10\omega + \frac{\omega - 3}{2} - 3j \pmod 3$

- 1. $8\omega + 3i + 6 \leq \frac{19\omega + 3}{2} < \frac{23\omega + 1}{2} \leq 12\omega - 1 - i$

  2. $10\omega - 3i \leq 10\omega < \frac{23\omega + 1}{2} \leq 12\omega - 1 - i$

  3. $10\omega + \frac{\omega - 3}{2} - 3j \leq \frac{21\omega - 3}{2} < \frac{23\omega + 1}{2} \leq 12\omega - 1 - i$

- $12\omega - i - 1 \leq 12\omega - 1 < 12\omega$

Based on these observations, we conclude that for $n > 15\omega + 1$ the construction in Table 4.7 gives vertex-disjoint triangles.

The same core construction will work for $\omega \equiv 3 \pmod 6$ by making the small adjustments due to parity given in Table 4.8 where once again $0 \leq i \leq \frac{\omega - 1}{2}$ and $0 \leq j \leq \frac{\omega - 3}{2}$.

If $\omega \equiv 0, 4 \pmod 6$, then we use the construction given in Table 4.9, where $0 \leq i \leq \frac{\omega - 1}{2}$.

Again, a few small adjustments allow us to use the same core construction for $\omega \equiv 2 \pmod 6$, as given in Table 4.10.■

Triangle with edge lengths	Vertices
$(2\omega + 4i + 1,\ 5\omega - 2i,\ 5\omega - 2i)$	$(i,\ 5\omega - i,\ 10\omega - 3i)$
$(2\omega + 4i + 2,\ \omega + 2i + 1,\ \omega + 2i + 1)$	$(6\omega - i + 3,\ 7\omega + i + 4,\ 8\omega + 3i + 5)$
$(2\omega + 4i + 3,\ 5\omega - 2i - 1,\ 5\omega - 2i - 1)$	$\left(\frac{\omega + 3}{2} + i,\ 5\omega + \frac{\omega + 1}{2} - i,\ 10\omega + \frac{\omega - 1}{2} - 3i\right)$
$(2\omega + 4i + 4,\ \omega + 2i + 2,\ \omega + 2i + 2)$	$(12\omega - i,\ \omega + i + 1,\ 2\omega + 3i + 3)$

Table 4.8: Construction for $n = 12\omega + 1$, $\omega \equiv 3 \pmod 6$, $\rho = 1$

Triangle with edge lengths	Vertices
$(2\omega + 4i + 1,\ 5\omega - 2i,\ 5\omega - 2i)$	$(i,\ 5\omega - i,\ 10\omega - 3i)$
$(2\omega + 4i + 2,\ \omega + 2i + 1,\ \omega + 2i + 1)$	$(6\omega - i + 2,\ 7\omega + i + 3,\ 8\omega + 3i + 4)$
$(2\omega + 4i + 3,\ 5\omega - 2i - 1,\ 5\omega - 2i - 1)$	$\left(\frac{\omega + 2}{2} + i,\ 5\omega + \frac{\omega}{2} - i,\ 10\omega + \frac{\omega - 2}{2} - 3i\right)$
$(2\omega + 4i + 4,\ \omega + 2i + 2,\ \omega + 2i + 2)$	$(12\omega - i,\ \omega + i + 1,\ 2\omega + 3i + 3)$

Table 4.9: Construction for $n = 12\omega + 1$, $\omega \equiv 0, 4 \pmod 6$, $\rho = 1$

Triangle with edge lengths	Vertices
$(2\omega + 4i + 1,\ 5\omega - 2i,\ 5\omega - 2i)$	$(i,\ 5\omega - i,\ 10\omega - 3i)$
$(2\omega + 4i + 2,\ \omega + 2i + 1,\ \omega + 2i + 1)$	$\left(6\omega + \frac{\omega}{2} - i,\ 7\omega + \frac{\omega}{2} + i + 1,\ 8\omega + \frac{\omega}{2} + 3i + 2\right)$
$(2\omega + 4i + 3,\ 5\omega - 2i - 1,\ 5\omega - 2i - 1)$	$(\omega + i + 1,\ 6\omega - i,\ 11\omega - 3i - 1)$
$(2\omega + 4i + 4,\ \omega + 2i + 2,\ \omega + 2i + 2)$	$(\omega - i - 1,\ 2\omega + i + 1,\ 3\omega + 3i + 3)$

Table 4.10: Construction for $n = 12\omega + 1$, $\omega \equiv 2 \pmod 6$, $\rho = 1$

4.3 Constructions for $\rho = 0$

The $\rho = 0$ case is special because it involves placing the blue edges with the largest and smallest edge lengths, which poses unique challenges. Our strategy is to place the largest blue edge lengths in such a way that it leaves two significant gaps to accommodate the smallest blue edges in groupings of odd and even edge lengths.

Lemma 4.4 *If $\lambda \geq 4$ is even, $n \equiv 1 \pmod{2\lambda}$, and $\rho = 0$ then there exist λ base triangles with edges of the lengths in $L_{\lambda,m}(1, \lambda)$ such that the triangles are vertex disjoint.*

Proof: If ω is odd, we use the construction in Table 4.11 where $0 \leq i \leq \frac{\omega - 1}{2}$ and $0 \leq j \leq \frac{\omega - 3}{2}$. In this construction, the largest gap starts at position $\frac{\omega + 1}{2}$, ends at position $m - \frac{5\omega}{2} + \frac{1}{2}$, and has length $m - 3\omega + 1$. There is a secondary gap which starts at position $m+1$, ends at position $2m - \frac{7\omega}{2} + \frac{5}{2}$, and has length $m - \frac{7\omega - 5}{2}$.

Triangle with edge lengths	Vertices
$(4i+1,\ m-2i,\ m-2i)$	$(i,\ m-i,\ 2m-3i)$
$(4j+3,\ m-2j-1,\ m-2j-1)$	$(2m-2\omega+j+2,\ m-2\omega-j,\ 2m-2\omega-3j-1)$

Table 4.11: Construction for $\rho = 0$ and $\omega \equiv 1 \pmod 2$

We prove that the triangles given in Table 4.11 are vertex disjoint by arguing that each set of positions (with variable i, j, k) are distinct from all others by establishing a chain of inqualities. We compare the largest of one set of positions with the smallest of those with the next biggest coefficient.

- $i \leq \frac{\omega - 1}{2} < \frac{2m - 6\omega + 3}{2} \leq m - 2\omega - j$
- $m - 2\omega - j \leq m - 2\omega < \frac{2m - \omega + 1}{2} \leq m - i$
- $m - i \leq m < \frac{4m - 7\omega + 7}{2} \leq 2m - 2\omega - 3j - 1$
- $2m - 2\omega - 3j - 1 \leq 2m - 2\omega - 1 < 2m - 2\omega + 2 \leq 2m - 2\omega + j + 2$
- $2m - 2\omega + j + 2 \leq \frac{4m - 3\omega + 1}{2} < \frac{4m - 3\omega + 3}{2} \leq 2m - 3i \leq 2m$

Based on these observations, we conclude that for $n > 15\omega + 1$ the construction in Table 4.11 gives vertex-disjoint triangles.

