Colourings of G-designs

by

© **Iren Darijani**

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Abstract

A G-decomposition of a graph H consists of a set V of vertices of H together with a set $\mathcal{B}$ of subgraphs (called blocks) of H, each isomorphic to G, that partition the edge set of H. A G-design of order n is a G-decomposition of the complete graph K_n on n vertices. A complete graph is a simple graph in which every pair of distinct vertices is connected by a unique edge. A G-design is said to be k-colourable if its vertex set can be partitioned into k sets (called colour classes) such that no block is monochromatic. It is k-chromatic if it is k-colourable but is not $(k-1)$-colourable.

The block intersection graph of a G-design with block set $\mathcal{B}$ is the graph with $\mathcal{B}$ as its vertex set such that two vertices are adjacent if and only if their associated blocks are not disjoint. The chromatic index of a graph H is the least number of colours that enable each edge of H to be assigned a single colour such that adjacent edges never have the same colour.

In this thesis we first investigate the chromatic index of block intersection graphs of Steiner triple systems. A Steiner triple system of order n is a K_3-design of order n. Next, we study k-colourings of e-star systems for all $k \geqslant 2$ and $e \geqslant 3$. An e-star is a complete bipartite graph $K_{1,e}$. An e-star system of order $n > 1$, is a G-design of order n when G is an e-star. We then study k-colourings of path systems for any $k \geqslant 2$. A path system is a G-design when G is a path.

Lay summary

A graph is a mathematical structure consisting of two sets that represent vertices and edges. A complete graph is a simple graph in which every pair of distinct vertices is connected by a unique edge. A G-decomposition of a graph is a partition of the edges of the graph into subgraphs isomorphic to G called blocks. A G-design of order n is a G-decomposition of the complete graph on n vertices. Colouring of a G-design is an assignment of colours to the vertices such that no block is monochromatic.

G-designs have been investigated to solve scheduling problems occuring in combinatorial mathematics. In combinatorial mathematics, there are famous problems such as the fifteen school girls problem. The fifteen school girls problem states: A school mistress has fifteen girl pupils and she wishes to take them on a daily walk. The girls are to walk in five rows of three girls each. It is required that no two girls should walk in the same row more than once per week. The solution of this problem can be obtained by constructing a G-design when G is a complete graph on three vertices.

In this thesis, we study colourings of some G-designs. Vertex colouring arises in a variety of scheduling applications. For instance, suppose that v delegates from k countries are to meet at round tables such that each delegate sits next to each other delegate exactly once. Also, each meeting must include at least two delegates from two different countries. This can be modelled as a colouring problem using k colours.

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Statement of contribution

Dr. David A. Pike proposed the research questions that were investigated throughout this thesis. Chapter 2 is a collaboration of work by Iren Darijani, Dr. David A. Pike, and Jonathan Poulin and it is based on a paper which is published in the Australasian Journal of Combinatorics. Chapters 3 and 4 are joint works with Dr. David A. Pike. Chapter 3 is based on a paper which is published in the Journal of Combinatorial Designs. Supervision of this thesis was done by Dr. David A. Pike. He also spent his valuable time to edit this thesis.

Table of contents

Chapter 1

Introduction

A *G-decomposition* of a graph H is a pair $(V, \mathcal{B})$ where V is the vertex set of H and $\mathcal{B}$ is a set of subgraphs of H, each isomorphic to G, whose edge sets partition the edge set of H. A *G-design* of order n is a G-decomposition of the complete graph K_n on n vertices. A *complete graph* is a simple graph in which every pair of distinct vertices is connected by a unique edge. A positive integer n is said to be *admissible* if there exists a G-design of order n. Given a G-design $(V, \mathcal{B})$, its associated *block intersection graph* is the graph on vertex set $\mathcal{B}$ for which two vertices B_1 and B_2 are adjacent if and only if $|V(B_1) \cap V(B_2)| \geqslant 1$. A *G-factor* of a graph H is a spanning subgraph of H, each component of which is isomorphic to G. If the edges of H can be partitioned into G-factors, then we say H has a *G-factorisation.* In this thesis, we study colourings of three kinds of G-designs.

When G is a complete graph on k vertices, a G-design of order n is better known as BIBD$(v, k, 1)$ with $v = n$. More generally, a *balanced incomplete block design* of order v, block size k and index λ, denoted as BIBD(v, k, λ), consists of a v-set V accompanied by a block set $\mathcal{B}$ which itself is a set (or multiset) of k-subsets of V such that each 2-subset of V is contained in exactly λ of the blocks of $\mathcal{B}$. A *Steiner triple system* of order v is a BIBD$(v, 3, 1)$, and is typically denoted by STS(v). A Steiner triple system is indeed a K_3-design. It is a classic result of Kirkman that a STS(v) exists if and only if $v \equiv 1$ or $3 \pmod 6$ [24]. For more information on Steiner triple systems and their properties, see [13].

A *bipartite graph* $G_{m,n}$ is a graph whose vertex set can be partitioned into two subsets V_1 and V_2 with m and n vertices respectively, such that every edge of $G_{m,n}$ joins a vertex of V_1 with a vertex of V_2. If $G_{m,n}$ contains every edge joining V_1 and V_2, then it is called a complete bipartite graph and denoted by $K_{m,n}$. An *e-star* is a complete bipartite graph $K_{1,e}$. When G is an e-star, a G-design of order n is called an *e-star system* of order n and denoted by $S_e(n)$. An e-star with star centre x and edges $\{x, a_1\}$, $\{x, a_2\}$, ... ,$\{x, a_e\}$ is denoted by either $\{x; a_1, \ldots, a_e\}$ or $\{x; A\}$ where $A = \{a_1, \ldots, a_e\}$. The necessary and sufficient conditions for the existence of an e-star system of order n are that e divides $\frac{n(n-1)}{2}$ and $n \geqslant 2e$ [37].

For any integer $m \geqslant 2$, the path with m vertices $\{a_1, \ldots, a_m\}$ and $m-1$ edges $\{a_1, a_2\}$, $\{a_2, a_3\}$, ..., $\{a_{m-1}, a_m\}$ is denoted by $P_m = (a_1, \ldots, a_m)$. When G is a P_m path, a G-design of order n is called a *P_m system* of order n and we denote it by $P_m(n)$. The necessary and sufficient conditions for the existence of a P_m system of order n are that $n = 1$ or $n \geqslant m$ and $n(n-1) \equiv 0 \pmod{2m-2}$ [34].

We will look at two kinds of colourings: edge-colourings of block intersection graphs of Steiner triple systems and k-colourings of star systems and path systems.

1.1 Edge colourings of graphs

An *edge-colouring* of a graph G is an assignment of colours to the edges of G so that no two adjacent edges have the same colour. The *chromatic index* χ' of a graph G is the least number of colours that are needed to edge-colour G.

Vizing's Theorem asserts that a simple graph G with maximum degree Δ either has chromatic index $\chi' = \Delta$ or $\chi' = \Delta + 1$ [18, 35]. A simple graph G is described as *Class 1* (resp. *Class 2*) if its chromatic index is Δ (resp. $\Delta + 1$), although determining which is the case is in general an NP-complete problem [19]. Nevertheless, for some types of graphs the situation is less difficult; for instance, a century-old result of König confirms that all bipartite graphs are Class 1 [26]. Cycles of odd length are examples of Class 2 graphs and Figure 1.1 is an example of a Class 1 graph which is not bipartite.

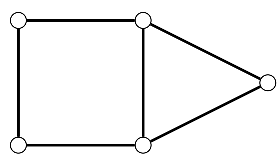

Figure 1.1: A Class 1 graph which is not bipartite

If G is a Δ-regular Class 1 graph of even order, then the edges of G can be partitioned into Δ 1-factors (*i.e.*, 1-regular spanning subgraphs); such a partition is called a *1-factorisation* of G. A *Hamilton decomposition* of a Δ-regular graph G consists of a set of Hamilton cycles (plus a 1-factor if Δ is odd) that partition the edges of G. Results pertaining to Hamilton decompositions date back to the nineteenth century when Walecki is described as having found elegant Hamilton decompositions of complete graphs [27]. For a survey paper about Hamilton decompositions of graphs, see [3]. In the event that G is a Hamilton decomposable graph of even order, then it is clear that G admits a 1-factorisation and hence G is Class 1. However, the converse does not hold as there are graphs with 1-factorisations that do not admit Hamilton decompositions (for an example, refer to Figure 1.2).

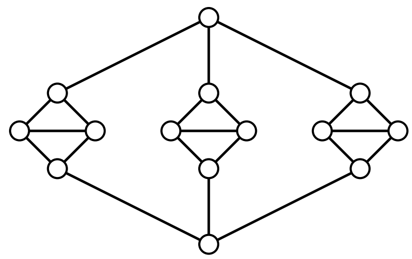

Figure 1.2: A graph with 1-factorisation that has no Hamilton decomposition

The *line graph* $L(G)$ of a graph G is the graph having the edge set of G as its vertex set, with two vertices of $L(G)$ being adjacent if and only if their corresponding edges in G are adjacent. Line graphs can be viewed as a special case of block intersection graphs, or, alternatively, block intersection graphs can be viewed as a generalisation of line graphs. There are a number of results in the literature concerning 1-factorisations and line graphs. For instance, Alspach proved that any complete graph K_n with an even number of edges has a Class 1 line graph [2]. More generally, Jaeger has established that if G is a Hamilton decomposable graph with an even number of edges then $L(G)$ is Class 1 [23].

One-factorisations are widely used within graph theory as well as in areas such as scheduling. For an in-depth review about the theory and application of 1-factorisations, readers can consult [36]. Among the various problems about 1-factorisations, particularly noteworthy is the 1-Factorisation Conjecture, one of the earliest references to which is [10]:

Conjecture 1.1.1. *If G is a simple Δ-regular graph on $2k$ vertices and $k \leqslant \Delta$ then G is Class 1.*

A stronger conjecture that implies the 1-Factorisation Conjecture was given by Nash-Williams [29]:

Conjecture 1.1.2. *If G is a simple Δ-regular graph on n vertices and $n \leqslant 2\Delta$ then G has a Hamilton decomposition.*

Both of these conjectures have recently been solved when the number of vertices is sufficiently large [14]. However, with regard to block intersection graphs of Steiner triple systems, the block intersection graph of a STS(v) has $\frac{v(v-1)}{6}$ vertices and is regular of degree $\frac{3v-9}{2}$. Hence it is only for $v \in \{9, 13\}$ (resp. admissible $v \leqslant 15$) that the block intersection graph of a STS(v) satisfies the hypothesis of the 1-Factorisation Conjecture (resp. Nash-Williams' Conjecture).

In 1987, Graham asked whether STS block intersection graphs have Hamilton cycles [5], a question that was subsequently affirmatively answered (see [5] and [20]). In addition to being Hamiltonian, block intersection graphs of BIBDs are also pancyclic [4, 28]. A graph is pancyclic if it contains cycles of all possible length from three up to the number of vertices of the graph. Even more recently the block intersection graphs of BIBDs have been shown to be cycle extendable, which in turn has enabled a polynomial-time algorithm to be developed for finding cycles of arbitrary specified length in them [1]. A graph is cycle extendable if every nonHamiltonian cycle is extendable. A nonHamiltonian cycle C is extendable if there exists another cycle in the graph that contain all the vertices of C plus one more vertex. In [32] it is reported that every STS(v) with order $v \leqslant 15$ has a Hamilton decomposable block intersection graph, but for admissible orders $v \geqslant 19$ the status remains undetermined.

Since the property of being Hamilton decomposable is (for graphs of even order) a stronger property than having a 1-factorisation, it is natural to initially consider the potentially easier question of deciding whether 1-factorisations exist for STS block intersection graphs. Any graph of order n and maximum degree Δ such that the number of edges exceeds $\Delta\lfloor \frac{n}{2} \rfloor$ is called *overfull* and is necessarily Class 2 [11]. Every regular graph of odd order is overfull and hence every STS(v) for which $v \equiv 3$ or 7 (mod 12) must have a Class 2 block intersection graph. In Chapter 2, we therefore only consider $v \equiv 1$ or 9 (mod 12), for which we offer this conjecture:

Conjecture 1.1.3. *Every STS(v) with $v \equiv 1$ or 9 (mod 12) has a Class 1 block intersection graph.*

In Chapter 2, we investigate the chromatic index of block intersection graphs of Steiner triple systems. A *Kirkman triple system* of order v, denoted KTS(v), is a STS(v) for which the blocks can be partitioned into $\frac{v-1}{2}$ sets called *parallel classes*, each of which consists of $\frac{v}{3}$ pairwise disjoint blocks. Kirkman triple systems are named after Kirkman who in 1850 posed the Kirkman school girls problem [25]. The problem states: Is it possible for a school mistress to take fifteen school girls on a walk each day of seven days of a week, walking in five rows of three girls so that each pair of girls walk together in the same row on exactly one day? A solution to this problem is equivalent to a KTS(15).

In Section 2.1 we show that Conjecture 1.1.3 is satisfied by every Kirkman triple system of order $v \equiv 9$ (mod 12). A Steiner triple system of order v is said to be *cyclic* if its automorphism group contains a cyclic subgroup of order v. In Section 2.2 we consider cyclic Steiner triple systems of order $v \equiv 1$ (mod 12) and we prove that Conjecture 1.1.3 holds for every such STS with $\frac{\varphi(v)}{v-1} \geqslant \frac{2}{3}$, where φ denotes Euler's totient function. In Section 2.3 we establish that Conjecture 1.1.3 is true for all cyclic Steiner triple systems of order $v \equiv 9$ (mod 12).

1.2 Colourings of designs

A G-design $(V, \mathcal{B})$ is said to be *weakly k-colourable* if its vertex set can be partitioned into k sets (called *colour classes*) such that no subgraph belonging to $\mathcal{B}$ is monochromatic. A G-design is *k-chromatic* if it is weakly k-colourable but is not weakly $(k-1)$-colourable. If a G-design is k-chromatic, we say that its *chromatic number* is k. A k-colouring of a G-design is said to be *equitable* if the cardinalities of the colour classes differ by at most one; it is *strongly equitable* if the colour classes are of the same size. A G-design will be called *(strongly) equitably k-chromatic* if it is k-chromatic and admits a (strongly) equitable k-colouring. If every k-colouring of a G-design $(V, \mathcal{B})$ can be obtained from some k-colouring ϕ by a permutation of the colours, we say that $(V, \mathcal{B})$ has a *unique* k-colouring, or that $(V, \mathcal{B})$ is *uniquely k-colourable.*

Weak colourings of G-designs were first studied for K_3-designs, i.e., Steiner triple systems. In particular, de Brandes, Phelps, and Rödl [7] proved that for any $k \geqslant 3$, there exists some integer v_k such that for all admissible $v \geqslant v_k$, there is a k-chromatic Steiner triple system of order v. Moreover, Burgess and Pike [8] showed that for all $k \geqslant 2$ and even $m \geqslant 4$ there exists a k-chromatic m-cycle system. An m-cycle system of order $n > 1$ is a partition of the edges of the complete graph K_n into m-cycles. Also, Horsley and Pike [21] showed that for all $k \geqslant 2$ and $m \geqslant 3$ with $(k, m) \neq (2, 3)$, there exist k-chromatic m-cycle systems of all admissible orders greater than or equal to some integer $n_{k,m}$.

1.2.1 Star systems

The existence of a $K_{1,e}$-factorisation of K_n was studied and completely settled in 1993 [38]. If e is even, a $K_{1,e}$-factorisation of K_n exists if and only if $n \equiv 1 \pmod{2e}$

and $n \equiv 0 \pmod{e+1}$. If e is odd, there does not exist a $K_{1,e}$-factorisation of K_n for any n.

In Chapter 3 of this thesis, we investigate k-colourings of e-star systems for $e \geqslant 3$. Note that a 3-star system of order n exists if and only if $n \equiv 0, 1 \pmod 3$ and $n \geqslant 6$ [37]. In Section 3.1 we first construct equitably 2-chromatic 3-star systems of all admissible orders. We then show that for all $k \geqslant 2$, if there exists a k-chromatic 3-star system of order n_0, then there exists a k-chromatic 3-star system of order n for all admissible $n > n_0$. Next, from a $(k-1)$-chromatic 3-star system, we construct a k-chromatic 3-star system, for all $k \geqslant 3$. Finally, we finish this section by showing that for any integer $k \geqslant 2$, there exists some integer n_k such that for all admissible $n \geqslant n_k$, there exists a k-chromatic 3-star system of order n.

In Section 3.2 we generalise the results in Section 3.1 for e-star systems for all $e \geqslant 3$. We first construct equitably 2-chromatic e-star systems of order $2e$ for all $e \geqslant 3$. We then show that for all $k \geqslant 2$ and $e \geqslant 4$, if there exists a k-chromatic e-star system of order n_0 such that $n_0 \equiv 0, 1 \pmod{2e}$, then there exists a k-chromatic e-star system for all $n > n_0$ such that $n \equiv 0, 1 \pmod{2e}$. Next, from a $(k-1)$-chromatic e-star system of order $n_{k-1} \equiv 0 \pmod{2e}$, we construct a k-chromatic e-star system of order $n_k \equiv 0 \pmod{2e}$, for all $k \geqslant 3$. Finally, we finish this section by showing that for any integer $k \geqslant 2$, there exists some integer n_k where $n_k \equiv 0 \pmod{2e}$ such that for all $n \geqslant n_k$ where $n \equiv 0, 1 \pmod{2e}$, there exists a k-chromatic e-star system of order n.