Otherwise, ω is even and we use the construction in Table 4.12 where $0 \leq i \leq \frac{\omega}{2} - 1$. This time, the largest gap begins at position $\frac{\omega}{2}$, ends at position $m - \frac{5\omega}{2} + 1$, and has length $m - 3\omega + 2$. The secondary gap begins at position $m + 1$, ends at position $2m - \frac{7\omega}{2} + 2$, and has length $m - \frac{7\omega}{2} + 2$.

Triangle with edge lengths	Vertices
$(4i+1,\, m-2i,\, m-2i)$	$(i,\, m-i,\, 2m-3i)$
$(4i+3,\, m-2i-1,\, m-2i-1)$	$(2m-2\omega+i+3,\, m-2\omega-i+1,$ $2m-2\omega-3i)$

Table 4.12: Construction for $\rho = 0$ and $\omega \equiv 0 \pmod 2$

So far, we have constructed vertex disjoint triangles such that all of the blue edges with odd lengths from 1 to $2\omega - 1$ occur exactly once and all of the red edges with lengths from $m - 2\omega + 1$ to m occur exactly twice. In order to complete construction of a $L_{\lambda,m}(1, \lambda)$, we must construct vertex disjoint triangles using all of the blue edges with even lengths from 2 to $2\omega - 1$ exactly once and all of the red edge lengths from 1 to ω exactly twice. Note that these triangles need to be vertex disjoint but also have vertices which are distinct from those in Table 4.11 or Table 4.12 depending on whether ω is even or odd.

Now for the blue edges with even length, we introduce recursive constructions to fit them into the two gaps. In fact, our constructions in this case can be viewed as extended odd and even Skolem-type sequences of multiplicity 3 (which may have more than the 'minimum' number of possible zeros). We denote the sequences with even entries by E_α and those with odd entries by O_α, where α is the largest term to be placed.

The construction for evens is given in Table 4.13 where α is even, $0 \leq i \leq \left\lfloor \frac{\alpha - 1}{3} \right\rfloor$, $0 \leq j \leq \frac{\alpha}{2} - \left\lfloor \frac{\alpha}{18} \right\rfloor - \left\lfloor \frac{\alpha - 1}{3} \right\rfloor - 2$, and $0 \leq k \leq \left\lfloor \frac{\alpha}{18} \right\rfloor - 1$. An example of a sequence constructed in this way is given in Example 4.5.

Example 4.5 The following sequence is a generalized extended even Skolem-type sequence of order 8 and multiplicity 3 constructed using Table 4.13.

12 10 8 6 0 0 0 0 0 6 8 10 12 0 0 6 0 0 8 0 0 10 4 0 12 2 4 2 0 2 4

We prove that the triangles given in Table 4.13 are vertex disjoint by arguing that each set of positions (with variable i, j, k) are distinct from all others by establishing a chain of inequalities. We compare the largest of one set of positions with the smallest of those with the next biggest coefficient. Once again, several of the positions have the same coefficient, so we must prove that they are not congruent modulo 3. Thereafter,

Triangle with edge lengths	Vertices
$(2\alpha - 4i, \alpha - 2i, \alpha - 2i)$	$(i, \alpha - i, 2\alpha - 3i)$
$\Big(2\alpha - 4\left\lfloor\frac{\alpha-1}{3}\right\rfloor - 4j - 4,$ $\alpha - 2\left\lfloor\frac{\alpha-1}{3}\right\rfloor - 2j - 2,$ $\alpha - 2\left\lfloor\frac{\alpha-1}{3}\right\rfloor - 2j - 2\Big)$	$\Big(\frac{\alpha}{2} + 3\left\lfloor\frac{\alpha}{18}\right\rfloor + 3\left\lfloor\frac{\alpha-1}{3}\right\rfloor\Big) + 3j + 7,$ $3\Big(\frac{\alpha}{2} + \left\lfloor\frac{\alpha}{18}\right\rfloor\Big) + \left\lfloor\frac{\alpha-1}{3}\right\rfloor + j + 5,$ $\frac{5\alpha}{2} + 3\left\lfloor\frac{\alpha}{18}\right\rfloor - \left\lfloor\frac{\alpha-1}{3}\right\rfloor - j + 3\Big)$
$\Big(4\left\lfloor\frac{\alpha}{18}\right\rfloor - 2k, 2\left\lfloor\frac{\alpha}{18}\right\rfloor - k, 2\left\lfloor\frac{\alpha}{18}\right\rfloor - k\Big)$	Placed in an $E_{2\lfloor\frac{\alpha}{18}\rfloor}$ beginning at position $\left\lfloor\frac{\alpha-1}{3}\right\rfloor + 1$.

Table 4.13: Even Skolem-type Construction

in order to maintain the chain of inequalities we need to show that the set of positions with the next smallest coefficient is smaller than all of them, and that they are smaller than the set of positions with the next largest coefficient.

The positions with the smallest coefficients are distinct since $0 \leq i \leq \left\lfloor \frac{\alpha-1}{3} \right\rfloor \leq \frac{\alpha-1}{3} < \frac{2\alpha+1}{3} = \alpha - \frac{\alpha-1}{3} \leq \alpha - \left\lfloor \frac{\alpha-1}{3} \right\rfloor \leq \alpha - i$.

The next largest sets of positions are also distinct because $\alpha - i \leq \alpha < \alpha + 1 = 2\alpha - 3\Big(\frac{\alpha-1}{3}\Big) \leq 2\alpha - 3\left\lfloor \frac{\alpha-1}{3} \right\rfloor \leq 2\alpha - 3i$.

Since $2\alpha - 3\left\lfloor \frac{\alpha-1}{3} \right\rfloor \leq 2\alpha - 3i$ and $\frac{\alpha}{2} + 3\left\lfloor \frac{\alpha}{18} \right\rfloor + 3\left\lfloor \frac{\alpha-1}{3} \right\rfloor + 3j + 7$ have similar coefficients we also require that $\alpha - i \leq \alpha < \frac{5\alpha}{3} \leq \frac{\alpha}{2} + 3\left\lfloor \frac{\alpha}{18} \right\rfloor + 3\left\lfloor \frac{\alpha-1}{3} \right\rfloor + 3j + 7$.

By rewriting $\frac{\alpha}{2} + 3\left\lfloor \frac{\alpha}{18} \right\rfloor + 3\left\lfloor \frac{\alpha-1}{3} \right\rfloor + 3j + 7$ as $2\alpha - 3(\frac{\alpha}{2} - \left\lfloor \frac{\alpha}{18} \right\rfloor - \left\lfloor \frac{\alpha-1}{3} \right\rfloor) + 7 + 3j$ it should become clear that $2\alpha - 3i \neq 2\alpha + 3\Big(\frac{\alpha}{2} - \left\lfloor \frac{\alpha}{18} \right\rfloor - \left\lfloor \frac{\alpha-1}{3} \right\rfloor\Big) + 7 + 3j \pmod 3$.

By using the facts that $\frac{\alpha-17}{18} \leq \left\lfloor \frac{\alpha}{18} \right\rfloor$ and $\frac{\alpha-3}{3} \leq \left\lfloor \frac{\alpha-1}{3} \right\rfloor$ we have that $2\alpha - 3i \leq 2\alpha < 2\alpha + 2 \leq \frac{3\alpha}{2} + 3\left\lfloor \frac{\alpha}{18} \right\rfloor + \left\lfloor \frac{\alpha-1}{3} \right\rfloor + j + 5$ as well as $\frac{\alpha}{2} + 3\left\lfloor \frac{\alpha}{18} \right\rfloor + 3\left\lfloor \frac{\alpha-1}{3} \right\rfloor + 3j + 7 \leq 2\alpha + 1 < 2\alpha + 2 \leq \frac{3\alpha}{2} + 3\left\lfloor \frac{\alpha}{18} \right\rfloor + \left\lfloor \frac{\alpha-1}{3} \right\rfloor + j + 5$.

Next we wish to argue that the sets of positions $3\Big(\frac{\alpha}{2} + \left\lfloor \frac{\alpha}{18} \right\rfloor\Big) + \left\lfloor \frac{\alpha-1}{3} \right\rfloor + j + 5$ and $\frac{5\alpha}{2} + 3\left\lfloor \frac{\alpha}{18} \right\rfloor - \left\lfloor \frac{\alpha-1}{3} \right\rfloor - j + 3$ are distinct. We know that α is positive, so we can say that $2\left\lfloor \frac{\alpha}{18} \right\rfloor + 2 > 0$, from which we can determine that $\alpha > \alpha - \left\lfloor \frac{\alpha}{18} \right\rfloor - 4 + 2$. Further arithmetic manipulation and the fact that $j \leq \frac{\alpha}{2} - \left\lfloor \frac{\alpha}{18} \right\rfloor - \left\lfloor \frac{\alpha-1}{3} \right\rfloor - 2$ gives us that $\alpha > 2\left\lfloor \frac{\alpha-1}{3} \right\rfloor + 2\Big(\frac{\alpha}{2} - \left\lfloor \frac{\alpha}{18} \right\rfloor - \left\lfloor \frac{\alpha-1}{3} \right\rfloor - 2\Big) + 2 \geq 2\left\lfloor \frac{\alpha-1}{3} \right\rfloor + 2j + 2$.

Reading from both ends of this inequality gives us that $\alpha > 2\left\lfloor \frac{\alpha-1}{3} \right\rfloor + 2j + 2$ which can be rearranged to give us our desired result that $3\left(\frac{\alpha}{2} + \left\lfloor \frac{\alpha}{18} \right\rfloor\right) + \left\lfloor \frac{\alpha-1}{3} \right\rfloor + j + 5 < \frac{5\alpha}{2} + 3\left\lfloor \frac{\alpha}{18} \right\rfloor - \left\lfloor \frac{\alpha-1}{3} \right\rfloor - j + 3$.

By considering the largest position in the sequence, this construction gives the following upper bound on the length of E_α:

$$|E_\alpha| \leq \frac{5\alpha}{2} + 3\left\lfloor \frac{\alpha}{18} \right\rfloor - \left\lfloor \frac{\alpha-1}{3} \right\rfloor + 3 - j \leq \frac{7\alpha}{3} + 4.$$

Furthermore the recursive part of the construction fits since $\frac{7}{3}\left(2\left\lfloor \frac{\alpha}{18} \right\rfloor\right) + 4 \leq \alpha - 2\left\lfloor \frac{\alpha-1}{3} \right\rfloor - 1$ which holds for $\alpha \geq \frac{13}{2}$, which is not a problem since there is no recursive part for $\alpha < 18$.

Based on these observations, we conclude that for $n > 15\omega + 1$ the construction in Table 4.13 gives vertex-disjoint triangles.

The construction for odds is given in Table 4.14 where α is the largest red edge length to be placed; $0 \leq i \leq \left\lfloor \frac{\alpha-1}{3} \right\rfloor$, $0 \leq j \leq \left\lfloor \frac{\alpha}{2} \right\rfloor - \left\lfloor \frac{\alpha-3}{18} \right\rfloor - \left\lfloor \frac{\alpha-1}{3} \right\rfloor - 2$, and $0 \leq k \leq \left\lfloor \frac{\alpha-3}{18} \right\rfloor - 1$. A sequence constructed according to Table 4.14 is given in Example 4.6.

Example 4.6 The following sequence is a generalized extended odd Skolem-type sequence of order 9 and multiplicity 3 constructed using Table 4.14.

9 7 5 1 1 1 0 5 7 9 0 0 5 0 0 7 0 0 9

Triangle with edge lengths	Vertices
$(2\alpha - 4i,\ \alpha - 2i,\ \alpha - 2i)$	$(i,\ \alpha - i,\ 2\alpha - 3i)$
$\Big(2\alpha - 4\left\lfloor \frac{\alpha-1}{3} \right\rfloor - 4j - 4,\ \alpha - 2\left\lfloor \frac{\alpha-1}{3} \right\rfloor - 2j - 2,\ \alpha - 2\left\lfloor \frac{\alpha-1}{3} \right\rfloor - 2j - 2\Big)$	$\Big(2\alpha - 3\Big(\left\lfloor \frac{\alpha}{2} \right\rfloor - \left\lfloor \frac{\alpha-3}{18} \right\rfloor - \left\lfloor \frac{\alpha-1}{3} \right\rfloor\Big) + 3j + 7,\ 3\Big(\left\lfloor \frac{\alpha}{2} \right\rfloor + \left\lfloor \frac{\alpha-3}{18} \right\rfloor\Big) + \left\lfloor \frac{\alpha-1}{3} \right\rfloor + j + 5,\ \left\lfloor \frac{5\alpha}{2} \right\rfloor + 3\left\lfloor \frac{\alpha-3}{18} \right\rfloor - \left\lfloor \frac{\alpha-1}{3} \right\rfloor - j + 3\Big)$
$\Big(4\left\lfloor \frac{\alpha-3}{18} \right\rfloor - 4k - 2,\ 2\left\lfloor \frac{\alpha-3}{18} \right\rfloor - 2k - 1,\ 2\left\lfloor \frac{\alpha-3}{18} \right\rfloor - 2k - 1\Big)$	Placed in an $O_{2\lfloor \frac{\alpha-21}{18} \rfloor}$ beginning at position $\left\lfloor \frac{\alpha-1}{3} \right\rfloor + 1$.