We also study unique colourings for e-star systems. The idea of uniquely colourable designs has already arisen in the context of Steiner triple systems. In 2003, Forbes [17] showed that for every admissible $v \geqslant 25$, there exists a 3-balanced Steiner triple system with a unique 3-colouring and also a Steiner triple system which has a unique, nonequitable 3-colouring. Note that a Steiner triple system is said to be *3-balanced* if every 3-colouring of it is equitable. In Section 3.3 we first construct a strongly equitably uniquely 2-chromatic e-star system. Next, we construct a strongly equitably k-chromatic e-star system from a strongly equitably uniquely $(k-1)$-chromatic e-star system. We then show how to construct a strongly equitably uniquely k-chromatic e-star system from a strongly equitably k-chromatic e-star system. Finally, we finish this section by showing that for any integer $k \geqslant 2$, there exists some integer n_k where $n_k \equiv 0 \pmod{2e}$ such that for all $n \geqslant n_k$ where $n \equiv 0, 1 \pmod{2e}$, there exists a uniquely k-chromatic e-star system of order n.

1.2.2 Path systems

In Chapter 4 of this thesis, we investigate k-colourings of path systems. In Section 4.1 we construct some small systems that are used as ingredients to construct larger systems in the next section. In Section 4.2 we first observe that there exists a k-chromatic P_m system for any $k \geqslant 2$ and $m \geqslant 4$ where m is even. We then show that there exists an equitably 2-chromatic P_4 system of each admissible order. Finally, we prove that for any integer $k \geqslant 3$, there exists some integer n_k such that for all admissible $n \geqslant n_k$, there exists a k-chromatic P_4 system of order n. Note that a P_4 system of order n exists if and only if $n \equiv 0, 1, 3, 4 \pmod 6$ [34].

Chapter 2

The chromatic index of STS block intersection graphs

In this chapter, we investigate the chromatic index of block intersection graphs of Kirkman triple systems and cyclic Steiner triple systems. We conjecture that whenever a Steiner triple system has a block intersection graph with an even number of vertices, the graph is Class 1. We prove this to be true for Kirkman triple systems and cyclic Steiner triple sytems of order $v \equiv 9 \pmod{12}$. We also prove that the conjecture holds for a large family of cyclic Steiner triple systems of order $v \equiv 1 \pmod{12}$. This is joint work with David A. Pike and Jonathan Poulin, which is published in the Australasian Journal of Combinatorics [15].

2.1 Kirkman triple systems

Recall that a Kirkman triple system is a Steiner triple system for which the blocks can be partitioned into sets of disjoint blocks such that the union of the blocks in each set of the partition is the whole set of points. Ray-Chaudhuri and Wilson proved in

1971 that a KTS(v) exists if and only if $v \equiv 3 \pmod 6$ [33]. Here we show that each Kirkman triple system for which $v \equiv 9 \pmod{12}$ gives rise to a block intersection graph that admits a 1-factorisation. Before proving the main result of this section, we state the following theorem which will be used throughout this thesis.

Theorem 2.1.1 (König [26])**.** *If G is a bipartite graph, then $\chi'(G) = \Delta(G)$.*

We now show that Conjecture 1.1.3 holds for KTS of order $v \equiv 9 \pmod{12}$.

Theorem 2.1.2. *A KTS(v) has a Class 1 block intersection graph if and only if $v \equiv 9 \pmod{12}$.*

Proof. It has already been observed in page 6 that every STS(v) with $v \equiv 3$ (mod 12) has a Class 2 block intersection graph, so we only need to now consider Kirkman triple systems with $v \equiv 9 \pmod{12}$. Suppose that $\mathcal{B}$ is the block set of a KTS(v) with $v \equiv 9 \pmod{12}$. Hence $v = 6m + 3$ where m is an odd integer, and the KTS has $\frac{v-1}{2} = 3m + 1$ parallel classes. Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_{3m+1}$ be the parallel classes.

There are $\binom{3m+1}{2}$ pairs of parallel classes, and we wish to partition this set of pairs into $3m$ sets $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_{3m}$ so that within the pairs of each set $\mathcal{S}_i$ each parallel class occurs exactly once. Such a partition can be easily obtained by making use of a 1-factorisation of K_{3m+1} on the vertex set $\{\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_{3m+1}\}$.

Consider two parallel classes, say $\mathcal{P}_x$ and $\mathcal{P}_y$, that are together as a pair in some set $\mathcal{S}_i$. The blocks of these two parallel classes induce a subgraph $G_{x,y}$ of the block intersection graph G of the KTS such that $G_{x,y}$ is a 3-regular bipartite graph with $2m+1$ vertices in each of its two parts. Recalling that bipartite graphs are Class 1 by Theorem 3.1.2, we can therefore edge-colour this cubic bipartite graph with the three colours $3i-2$, $3i-1$ and $3i$. Moreover, we can use these same three colours for each subgraph of G induced by a pair of parallel classes in $\mathcal{S}_i$, since each such pair induces a subgraph of G that is disjoint from the subgraphs induced by the other pairs of $\mathcal{S}_i$.

We have now exhibited a proper colouring of the edges of the block intersection graph that uses only $9m$ colours, and hence we conclude that the KTS has a Class 1 block intersection graph. □

2.2 Cyclic Steiner triple systems of order $v \equiv 1 \pmod{12}$

In this section we investigate Steiner triple systems of order $v \equiv 1 \pmod{12}$ for which the block set $\mathcal{B}$ can be generated from a set of $\frac{v-1}{6}$ base blocks through the repeated application of a permutation σ of the points of V. More succinctly, we consider Steiner triple systems that are *cyclic*, which are known to exist if and only if $v \equiv 1$ or $3 \pmod 6$ and $v \neq 9$ [30]. Without loss of generality we may assume that $V = \mathbb{Z}_v$ and that the permutation σ is $(0, 1, 2, \ldots, v-1)$. We prefer an additive notation for this permutation, so that for any block B, we define $B + i = \{x + i \pmod v : x \in B\}$.

Here we will prove that many cyclic Steiner triple systems of order $v \equiv 1 \pmod{12}$ have Class 1 block intersection graphs. As with the Kirkman triple systems of Section 2.1 we will find a means of decomposing the block intersection graph into several subgraphs that themselves are Class 1. Many of these subgraphs will be bipartite, but there is another type of subgraph that will also be helpful to us.

Given positive integers n and k, we denote by $P(n, k)$ the *generalised Petersen graph* on $2n$ vertices and $3n$ edges. Specifically, the vertex set of $P(n, k)$ is $\{w_0, w_1, \ldots, w_{n-1}\} \cup \{x_0, x_1, \ldots, x_{n-1}\}$, and for each $i \in \mathbb{Z}_n$ let $\{w_i, w_{i+1}\}$, $\{w_i, x_i\}$ and $\{x_i, x_{i+k}\}$ be edges. The familiar Petersen graph is therefore $P(5, 2)$. Although the Petersen graph is Class 2, Castagna and Prins have shown that it is the unique generalised Petersen graph with this property [9].

Given a block B in a cyclic STS(v), we will refer to the set of blocks $\{B, B+1, B+2, \ldots, B+(v-1)\}$ as its *orbit*. Note that if, in the block intersection graph of a cyclic STS, B is adjacent to $B+i$, then B is also adjacent to $B-i$; the edges $\{B, B+i\}$ and $\{B, B-i\}$ are said to have *difference* i with respect to the orbit of B. If the six neighbours that a block B has within its orbit are $B \pm i$, $B \pm j$ and $B \pm k$, then we will refer to the three least positive elements of the 6-set $\{\pm i \pmod{v}, \pm j \pmod{v}, \pm k \pmod{v}\}$ as the *orbital differences* for the orbit of B. Note that for any integer $n \geqslant 3$, the cycle with n vertices $\{a_1, \ldots, a_n\}$ and n edges $\{a_1, a_2\}$, $\{a_2, a_3\}$, $\ldots$, $\{a_{n-1}, a_n\}$,$\{a_n, a_1\}$ is denoted by $(a_1, \ldots, a_n)$.

Lemma 2.2.1. *Given two orbits of size v in a cyclic STS(v), if one of them has an orbital difference d that is co-prime to v, and e is an orbital difference for the other orbit, then a $P(v, d^{-1}e)$ is formed by the edges of difference d in the first orbit, the edges of difference e in the second orbit and a suitably chosen 1-factor between the two orbits.*

Proof. Let B_0 be a block having an orbit with a difference d that is co-prime to v. For each $i \in \mathbb{Z}_v$ let $B_i = B_0 + i$. Then there exists a v-cycle $(B_0, B_d, B_{2d}, \ldots, B_{(v-1)d})$. B_0 is adjacent to nine vertices in the other orbit of the lemma's hypothesis. Let A_0 be one of these nine neighbours of B_0. We now obtain a set of v edges of the form $\{B_i, A_i\}$, yielding the spokes of a generalised Petersen graph. Since e is an orbital difference for the orbit of A_0, there exists an edge $\{A_{ie}, A_{(i+1)e}\}$ for each $i \in \mathbb{Z}_v$. Relabel $B_0, B_d, B_{2d}, \ldots, B_{(v-1)d}$ (resp. $A_0, A_d, A_{2d}, \ldots, A_{(v-1)d}$) as $w_0, w_1, w_2, \ldots, w_{v-1}$ (resp. $x_0, x_1, x_2, \ldots, x_{v-1}$), respectively. Then we find that x_j is adjacent to x_{j+k} if and only if A_j is adjacent to A_{j+kd}. In the graph that we have constructed, the three neighbours of A_j are A_{j+e}, A_{j-e} and B_j and so we obtain $e \equiv kd \pmod{v}$. Hence

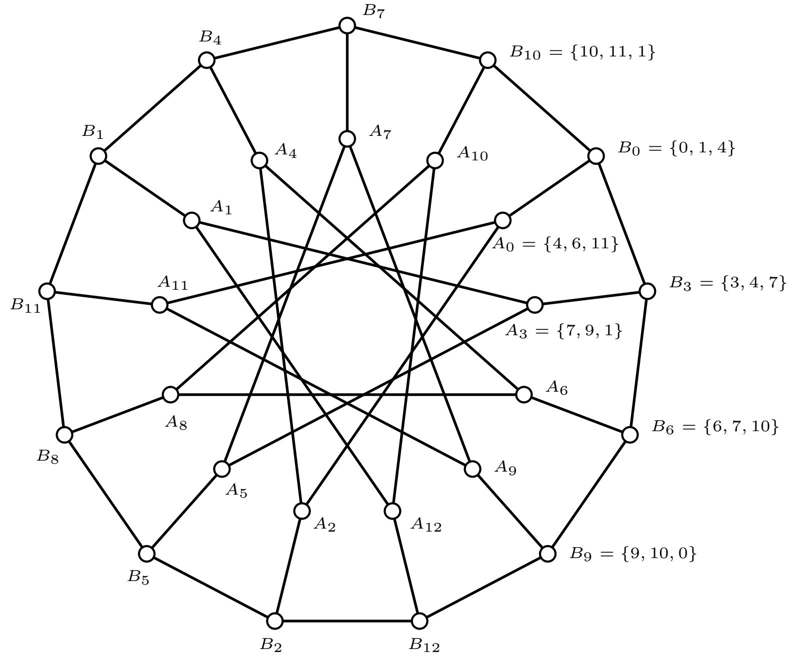

Figure 2.1: A $P(13, 5)$ subgraph in the block intersection graph of a cyclic STS(13)

$k \equiv d^{-1}e \pmod{v}$ and thus the edges that we have described constitute a $P(v, d^{-1}e)$ that is a subgraph of the block intersection graph of the cyclic STS(v). □

As an example of such a design, let $v = 13$ and observe that $\mathcal{B} = \{\{0, 1, 4\} + i : i \in \mathbb{Z}_{13}\} \cup \{\{0, 2, 7\} + i : i \in \mathbb{Z}_{13}\}$ is the block set of a cyclic STS(13) having $\{0, 1, 4\}$ and $\{0, 2, 7\}$ as base blocks. For each $i \in \mathbb{Z}_{13}$, let A_i be the block $\{4, 6, 11\} + i$ and similarly let $B_i = \{0, 1, 4\} + i$. The orbital differences arising from A_0 are 2, 5 and 6, while those arising from B_0 are 1, 3 and 4. Using orbital difference $d = 3$ (which is co-prime to v and has multiplicative inverse $d^{-1} \equiv 9$) we obtain a 13-cycle $(B_0, B_3, B_6, \ldots, B_{10})$. B_0 is adjacent to nine vertices in the orbit of A_0, one of which is A_0 itself. We thus have a set of 13 edges of the form $\{B_i, A_i\}$, yielding the spokes of a generalised Petersen graph. Since $e = 2$ is an orbital difference for the orbit of A_0, we include in our graph the edge $\{A_{2i}, A_{2(i+1)}\}$ for each $i \in \mathbb{Z}_{13}$. The result is the $P(13, 5)$ shown in Figure 2.1.

To foreshadow results that are yet to come, we can continue with this example and demonstrate how to obtain a 1-factorisation of the block intersection graph of this cyclic STS(13). Since $P(13,5)$ is a generalised Petersen graph other than the Petersen graph, then we can properly colour its edges with three colours, say 1, 2 and 3. Using orbital differences $d = 1$ and $e = 6$, along with a 1-factor of spokes such as edges of the form $\{A_i, B_{i+3}\}$ we obtain a $P(13,6)$ which we can edge-colour with colours 4, 5 and 6. The orbital differences $d = 4$ and $e = 5$, along with edges of the form $\{A_i, B_{i+7}\}$ produce another $P(13,6)$ which we can edge-colour with colours 7, 8 and 9. The remaining uncoloured edges from the block intersection graph of the cyclic STS(13) induce a 6-regular bipartite graph, which can be properly edge-coloured with colours 10 through 15.

Observe that any cyclic STS(v) for which $v \equiv 1 \pmod{12}$ has $\{1, 2, \ldots, \frac{v-1}{2}\}$ as its set of orbital differences. We will want to partition these $\frac{v-1}{2}$ differences into $\frac{v-1}{4}$ pairs of differences, and for each such pair we will want to construct a generalised Petersen graph of order $2v$. The hypothesis of Lemma 2.2.1 requires that one of the two orbital differences of the pair be co-prime to v. If we let φ denote Euler's totient function, then the proportion of the orbital differences that are co-prime to v is $\frac{\varphi(v)}{v-1}$. It is necessary that $\frac{\varphi(v)}{v-1}$ be at least one half in order for our approach to showing that a cyclic STS(v) has a Class 1 block intersection graph to work. We are able to show that $\frac{\varphi(v)}{v-1} \geqslant \frac{2}{3}$ is sufficient.

Theorem 2.2.1. *Any cyclic STS(v) with $v \equiv 1 \pmod{12}$ and $\frac{\varphi(v)}{v-1} \geqslant \frac{2}{3}$ has a Class 1 block intersection graph.*

Proof. First note that when $v \equiv 1 \pmod{12}$, the blocks of a cyclic STS(v) give rise to $N = \frac{v-1}{6}$ distinct orbits. Observe that N is even and consider the complete graph K_N in which each vertex denotes an orbit of the STS. Since N is even, K_N admits a 1-factorisation in which each 1-factor consists of a set of pairs of orbits.

If, for just one of these 1-factors, say $\mathcal{F}_0$, it is the case that each pair of orbits has at least three (of its six) orbital differences that are co-prime to v, then we can use the technique of Lemma 2.2.1 to construct three edge-disjoint generalised Petersen graphs for each pair of orbits corresponding to an edge of the 1-factor. For the first of the three generalised Petersen graphs constructed from each such pair of orbits, a proper edge-colouring with colours 1, 2 and 3 is possible. For the second (resp. third) generalised Petersen graph arising from each of these pair of orbits, colours 4, 5 and 6 (resp. 7, 8 and 9) can be used in a proper edge-colouring. Since for each pair of orbits, the edges between blocks of the same orbits have already been coloured, the remaining edges are the edges between these pairs of orbits. These edges induce a 6-regular bipartite graph, which is Class 1 by Theorem 2.1.1 and hence can be properly edge-coloured with colours 10 through 15. Let $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_{N-2}$ denote the remaining 1-factors in the 1-factorisation of K_N. Since all edges that join vertices of the same orbit have already been coloured (with one of the colours in the set $\{1, 2, \ldots, 9\}$), then each edge of these 1-factors corresponds to a pair of orbits for which the uncoloured edges induce a 9-regular bipartite graph. For each $i \in \{1, 2, \ldots, N-2\}$ and each edge of $\mathcal{F}_i$, we use the colours of $\{9i+7, 9i+8, \ldots, 9i+15\}$ to properly colour the edges of these 9-regular bipartite graphs. The result is a Class 1 colouring for the block intersection graph of the STS.

All that now remains is to prove that there is some partition of the N orbits into $\frac{N}{2}$ pairs such that at least three of the orbital differences of each of these pairs are co-prime to v (*i.e.*, we need to prove that there is a way to select the initial 1-factor $\mathcal{F}_0$ of K_N).

For each $j \in \{0, 1, 2, 3\}$ let c_j denote the number of orbits having exactly j orbital differences that are co-prime to v. Clearly $c_0 + c_1 + c_2 + c_3 = N$. Since each orbit with no differences that are co-prime to v must be paired with an orbit having three differences that are co-prime to v, and each orbit with one such difference must be paired with an orbit having at least two, it is readily evident that the following two conditions are necessary for a suitable 1-factor $\mathcal{F}_0$ to exist:

$$c_3 \geqslant c_0 \tag{2.1}$$

$$c_2 + (c_3 - c_0) \geqslant c_1 \tag{2.2}$$

Moreover, these two conditions are sufficient since the c_0 orbits having no differences that are co-prime to v can be arbitrarily paired with c_0 of the c_3 orbits having three co-prime differences. The c_1 orbits with one co-prime difference can then be arbitrarily paired with c_1 of the remaining $c_2+(c_3-c_0)$ unpaired orbits that have at least two co-prime differences. And lastly, all of the remaining unpaired orbits can be arbitrarily paired together, yielding a suitable 1-factor $\mathcal{F}_0$.