Table 4.14: Odd Skolem-type Construction

The construction gives a bound on the length of O_α:

$$|O_\alpha| \leq \frac{5\alpha}{2} + 3\left\lfloor \frac{\alpha - 3}{18} \right\rfloor - \left\lfloor \frac{\alpha - 1}{3} \right\rfloor - j + 3 \leq \frac{7\alpha}{3} + \frac{7}{2}.$$

And again the recursive part of the construction fits because $\frac{7(2\left\lfloor \frac{\alpha-3}{18} \right\rfloor - 1)}{3} + \frac{7}{2} \leq \alpha - 2\left\lfloor \frac{\alpha - 1}{3} \right\rfloor - 1$ which holds for $\alpha \geq \frac{39}{16}$, and is again not a problem since there is no recursive part for $n < 18$.

Now we consider the problem of placing these odd and even Skolem type sequences into the gaps from the constructions in Tables 4.11 and 4.12.

First, suppose that ω is even. We wish to place an E_ω which has length $\frac{7\omega}{3} + 4$ and an $O_{\omega-1}$ which has length $\frac{7\omega}{3} + \frac{7}{6}$. The odd sequence always fits in the smaller gap since $\frac{7\omega}{3} + \frac{7}{6} \leq m - \frac{7\omega}{2} + 2$. The even sequence fits in the longer gap when $\frac{7\omega}{3} + 4 \leq m - 3\omega + 2$ which holds when $n \geq \frac{32\omega}{3} + 5$. Noting that we have $n \equiv 1 \pmod{4\omega}$ and that $12\omega + 1 \geq \frac{32\omega}{3} + 5$ when $\omega \geq 3$, the only missing cases are $n = 12\omega + 1$ when $\omega = 2$ and $n = 8\omega + 1$. When $\omega = 2$, an E_2 is given by the sequence 2 0 2 0 2 which has length 5, while under the further assumption that $n = 12\omega + 1$ the gap has size $m - 3\omega + 2 = 6\omega - 3\omega + 2 = 8$, so this sequence fits in this gap as well. The construction from Table 4.12 gives the blue edges with odd lengths from 1 to $\omega - 2$ and the constructions from Tables 4.13 and 4.14 placed in the gaps as described here give vertex disjoint triangles for the blue edges with even lengths from 2 to ω so this gives an $L_{\lambda,m}(1,\lambda)$ for ω even.

Now suppose that ω is odd. We wish to place an $E_{\omega-1}$ which has length $\dfrac{7\omega}{3}+\dfrac{5}{3}$ and an O_ω which has length $\dfrac{7\omega}{3}+\dfrac{7}{2}$. The even sequence always fits in the smaller gap since $\dfrac{7\omega}{3}+\dfrac{5}{3} \leq m - \dfrac{7\omega-5}{2}$. The even sequence fits in the longer gap when $\dfrac{7\omega}{3}+\dfrac{7}{2} \leq m-3\omega+1$ which once again holds when $n \geq \dfrac{32\omega}{3}+5$. Since $\omega=1$ is covered by Theorem 3.2, the only missing cases are $n = 12\omega+1$ when $\omega = 3$ and $n = 8\omega+1$. When $\omega = 3$, an O_3 is given by the sequence 3 0 0 3 0 0 3 1 1 1 which has length 10, while under the further assumption that $n = 12\omega + 1$ the gap has size $m - 3\omega + 2 = 6\omega - 3\omega + 2 = 10$, so this sequence fits in this gap as well. The construction from Table 4.11 gives the blue edges with odd lengths from 1 to $\omega - 2$ and the constructions from Tables 4.13 and 4.14 placed in the gaps as described here give vertex disjoint triangles for the blue edges with even lengths from 2 to ω so this gives an $L_{\lambda,m}(1,\lambda)$ for ω odd. ■

4.4 Main Results

We now have the tools to present the main result.

Theorem 4.7 *If* $\lambda = 2\omega$ *and* $n \equiv 1 \pmod{2\lambda}$ *then a cyclic perfect* $T(\cup_\lambda P_2)$ *triple system of order* n *exists, except possibly when* $n = 8\omega + 1$.

Proof: If λ is even, $n \equiv 1 \pmod{2\lambda}$, and $n \neq 8\omega + 1$ then by Lemma 4.1 there exist vertex disjoint triangles with edge lengths from $L_{\lambda,m}(\lambda\rho + 1, \lambda(\rho + 1))$. Lemmas 4.2 and 4.3 guarantee the existence of vertex disjoint triangles with edge lengths given by $L_{\lambda,m}(\lambda+1, 2\lambda)$. The edge lengths from $L_{\lambda,m}(1, \lambda)$ are given in vertex disjoint triangles by Lemma 4.4.

All that remains to be shown is that taking the collection of the edge lengths in this way together with their orbits generates an appropriate decomposition.

Fix $\rho = 0$ and iterate i. The differences in the first positions of the edge length triples, which will be the lengths of the blue edges, are $1, 2, \ldots, 2\omega$. Now set $\rho = 1$ and iterate i again from 0 to obtain the lengths $2\omega + 1, 2\omega + 2, \ldots, 4\omega$ for the blue edges. Continuing in this manner, we obtain the lengths $1, 2, \ldots, \dfrac{n-1}{2}$ for the blue edges. Since the orbit of such an edge contains $n - 1$ other edges of the same length, we obtain the $\dfrac{n(n-1)}{2}$ edges of a copy of K_n coloured blue.

Now looking at the second and third positions of the edge length triples, which give the lengths of the red edges, if we fix $\rho = 0$ and iterate i we get the edge lengths $1, 2, \ldots, \omega - 1, \omega, m - \omega + 1, m - \omega + 2, \ldots, m - 1, m$. Setting $\rho = 1$ and iterating i produces the edge lengths $\omega+1, \omega+2, \ldots, 2\omega-1, 2\omega, m-2\omega+1, m-2\omega+2, \ldots, m-\omega-1, m-\omega$. If we continue to repeat this process then we obtain the red edge lengths $1, 2, \ldots, \omega\left(\dfrac{n-4\omega-1}{4\omega}\right)+\omega = \dfrac{n-1}{4}$ and $m, m-1, \ldots, m-\omega\left(\dfrac{n-4\omega-1}{4\omega}\right)-\omega+1 =$

$\frac{n-1}{2} - \frac{n-1}{4} + 1 = \frac{n-1}{4} + 1 = \frac{n+1}{4}$, which is all of the red edge lengths from 1 to $m = \frac{n-1}{2}$.

Thus, every possible edge length occurs once as a blue edge length and twice as a red edge length.

Now consider what happens when we take the orbit of a set of base triangles under $\epsilon = (0\ 1\ \ldots\ n-1)$. This new set of triangles will once again consist of disjoint 3-cycles with the same lengths as before, but covering different edges.

Consider a single edge of length d_0 in a given set of base triangles. The orbit of this edge contains n distinct edges with the same length d_0 as the original edge. This is in fact, one copy of all possible edges of length d_0.

Now we count how many differences are given by the construction. Each $L_{\lambda,m}(\lambda\rho+1, \lambda(\rho+1))$, $0 \leq \rho \leq \frac{n-4\omega-1}{4\omega}$, gives exactly 6λ edges which are translated n times. Overall, this means the construction produces $3n\left(\frac{n-1}{2}\right)$ edges. However, in $3K_n$ each of the $\frac{n-1}{2}$ distinct least positive edge lengths occurs 3 times for every edge by n times for translations around, so the construction produces the same number of edge lengths as the total number in $3K_n$. Since we have shown that each edge length occurs at least 3 times, they must each occur exactly 3 times.■

We illustrate the construction from Table 4.2 with an example.

Example 4.8 Consider $n = 25$ and $\lambda = 4$ so that $m = 12$ and $\omega = 2$. We construct $L_{4,12}(9,12)$ using Table 4.2, $L_{4,12}(5,8)$ using Table 4.10, and $L_{4,12}(1,4)$ using Tables 4.12, 4.13, and 4.14. The complete construction for a perfect $T(\cup_\lambda P_2)$ triple system of order 25 is given in Table 4.15, where the double lines separate the sets of vertex disjoint triangles, and is illustrated in Figures 4.1, 4.2, and 4.3.

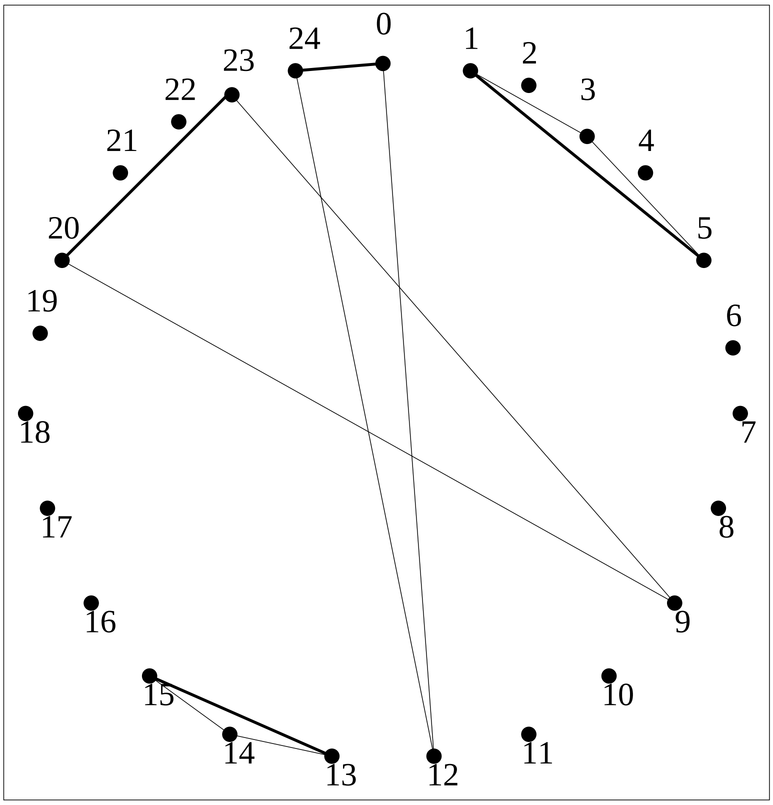

Figure 4.1: The base triangles for $\rho = 0$ in a $T(\cup_4 P_2)$ triple system with $n = 25$

Triangle with edge lengths	Vertices
(1, 12, 12)	(0, 12, 24)
(2, 1, 1)	(13, 14, 15)
(3, 11, 11)	(23, 9, 20)
(4, 2, 2)	(1, 3, 5)
(5, 10, 10)	(0, 10, 20)
(6, 3, 3)	(13, 16, 19)
(7, 9, 9)	(3, 12, 21)
(8, 4, 4)	(1, 5, 9)
(9, 8, 8)	(3, 12, 20)
(10, 5, 5)	(0, 5, 10)
(11, 7, 7)	(4, 15, 22)
(12, 6, 6)	(1, 7, 13)

Table 4.15: Construction of $T(\cup_4 P_2)$ for $n = 25$

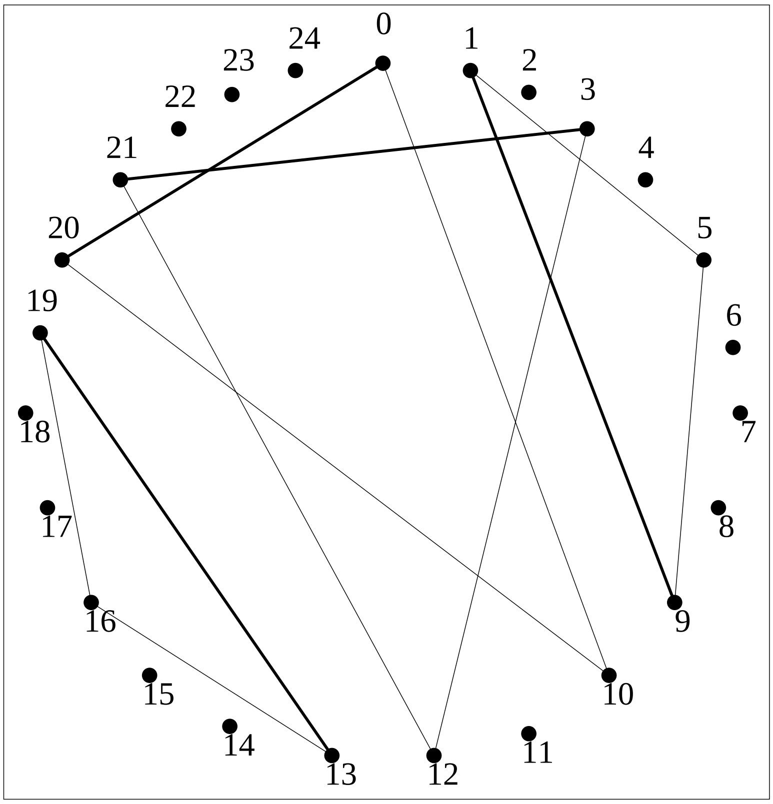

Figure 4.2: The base triangles for $\rho = 1$ in a $T(\cup_4 P_2)$ triple system with $n = 25$

Corollary 4.9 shows that if λ is any power of 2 then a perfect $T(\cup_\lambda P_2)$ triple system of order n exists if and only if $n \equiv 1 \pmod{2\lambda}$ and $n \geq 3\lambda$, except possibly when $n = 4\lambda + 1$.