The total number of orbital differences arising from the N orbits is $3N = 3(c_0 + c_1 + c_2 + c_3)$ and the number of orbital differences that are not co-prime to v is $3c_0 + 2c_1 + c_2$. Given that $\frac{\varphi(v)}{v-1} \geqslant \frac{2}{3}$, it follows that $\frac{3c_0+2c_1+c_2}{3(c_0+c_1+c_2+c_3)} \leqslant \frac{1}{3}$. Therefore, $3c_0 + 2c_1 + c_2 \leqslant c_0 + c_1 + c_2 + c_3$, which implies that $c_3 \geqslant c_0$ (satisfying condition (2.1)) and also that $c_1 - c_3 \leqslant -2c_0$. It is clear that $c_0 + c_2$ is nonnegative and thus $-2c_0 \leqslant c_2 - c_0$. Therefore $c_1 - c_3 \leqslant c_2 - c_0$ and so condition (2.2) is satisfied. □

While Theorem 2.2.1 shows that Conjecture 1.1.3 holds for any cyclic STS(v) with $v \equiv 1 \pmod{12}$ and $\frac{\varphi(v)}{v-1} \geqslant \frac{2}{3}$, we note that our technique of decomposing the block intersection graph of a cyclic STS into generalised Petersen graphs and bipartite graphs can also often be used when $\frac{1}{2} \leqslant \frac{\varphi(v)}{v-1} < \frac{2}{3}$. The crucial requirement is that the orbital differences be distributed among the orbits of the STS in such a manner that conditions (2.1) and (2.2) are satisfied. Cyclic Steiner triple systems for which

$\frac{1}{2} \leqslant \frac{\varphi(v)}{v-1} < \frac{2}{3}$ do exist, although they appear to be somewhat sporadic. The smallest order $v \equiv 1 \pmod{12}$ for which $\frac{\varphi(v)}{v-1} < \frac{2}{3}$ is $v = 385$, for which $\frac{\varphi(v)}{v-1} = \frac{5}{8}$.

Cyclic Steiner triple systems for which our technique is certain to fail also exist, namely those for which fewer than half of the orbital differences are co-prime to v. The smallest such order $v \equiv 1 \pmod{12}$ is $v = 37182145$, for which $\frac{\varphi(v)}{v-1} = \frac{95040}{193657}$. If Conjecture 1.1.3 holds in general, then there must be some means of obtaining a Class 1 edge-colouring of the block intersection graph of a cyclic STS(37182145) other than by the method that we have described.

Interestingly, both 385 and 37182145 are products of consecutive prime numbers. We now investigate some of the conditions on v that pertain to when $\frac{\varphi(v)}{v-1}$ satisfies the hypothesis of Theorem 2.2.1. Note in particular that if $\frac{\varphi(v)}{v} \geqslant \frac{2}{3}$, then $\frac{\varphi(v)}{v-1} > \frac{2}{3}$. If v has prime factorisation $v = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t}$, then $\frac{\varphi(v)}{v} = (1 - \frac{1}{p_1})(1 - \frac{1}{p_2}) \cdots (1 - \frac{1}{p_t})$, which is independent of the exponents $\alpha_1, \alpha_2, \ldots, \alpha_t$ of the prime factorisation. We can therefore concentrate our attention on the prime factors of v.

Corollary 2.2.1. *If $v \equiv 1 \pmod{12}$ is a prime power, then any cyclic STS(v) has a Class 1 block intersection graph.*

Proof. As already observed, it suffices to prove that $\frac{\varphi(v)}{v} \geqslant \frac{2}{3}$. Let $v = p^\alpha$ and note that since $v \equiv 1 \pmod{12}$, we have $p \neq 2$ and $p \neq 3$. Hence $\frac{\varphi(v)}{v} = 1 - \frac{1}{p} \geqslant 1 - \frac{1}{5} > \frac{2}{3}$.
□

Corollary 2.2.2. *If $v \equiv 1 \pmod{12}$ has only two prime divisors, then any cyclic STS(v) has a Class 1 block intersection graph.*

Proof. Suppose $v = p_1^{\alpha_1} p_2^{\alpha_2}$ where $p_1 < p_2$. Then $p_1 \geqslant 5$ and $p_2 \geqslant 7$. Therefore $\frac{\varphi(v)}{v} \geqslant (1 - \frac{1}{5})(1 - \frac{1}{7}) > \frac{2}{3}$. □

Corollary 2.2.3. *If $v \equiv 1 \pmod{12}$ has only three prime divisors, and these three divisors are not one of the trios in the set $\mathcal{T} = \big\{\{5,7,11\}, \{5,7,13\}, \{5,7,17\}, \{5,7,19\}, \{5,7,23\}, \{5,7,29\}, \{5,7,31\}\big\}$, then any cyclic STS($v$) has a Class 1 block intersection graph.*

Proof. Observe that if v were to equal $p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}$ where $\{p_1, p_2, p_3\} \in \mathcal{T}$, then $\frac{\varphi(v)}{v}$ would be less than $\frac{2}{3}$, which would not imply the hypothesis of Theorem 2.2.1. However, with these cases excluded from consideration, we obtain $\frac{\varphi(v)}{v} > \frac{2}{3}$. □

2.3 Cyclic Steiner triple systems of order $v \equiv 9 \pmod{12}$

In this section we will prove that all cyclic Steiner triple systems of order $v \equiv 9 \pmod{12}$ have Class 1 block intersection graphs. Similar to cyclic Steiner triple systems of order $v \equiv 1 \pmod{12}$, we will find a decomposition of the block intersection graph into several subgraphs that themselves are Class 1.

Cyclic Steiner triple systems of order $v \equiv 9 \pmod{12}$ can be generated from a set of base blocks which are a solution of Heffter's second difference problem, through the repeated application of a permutation of the points of V. Recall that Heffter's second difference problem is as follows: given $v = 6n + 3$, is it possible to partition the set $\{1, 2, \ldots, \frac{v-1}{2} = 3n+1\} \setminus \{2n+1\}$ into n triples $\{x, y, z\}$ such that $x + y = \pm z \pmod{v}$? For $n \in \mathbb{Z}$ such a partition is always possible [30] . Given a solution to Heffter's second difference problem, one can construct a difference triple $\{0, x, x + y\}$ from each triple $\{x, y, z\}$, and then the set of all difference triples along with $\{0, 2n+1, 4n+2\}$ serve as a collection of base blocks. If $\mathcal{B}(v)$ is a collection of base blocks obtained from a solution of Heffter's second difference problem, then a cyclic STS(v) $(\{0, 1, \ldots, v-1\}, \mathcal{T})$ of order $v \equiv 9 \pmod{12}$ is constructed as follows:

$$\mathcal{T} = \Big\{\{i, x+i, x+y+i\} \mid 0 \leqslant i \leqslant v-1, \{0, x, x+y\} \in \mathcal{B}(v)\Big\}$$
$$\bigcup \Big\{\{i, 2n+1+i, 4n+2+i\} \mid 0 \leqslant i \leqslant 2n\Big\}.$$

For each base block $B \in \mathcal{B}(v)$, we call the set of blocks it generates a *full orbit* and for the base block $\{0, 2n+1, 4n+2\}$, the set of blocks it generates is called a *short orbit*.

Given a set of blocks $\mathcal{B}$, we denote by $G_{\mathcal{B}}$ the block intersection graph induced by the blocks of $\mathcal{B}$. If $\hat{\mathcal{B}} \subseteq \mathcal{B}$, then $G_{\mathcal{B}}[\hat{\mathcal{B}}]$ denotes the subgraph of $G_{\mathcal{B}}$ induced by $\hat{\mathcal{B}}$.

Lemma 2.3.1. *Let $(V, \mathcal{B})$ be a cyclic STS(v) of order $v \equiv 9 \pmod{12}$ and d be an orbital difference of the full orbit $\mathcal{O} = \big\{B_0, B_1, \ldots, B_{v-1}\big\}$. Then the edges between blocks of difference d in $G_{\mathcal{B}}[\mathcal{O}]$, form* $\gcd(v,d)$ *cycles $(B_i, B_{i+d}, \ldots, B_{i+(\ell-1)d})$, of length $\ell = \frac{v}{\gcd(v,d)}$, where $B_{i+\ell d} = B_i$.*

Proof. Without loss of generality we may assume that $i = 0$ and let $B_0, B_d, \ldots, B_{(\ell-1)d}$ be distinct elements and $B_{\ell d}$ be the first element such that $B_{\ell d} = B_{jd}$ for some $0 \leqslant j \leqslant \ell - 1$. Suppose that $B_{\ell d} = B_{jd}$ for some $1 \leqslant j \leqslant \ell - 1$. Then we have $\ell d \equiv jd \pmod{v}$ and as a result, $(\ell - j)d \equiv 0 \pmod{v}$ which yields $B_{(\ell-j)d} = B_0$ where $0 \leqslant \ell - j \leqslant \ell - 1$; this is a contradiction. Therefore, $B_{\ell d} = B_0$ and consequently $\ell d \equiv 0 \pmod{v}$. So ℓ is the least integer such that ℓd is a multiple of v, which is clearly $\frac{v}{\gcd(v,d)}$. Hence the number of cycles of this form is $\frac{v}{\ell} = \gcd(v,d)$. □

Theorem 2.3.1. *Any cyclic STS(v) with $v \equiv 9 \pmod{12}$ has a Class 1 block intersection graph.*

Proof. First, observe that in a cyclic STS(v) with $v \equiv 9 \pmod{12}$ the number of points and the number of blocks are $v = 6t+3$ and $b = t(6t+3) + (2t+1)$ respectively, where t is an odd integer. Hence we have t full orbits of size $v = 6t+3$

and one short orbit of size $\frac{v}{3} = 2t+1$, which give rise to $N = t+1$ distinct orbits. Let $\mathcal{O}_0, \mathcal{O}_1, \ldots, \mathcal{O}_{N-2}$ be the full orbits and $\mathcal{S}$ be the short orbit. We construct a complete graph K_N on N vertices where each vertex represents an orbit of the STS. Since N is even, K_N admits a 1-factorisation and hence we can partition the set of all pairs of orbits into $N-1$ sets $\mathcal{F}_0, \ldots, \mathcal{F}_{N-2}$ such that each orbit occurs exactly once in each set. For each one of these 1-factors, we have $\frac{N}{2} - 1$ pairs of full orbits and one pair consisting of a full orbit and a short orbit.

Let $\mathcal{F}$ be a 1-factor of K_N. If $(\mathcal{O}_i, \mathcal{O}_j) \in \mathcal{F}$, then we define $G_{\mathcal{F}}[\mathcal{O}_i, \mathcal{O}_j]$ to be the graph on vertex set $\mathcal{O}_i \cup \mathcal{O}_j$ and edge set $E(G_{\mathcal{F}}[\mathcal{O}_i, \mathcal{O}_j]) = \big\{\{B, \hat{B}\} \mid B \in \mathcal{O}_i, \hat{B} \in \mathcal{O}_j, B \cap \hat{B} \neq \emptyset\big\}$. Let $G_{\mathcal{F}} = \big(\bigcup_{(\mathcal{O}_i,\mathcal{O}_j)\in\mathcal{F}} G_{\mathcal{F}}[\mathcal{O}_i, \mathcal{O}_j]\big) \cup G_{\mathcal{F}}[\mathcal{O} \cup \mathcal{S}]$, where $(\mathcal{O}, \mathcal{S}) \in \mathcal{F}$. For each $\mathcal{F}$, we show how to colour $G_{\mathcal{F}}$, starting with $G_{\mathcal{F}_0}$.

Without loss of generality we may assume that $\mathcal{F}_0 = \{(\mathcal{O}_0, \mathcal{O}_1), \ldots, (\mathcal{O}_{N-4}, \mathcal{O}_{N-3}), (\mathcal{O}_{N-2}, \mathcal{S})\}$. To colour $G_{\mathcal{F}_0}$, we first properly colour the edges in $G_{\mathcal{F}_0}[\mathcal{O}_0, \mathcal{O}_1], \ldots, G_{\mathcal{F}_0}[\mathcal{O}_{N-4}, \mathcal{O}_{N-3}]$, which induce a 9-regular bipartite graph and hence is 9-edge-colourable by Theorem 2.1.1, using colours $1, 2, \ldots, 9$. If it is possible to colour the 9-regular subgraph $G_{\mathcal{B}}[\mathcal{O}_{N-2} \cup \mathcal{S}]$ using the same nine colours, then we can similarly properly colour each $G_{\mathcal{F}_i}$, where $i \in \{1, 2, \ldots, N-2\}$, using colours $\{9i+1, 9i+2, \ldots, 9i+9\}$. All that remains is to colour $G_{\mathcal{B}}[\mathcal{O}_{N-2} \cup \mathcal{S}]$ with nine colours.

Now, suppose that $\mathcal{O}_{N-2} = \{B_0, B_1, \ldots, B_{v-1}\}$ with orbital differences d_1, d_2, and d_3 and $\mathcal{S} = \{A_0, A_1, \ldots, A_{\frac{v}{3}-1}\}$. As we showed in Lemma 2.3.1, for each orbital difference d_k, $k \in \{1, 2, 3\}$, there exist $\gcd(v, d_k)$ cycles of length $\ell_k = \frac{v}{\gcd(v,d_k)}$ of the form $(B_i, B_{i+d_k}, \ldots, B_{i+(\ell_k-1)d_k})$ in $G_{\mathcal{F}_0}[\mathcal{O}_{N-2}]$. In $G_{\mathcal{F}_0}[\mathcal{O}_{N-2} \cup \mathcal{S}]$, each B_i, $i \in \{0, 1, \ldots, v-1\}$ is adjacent to three vertices in $\mathcal{S}$. Let A_i be one of these three neighbours of B_i. We then obtain a set of ℓ_k edges $\{B_i, A_i\}, \{B_{i+d_k}, A_{i+d_k}\}, \ldots, \{B_{i+(\ell_k-1)d_k}, A_{i+(\ell_k-1)d_k}\}$, where the indices of the blocks of the full orbit are reduced modulo v and the indices of the blocks of the short orbit are reduced modulo $\frac{v}{3}$. We call each cycle $(B_i, B_{i+d_k}, \ldots, B_{i+(\ell_k-1)d_k})$ along with the set of edges $\{B_i, A_i\}, \{B_{i+d_k}, A_{i+d_k}\}, \ldots,$ $\{B_{i+(\ell_k-1)d_k}, A_{i+(\ell_k-1)d_k}\}$ a configuration. We also call the union of these $\gcd(v, d_k)$ configurations, the subgraph of $G_{\mathcal{B}}[\mathcal{O}_{N-2} \cup \mathcal{S}]$ induced by orbital difference d_k. Now, let G_{d_k} be the subgraph of $G_{\mathcal{B}}[\mathcal{O}_{N-2} \cup \mathcal{S}]$ induced by the orbital difference d_k, $k \in \{1, 2, 3\}$. Clearly, we have $\gcd(v, d_k)$ configurations for orbital difference d_k, which decompose G_{d_k} into edge-disjoint subgraphs. Since the indices of the blocks of the short orbit are reduced modulo $\frac{v}{3}$, each block of the short orbit occurs thrice among the $\gcd(v, d_k)$ configurations. Hence G_{d_k} is a 3-regular graph which has a decomposition into edge-disjoint configurations. It now suffices to properly colour G_{d_1} using colours 1,2,3 and G_{d_2} using colours 4,5,6 and finally G_{d_3} using colours 7,8,9. Note that

$$V_1 = \left\{B_0, B_1, \ldots, B_{v-1}, A_0, A_1, \ldots, A_{\frac{v}{3}-1}\right\}$$

is the set of vertices and

$$E_1 = \Big\{\{B_0, A_0\}, \{B_{d_1}, A_{d_1}\}, \ldots, \{B_{(\ell_1-1)d_1}, A_{(\ell_1-1)d_1}\}, \{B_1, A_1\}, \{B_{1+d_1}, A_{1+d_1}\}, \ldots,$$
$$\{B_{1+(\ell_1-1)d_1}, A_{1+(\ell_1-1)d_1}\}, \ldots, \{B_n, A_n\}, \{B_{n+d_1}, A_{n+d_1}\}, \ldots, \{B_{n+(\ell_1-1)d_1}, A_{n+(\ell_1-1)d_1}\}\Big\} \cup$$
$$E(C_0) \cup E(C_1) \cup \cdots \cup E(C_n)$$

is the set of edges of G_{d_1}, where

$$C_0 = (B_0, B_{d_1}, \ldots, B_{(\ell_1-1)d_1}), C_1 = (B_1, B_{1+d_1}, \ldots, B_{1+(\ell_1-1)d_1}), \ldots,$$

$$C_n = (B_n, B_{n+d_1}, \ldots, B_{n+(\ell_1-1)d_1}),$$

are the $\gcd(v, d_1)$ cycles of length $\ell_1 = \frac{v}{\gcd(v,d_1)}$, and for each $0 \leqslant i \leqslant n$, $E(C_i)$ is the set of edges of C_i. Note that we have $n = \gcd(v, d_1) - 1$.

Suppose $A_0, A_{d_1}, \ldots, A_{(\ell'_1-1)d_1}$ are distinct elements and $A_{\ell'_1 d_1} = A_0$. It is easy to see that $\ell'_1 = \frac{\frac{v}{3}}{\gcd(\frac{v}{3},d_1)}$. Note that we have two cases: 1) $\ell'_1 = \ell_1$ and 2) $\ell'_1 < \ell_1$.