Corollary 4.9 *If $\lambda = 2^\nu$, where ν is any positive integer, then $3K_n$ has a perfect decomposition into $T(\cup_\lambda P_2)$ if and only if $n \equiv 1 \pmod{2^{\nu+1}}$ and $n \geq 3\lambda$, except possibly when $n = 4\lambda + 1$.*

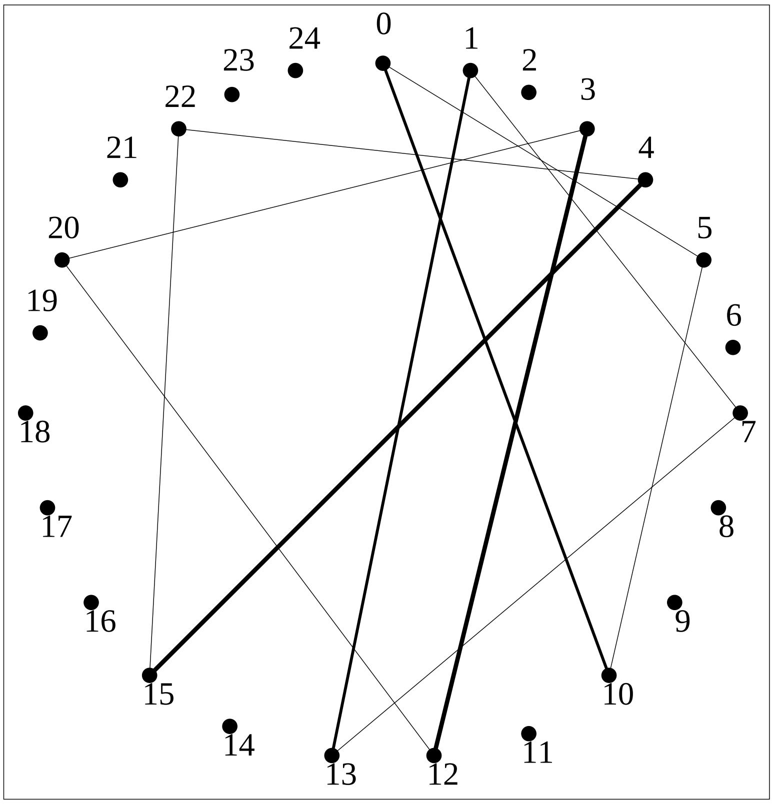

Figure 4.3: The base triangles for $\rho = 2$ in a $T(\cup_4 P_2)$ triple system with $n = 25$

PROOF: If λ is a power of 2 then it is also even, so the existence of a perfect $T(\cup_\lambda P_2)$ triple system of order n given that $n \equiv 1 \pmod{2^{\nu+1}}$, $n \geq 3\lambda$, $n \neq 8\omega + 1$, follows from Theorems 4.7 and 3.2.

We also know from Proposition 2.2 that such a decomposition exists only if $n \geq 3\lambda$, $n \equiv 1 \pmod 2$ and $n(n-1) \equiv 0 \pmod{2\lambda}$, so suppose that $n^2 - n \equiv 0 \pmod{2^{\nu+1}}$. Since n is odd we can write $n \equiv [2x+1] \pmod{2^{\nu+1}}$ where x is a natural number.

Now we must have $(2x+1)^2 - (2x+1) \equiv 0 \pmod{2^{\nu+1}}$, from which it follows that $2x(2x+1) \equiv 0 \pmod{2^{\nu+1}}$ and hence $x(2x+1) \equiv 0 \pmod{2^{\nu}}$. However, if $x \geq 1$ then $(2x+1)|2^{\nu}$ which is a contradiction, so $x = 0$ and $n \equiv 1 \pmod{2^{\nu+1}}$. ■

Chapter 5

Discussion and open problems

We have determined the spectra of perfect $T(\cup_\lambda P_2)$ triple systems for the case when λ is even and $n \equiv 1 \pmod{2\lambda}$, and have constructed such decompositions for all possible values except for $n = 4\lambda + 1$. One generalization would be proving an analogous result for odd λ. Constructing perfect $T(\cup_\lambda P_2)$ triple systems for other solutions to the necessary condition is also an open problem. In particular, we believe that if p is an odd prime then the problem of determining the spectrum of perfect $T(\cup_p P_2)$ triple systems would be tractable, since then the only other solutions to the necessary condition would be $n \equiv p \pmod{2\lambda}$. Based on our results, we make Conjecture 5.1.

Conjecture 5.1 *A perfect $T(\cup_\lambda P_2)$ triple system of order n exists if and only if $n(n-1) \equiv 0 \pmod{2\lambda}$, $n \equiv 1 \pmod{2}$ and $n \geq 3\lambda$.*

All of our constructions have been cyclic in nature. It is natural to ask whether or not there exist any perfect $T(\cup_\lambda P_2)$ triple systems which are not cyclic.

Open Problem 5.2 *For what values of n and λ do perfect $T(\cup_\lambda P_2)$ triple systems which are not cyclic exist?*

Other open problems include determining the spectra when G is a path or a cycle respectively, since these decomposition problems have been studied in other cases before.

Open Problem 5.3 *What are the spectra for perfect $T(P_m)$ and $T(C_m)$ triple systems of order n respectively?*

Appendix A

Constructing additional perfect $T(P_2 \cup P_2 \cup P_2)$ triple systems

A.1 Additional constructions for $n \equiv 1 \pmod 6$

Triangle with edge lengths	Vertices
(1, 6, 6)	(0, 1, 7)
(2, 1, 1)	(8, 9, 10)
(3, 5, 5)	(3, 6, 11)

Table A.1: Construction for $n = 13$ for blue edges of lengths $6i+1$, $6i+2$, and $6i+3$

A.2 Additional constructions for $n \equiv 3 \pmod 6$

Triangle with edge lengths	Vertices
(2, 1, 3)	(6, 8, 5)
(3, 1, 2)	(0, 3, 1)
(4, 2, 3)	(2, 7, 4)

Table A.2: Construction for $n = 9$ with blue edges of lengths 2, 3, and 4

Triangle with edge lengths	Vertices
(2, 1, 1)	(7, 8, 9)
(3, 6, 6)	(0, 6, 12)
(4, 2, 2)	(1, 3, 5)

Table A.3: Construction for $n = 15$ with blue edges of lengths 2, 3, and 4

Triangle with edge lengths	Vertices
(2, 1, 1)	(7, 8, 9)
(3, 7, 7)	(0, 3, 10)
(4, 2, 2)	(2, 4, 6)

Table A.4: Construction for $n = 21$ with blue edges of lengths 2, 3, and 4

Triangle with edge lengths	Vertices
(5, 5, 5)	(0, 5, 10)
(6, 3, 3)	(3, 6, 9)
(7, 4, 4)	(1, 8, 12)

Table A.5: Construction for $n = 15$ with blue edges of lengths 5, 6, and 7

Appendix B

Constructions for $n \equiv 1 \pmod{2\lambda}$, $\rho = 0$, and $\rho = 1$ when λ is small

We give explicit constructions for $\rho = 0$ and $\rho = 1$ for $n \equiv 1 \pmod{2\lambda}$ when $\lambda = 4, 6, 8$. This includes constructions for the case $n = 8\omega+1$, which we have not solved in general, but based in part on this evidence we would conjecture that perfect $T(\cup_\lambda P_2)$ triple systems of order $n = 4\lambda + 1$ do in fact exist.

B.1 Additional constructions for $\lambda = 4$

Since $\lambda = 4 = 2^2$, $n(n-1) \equiv 0 \pmod{2\lambda} \equiv 0 \pmod{2^3}$ if and only if $n \equiv 1 \pmod 8$. We also require that $n \geq 12$.

B.1.1 Constructions for $\rho = 0$

The construction given in Table B.1 works for $n > 17$.

Triangle with edge lengths	Vertices
$(1,\ m,\ m)$	$(0,\ 1,\ m+1)$
$(2,\ 1,\ 1)$	$(m+6,\ m+7,\ m+8)$
$(3,\ m-1,\ m-1)$	$(3,\ 6,\ m+5)$
$(4,\ 2,\ 2)$	$(5,\ 7{,}9)$

Table B.1: Construction for $\lambda = 4$ and $\rho = 0$

If $n = 17$ then we use the construction in Table B.2.

B.1.2 Constructions for $\rho = 1$

Certainly, if $m > 14$ then $n > 29$ and the construction in Table B.3 consists of distinct vertices. However, if $n = 17$ the construction still works since $m = 8$ and $m + 3 = 11$

Triangle with edge lengths	Vertices
(1, 8, 8)	(0, 1, 9)
(2, 1, 1)	(14, 15, 16)
(3, 7, 7)	(3, 6, 13)
(4, 2, 2)	(8, 10, 12)

Table B.2: Construction for $\lambda = 4$ and $\rho = 0$ and $n = 17$

are distinct from the constant vertices 6, 10, 14, etc. Similarly, if $n = 25$ then $m = 12$ is distinct from the constant vertices.

Triangle with edge lengths	Vertices
$(5, m-2, m-2)$	$(0, 5, m+3)$
(6, 3, 3)	(1, 4, 7)
$(7, m-3, m-3)$	$(3, m, m+7)$
(8, 4, 4)	(2, 6, 10)

Table B.3: Construction for $\lambda = 4$ and $\rho = 1$

B.2 Additional constructions for $\lambda = 6$ and $n \equiv 1 \pmod{12}$

We consider $n \equiv 1 \pmod{12}$ and $n \geq 25$.

B.2.1 Constructions for $\rho = 0$

The construction in Table B.4 works for all $n \geq 25$.

Triangle with edge lengths	Vertices
$(1, m, m)$	$(0, 1, m+1)$
$(2, 1, 1)$	$(m+10, m+11, m+12)$
$(3, m-1, m-1)$	$(3, 6, m+5)$
$(4, 2, 2)$	$(7, 9, 11)$
$(5, m-2, m-2)$	$(5, 10, m+8)$
$(6, 3, 3)$	$(m+3, m+6, m+9)$

Table B.4: Construction for $\lambda = 6$ and $\rho = 0$

B.2.2 Constructions for $\rho = 1$

The construction in Table B.5 works for all $n > 25$.

Triangle with edge lengths	Vertices
$(7,\ m-3,\ m-3)$	$(4,\ m+1,\ m+8)$
$(8,\ 4,\ 4)$	$(3,\ 7,\ 11)$
$(9,\ m-4,\ m-4)$	$(6,\ m+2,\ m+11)$
$(10,\ 5,\ 5)$	$(0,\ 5,\ 10)$
$(11,\ m-5,\ m-5)$	$(1,\ 12,\ m+7)$
$(12,\ 6,\ 6)$	$(m,\ m+6,\ m+12)$

Table B.5: Construction for $\lambda = 6$ and $\rho = 1$

This construction does not work for $n = 25$ in which case we use the construction in Table B.6.

B.3 Additional constructions for $\lambda = 8$

The necessary conditions are that $n \equiv 1 \pmod{16}$ and that $n \geq 24$ so we require that $n \geq 33$.

B.3.1 Constructions for $\rho = 0$

The construction in Table B.7 works for all $n > 25$.

Triangle with edge lengths	Vertices
(7, 9, 9)	(6, 15, 22)
(8, 4, 4)	(13, 17, 21)
(9, 8, 8)	(3, 11, 20)
(10, 5, 5)	(9, 14, 19)
(11, 7, 7)	(0, 7, 18)
(12, 6, 6)	(4, 10, 16)

Table B.6: Construction for $\lambda = 6$ and $\rho = 1$ and $n = 25$

Triangle with edge lengths	Vertices
$(1,\ m,\ m)$	$(0,\ 1,\ m+1)$
(2, 1, 1)	$(m+11,\ m+12,\ m+13)$
$(3,\ m-1,\ m-1)$	$(2,\ 5,\ m+4)$
(4, 2, 2)	(8, 10, 12)
$(5,\ m-2,\ m-2)$	$(4,\ 9,\ m+7)$
(6, 3, 3)	$(m+2,\ m+5,\ m+8)$
$(7,\ m-3,\ m-3)$	$(6,\ 13,\ m+10)$
(8, 4, 4)	(3, 7, 11)

Table B.7: Construction for $\lambda = 8$ and $\rho = 0$

B.3.2 Constructions for $\rho = 1$

Tthe construction in Table B.8 works for all $n > 31$.

Triangle with edge lengths	Vertices
$(9,\ m-4,\ m-4)$	$(4,\ 13,\ m+9)$
$(10,\ 5,\ 5)$	$(2,\ 7,\ 12)$
$(11,\ m-5,\ m-5)$	$(0,\ 11,\ m+6)$
$(12,\ 6,\ 6)$	$(m+1,\ m+7,\ m+13)$
$(13,\ m-6,\ m-6)$	$(8,\ m+2,\ m+15)$
$(14,\ 7,\ 7)$	$(m-2,\ m+5,\ m+12)$
$(15,\ m-7,\ m-7)$	$(6,\ m-1,\ m+14)$
$(16,\ 8,\ 8)$	$(m,\ m+8,\ m+16)$

Table B.8: Construction for $\lambda = 8$ and $\rho = 1$

Appendix C

Additional constructions for $n = 4\lambda + 1$ and $\rho = 1$

We note that in this case, since $n = 8w+1$ and $\rho = 1$ includes the indexing differences $\lambda + 1 = 2\omega + 1, 2\omega + 2, \ldots, 4\omega = 2\lambda = \dfrac{n-1}{2}$, the non-indexing differences are in fact 2ω consecutive integers. This leads to more elegant constructions if $\omega \equiv 0, 2 \pmod 3$; but causes considerable difficulty in the case $\omega \equiv 1 \pmod 3$.

We begin with the constructions for $\omega \equiv 0, 2 \pmod 6$, which are given in Table C.1 where $0 \leq i \leq \frac{\omega-2}{2}$.