Case 1 ($\ell'_1 = \ell_1$): If $\ell'_1 = \ell_1 = \frac{v}{3}$, then there will be three configurations such that each A_j, $j \in \{0, 1, \ldots, \frac{v}{3} - 1\}$, is adjacent to exactly one block in each configuration. Since the configuration is a cycle with pendant edges, it is easy to colour the first configuration using three colours 1, 2, and 3. To colour the second configuration, first colour the edges incident to each A_j, $j \in \{0, 1, \ldots, \frac{v}{3} - 1\}$. If the edge incident to A_j is coloured by colour i in the first configuration, then colour the edge incident to A_j in the second configuration by colour $\sigma(i)$, where σ is the permutation $(1\ 2\ 3)$. The remaining edges in the second configuration can be easily coloured properly using colours 1, 2, and 3. Similarly, to colour the third configuration, we permute the colours in the first configuration by $\sigma^2 = (1\ 3\ 2)$. Since each vertex A_i has received a distinct colour in each of the three configurations, then G_{d_1} has been properly 3-edge-coloured.

If $\ell'_1 = \ell_1 < \frac{v}{3}$, then there will be more than three configurations. Note that we call the union of configurations in which each block of the short orbit occurs exactly once a layer. Clearly, there are three layers. Since in every layer, each A_j, $j \in \{0, 1, \ldots, \frac{v}{3}-1\}$, occurs exactly once and a layer is a union of configurations, we can easily colour the layer using three colours 1, 2, and 3. To colour the second and the third layers, again permute the colours in the first layer by $\sigma = (1\ 2\ 3)$ and $\sigma^2 = (1\ 3\ 2)$ respectively.

Case 2 ($\ell'_1 < \ell_1$): In this case we have $n + 1$ configurations $\mathcal{C}_1, \ldots, \mathcal{C}_{n+1}$, where $n + 1 \geqslant 1$, and a partition of the set of the blocks of the short orbit into $n + 1$ subsets $\mathcal{A}_1, \ldots, \mathcal{A}_{n+1}$ such that each block in the set $\mathcal{A}_i = \{A_{i_1}, \ldots, A_{i_{\ell'_1}}\}$ occurs three times in exactly one of the configurations, say $\mathcal{C}_i$, $i \in \{1, \ldots, n\}$. In other words, each block in $\mathcal{A}_i$ has degree three in $\mathcal{C}_i$ and degree zero in the other configurations. To colour the configuration $\mathcal{C}_i$, first colour the three edges incident to each block in $\mathcal{A}_i$ by colours 1, 2, and 3. It is then straightforward to properly colour the remaining edges in the configuration which form a cycle using colours 1, 2, and 3.

Similarly, it is possible to colour G_{d_2} using colours 4, 5, 6 and G_{d_3} using colours 7, 8, 9. $\square$

Chapter 3

Colourings of star systems

In this chapter, we investigate k-colourings of e-star systems for any $k \geqslant 2$ and $e \geqslant 3$. We first show that for any integer $k \geqslant 2$, there exists a k-chromatic 3-star system of order n for all sufficiently large admissible n. Next, we generalise this result for e-star systems for any $e \geqslant 3$. We show that for all $k \geqslant 2$ and $e \geqslant 3$, there exists a k-chromatic e-star system of order n for all sufficiently large n such that $n \equiv 0, 1 \pmod{2e}$. Finally, we prove that for all $k \geqslant 2$ and $e \geqslant 3$, there exists a uniquely k-chromatic e-star system of order n for all sufficiently large n such that $n \equiv 0, 1 \pmod{2e}$. This is joint work with David A. Pike, which is published in the Journal of Combinatorial Designs [16].

3.1 k-colourings of 3-star systems

We initially concentrate on 3-star systems and we show that for any integer $k \geqslant 2$, there exists some integer n_k such that for all admissible $n \geqslant n_k$, there exists a k-chromatic 3-star system of order n. We first construct an equitably 2-chromatic 3-star system of order n for all admissible $n \equiv 0, 1 \pmod 3$. Note that from here on, by decomposing the edges of some set E into stars, we mean decomposing the graph induced by the set of edges of E into stars.

Theorem 3.1.1. *For each admissible order n, there exists an equitably 2-chromatic 3-star system of order n.*

Proof. We first construct an equitably 2-chromatic 3-star system $S_3(6)$. Let $V = \{1,2,3,4,5,6\}$, $\mathcal{B} = \big\{\{1;3,5,6\}, \{2;1,3,6\}, \{4;1,2,3\}, \{5;2,3,4\}, \{6;3,4,5\}\big\}$, $R = \{1,3,5\}$, and $Y = \{2,4,6\}$. Then $(V,\mathcal{B})$ is an equitably 2-chromatic 3-star system of order six with colour classes R and Y.

Now, suppose that there exists an equitably 2-chromatic 3-star system $S_3(3t)$, $(V,\mathcal{B})$, where $t \geqslant 2$, $V = \{1,\dots,3t\}$ is the set of points which is partitioned into two subsets R and Y, and $\mathcal{B}$ is the set of blocks. Without loss of generality, if $3t$ is odd, we can let $R = \{1,3,5,\dots,3t\}$, $Y = \{2,4,6,\dots,3t-1\}$ and if $3t$ is even, $R = \{1,3,5,\dots,3t-1\}$, $Y = \{2,4,6,\dots,3t\}$.

Next, we construct an equitably 2-chromatic 3-star system $S_3(3t+1)$, $(\hat{V},\hat{\mathcal{B}})$, from $(V,\mathcal{B})$. Let $\hat{V} = V \cup \{3t+1\}$ and $\hat{\mathcal{B}} = \mathcal{B} \cup \big\{\{3t+1;1,2,3\},\{3t+1;4,5,6\},\dots,\{3t+1;3t-2,3t-1,3t\}\big\}$. Let $R = \{1,3,5,\dots,3t\}$, $Y = \{2,4,6,\dots,3t+1\}$ if $3t$ is odd, and $R = \{1,3,5,\dots,3t+1\}$, $Y = \{2,4,6,\dots,3t\}$ if $3t$ is even. Then $(\hat{V},\hat{\mathcal{B}})$ is an equitably 2-chromatic 3-star system of order $3t+1$ with colour classes R and Y.

Finally, we construct an equitably 2-chromatic 3-star system $S_3(3t+3)$, $(\tilde{V},\tilde{\mathcal{B}})$, from $(V,\mathcal{B})$. Let $\tilde{V} = V \cup \{3t+1,3t+2,3t+3\}$ and $\tilde{\mathcal{B}} = \mathcal{B} \cup \big\{\{3t+1;1,2,3\},\dots,\{3t+1;3t-5,3t-4,3t-3\},\{3t+2;1,2,3\},\dots,\{3t+2;3t-5,3t-4,3t-3\},\{3t+3;1,2,3\},\dots,\{3t+3;3t-5,3t-4,3t-3\},\{3t+1;3t-2,3t-1,3t+2\},\{3t+2;3t-2,3t-1,3t+3\},\{3t+3;3t-2,3t-1,3t+1\},\{3t;3t+1,3t+2,3t+3\}\big\}$. Let R be the set of odd elements of $\tilde{V}$ and Y be the set of even elements of $\tilde{V}$. Then $(\tilde{V},\tilde{\mathcal{B}})$ is an equitably 2-chromatic 3-star system of order $3t+3$ with colour classes R and Y. $\square$

We now show how to construct a k-chromatic 3-star system from a smaller k-chromatic 3-star system.

Theorem 3.1.2. *Let $k \geqslant 2$. If there exists a k-chromatic 3-star system of order n_0, then there exists a k-chromatic 3-star system of order n for all admissible $n > n_0$.*

Proof. Suppose that there exists a k-chromatic 3-star system $(V, \mathcal{B})$, of order n_0, where $V = \{1, \ldots, n_0\}$ is the set of points and $\mathcal{B}$ is the set of blocks. Given a k-colouring of $(V, \mathcal{B})$ with colours $1, 2, \ldots, k$ and colour classes $C_1, C_2, \ldots, C_k$, let $R = \bigcup_{i=1}^{\ell} C_i$ and $Y = \bigcup_{i=\ell+1}^{k} C_i$ for some integer ℓ such that $1 \leqslant \ell < k$. Let $r_1, r_2, \ldots, r_{|R|}$ be the elements of R and $y_1, y_2, \ldots, y_{|Y|}$ be the elements of Y. Observe that R and Y form a partition of V. Without loss of generality assume that $|R| \geqslant |Y|$.

Case 1. Suppose that $n_0 \equiv 0 \pmod 3$. Then $n_0 = 3t$, $t \geqslant 2$.

We construct a k-chromatic $S_3(3t+1)$, $(\hat{V}, \hat{\mathcal{B}})$, from $(V, \mathcal{B})$. Let $\hat{V} = V \cup \{3t+1\}$.

If $|R| = |Y|$, let $\hat{\mathcal{B}} = \mathcal{B} \cup \mathcal{T}$ where $\mathcal{T}$ is the set $\big\{\{3t+1; r_1, r_2, y_1\}, \{3t+1; y_2, y_3, r_3\}, \ldots, \{3t+1; r_{|R|-2}, r_{|R|-1}, y_{|Y|-2}\}, \{3t+1; y_{|Y|-1}, y_{|Y|}, r_{|R|}\}\big\}$.

Otherwise, $|R| > |Y|$. If $|Y|$ is even, then let $m = \frac{|Y|}{2}$ and $\hat{\mathcal{B}} = \mathcal{B} \cup \mathcal{T}$ where $\mathcal{T}$ is the set $\big\{\{3t+1; y_1, y_2, r_1\}, \ldots, \{3t+1; y_{|Y|-1}, y_{|Y|}, r_m\}, \{3t+1; r_{m+1}, r_{m+2}, r_{m+3}\}, \ldots, \{3t+1; r_{|R|-2}, r_{|R|-1}, r_{|R|}\}\big\}$. If $|Y|$ is odd then let $m = \frac{|Y|+3}{2}$ and $\hat{\mathcal{B}} = \mathcal{B} \cup \mathcal{T}$ where $\mathcal{T}$ is the set $\big\{\{3t+1; y_1, y_2, r_1\}, \ldots, \{3t+1; y_{|Y|-2}, y_{|Y|-1}, r_{m-2}\}, \{3t+1; y_{|Y|}, r_{m-1}, r_m\}, \{3t+1; r_{m+1}, r_{m+2}, r_{m+3}\}, \ldots, \{3t+1; r_{|R|-2}, r_{|R|-1}, r_{|R|}\}\big\}$.

Note that $(\hat{V}, \hat{\mathcal{B}})$ is not $(k-1)$-colourable because it contains a copy of $(V, \mathcal{B})$. Observe that $C_1, \ldots, C_{k-1}, C_k \cup \{3t+1\}$ are the colour classes of a k-colouring of $(\hat{V}, \hat{\mathcal{B}})$. Therefore, $(\hat{V}, \hat{\mathcal{B}})$ is a k-chromatic 3-star system of order $3t+1$.

Case 2. Suppose that $n_0 \equiv 1 \pmod 3$. Then $n_0 = 3t+1$, $t \geqslant 2$. We construct a k-chromatic $S_3(3t+3)$, $(\hat{V}, \hat{\mathcal{B}})$, from $(V, \mathcal{B})$. Let $\hat{V} = V \cup \{3t+2, 3t+3\}$. Note that

$|R| \geqslant 4$ since $n_0 \geqslant 7$ and $|R| \geqslant |Y|$. So without loss of generality, we may assume that $\{3t, 3t+1\} \subset R$. Moreover, since the edge $\{3t, 3t+1\}$ must be in some 3-star, we may also assume that there exists a 3-star $S = \{3t+1; 3t, 3t-1, 3t-2\} \in \mathcal{B}$. We will dismantle the 3-star S in each of the following subcases:

Case 2.1. All the vertices of the 3-star S belong to the set R. Hence R is the union of $\ell \geqslant 2$ distinct colour classes. Let $R' = R \setminus \{3t+1, 3t, 3t-1, 3t-2\}$.

Let $\mathcal{T}$ be the set $\big\{\{3t+2; 3t+1, 3t, 3t-1\}, \{3t+3; 3t+2, 3t, 3t-1\}, \{3t-2; 3t+2, 3t+1, 3t+3\}, \{3t+1; 3t+3, 3t, 3t-1\}\big\}$.

If $|R'| = |Y|$, let $\hat{\mathcal{B}} = (\mathcal{B} \setminus \{S\}) \cup \mathcal{T} \cup (\bigcup_{i=1}^{i=2} \mathcal{T}_i)$ where $\mathcal{T}_i$ is the set $\big\{\{3t+i; r_1, r_2, y_1\}, \{3t+i; y_2, y_3, r_3\}, \ldots, \{3t+i; r_{|R|-6}, r_{|R|-5}, y_{|Y|-2}\}, \{3t+i; y_{|Y|-1}, y_{|Y|}, r_{|R|-4}\}\big\}$ for $i \in \{1, 2\}$.

If $|R'| > |Y|$ and $|Y|$ is even, then let $m = \frac{|Y|}{2}$ and $\hat{\mathcal{B}} = (\mathcal{B} \setminus \{S\}) \cup \mathcal{T} \cup (\bigcup_{i=1}^{i=2} \mathcal{T}_i)$ where $\mathcal{T}_i$ is the set $\big\{\{3t+i; y_1, y_2, r_1\}, \ldots, \{3t+i; y_{|Y|-1}, y_{|Y|}, r_m\}, \{3t+i; r_{m+1}, r_{m+2}, r_{m+3}\}, \ldots, \{3t+i; r_{|R|-6}, r_{|R|-5}, r_{|R|-4}\}\big\}$ for $i \in \{1, 2\}$. The edges can be decomposed into 3-stars in a similar manner when $|R'| > |Y|$ and $|Y|$ is odd.

Otherwise $|R'| < |Y|$. Then we have $|R'| = |Y| - 1$ or $|R'| = |Y| - 2$ or $|R'| = |Y| - 3$ or $|R'| = |Y| - 4$. If $|R'| = 0$, then $Y = \{1, 2, 3\}$ and $R = \{4, 5, 6, 7\}$. Let $\hat{\mathcal{B}} = (\mathcal{B} \setminus \{S\}) \cup \mathcal{T}_0$ where $\mathcal{T}_0$ is the set $\big\{\{8; 1, 2, 3\}, \{9; 1, 2, 3\}, \{8; 7, 6, 5\}, \{9; 8, 6, 5\}, \{4; 8, 7, 9\}, \{7; 9, 6, 5\}\big\}$. If $|R'| \geqslant 1$ and $|R'| = |Y| - 4$, let $\hat{\mathcal{B}} = (\mathcal{B} \setminus \{S\}) \cup \mathcal{T} \cup (\bigcup_{i=1}^{i=2} \mathcal{T}_i)$ where $\mathcal{T}_i$ is the set $\big\{\{3t+i; r_1, y_1, y_2\}, \{3t+i; r_2, y_3, y_4\}, \{3t+i; r_3, y_5, y_6\}, \{3t+i; r_4, y_7, y_8\}, \{3t+i; r_5, r_6, y_9\}, \{3t+i; y_{10}, y_{11}, r_7\}, \ldots, \{3t+i; r_{|R|-6}, r_{|R|-5}, y_{|Y|-2}\}, \{3t+i; y_{|Y|-1}, y_{|Y|}, r_{|R|-4}\}\big\}$ for $i \in \{1, 2\}$. The edges can be decomposed into 3-stars in a similar manner when $1 \leqslant |R'| < |Y|$ and $|R'| = |Y| - j$, $1 \leqslant j \leqslant 3$.

Note that $(\hat{V}, \hat{\mathcal{B}})$ is not $(k-1)$-colourable because it contains a copy of $(V, \mathcal{B})$. Observe that $C_1, \ldots, C_{k-1}, C_k \cup \{3t+2, 3t+3\}$ are the colour classes of a k-colouring of $(\hat{V}, \hat{\mathcal{B}})$. Therefore, $(\hat{V}, \hat{\mathcal{B}})$ is a k-chromatic 3-star system of order $3t+3$.

Case 2.2. Exactly three vertices of the 3-star S belong to the set R. In this case, we let R' to be set R excluding the three vertices of S that belong to the set R and Y' to be set Y excluding the one vertex of S that belong to the set Y. The edges can be decomposed into 3-stars in a manner similar to the last case for when $|R'| = |Y'|$, $|R'| > |Y'|$, and $|R'| < |Y'|$. The only difference is that in this case $|R'| = |R| - 3$, but in the last case $|R'| = |R| - 4$. This difference does not have impact on the process of decomposition.

Case 2.3. Exactly two vertices of the 3-star S belong to the set R. Let $R' = R \setminus \{3t, 3t+1\}$ and $Y' = Y \setminus \{3t-1, 3t-2\}$.

If $|R'| > |Y'|$ and $|Y'|$ is even, then let $m = \frac{|Y'|}{2}$ and $\hat{\mathcal{B}} = (\mathcal{B} \setminus \{S\}) \cup \mathcal{T} \cup (\bigcup_{i=1}^{i=2} \mathcal{T}_i)$ where $\mathcal{T}_i$ is the set $\big\{\{3t+i; y_1, y_2, r_1\}, \ldots, \{3t+i; y_{|Y|-3}, y_{|Y|-2}, r_m\}, \{3t+i; r_{m+1}, r_{m+2}, r_{m+3}\}, \ldots, \{3t+i; r_{|R|-4}, r_{|R|-3}, r_{|R|-2}\}\big\}$ for $i \in \{1, 2\}$ and $\mathcal{T}$ is the set $\big\{\{3t+2; 3t+1, 3t, 3t-1\}, \{3t+3; 3t+2, 3t, 3t-1\}, \{3t-2; 3t+2, 3t+1, 3t+3\}, \{3t+1; 3t+3, 3t, 3t-1\}\big\}$. The edges can be decomposed into 3-stars in a similar manner when $|R'| > |Y'|$ and $|Y'|$ is odd; likewise when $|R'| = |Y'|$.