Here is a justification that the positions are distinct:

- $i < \omega - i - 1 < \omega + i < 2\omega - i + 1 < 2\omega + i + 1 < 3\omega + 3i + 3$

Triangle with edge lengths	Vertices
$(2\omega + 1 + 4i,\ 3\omega - 2i,\ 3\omega - 2i)$	$(i,\ 3\omega - i,\ 6\omega - 3i)$
$(2\omega + 2 + 4i,\ \omega + 2i + 1,\ \omega + 2i + 1)$	$(7\omega - 3i - 1,\ 8\omega - i,\ \omega + i)$
$(2\omega + 3 + 4i,\ 3\omega - 1 - 2i,\ 3\omega - 1 - 2i)$	$(4\omega + 3i + 2,\ 7\omega + i + 1,\ 2\omega - i - 1)$
$(2\omega + 4 + 4i,\ \omega + 2i + 2,\ \omega + 2i + 2)$	$(\omega - i - 1,\ 2\omega + i + 1,\ 3\omega + 3i + 3)$

Table C.1: Construction for $n = 8\omega + 1$, $\omega \equiv 0, 2 \pmod 6$, $\rho = 1$

- $2\omega + i + 1 < 4\omega + 3i + 2$
- $3\omega + 3i + 3 \not\equiv 4\omega + 3i + 2 \pmod 3$
- $3\omega + 3i + 3 \leq \frac{9\omega}{2} < \frac{9\omega+2}{2} \leq 6\omega - 3i$
- $4\omega + 3i + 2 \not\equiv 6\omega - 3i \pmod 3$
- $4\omega + 3i + 2 \leq \frac{11\omega-2}{2} < \frac{11\omega+4}{2} \leq 7\omega - 3i - 1$
- $6\omega - 3i \not\equiv 7\omega - 3i - 1 \pmod 3$
- $6\omega - 3i < 7\omega + i + 1$
- $7\omega - 3i - 1 < 7\omega + i + 1 < 8\omega - i \leq 8\omega$

The same construction will also work for $\omega \equiv 3, 5 \pmod 6$, with only a slight modification because $\lambda \equiv 2 \pmod 4$; this is given in Table C.2 where $0 \leq i \leq \frac{\omega-1}{2}$

and $0 \leq j \leq \frac{\omega-3}{2}$.

Triangle with edge lengths	Vertices
$(2\omega + 1 + 4i,\ 3\omega - 2i,\ 3\omega - 2i)$	$(i,\ 3\omega - i,\ 6\omega - 3i)$
$(2\omega + 2 + 4i,\ \omega + 2i + 1,\ \omega + 2i + 1)$	$(7\omega - 3i - 1,\ 8\omega - i,\ \omega + i)$
$(2\omega + 3 + 4j,\ 3\omega - 1 - 2j,\ 3\omega - 1 - 2j)$	$(4\omega + 3j + 2,\ 7\omega + j + 1,\ 2\omega - j - 1)$
$(2\omega + 4 + 4j,\ \omega + 2j + 2,\ \omega + 2j + 2)$	$(\omega - j - 1,\ 2\omega + j + 1,\ 3\omega + 3j + 3)$

Table C.2: Construction for $n = 8\omega + 1$, $\omega \equiv 3, 5 \pmod 6$, $\rho = 1$

Unfortunately $\omega \equiv 1, 4 \pmod 6$ requires a more nuanced approach. We follow a similar pattern to the previous constructions in this subsection, except we have to shift one set of differences to the end. The constructions for $\omega \equiv 1 \pmod 6$ and $\omega \equiv 4 \pmod 6$ are given in Table C.3, with $0 \leq \ell \leq \frac{\omega-1}{2}$, $0 \leq i \leq \frac{\omega-3}{2}$, $1 \leq j \leq \frac{\omega-3}{2}$; and in Table C.4, with $0 \leq i \leq \frac{\omega}{2} - 1$, $1 \leq j \leq \frac{\omega}{2} - 1$ respectively.

Since this construction is significantly different from the last, we once again argue that the positions are distinct.

- $i < \omega - \ell - 1 < \omega + j - 1 < 2\omega - \ell - 1 < 2\omega + \ell < 3\omega - i - 1 < 3\omega + 3\ell + 1$
- $3\omega - i - 1 < 4\omega + 3\ell - 1$
- $3\omega + 3\ell + 1 \not\equiv 4\omega + 3\ell - 1 \pmod 3$

Triangle with edge lengths	Vertices
$(2\omega + 1 + 4\ell,\, 3\omega - 2\ell,\, 3\omega - 2\ell)$	$(4\omega - 1 + 3\ell,\, 7\omega - 1 + \ell,\, 2\omega - 1 - \ell)$
$(2\omega + 2 + 4\ell,\, \omega + 1 + 2\ell,\, \omega + 1 + 2\ell)$	$(\omega - 1 - \ell,\, 2\omega + \ell,\, 3\omega + 1 + 3\ell)$
$(2\omega + 3 + 4i,\, 3\omega - 1 - 2i,\, 3\omega - 1 - 2i)$	$(i,\, 3\omega - 1 - i,\, 6\omega - 2 - 3i)$
$(2\omega + 4 + 4j,\, \omega + 2 + 2j,\, \omega + 2 + 2j)$	$(7\omega - 4 - 3j,\, 8\omega - 2 - j,\, \omega - 1 + j)$
$(2\omega + 4,\, \omega + 2,\, \omega + 2)$	$(6\omega - 4,\, 7\omega - 2,\, 8\omega)$

Table C.3: Construction for $n = 8\omega + 1$, $\omega \equiv 1 \pmod 6$, $\rho = 1$

- $3\omega + 3\ell + 1 \leq \frac{9\omega-1}{2} < \frac{9\omega+5}{2} \leq 6\omega - 3i - 2$
- $4\omega + 3\ell - 1 \not\equiv 6\omega - 3i - 2 \pmod 3$
- $4\omega + 3\ell - 1 \leq \frac{11\omega-5}{2} < \frac{11\omega+1}{2} \leq 7\omega - 3j - 4$
- $6\omega - 3i - 2 \not\equiv 7\omega - 3j - 4 \pmod 3$
- $6\omega - 3i - 2 < 6\omega - 4$
- $7\omega - 3j - 4 \not\equiv 6\omega - 4 < 7\omega + \ell - 1$
- $7\omega - 3i - 1 < 7\omega + \ell - 1$
- $7\omega - 4 - 3j < 7\omega - 2 < 8\omega - j - 2$
- $7\omega + \ell - 1 < 8\omega - j - 2 < 8\omega$

Triangle with edge lengths	Vertices
$(2\omega + 1 + 4i,\ 3\omega - 2i,\ 3\omega - 2i)$	$(4\omega - 1 + 3i,\ 7\omega - 1 + i,\ 2\omega - 2 - i)$
$(2\omega + 2 + 4i,\ \omega + 2i + 1,\ \omega + 2i + 1)$	$(\omega - 1 - i,\ 2\omega + i,\ 3\omega + 1 + 3i)$
$(2\omega + 3 + 4i,\ 3\omega - 1 - 2i,\ 3\omega - 1 - 2i)$	$(i,\ 3\omega - i - 1,\ 6\omega - 2 - 3i)$
$(2\omega + 4 + 4j,\ \omega + 2j + 2,\ \omega + 2j + 2)$	$(7\omega - 4 - 3j,\ 8\omega - 2 - j,\ \omega - 1 + j)$
$(2\omega + 4,\ \omega + 2,\ \omega + 2)$	$(6\omega - 4,\ 7\omega - 2,\ 8\omega)$

Table C.4: Construction for $n = 8\omega + 1$, $\omega \equiv 4 \pmod 6$, $\rho = 1$

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