Note that $(\hat{V}, \hat{\mathcal{B}})$ is not $(k-1)$-colourable because it contains a copy of $(V, \mathcal{B})$. Observe that $C_1, \ldots, C_{k-1}, C_k \cup \{3t+2, 3t+3\}$ are the colour classes of a k-colouring of $(\hat{V}, \hat{\mathcal{B}})$. Therefore, $(\hat{V}, \hat{\mathcal{B}})$ is a k-chromatic 3-star system of order $3t+3$. □

The following theorem from [6] is used in the proof of some theorems in this chapter. We denote the complete h-uniform hypergraph of n vertices, $V = \{v_1, \ldots, v_n\}$, which is the set of all h-element subsets of V, by K_n^h.

Theorem 3.1.3. *[6] Let $a_1, \ldots, a_s$ be natural numbers such that $\sum_{i=1}^{s} a_i = \binom{n}{h}$. Then* *the edges of* K_n^h *can be partitioned in almost regular hypergraphs* X_i*, so that the number of edges of* X_i *is* a_i*,* $1 \leqslant i \leqslant s$*.*

We now show how to iteratively construct a k-chromatic 3-star system from a $(k-1)$-chromatic 3-star system.

Theorem 3.1.4. *Let* $k \geqslant 3$*. If there exists a* $(k-1)$*-chromatic 3-star system of order* n_{k-1}*, then there exists a* k*-chromatic 3-star system of order* n_k*, for some admissible order* n_k*.*

Proof. Given that there exists a $(k-1)$-chromatic 3-star system, then by Theorem 3.1.2, there exists a $(k-1)$-chromatic 3-star system of order n_{k-1} for some $n_{k-1} \in \mathbb{Z}$ such that $n_{k-1} \equiv 0 \pmod 3$. Let $(U_0, \mathcal{A}_0)$ be a $(k-1)$-chromatic 3-star system of order n_{k-1} with colour classes $C_1, \ldots, C_{k-1}$ such that $|C_1| \leqslant \cdots \leqslant |C_{k-1}|$ and each vertex of C_s has colour s, $1 \leqslant s \leqslant k-1$. For each $s \in \{1, \ldots, k-1\}$, let $C_s = \{c_1^s, \ldots, c_{|C_s|}^s\}$. For a positive integer ℓ that will be fixed later and for each $i \in \{1, \ldots, \ell\}$, let $U_i = U_0 \times \{i\}$ and $\mathcal{A}_i = \mathcal{A}_0 \times \{i\}$ where $\mathcal{A}_0 \times \{i\}$ denotes the set $\{S \times \{i\} \mid S \in \mathcal{A}_0\}$ and when $S = \{x; a, b, c\}$, $S \times \{i\}$ denotes $\big\{x \times \{i\}; a \times \{i\}, b \times \{i\}, c \times \{i\}\big\}$. So $(U_i, \mathcal{A}_i)$ has colour classes $C_1 \times \{i\}, \ldots, C_{k-1} \times \{i\}$. If $2k \equiv 0 \pmod 3$ then let $V = \{1, 2, 3, \ldots, 2k\}$; otherwise let $V = \{1, 2, 3, \ldots, 2k-1\}$. Also let $U = \bigcup_{i=1}^{\ell} U_i$, so that $V \cap U = \emptyset$. We will embed $(U_1, \mathcal{A}_1), \ldots, (U_\ell, \mathcal{A}_\ell)$ into a k-chromatic 3-star system $(\hat{V}, \hat{\mathcal{B}})$ where $\hat{V} = V \cup U$. Since the edges of the complete graph on the set V admits a decomposition into 3-stars, we let $(V, \mathcal{B})$ be an arbitrary 3-star system of order $2k-1$ or $2k$. We now need to decompose the edges between V and U and the edges between U_i and U_j, for $1 \leqslant i < j \leqslant \ell$ into 3-stars such that the resulting 3-star system $(\hat{V}, \hat{\mathcal{B}})$ is k-chromatic. To do so, we will decompose the edges between V and U into 3-stars in a way such that no 3-subset in V is monochromatic in any putative $(k-1)$-colouring. Three cases arise.

Case 1: $2k-1 \equiv 0 \pmod 3$. Then $2k-1 = 3t$ for some $t \in \mathbb{Z}^+$. The number of 3-subsets of the set V is $\binom{2k-1}{3} = t(k-1)(2k-3)$. We now fix $\ell = (k-1)(2k-3)$. Partition the set of all 3-subsets of V into ℓ sets $\mathbb{T}_1, \ldots, \mathbb{T}_\ell$ each consisting of t mutually disjoint 3-subsets. This partition is known to exist by Theorem 3.1.3. Let $\mathbb{T}_i = \big\{\{x_1^i, y_1^i, z_1^i\}, \ldots, \{x_t^i, y_t^i, z_t^i\}\big\}$, $1 \leqslant i \leqslant \ell$. Decompose the edges between V and U_i into the 3-stars of the set $\mathcal{T}_i = \bigcup_{u \in U_i} \big\{\{u; x_1^i, y_1^i, z_1^i\}, \ldots, \{u; x_t^i, y_t^i, z_t^i\}\big\}$, where $1 \leqslant i \leqslant \ell$.

We now begin to decompose the edges between U_i and U_j for $1 \leqslant i < j \leqslant \ell$ into 3-stars. If $|C_1|$ is even, then let $r = \frac{|C_1|}{2}$. Observe that $2r = |C_1| \leqslant |C_2|$ and so for each $p \in \{1, 2, \ldots, |C_1|\}$ we decompose the edges between $c_p^1 \times \{i\}$ and U_j into the 3-stars of the set: $\mathcal{S}_p^1 = \big\{\{c_p^1 \times \{i\}; c_1^1 \times \{j\}, c_2^1 \times \{j\}, c_1^2 \times \{j\}\}, \{c_p^1 \times \{i\}; c_3^1 \times \{j\}, c_4^1 \times \{j\}, c_2^2 \times \{j\}\}, \ldots, \{c_p^1 \times \{i\}; c_{|C_1|-1}^1 \times \{j\}, c_{|C_1|}^1 \times \{j\}, c_r^2 \times \{j\}\}\big\} \cup \big\{\{c_p^1 \times \{i\}; x_t, y_t, z_t\} \mid 1 \leqslant t \leqslant \frac{n_{k-1}-3r}{3}\big\}$, where the sets $\{x_t, y_t, z_t\}$, $1 \leqslant t \leqslant \frac{n_{k-1}-3r}{3}$, form a partition of $U_j \backslash \big((C_1 \times \{j\}) \cup \{c_1^2 \times \{j\}, c_2^2 \times \{j\}, \ldots, c_r^2 \times \{j\}\}\big)$. If $|C_1|$ is odd, then let $r = \frac{|C_1|+1}{2}$ and decompose the edges between $c_p^1 \times \{i\}$ and U_j into the 3-stars of the set: $\mathcal{S}_p^1 = \big\{\{c_p^1 \times \{i\}; c_1^1 \times \{j\}, c_2^1 \times \{j\}, c_1^2 \times \{j\}\}, \{c_p^1 \times \{i\}; c_3^1 \times \{j\}, c_4^1 \times \{j\}, c_2^2 \times \{j\}\}, \ldots, \{c_p^1 \times \{i\}; c_{|C_1|-2}^1 \times \{j\}, c_{|C_1|-1}^1 \times \{j\}, c_{r-1}^2 \times \{j\}\}, \{c_p^1 \times \{i\}; c_{|C_1|}^1 \times \{j\}, c_r^2 \times \{j\}, c_{r+1}^2 \times \{j\}\}\big\} \cup \big\{\{c_p^1 \times \{i\}; x_t, y_t, z_t\} \mid 1 \leqslant t \leqslant \frac{n_{k-1}-3r}{3}\big\}$, where the sets $\{x_t, y_t, z_t\}$, $1 \leqslant t \leqslant \frac{n_{k-1}-3r}{3}$, form a partition of $U_j \setminus \big((C_1 \times \{j\}) \cup \{c_1^2 \times \{j\}, c_2^2 \times \{j\}, \ldots, c_{r+1}^2 \times \{j\}\}\big)$. The edges between $(C_2 \times \{i\}) \cup \cdots \cup (C_{k-2} \times \{i\})$ and U_j are decomposed into sets of 3-stars $\mathcal{S}_1^2, \ldots, \mathcal{S}_{|C_2|}^2, \ldots, \mathcal{S}_1^{k-2}, \ldots, \mathcal{S}_{|C_{k-2}|}^{k-2}$ in a similar manner.

Next, we decompose the edges between $C_{k-1} \times \{i\}$ and U_j into 3-stars. If $|C_{k-1}| = 2n$ for some $n \in \mathbb{Z}^+$, then we let $C_k = \{c_{n+1}^{k-1}, \ldots, c_{|C_{k-1}|}^{k-1}\}$. If $2n$ is a multiple of 3, then for each $p \in \{1, 2, \ldots, |C_{k-1}|\}$ we decompose the edges between c_p^{k-1} and U_j into the 3-stars of the set: $\mathcal{S}_p^{k-1} = \big\{\{c_p^{k-1} \times \{i\}; c_1^{k-1} \times \{j\}, c_2^{k-1} \times \{j\}, c_{n+1}^{k-1} \times \{j\}\}, \{c_p^{k-1} \times \{i\}; c_{n+2}^{k-1} \times \{j\}, c_{n+3}^{k-1} \times \{j\}, c_3^{k-1} \times \{j\}\}, \ldots, \{c_p^{k-1} \times \{i\}; c_{n-2}^{k-1} \times \{j\}, c_{n-1}^{k-1} \times \{j\}, c_{|C_{k-1}|-2}^{k-1} \times$

$\{j\}\}$, $\{c_p^{k-1}\times\{i\}; c_{|C_{k-1}|-1}^{k-1}\times\{j\}, c_{|C_{k-1}|}^{k-1}\times\{j\}, c_n^{k-1}\times\{j\}\}\big\}\cup\big\{\{c_p^{k-1}\times\{i\}; x_t, y_t, z_t\} \mid 1\leqslant t\leqslant \frac{n_{k-1}-|C_{k-1}|}{3}\big\}$, where the sets $\{x_t, y_t, z_t\}$, $1\leqslant t\leqslant \frac{n_{k-1}-|C_{k-1}|}{3}$, form a partition of $\bigcup_{s=1}^{k-2}(C_s\times\{j\})$. If $2n$ is congruent to 1 modulo 3, then the edges are decomposed into the 3-stars of the set: $\mathcal{S}_p^{k-1}=\big\{\{c_p^{k-1}\times\{i\}; c_1^{k-1}\times\{j\}, c_2^{k-1}\times\{j\}, c_{n+1}^{k-1}\times\{j\}\}$, $\{c_p^{k-1}\times\{i\}; c_{n+2}^{k-1}\times\{j\}, c_{n+3}^{k-1}\times\{j\}, c_3^{k-1}\times\{j\}\}$, $\ldots$, $\{c_p^{k-1}\times\{i\}; c_{n-2}^{k-1}\times\{j\}, c_{n-1}^{k-1}\times\{j\}, c_{|C_{k-1}|-3}^{k-1}\times\{j\}\}$, $\{c_p^{k-1}\times\{i\}; c_{|C_{k-1}|-2}^{k-1}\times\{j\}, c_{|C_{k-1}|-1}^{k-1}\times\{j\}, c_n^{k-1}\times\{j\}\}$, $\{c_p^{k-1}\times\{i\}; c_{|C_{k-1}|}^{k-1}\times\{j\}, a, b\}\big\}\cup\big\{\{c_p^{k-1}\times\{i\}; x_t, y_t, z_t\} \mid 1\leqslant t\leqslant \frac{n_{k-1}-(|C_{k-1}|+2)}{3}\big\}$, where a and b are distinct vertices of $\bigcup_{s=1}^{k-2}(C_s\times\{j\})$ and the sets $\{x_t, y_t, z_t\}$, $1\leqslant t\leqslant \frac{n_{k-1}-(|C_{k-1}|+2)}{3}$, form a partition of $\bigcup_{s=1}^{k-2}(C_s\times\{j\})\setminus\{a,b\}$. The edges are decomposed into 3-stars in a similar manner when $2n$ is congruent to 2 modulo 3. If $|C_{k-1}|=2n+1$ for some $n\in\mathbb{Z}^+$, then we let $C_k=\{c_{n+1}^{k-1}\ \ldots, c_{|C_{k-1}|}^{k-1}\}$ and decompose the edges into 3-stars in a similar manner. Now $\mathcal{U}_j^i=(\bigcup_{a=1}^{|C_1|}\mathcal{S}_a^1)\cup\cdots\cup(\bigcup_{a=1}^{|C_{k-1}|}\mathcal{S}_a^{k-1})$ forms a 3-star decomposition of the edges between U_i and U_j, $1\leqslant i<j\leqslant \ell$.

Let $\hat{\mathcal{B}}=\mathcal{B}\cup(\bigcup_{i=1}^{\ell}\mathcal{A}_i)\cup(\bigcup_{i=1}^{\ell}\mathcal{T}_i)\cup(\bigcup_{j=2}^{\ell}\mathcal{U}_j^1)\cup(\bigcup_{j=3}^{\ell}\mathcal{U}_j^2)\cup\cdots\cup\mathcal{U}_\ell^{\ell-1}$. The k colour classes $\{1,2\}\cup\bigcup_{i=1}^{\ell}(C_1\times\{i\})$, $\{3,4\}\cup\bigcup_{i=1}^{\ell}(C_2\times\{i\})$, $\cdots$, $\{2k-1,2k-2\}\cup\bigcup_{i=1}^{\ell}(C_{k-1}\setminus C_k)\times\{i\})$, $\{2k-1\}\cup\bigcup_{i=1}^{\ell}(C_k\times\{i\})$ exhibit a k-colouring of $(\hat{V},\hat{\mathcal{B}})$, because no star is monochromatic in this k-colouring. Since $(U_i,\mathcal{A}_i)$, $1\leqslant i\leqslant\ell$ is a $(k-1)$-chromatic 3-star system, there are vertices $u_1^i,\ldots,u_{k-1}^i$ in U_i such that $u_s^i\in C_s\times\{i\}$ for each $1\leqslant s\leqslant k-1$. Since $|V|=2k-1>2(k-1)$ then when attempting to colour V with $k-1$ colours, some colour must occur at least thrice within V. Thus, for some $j\in\{1,\ldots,t\}$ and some $i\in\{1,\ldots,\ell\}$ and for some $s\in\{1,\ldots,k-1\}$, the three vertices x_j^i, y_j^i and z_j^i are each coloured with the same colour s. Then the 3-star $\{u_s^i; x_j^i, y_j^i, z_j^i\}$ would be monochromatic which is a contradiction. Hence $(\hat{V},\hat{\mathcal{B}})$ cannot be coloured with $k-1$ colours.

Therefore, $(\hat{V}, \hat{\mathcal{B}})$ is a k-chromatic 3-star system of order $n_k = (2k-1) + n_{k-1}(k-1)(2k-3)$.

Case 2: $2k-1 \equiv 1 \pmod 3$. Then $2k-1 = 3t+1$ for some $t \in \mathbb{Z}^+$. The number of 3-subsets of the set V is $\binom{2k-1}{3} = \frac{t}{2}(2k-1)(2k-3)$. Note that since $\binom{2k-1}{3} = \frac{t}{2}(2k-1)(2k-3)$ is an integer and $2k-1$ and $2k-3$ are not divisible by 2, t must be even. Let $t = 2t'$. Partition the set of all 3-subsets of V into $\ell' = (2k-1)(k-2)$ sets $\mathbb{T}_1, \ldots, \mathbb{T}_{\ell'}$ of size t of disjoint 3-subsets and $\ell'' = 2k-1$ sets $\mathbb{T}_{\ell'+1}, \ldots, \mathbb{T}_{\ell'+\ell''}$ of size t' of disjoint 3-subsets. Such a partition is known to exist by Theorem 3.1.3. We now fix $\ell = \ell' + \ell''$. Let $\mathbb{T}_i = \big\{\{x_1^i, y_1^i, z_1^i\}, \ldots, \{x_t^i, y_t^i, z_t^i\}\big\}$, $1 \leqslant i \leqslant \ell'$ and $\mathbb{T}_i = \big\{\{x_1^i, y_1^i, z_1^i\}, \ldots, \{x_{t'}^i, y_{t'}^i, z_{t'}^i\}\big\}$, $\ell'+1 \leqslant i \leqslant \ell'+\ell''$.

Now, we decompose the edges between V and U_i, $1 \leqslant i \leqslant \ell'$, into 3-stars. Let v be the single vertex of the set $V \setminus \bigcup_{j=1}^{t}\{x_j^i, y_j^i, z_j^i\}$. The edges between v and U_i, $1 \leqslant i \leqslant \ell'$, are decomposed into 3-stars in a manner similar to the decomposition of the edges between a vertex $u \in U_i$ and U_j, $1 \leqslant i < j \leqslant \ell$, in Case 1, by labelling u with v. Let $\mathcal{T}_i^1$ be the decomposition of the edges between v and U_i into 3-stars, $\mathcal{T}_i^2 = \bigcup_{u \in U_i} \big\{\{u; x_1^i, y_1^i, z_1^i\}, \ldots, \{u; x_t^i, y_t^i, z_t^i\}\big\}$ and $\mathcal{T}_i = \mathcal{T}_i^1 \cup \mathcal{T}_i^2$, $1 \leqslant i \leqslant \ell'$. Then $\mathcal{T}_i$ is a decomposition of the edges between V and U_i, $1 \leqslant i \leqslant \ell'$, into 3-stars. The edges between V and U_i are decomposed into 3-stars $\mathcal{T}_i$ in a similar manner for $\ell'+1 \leqslant i \leqslant \ell'+\ell''$.

The edges between U_i and U_j, $1 \leqslant i < j \leqslant \ell$, are decomposed into sets of 3-stars $\mathcal{U}_j^i$ in a manner similar to Case 1.

Let $\hat{\mathcal{B}} = \mathcal{B} \cup (\bigcup_{i=1}^{\ell} \mathcal{A}_i) \cup (\bigcup_{i=1}^{\ell} \mathcal{T}_i) \cup (\bigcup_{j=2}^{\ell} \mathcal{U}_j^1) \cup (\bigcup_{j=3}^{\ell} \mathcal{U}_j^2) \cup \cdots \cup \mathcal{U}_\ell^{\ell-1}$. The k colour classes $\{1,2\} \cup \bigcup_{i=1}^{\ell}(C_1 \times \{i\})$, $\{3,4\} \cup \bigcup_{i=1}^{\ell}(C_2 \times \{i\})$, $\cdots$, $\{2k-1, 2k-2\} \cup \bigcup_{i=1}^{\ell}(C_{k-1} \setminus C_k) \times \{i\})$, $\{2k-1\} \cup \bigcup_{i=1}^{\ell}(C_k \times \{i\})$ exhibit a k-colouring of $(\hat{V}, \hat{\mathcal{B}})$, because no star is monochromatic in this k-colouring. It is seen that $(\hat{V}, \hat{\mathcal{B}})$ is a k-chromatic 3-star system of order $n_k = (2k-1) + n_{k-1}(2k-1)(k-1)$ in a manner similar to Case 1.

Case 3: $2k-1 \equiv 2 \pmod 3$. Then $2k = 3t$ for some $t \in \mathbb{Z}^+$. The number of 3-subsets of the set V is $\binom{2k}{3} = t(2k-1)(k-1)$. We now fix $\ell = (2k-1)(k-1)$. Partition the set of all 3-subsets of V into ℓ sets $\mathbb{T}_1, \dots, \mathbb{T}_\ell$ of size t of disjoint 3-subsets. Such a partition is known to exist by Theorem 3.1.3. Let $\mathbb{T}_i = \big\{\{x_1^i, y_1^i, z_1^i\}, \dots, \{x_t^i, y_t^i, z_t^i\}\big\}$, $1 \leqslant i \leqslant \ell$. We decompose the edges between V and U_i into the 3-stars of the set $\mathcal{T}_i = \bigcup_{u \in U_i} \big\{\{u; x_1^i, y_1^i, z_1^i\}, \dots, \{u; x_t^i, y_t^i, z_t^i\}\big\}$, where $1 \leqslant i \leqslant \ell$. The edges between U_i and U_j, $1 \leqslant i < j \leqslant \ell$, are decomposed into 3-stars $\mathcal{U}_j^i$ in a manner similar to Case 1. Let $\hat{\mathcal{B}} = \mathcal{B} \cup (\bigcup_{i=1}^{\ell} \mathcal{A}_i) \cup (\bigcup_{i=1}^{\ell} \mathcal{T}_i) \cup (\bigcup_{j=2}^{\ell} \mathcal{U}_j^1) \cup (\bigcup_{j=3}^{\ell} \mathcal{U}_j^2) \cup \cdots \cup \mathcal{U}_\ell^{\ell-1}$. The k colour classes $\{1,2\} \cup \bigcup_{i=1}^{\ell} (C_1 \times \{i\})$, $\{3,4\} \cup \bigcup_{i=1}^{\ell} (C_2 \times \{i\})$, $\cdots$, $\{2k-1, 2k-2\} \cup \bigcup_{i=1}^{\ell} (C_{k-1} \setminus C_k) \times \{i\})$, $\{2k-1, 2k\} \cup \bigcup_{i=1}^{\ell} (C_k \times \{i\})$ exhibit a k-colouring of $(\hat{V}, \hat{\mathcal{B}})$. It is seen that $(\hat{V}, \hat{\mathcal{B}})$ is a k-chromatic 3-star system of order $n_k = 2k + n_{k-1}(2k-1)(k-1)$ in a manner similar to Case 1. □

We finish this section with the following corollary.

Corollary 3.1.1. *For any integer $k \geqslant 2$, there exists some integer n_k such that for all admissible $n \geqslant n_k$, there exists a k-chromatic 3-star system of order n.*

Proof. By Theorem 3.1.1, there exists a 2-chromatic 3-star system of all admissible orders. For $k \geqslant 3$, apply Theorem 3.1.2 and Theorem 3.1.4 to recursively construct a k-chromatic 3-star system of order n for all sufficiently large admissible n. □

Observe that the cardinalities of the colour classes of the k-chromatic 3-star systems from Theorem 3.1.2 and Theorem 3.1.4 are not excessively imbalanced. The cardinalities of the colour classes of the k-chromatic 3-star systems either differ by at most a small number or the cardinality of the largest colour class of the k-chromatic 3-star systems is almost two times that of the cardinality of the smallest colour class.

3.2 k-colourings of e-star systems

In this section, we generalise the results of the last section to e-star systems for all $e \geqslant 3$. For any arbitrary integers $e \geqslant 3$ and $k \geqslant 2$, we show that there exists some integer n_k where $n_k \equiv 0 \pmod{2e}$ such that for all $n \geqslant n_k$ where $n \equiv 0, 1 \pmod{2e}$, there exists a k-chromatic e-star system of order n. We first construct a strongly equitably 2-chromatic e-star system of order $2e$ for all $e \geqslant 3$.

Theorem 3.2.1. *There exists a strongly equitably 2-chromatic e-star system of order $2e$ for all $e \geqslant 3$.*

Proof. By Theorem 3.1.1, there exists an equitably 2-chromatic 3-star system of order six, $(V, \mathcal{B})$, where $V = \{1, 2, 3, 4, 5, 6\}$ and $\mathcal{B} = \big\{\{1; 3, 5, 6\}, \{2; 1, 3, 6\}, \{4; 1, 2, 3\}, \{5; 2, 3, 4\}, \{6; 3, 4, 5\}\big\}$ with colour classes $R = \{1, 3, 5\}$ and $Y = \{2, 4, 6\}$. We construct a 2-chromatic 4-star system of order eight, $(\hat{V}, \hat{\mathcal{B}})$, from $(V, \mathcal{B})$. Let $\hat{V} = V \cup \{7, 8\}$ and $\hat{\mathcal{B}} = \big\{\{1; 3, 5, 6, 8\}, \{2; 1, 3, 6, 8\}, \{4; 1, 2, 3, 8\}, \{5; 2, 3, 4, 7\}, \{6; 3, 4, 5, 7\}, \{7; 1, 2, 3, 4\}, \{8; 3, 5, 6, 7\}\big\}$, $\hat{R} = \{1, 3, 5, 7\}$ and $\hat{Y} = \{2, 4, 6, 8\}$. Then $(\hat{V}, \hat{\mathcal{B}})$ is an equitably 2-chromatic 4-star system of order 8 with colour classes $\hat{R}$ and $\hat{Y}$.

We now generalise this construction, which can be used in an iterative manner. Note that for each vertex $v \in V \setminus \{3\}$, exactly one star has v as its centre. Suppose that there exists an equitably 2-chromatic e-star system, $(V, \mathcal{B})$, of order $2e$, where $V = \{1, \dots, 2e\}$ is the set of points which is partitioned into two subsets R and Y and $\mathcal{B} = \big\{\{1; a_1^1, \dots, a_e^1\}, \{2; a_1^2, \dots, a_e^2\}, \{4; a_1^4, \dots, a_e^4\}, \{5; a_1^5, \dots, a_e^5\}, \dots, \{2e; a_1^{2e}, \dots, a_e^{2e}\}\big\}$ is the set of blocks. We construct an equitably 2-chromatic $(e+1)$-star system, $(\hat{V}, \hat{\mathcal{B}})$, of order $2e+2$, from $(V, \mathcal{B})$. Let $\hat{V} = V \cup \{2e+1, 2e+$

$2\}$ and $\hat{\mathcal{B}} = \big\{\{1; a_1^1, \ldots, a_e^1, 2e+2\}, \{2; a_1^2, \ldots, a_e^2, 2e+2\}, \{4; a_1^4, \ldots, a_e^4, 2e+2\}, \{5; a_1^5, \ldots, a_e^5, 2e+2\}, \ldots, \{e+1; a_1^{e+1}, \ldots, a_e^{e+1}, 2e+2\}, \{e+2; a_1^{e+2}, \ldots, a_e^{e+2}, 2e+1\}, \{e+3; a_1^{e+3}, \ldots, a_e^{e+3}, 2e+1\}, \ldots, \{2e; a_1^{2e}, \ldots, a_e^{2e}, 2e+1\}, \{2e+1; 1, 2, \ldots, e+1\}, \{2e+2; 3, e+2, e+3, \ldots, 2e, 2e+1\}\big\}$. Then $(\hat{V}, \hat{\mathcal{B}})$ is a strongly equitable 2-chromatic $(e+1)$-star system of order $2e+2$ with colour classes $\hat{R} = \{1, 3, 5, \ldots, 2e+1\}$ and $\hat{Y} = \{2, 4, 6, \ldots, 2e+2\}$. □

We now show how to construct a k-chromatic e-star system from a smaller k-chromatic e-star system.

Theorem 3.2.2. *Let $k \geqslant 2$ and $e \geqslant 3$. If there exists a k-chromatic e-star system of order n_0 such that $n_0 \equiv 0, 1 \pmod{2e}$, then there exists a k-chromatic e-star system of order n, for all $n > n_0$ such that $n \equiv 0, 1 \pmod{2e}$.*

Proof. Suppose that there exists a k-chromatic e-star system, $(V, \mathcal{B})$, where $V = \{1, \ldots, n_0\}$ is the set of points and $\mathcal{B}$ is the set of blocks and $n_0 \equiv 0$ or $1 \pmod{2e}$. Given a k-colouring of $(V, \mathcal{B})$, with colours $1, 2, \ldots, k$ and colour classes $C_1, C_2, \ldots, C_k$, let $R = \bigcup_{i=1}^{\ell} C_i$ and $Y = \bigcup_{i=\ell+1}^{k} C_i$ for some integer ℓ such that $1 \leqslant \ell < k$. Let $r_1, r_2, \ldots, r_{|R|}$ be the elements of R and $y_1, y_2, \ldots, y_{|Y|}$ be the elements of Y. Observe that R and Y form a partition of V. Without loss of generality assume that $|R| \geqslant |Y|$.

Case 1. Suppose that $n_0 \equiv 0 \pmod{2e}$. Then $n_0 = 2et$, $t \geqslant 1$.

First, we construct a k-chromatic $S_e(2et+1)$, $(\hat{V}, \hat{\mathcal{B}})$, from $(V, \mathcal{B})$. Let $\hat{V} = V \cup \{2et+1\}$.

If $|R| = |Y|$, let $\hat{\mathcal{B}} = \mathcal{B} \cup \mathcal{T}$ where $\mathcal{T}$ is the set $\big\{\{2et+1; r_1, \ldots, r_{e-1}, y_1\}, \{2et+1; y_2, \ldots, y_e, r_e\}, \ldots, \{2et+1; r_{|R|-(e-1)}, \ldots, r_{|R|-1}, y_{|Y|-(e-1)}\}, \{2et+1; y_{|Y|-e+2}, \ldots, y_{|Y|}, r_{|R|}\}\big\}$.

Otherwise, $|R| > |Y|$. Then let $\hat{\mathcal{B}} = \mathcal{B} \cup \mathcal{T}$ where $\mathcal{T}$ is the set $\big\{\{2et+1; y_1, \ldots, y_{e-1}, r_1\}, \ldots, \{2et+1; y_{t'e-(t'-1)}, \ldots, y_{(t'+1)e-t'-1}, r_{t'+1}\}, \{2et+1; y_{(t'+1)e-t'}, \ldots, y_{|Y|}, r_{t'+2}, \ldots, r_{t'+i-1}\},$

$\{2et+1; r_{t'+i}, \ldots, r_{t'+i+e-1}\}$, $\ldots$, $\{2et+1; r_{|R|-(e-1)}, \ldots, r_{|R|}\}\}$, where $t' = \lfloor \frac{|Y|}{e-1} \rfloor - 1$ and $i = (t'+2)e - |Y| - t' + 1$.

Note that $(\hat{V}, \hat{\mathcal{B}})$ is not $(k-1)$-colourable because it contains a copy of $(V, \mathcal{B})$. Observe that $C_1, \ldots, C_{k-1}, C_k \cup \{2et+1\}$ are the colour classes of a k-colouring of $(\hat{V}, \hat{\mathcal{B}})$. Therefore, $(\hat{V}, \hat{\mathcal{B}})$ is a k-chromatic e-star system of order $2et+1$.

Next, we construct a k-chromatic $S_e(2et+2e)$, $(\tilde{V}, \tilde{\mathcal{B}})$, from $(V, \mathcal{B})$. Let $\tilde{V} = V \cup V'$ where $V' = \{2et+1, \ldots, 2et+2e\}$ and let $(V', \mathcal{B}')$ be a 2-chromatic e-star system of order $2e$ with colour classes $R' = \{2et+1, 2et+3, \ldots, 2et+2e-1\}$ and $Y' = \{2et+2, 2et+4, \ldots, 2et+2e\}$, constructed from Theorem 3.2.1. Let $\tilde{\mathcal{B}} = \mathcal{B} \cup \mathcal{B}' \cup \mathcal{T}$ where $\mathcal{T} = \bigcup_{v \in V} \{\{v; 2et+1, \ldots, 2et+e\}, \{v; 2et+e+1, \ldots, 2et+2e\}\}$. Note that $(\tilde{V}, \tilde{\mathcal{B}})$ is not $(k-1)$-colourable because it contains a copy of $(V, \mathcal{B})$. Observe that $C_1 \cup R', C_2 \cup Y', C_3, \ldots, C_k$ are the colour classes of a k-colouring of $(\tilde{V}, \tilde{\mathcal{B}})$. Therefore, $(\tilde{V}, \tilde{\mathcal{B}})$ is a k-chromatic e-star system of order $2et+2e$.

Case 2. Suppose that $n_0 \equiv 1 \pmod{2e}$. Then $n_0 = 2et+1$, $t \geqslant 1$. We construct a k-chromatic $S_e(2et+2e)$, $(\hat{V}, \hat{\mathcal{B}})$, from $(V, \mathcal{B})$. Let $\hat{V} = V \cup V_1 \cup V_2$ where $V_1 = \{2et+2, \ldots, 2et+e+1\}$ and $V_2 = \{2et+e+2, \ldots, 2et+2e\}$. Let $v_0 = r_{|R|}$ and $R' = R \setminus \{v_0\}$.

If $|R'| = |Y|$, let $\mathcal{T}_1 = \bigcup_{v \in V_2} \{\{v; r_1, \ldots, r_{e-1}, y_1\}$, $\{v; y_2, \ldots, y_e, r_e\}, \ldots, \{v; r_{|R|-e}, \ldots, r_{|R|-2}, y_{|Y|-e+1}\}$, $\{v; y_{|Y|-e+2}, \ldots, y_{|Y|}, r_{|R|-1}\}\}$.

If $|R'| = |Y| - 1$, decompose the edges between V_2 and $V \setminus \{v_0\}$ into a set of e-stars $\mathcal{T}_1$ in a manner similar to the case $|R'| = |Y|$. The only difference is that in this case R' has one vertex less than the last case. This difference does not have impact on the process of decomposition.

If $|R'| > |Y|$, decompose the edges between V_2 and $V \setminus \{v_0\}$ into a set of e-stars $\mathcal{T}_1$ in a manner similar to the Case 1 when $|R| > |Y|$, by labelling R with R' and $2et+1$ with a vertex $v \in V_2$.

Let $V' = V_1 \cup V_2 \cup \{v_0\}$ and $(V', \mathcal{B}')$ be a 2-chromatic e-star system of order $2e$ with colour classes $R' = \{v_0\} \cup \{2et+3, 2et+4, \ldots, 2et+e+1\}$ and $Y' = \{2et+2, 2et+e+2, 2et+e+3, \ldots, 2et+2e\}$, constructed from Theorem 3.2.1. Let $\mathcal{T}_2 = \bigcup_{v \in V \setminus \{v_0\}} \big\{\{v; 2et+2, \ldots, 2et+e+1\}\big\}$. Let $\hat{\mathcal{B}} = \mathcal{B} \cup \mathcal{B}' \cup \mathcal{T}_1 \cup \mathcal{T}_2$. Note that $(\hat{V}, \hat{\mathcal{B}})$ is not $(k-1)$-colourable because it contains a copy of $(V, \mathcal{B})$. Observe that $C_1 \cup R', C_2, \ldots, C_{k-1}, C_k \cup Y'$ are the colour classes of a k-colouring of $(\hat{V}, \hat{\mathcal{B}})$. Therefore, $(\hat{V}, \hat{\mathcal{B}})$ is a k-chromatic e-star system of order $2et+2e$. □

We now show how to iteratively construct a k-chromatic e-star system from a $(k-1)$-chromatic e-star system.

Theorem 3.2.3. *Let $k \geqslant 3$ and $e \geqslant 3$. If there exists a $(k-1)$-chromatic e-star system of order $n_{k-1} \equiv 0 \pmod{2e}$, then there exists a k-chromatic e-star system of order n_k, for some $n_k \equiv 0 \pmod{2e}$.*

Proof. Let $(U_0, \mathcal{A}_0)$ be a $(k-1)$-chromatic e-star system of order n_{k-1} such that $n_{k-1} \equiv 0 \pmod{2e}$, with colour classes $C_1, \ldots, C_{k-1}$ $|C_1| \leqslant \cdots \leqslant |C_{k-1}|$ and each vertex of C_s has colour s, for $1 \leqslant s \leqslant k-1$. For each $s \in \{1, \ldots, k-1\}$, let $C_s = \{c_1^s, \ldots, c_{|C_s|}^s\}$. For a positive integer ℓ that will be fixed later and for each $i \in \{1, \ldots, \ell\}$, let $U_i = U_0 \times \{i\}$ and $\mathcal{A}_i = \mathcal{A}_0 \times \{i\}$ where $\mathcal{A}_0 \times \{i\}$ denotes the set $\{S \times \{i\} \mid S \in \mathcal{A}_0\}$. So $(U_i, \mathcal{A}_i)$ has colour classes $C_1 \times \{i\}, \ldots, C_{k-1} \times \{i\}$. Let $W = \{w_1, w_2, \ldots, w_{(e-1)(k-1)+1}\}$, $B = \{b_1, b_2, \ldots, b_{k-2}\}$ and $D = \{d_1, d_2, \ldots, d_e\}$ be pairwise disjoint sets. If $k-1$ is even then let $V = W \cup B$; otherwise let $V = W \cup B \cup D$. Also let $U = \bigcup_{i=1}^{\ell} U_i$, so that $V \cap U = \emptyset$. We will embed $(U_1, \mathcal{A}_1), \ldots, (U_\ell, \mathcal{A}_\ell)$ into a k-chromatic e-star system $(\hat{V}, \hat{\mathcal{B}})$ where $\hat{V} = V \cup U$. Let $n_k = |\hat{V}|$ and observe that $n_k \equiv 0 \pmod{2e}$. By Theorem 3.2.1 and Theorem 3.2.2, the complete graph on the set V admits a decomposition into e-stars which is 2-chromatic. Let $(V, \mathcal{B})$ be a 2-chromatic e-star system on the set V (so $(V, \mathcal{B})$ will not obstruct the k-colouring that we will construct). We now need to decompose the edges between V and U and the edges between U_i and U_j, for $1 \leqslant i < j \leqslant \ell$ into e-stars such that the resulting e-star system $(\hat{V}, \hat{\mathcal{B}})$ is k-chromatic. To do so, we will decompose the edges between V and U into e-stars in a way such that no e-subset in V is monochromatic in any putative $(k-1)$-colouring.

The number of e-subsets of the set W is $N = \binom{(e-1)(k-1)+1}{e}$. Let $a = (k-2) + \lfloor \frac{-k+2}{e} \rfloor$ and $N = aq + r$ where q is a positive integer and $0 \leqslant r < a$ is an integer. Partition the set of all e-subsets of W into q sets $\mathbb{T}_1, \ldots, \mathbb{T}_q$ of size a of disjoint e-subsets and one set $\mathbb{T}_{q+1}$ of size r of disjoint e-subsets . Such a partition is known to exist by Theorem 3.1.3. We now fix $\ell = q + 1$ if $r > 0$ and $\ell = q$ if $r = 0$. Let $\mathbb{T}_i = \big\{\{(x_1)^i_1, \ldots, (x_e)^i_1\}, \ldots, \{(x_1)^i_a, \ldots, (x_e)^i_a\}\big\}$, $1 \leqslant i \leqslant q$ and $\mathbb{T}_{q+1} = \big\{\{(x_1)^{q+1}_1, \ldots, (x_e)^{q+1}_1\}, \ldots, \{(x_1)^{q+1}_r, \ldots, (x_e)^{q+1}_r\}\big\}$ if $r > 0$, and $\mathbb{T}_{q+1} = \emptyset$ if $r = 0$. Let $W_i = W \setminus \bigcup_{j=1}^{a} \big(\{(x_1)^i_j, \ldots, (x_e)^i_j\}\big)$ for $1 \leqslant i \leqslant q$ and $W_{q+1} = W \setminus \bigcup_{j=1}^{r} \big(\{(x_1)^{q+1}_j, \ldots, (x_e)^{q+1}_j\}\big)$ if $r > 0$ and $W_{q+1} = \emptyset$ if $r = 0$.

For each $i \in \{1, 2, \ldots, \ell\}$ and for each $w \in W_i$, we decompose the edges between w and U_i. First, we decompose all the edges between w and $C_1 \times \{i\}$ into e-stars. Let $\mathcal{R}^{1,2}_w = \big\{\{w; c^1_1 \times \{i\}, \ldots, c^1_{e-1} \times \{i\}, c^2_1 \times \{i\}\}, \ldots, \{w; c^1_{(t-1)(e-1)+1} \times \{i\}, \ldots, c^1_{t(e-1)} \times \{i\}, c^2_t \times \{i\}\}, \{w; c^1_{t(e-1)+1} \times \{i\}, \ldots, c^1_{|C_1|} \times \{i\}, c^2_{t+1} \times \{i\}, \ldots, c^2_{t+t'} \times \{i\}\}\big\}$ where t and t' are integers such that $t = \lfloor \frac{|C_1|}{e-1} \rfloor$ and $t' = e - (|C_1| - t(e-1))$. Then $\mathcal{R}^{1,2}_w$ is a decomposition of all the edges between w and $C_1 \times \{i\}$ along with the edges between w and $C'_2 \times \{i\}$ into e-stars where $C'_2 = \big\{c^2_1, \ldots, c^2_{t+t'}\big\} \subseteq C_2$. For each $j = 2, 3, \ldots, k-2$, iteratively proceed in a similar manner to decompose all the edges between w and $(C_j \setminus C'_j) \times \{i\}$ into a set $\mathcal{R}^{j,j+1}_w$, of e-stars . So $\bigcup_{j=1}^{k-2} \mathcal{R}^{j,j+1}_w$ gives a decomposition of the edges between w and $(C_1 \cup \cdots \cup C_{k-2} \cup C'_{k-1}) \times \{i\}$ where $C'_{k-1} \subseteq C_{k-1}$. Let $m = \frac{|C_{k-1} \setminus C'_{k-1}|}{e}$ and let $C''_{k-1} = C_{k-1} \setminus C'_{k-1}$ and without loss of generality we can let $C''_{k-1} = \{c_1^{k-1}, \ldots, c_{me}^{k-1}\}$. Let $C_k = \{c_1^{k-1}, c_{e+1}^{k-1}, c_{2e+1}^{k-1}, \ldots, c_{(m-1)e+1}^{k-1}\}$ and decompose the edges between w and $C''_{k-1} \times \{i\}$ into e-stars of the set: $\mathcal{R}_w^{k-1} = \{\{w; c_1^{k-1} \times \{i\}, c_2^{k-1} \times \{i\}, \ldots, c_e^{k-1} \times \{i\}\}, \{w; c_{e+1}^{k-1} \times \{i\}, c_{e+2}^{k-1} \times \{i\}, \ldots, c_{2e}^{k-1} \times \{i\}\}, \ldots, \{w; c_{(m-1)e+1}^{k-1} \times \{i\}, c_{(m-1)e+2}^{k-1} \times \{i\}, \ldots, c_{me}^{k-1} \times \{i\}\}\}$.

For each $i \in \{1, 2, \ldots, q\}$, let $\mathcal{T}_i = \bigcup_{u \in U_i} \{\{u; (x_1)_1^i, \ldots, (x_e)_1^i\}, \ldots, \{u; (x_1)_a^i, \ldots, (x_e)_a^i\}\}$ and $\mathcal{T}_{q+1} = \{\{u; (x_1)_1^{q+1}, \ldots, (x_e)_1^{q+1}\}, \ldots, \{u; (x_1)_r^{q+1}, \ldots, (x_e)_r^{q+1}\}\}$ if $r > 0$ and $\mathcal{T}_{q+1} = \emptyset$ if $r = 0$. Then $\mathcal{P}_i = \mathcal{T}_i \cup \bigcup_{w \in W_i} (\mathcal{R}_w^{k-1} \cup (\bigcup_{j=1}^{k-2} \mathcal{R}_w^{j,j+1}))$ is a decomposition of the edges between W and U_i into e-stars. The edges between U_i and U_j, $1 \leqslant i < j \leqslant \ell$ are decomposed into sets $\mathcal{U}_j^i$ of e-stars in a manner similar to the decomposition of the edges between W_i and U_i, $1 \leqslant i \leqslant \ell$, by labelling each vertex $w \in W_i$ with a vertex $u \in U_i$.

We have two cases.

Case 1. $k-1$ is even. Then $V = W \cup B$. The edges between B and U_i, $1 \leqslant i \leqslant \ell$ are decomposed into sets of e-stars $\mathcal{F}_i$ in a manner similar to the decomposition of the edges between W_i and U_i, $1 \leqslant i \leqslant \ell$. Therefore, $(\hat{V}, \hat{\mathcal{B}})$ where $\hat{\mathcal{B}} = \mathcal{B} \cup (\bigcup_{i=1}^{\ell} \mathcal{A}_i) \cup (\bigcup_{i=1}^{\ell} \mathcal{P}_i) \cup (\bigcup_{i=1}^{\ell} \mathcal{F}_i) \cup (\bigcup_{j=2}^{\ell} \mathcal{U}_j^1) \cup (\bigcup_{j=3}^{\ell} \mathcal{U}_j^2) \cup \cdots \cup \mathcal{U}_\ell^{\ell-1}$ is an e-star system of order $n_k = (k-1)e + n_{k-1}\ell$. A k-colouring is given by $(\bigcup_{i=1}^{\ell} C_1 \times \{i\}) \cup \{w_1, \ldots, w_{e-1}, b_1\}$, $(\bigcup_{i=1}^{\ell} C_2 \times \{i\}) \cup \{w_e, \ldots, w_{2e-2}, b_2\}, \ldots, (\bigcup_{i=1}^{\ell} C_{k-2} \times \{i\}) \cup \{w_{(e-1)(k-1)-2e-1}, \ldots, w_{(e-1)(k-1)-(e-1)}, b_{k-2}\}$, $(\bigcup_{i=1}^{\ell} (C_{k-1} \setminus C_k) \times \{i\}) \cup \{w_{(e-1)(k-1)-e+2}, \ldots, w_{(e-1)(k-1)}\}$, $(\bigcup_{i=1}^{\ell} C_k \times \{i\}) \cup \{w_{(e-1)(k-1)+1}\}$.

Case 2. $k-1$ is odd. Then $V = W \cup B \cup D$. The edges between D and U_i, $1 \leqslant i \leqslant \ell$ are decomposed into sets $\mathcal{H}_i$ of e-stars in a manner similar to the decomposition of the edges between W_i and U_i, $1 \leqslant i \leqslant \ell$. Therefore, $(\hat{V}, \hat{\mathcal{B}})$ where

$\hat{\mathcal{B}} = \mathcal{B} \cup (\bigcup_{i=1}^{\ell} \mathcal{A}_i) \cup (\bigcup_{i=1}^{\ell} \mathcal{P}_i) \cup (\bigcup_{i=1}^{\ell} \mathcal{F}_i) \cup (\bigcup_{i=1}^{\ell} \mathcal{H}_i) \cup (\bigcup_{j=2}^{\ell} \mathcal{U}_j^1) \cup (\bigcup_{j=3}^{\ell} \mathcal{U}_j^2) \cup \cdots \cup \mathcal{U}_\ell^{\ell-1}$ is an e-star system of order $n_k = ek + n_{k-1}\ell$. A k-colouring is given by $(\bigcup_{i=1}^{\ell} C_1 \times \{i\}) \cup \{w_1, \ldots, w_{e-1}, b_1\}$, $(\bigcup_{i=1}^{\ell} C_2 \times \{i\}) \cup \{w_e, \ldots, w_{2e-2}, b_2\}$ $\ldots$, $(\bigcup_{i=1}^{\ell} C_{k-2} \times \{i\}) \cup \{w_{(e-1)(k-1)-2e-1}, \ldots, w_{(e-1)(k-1)-(e-1)}, b_{k-2}\}$, $\big(\bigcup_{i=1}^{\ell} (C_{k-1} \backslash C_k) \times \{i\}\big) \cup \{w_{(e-1)(k-1)-e+2}, \ldots, w_{(e-1)(k-1)}, d_1\}$, $(\bigcup_{i=1}^{\ell} C_k \times \{i\}) \cup \{w_{(e-1)(k-1)+1}, d_2, \ldots, d_e\}$.

Since $(U_i, \mathcal{A}_i)$, $1 \leqslant i \leqslant \ell$, is a $(k-1)$-chromatic e-star system, there are vertices $u_1^i, \ldots, u_{k-1}^i$ in U_i such that $u_s^i \in C_s \times \{i\}$ for each $1 \leqslant s \leqslant k-1$. Since $|W| = (e-1)(k-1)+1 > (e-1)(k-1)$, when attempting to colour W with $k-1$ colours, some colour must occur at least e times within W. Thus, for some $i \in \{1, \ldots, q\}$ and some $j \in \{1, \ldots, a\}$ and for some $s \in \{1, \ldots, k-1\}$, the vertices $(x_1)_j^i, (x_2)_j^i, \ldots, (x_e)_j^i$ are each coloured with the same colour s, or else for $i = q+1$ and some $j \in \{1, \ldots, r\}$ and for some $s \in \{1, \ldots, k-1\}$, the vertices $(x_1)_j^i, (x_2)_j^i, \ldots, (x_e)_j^i$ are each coloured with the same colour s. Then the e-star $\{u_s^i; (x_1)_j^i, (x_2)_j^i, \ldots, (x_e)_j^i\}$ would be monochromatic which is a contradiction. Hence in either case $(\hat{V}, \hat{\mathcal{B}})$ cannot be coloured with $k-1$ colours. □

We summarise this section with the following corollary.

Corollary 3.2.1. *For any integers $k \geqslant 2$ and $e \geqslant 3$, there exists some integer n_k where $n_k \equiv 0$ (mod $2e$) such that for all admissible $n \geqslant n_k$ where $n \equiv 0, 1$ (mod $2e$), there exists a k-chromatic e-star system of order n.*

Proof. By Theorem 3.2.1, there exists a strongly equitably 2-chromatic e-star system of order $2e$. For $k \geqslant 2$, apply Theorem 3.2.2 and Theorem 3.2.3 to recursively construct a k-chromatic e-star system of order n for all sufficiently large n such that $n \equiv 0, 1$ (mod $2e$). □

3.3 Unique colourings of e-star systems

We now investigate uniquely k-chromatic e-star systems for any $e \geqslant 3$. We commence by showing that such designs do indeed exist for $k = 2$.

Theorem 3.3.1. *For any $e \geqslant 3$, there exists a strongly equitably uniquely 2-chromatic e-star system of order n for some $n \equiv 0 \pmod{2e}$.*

Proof. Let $(U_0, \mathcal{A}_0)$ be a strongly equitable 2-chromatic e-star system of order $2e$ constructed from Theorem 3.2.1, with colour classes C_1 and C_2 such that each vertex of C_s has colour s, for $s = 1, 2$. For a positive integer ℓ that will be fixed later and for each $i \in \{1, \dots, \ell\}$, let $U_i = U_0 \times \{i\}$ and $\mathcal{A}_i = \mathcal{A}_0 \times \{i\}$ where $\mathcal{A}_0 \times \{i\}$ denotes the set $\{S \times \{i\} \mid S \in \mathcal{A}_0\}$. So $(U_i, \mathcal{A}_i)$ has colour classes $C_1 \times \{i\}$ and $C_2 \times \{i\}$. Let $U = \bigcup_{i=1}^{\ell} U_i$. Let $A^1 = \{a_1^1, \dots, a_e^1\}$, $A^2 = \{a_1^2, \dots, a_e^2\}$ and $A = A^1 \cup A^2$. Similarly let $F^1 = \{f_1^1, \dots, f_e^1\}$, $F^2 = \{f_1^2, \dots, f_e^2\}$, $F = F^1 \cup F^2$, $G^1 = \{g_1^1, \dots, g_e^1\}$, $G^2 = \{g_1^2, \dots, g_e^2\}$, $G = G^1 \cup G^2$, $H^1 = \{h_1^1, \dots, h_e^1\}$, $H^2 = \{h_1^2, \dots, h_e^2\}$, $H = H^1 \cup H^2$, $K^1 = \{k_1^1, \dots, k_e^1\}$, $K^2 = \{k_1^2, \dots, k_e^2\}$ and $K = K^1 \cup K^2$. We construct a uniquely 2-chromatic e-star system $(\hat{V}, \hat{\mathcal{B}})$ where $\hat{V} = A \cup F \cup G \cup H \cup U \cup K$.

We need to decompose the edges between the subsets of $\hat{V}$ into e-stars such that the resulting e-star system $(\hat{V}, \hat{\mathcal{B}})$ is uniquely 2-chromatic.

First, we decompose the edges between A and U into e-stars in a way such that no e-subset in A other than the e-subsets A^1 and A^2 is monochromatic in any putative 2-colouring of $(\hat{V}, \hat{\mathcal{B}})$. Let $\mathcal{D}$ be the set of all e-subsets of A except A^1 and A^2. The number of e-subsets of the set A is $N = \binom{2e}{e}$ and $|\mathcal{D}| = N - 2$. Let T_i, $1 \leqslant i \leqslant |\mathcal{D}|$, be the elements of $\mathcal{D}$. Let $\tilde{A}_i = A \setminus T_i$, $1 \leqslant i \leqslant |\mathcal{D}|$. We now fix $\ell = |\mathcal{D}|$.

Partition U_0 into e-subsets E_1 and E_2 such that $E_j \cap C_1 \neq \emptyset$ and $E_j \cap C_2 \neq \emptyset$, $j = 1, 2$. For each $i \in \{1, 2, \dots, \ell\}$ and for each $w \in \tilde{A}_i$, we decompose the edges between w

and U_i into the set of e-stars $\mathcal{R}_w = \bigcup_{j=1}^{2} \big\{\{w; E_j \times \{i\}\}\big\}$. For each $i \in \{1, \ldots, \ell\}$, let $\mathcal{T}_i = \bigcup_{u \in U_i} \big\{\{u; T_i\}\big\}$. Then $\mathcal{P}_i = \mathcal{T}_i \cup \big(\bigcup_{w \in \tilde{A}_i} \mathcal{R}_w \big)$ is a decomposition of the edges between A and U_i, and $\bigcup_{i=1}^{\ell} \mathcal{P}_i$ is a decomposition of the edges between A and U into e-stars.

Recall that $(U_i, \mathcal{A}_i)$ is 2-chromatic. So for any 2-colouring of $(U_i, \mathcal{A}_i)$ there must be a vertex u_s^i of colour s in U_i, for each $1 \leqslant s \leqslant 2$. Note that for each $i \in \{1, \ldots, |\mathcal{D}|\}$, $\{u_s^i; T_i\} \in \mathcal{T}_i$ and so to ultimately obtain a valid 2-colouring of $(\hat{V}, \hat{\mathcal{B}})$, the e vertices of T_i must not all have colour s, $1 \leqslant s \leqslant 2$. If a colour s occurs on more than e vertices of A then there exists a T_i with only colour s and then $\{u_s^i; T_i\}$ would be monochromatic. So in any valid 2-colouring of $(\hat{V}, \hat{\mathcal{B}})$, each colour must occur on exactly e points of A. If the points of colour s are not all in A^1 or A^2, then they are the leaves of a monochromatic e-star with centre u_s^i. Without loss of generality, we assign colour s to all e points of A^s, $1 \leqslant s \leqslant 2$.

We now begin to decompose the edges between A and F into e-stars. For each $i \in \{1, \ldots, e\}$, let $\mathcal{K}_i^1 = \big\{\{f_i^1; A^2\}\big\}$ and $\mathcal{K}_i^2 = \big\{\{f_i^2; A^1\}\big\}$. Since each vertex of A^2 has colour 2, each vertex of F^1 must have colour 1 or else a monochromatic e-star would now exist. Likewise, each vertex of F^2 must have colour 2. Let $\mathcal{K}^1 = \bigcup_{i=1}^{e} \mathcal{K}_i^1$ and $\mathcal{K}^2 = \bigcup_{i=1}^{e} \mathcal{K}_i^2$. We decompose the edges between A^i and F^i, $1 \leqslant i \leqslant 2$, later. Decompose the edges between A^1 and G^2 and between A^2 and G^1 into a set of e-stars $\mathcal{L}$ in a similar manner, forcing the vertices of the subsets G^1 and G^2 to have colours 1 and 2 respectively. We decompose the edges between A^i and G^i, $1 \leqslant i \leqslant 2$, later. Also, decompose the edges between K^1 and G^2 and between K^2 and G^1 into a set of e-stars $\mathcal{C}$ in a similar manner, forcing the vertices of the subsets K^1 and K^2 to have colours 1 and 2 respectively. We decompose the edges between K^i and G^i, $1 \leqslant i \leqslant 2$, later.

Next, we decompose the edges between H and $F \cup G$. For each $i \in \{1, \ldots, e\}$, let $\mathcal{E}_i^1 = \big\{\{h_i^1; f_1^2, \ldots, f_e^2\}, \{h_i^1; f_1^1, \ldots, f_{e-1}^1, g_1^2\}, \{h_i^1; f_e^1, g_1^1, \ldots, g_{e-2}^1, g_2^2\}, \{h_i^1; g_{e-1}^1, g_e^1, g_3^2, \ldots, g_e^2\}\big\}$, and $\mathcal{E}^1 = \bigcup_{i=1}^{e} \mathcal{E}_i^1$. Then $\mathcal{E}^1$ is a decomposition of the edges between H^1 and $F \cup G$ into e-stars such that all the vertices in H^1 are forced to have colour 1. Decompose the edges between H^2 and $F \cup G$ into a set of e-stars $\mathcal{E}^2$ in a similar manner, forcing every vertex in H^2 to have unique colour 2. Let $\mathcal{E} = \bigcup_{j=1}^{2} \mathcal{E}^j$.

Partition the set $F^1 \cup G^1 \cup H \cup K \setminus \{h_1^2, h_2^2, h_3^2, k_1^2, k_2^2, k_3^2\}$ of $6e-6$ vertices into six sets $S_1, \ldots, S_6$ of size $e-1$. For each $1 \leqslant i \leqslant e$, let $\mathcal{M}_i^1 = \big\{\{a_i^1; S_1 \cup \{h_1^2\}\}, \{a_i^1; S_2 \cup \{h_2^2\}\}, \{a_i^1; S_3 \cup \{h_3^2\}\}, \{a_i^1; S_4 \cup \{k_1^2\}\}, \{a_i^1; S_5 \cup \{k_2^2\}\}, \{a_i^1; S_6 \cup \{k_3^2\}\}\big\}$. Then $\mathcal{M}^1 = \bigcup_{i=1}^{e} \mathcal{M}_i^1$ is a decomposition of the edges between A^1 and $F^1 \cup G^1 \cup H \cup K$ into e-stars, none of which are monochromatic. Decompose the edges between A^2 and $F^2 \cup G^2 \cup H \cup K$ into a set of e-stars $\mathcal{M}^2$ in a similar manner. Let $\mathcal{M} = \bigcup_{j=1}^{2} \mathcal{M}^j$.

Partition the set $G^1 \cup F \cup H \setminus \{f_1^2, f_2^2, f_3^2, h_1^2, h_2^2\}$ of $5e-5$ vertices into five sets $R_1, \ldots, R_5$ of size $e-1$. For each $1 \leqslant i \leqslant e$, let $\mathcal{F}_i^1 = \big\{\{k_i^1; R_1 \cup \{f_1^2\}\}, \{k_i^1; R_2 \cup \{f_2^2\}\}, \{k_i^1; R_3 \cup \{f_3^2\}\}, \{k_i^1; R_4 \cup \{h_1^2\}\}, \{k_i^1; R_5 \cup \{h_2^2\}\}\big\}$. Then $\mathcal{F}^1 = \bigcup_{i=1}^{e} \mathcal{F}_i^1$ is a decomposition of the edges between K^1 and $G^1 \cup F \cup H$ into e-stars, none of which are monochromatic. Decompose the edges between K^2 and $G^2 \cup F \cup H$ into a set $\mathcal{F}^2$ of e-stars in a similar manner.

Next, for each $1 \leqslant i \leqslant \ell$, we decompose the edges between $F \cup G$ and U_i into e-stars. Let $u_1^s, \ldots, u_e^s$ be the vertices of colour s, $1 \leqslant s \leqslant 2$, in an equitable 2-colouring of $(U_0, \mathcal{A}_0)$. For each $j \in \{1, \ldots, e\}$, let $\mathcal{N}_j^1 = \big\{\{u_j^1 \times \{i\}; f_1^2, \ldots, f_e^2\}, \{u_j^1 \times \{i\}; f_1^1, \ldots, f_{e-1}^1, g_1^2\}, \{u_j^1 \times \{i\}; f_e^1, g_1^1, \ldots, g_{e-2}^1, g_2^2\}, \{u_j^1 \times \{i\}; g_{e-1}^1, g_e^1, g_3^2, \ldots, g_e^2\}\big\}$ and observe that $u_j^1 \times \{i\}$ in now forced to have colour 1.

For each $j \in \{1, \ldots, e\}$, decompose the edges between $u_j^2 \times \{i\}$ and $F \cup G$ into a set of e-stars $\mathcal{N}_j^2$ in a similar manner so that $u_j^2 \times \{i\}$ is forced to have colour 2.

Let $\mathcal{N}^s = \bigcup_{j=1}^{e} \mathcal{N}_j^s$ and $\mathcal{O}_i = \bigcup_{s=1}^{2} \mathcal{N}^s$. Then $\mathcal{O}_i$ is a decomposition of the edges between $F \cup G$ and U_i into e-stars which forces every vertex of U_i to have a unique colour. Let $\mathcal{O} = \bigcup_{i=1}^{\ell} \mathcal{O}_i$.

At this point every vertex of V is coloured, and the colouring is unique (up to a permutation of the colours). Moreover, the two colour classes are equal in size and so the colouring is strongly equitable. All that remains is to complete the decomposition without introducing monochromatic e-stars.

For each $1 \leqslant i \leqslant e$, we begin to decompose the edges between F and G into e-stars. Let $\mathcal{Q}_i^1 = \big\{\{f_i^1; g_1^1, \ldots, g_{e-1}^1, g_1^2\}, \{f_i^1; g_e^1, g_2^2, \ldots, g_e^2\}\big\}$. Then $\mathcal{Q}^1 = \bigcup_{i=1}^{e} \mathcal{Q}_i^1$ is a decomposition of the edges between F^1 and G into e-stars, none of which are monochromatic. Decompose the edges between F^2 and G into a set of e-stars $\mathcal{Q}^2$ in a similar manner. Let $\mathcal{Q} = \bigcup_{j=1}^{2} \mathcal{Q}^j$.

Next, for each $1 \leqslant i \leqslant \ell$, we decompose the edges between H and U_i into e-stars. Recall that $u_1^s, \ldots, u_e^s$ are vertices of colour s, $1 \leqslant s \leqslant 2$. Let $\mathcal{J}_j^1 = \big\{\{u_j^1 \times \{i\}; h_1^1, \ldots, h_{e-1}^1, h_1^2\}, \{u_j^1 \times \{i\}; h_e^1, h_2^2, \ldots, h_e^2\}\big\}$, $1 \leqslant j \leqslant e$. For each $j \in \{1, \ldots, e\}$, decompose the edges between $u_j^2 \times \{i\}$ and H into a set of e-stars $\mathcal{J}_j^2$ in a similar manner. Let $\mathcal{J}^s = \bigcup_{j=1}^{e} \mathcal{J}_j^s$ and $\mathcal{S}_i = \bigcup_{s=1}^{2} \mathcal{J}^s$. Then $\mathcal{S}_i$ is a decomposition of the edges between H and U_i into e-stars. Let $\mathcal{S} = \bigcup_{i=1}^{\ell} \mathcal{S}_i$. For each $1 \leqslant i \leqslant \ell$, decompose the edges between K and U_i into a set of e-stars $\mathcal{H}_i$ in a similar manner so that $\bigcup_{i=1}^{\ell} \mathcal{H}_i$ decomposes all of the edges between K and U. The edges between U_i and U_j, $1 \leqslant i < j \leqslant \ell$, are decomposed into set $\mathcal{U}_j^i$ of e-stars in a similar manner.

Note that $|A| = |F| = |G| = |H| = |K| = 2e$ and so we let each of $(A, \mathcal{A})$, $(F, \mathcal{F})$, $(G, \mathcal{G})$, $(H, \mathcal{H})$, and $(K, \mathcal{K})$ be an equitable 2-chromatic e-star system on the sets A, F, G, H, and K, respectively, such that the colour classes are in agreement with the colouring that we have forced upon V.

Finally, let $\hat{\mathcal{B}} = \mathcal{A} \cup \mathcal{F} \cup \mathcal{G} \cup \mathcal{H} \cup \mathcal{K} \cup \big(\bigcup_{i=1}^{\ell} \mathcal{A}_i\big) \cup \big(\bigcup_{i=1}^{\ell} \mathcal{P}_i\big) \cup \big(\bigcup_{j=2}^{\ell} \mathcal{U}_j^1\big) \cup \big(\bigcup_{j=3}^{\ell} \mathcal{U}_j^2\big) \cup \ldots \cup \big(\mathcal{U}_\ell^{\ell-1}\big) \cup \big(\bigcup_{j=1}^{2} \mathcal{K}^j\big) \cup \mathcal{L} \cup \mathcal{E} \cup \mathcal{M} \cup \mathcal{O} \cup \mathcal{Q} \cup \mathcal{S} \cup \mathcal{C} \cup \big(\bigcup_{j=1}^{2} \mathcal{F}^j\big) \cup \big(\bigcup_{i=1}^{\ell} \mathcal{H}_i\big)$. Then $(\hat{V}, \hat{\mathcal{B}})$ is an e-star system of order $n = 10e + \ell n_0$ which is strongly equitably uniquely 2-chromatic.

□

Observe that each of the cardinalities of the colour classes of the uniquely 2-chromatic e-star systems constructed from Theorem 3.3.1 is greater than e. We tacitly use this property to construct a uniquely 2-chromatic e-star system from a smaller uniquely 2-chromatic e-star system.

Theorem 3.3.2. *Let $(V, \mathcal{B})$ be a strongly equitable uniquely 2-chromatic e-star system of order n_0 constructed from Theorem 3.3.1, with colour classes C_1 and C_2. Then there exists a uniquely 2-chromatic e-star system of order n for all $n > n_0$ such that $n \equiv 0, 1 \pmod{2e}$.*

Proof. For $s \in \{1, 2\}$, let $C_s = \{c_1^s, \ldots, c_{|C_s|}^s\}$. Since $n_0 \equiv 0 \pmod{2e}$, then let $n_0 = 2et$, $t \geqslant 1$.

First, we construct a uniquely 2-chromatic $S_e(2et+1)$, $(\hat{V}, \hat{\mathcal{B}})$, from $(V, \mathcal{B})$. Let $\hat{V} = V \cup \{2et+1\}$. Let $\mathcal{T}_1^{2et+1} = \big\{\{2et+1; c_1^2, \ldots, c_e^2\}\big\}$ and $\mathcal{T}_2^{2et+1} = \big\{\{2et+1; c_1^1, \ldots, c_{e-1}^1, c_{e+1}^2\}, \{2et+1; c_e^1, \ldots, c_{2e-2}^1, c_{e+2}^2\}, \ldots, \{2et+1; c_{(t-1)(e-1)+1}^1, \ldots, c_{t(e-1)}^1, c_{e+t}^2\}, \{2et+1; c_{t(e-1)+1}^1, \ldots, c_{|C_1|}^1, c_{e+t+1}^2, \ldots, c_{e+t+r}^2\}\big\}$ where $t = \lfloor \frac{|C_1|}{e-1} \rfloor$ and $r = e - (|C_1| - t(e-1))$.

Partition the set $C_2 \setminus \{c_1^2, \ldots, c_e^2, c_{e+1}^2, c_{e+2}^2, \ldots, c_{e+t+r}^2\}$ into a set $\mathbb{E}$ of m disjoint e-subsets $X_1, \ldots, X_m$ where $m = \frac{|C_2|-(e+t+r)}{e}$. Let $\mathcal{T}_3^{2et+1} = \bigcup_{i=1}^{m} \big\{\{2et+1; X_i\}\big\}$.

Let $\hat{\mathcal{B}} = \mathcal{B} \cup \mathcal{T}_1^{2et+1} \cup \mathcal{T}_2^{2et+1} \cup \mathcal{T}_3^{2et+1}$. Then $(\hat{V}, \hat{\mathcal{B}})$ is a uniquely 2-chromatic e-star system of order $2et+1$ with colour classes $C_1 \cup \{2et+1\}$ and C_2.

Next, we construct a uniquely 2-chromatic $S_e(2et+2e)$, $(\mathring{V}, \mathring{B})$, from $(\hat{V}, \hat{\mathcal{B}})$. Let $\mathring{V} = \hat{V} \cup V'$ where $V' = \{2et+2, 2et+3, \ldots, 2et+2e\}$ and let $v_0 \in C_1 \setminus \{c_1^1, \ldots, c_e^1\}$. Let $(V' \cup v_0, \mathcal{B}')$ be a 2-chromatic e-star system of order $2e$ constructed from Theorem 3.2.1. Let $\mathcal{T}_1^i = \bigcup_i \{\{i; c_1^2, \ldots, c_e^2\}\}$, $i \in \{2et+3, 2et+5, \ldots, 2et+2e-1\}$ and $\mathcal{T}_1^j = \bigcup_j \{\{j; c_1^1, \ldots, c_e^1\}\}$, $j \in \{2et+2, 2et+4, \ldots, 2et+2e\}$. For any $k \in \{2et+2, 2et+3, \ldots, 2et+2e\}$, decompose the remaining edges between vertex k and set $V \setminus \{v_0\}$ into sets of e-stars $\mathcal{T}_2^k$ and $\mathcal{T}_3^k$ in a manner similar to the construction of $(\hat{V}, \hat{\mathcal{B}})$ from $(V, \mathcal{B})$.

Let $\mathring{\mathcal{B}} = \hat{\mathcal{B}} \cup \mathcal{B}' \cup \big(\bigcup_{k=2et+2}^{2et+2e} \mathcal{T}_1^k\big) \cup \big(\bigcup_{k=2et+2}^{2et+2e} \mathcal{T}_2^k\big) \cup \big(\bigcup_{k=2et+2}^{2et+2e} \mathcal{T}_3^k\big)$. Then $(\mathring{V}, \mathring{\mathcal{B}})$ is a uniquely 2-chromatic e-star system of order $2et+2e$ with colour classes $C_1 \cup \{2et+3, 2et+5, \ldots, 2et+2e-1\}$ and $C_2 \cup \{2et+2, 2et+4, \ldots, 2et+2e\}$. □

We then obtain the following corollary.

Corollary 3.3.1. *Let $e \geqslant 3$. There exists some integer n_0 where $n_0 \equiv 0$ (mod $2e$) such that for all admissible $n \geqslant n_0$ where $n \equiv 0, 1$ (mod $2e$), there exists a uniquely 2-chromatic e-star system of order n.*

Proof. Apply Theorem 3.3.1 and Theorem 3.3.2. □

We now show how to construct a strongly equitable k-chromatic e-star system from a strongly equitable uniquely $(k-1)$-chromatic e-star system.

Theorem 3.3.3. *Let $k \geqslant 3$ and $e \geqslant 3$. If there exists a strongly equitably uniquely $(k-1)$-chromatic e-star system of order $n_{k-1} \equiv 0$ (mod $2e$) with colour classes $C_1, \ldots, C_{k-1}$ such that $|C_i| > e$, $1 \leqslant i \leqslant k-1$, then there exists a strongly equitably k-chromatic e-star system of order n_k for some $n_k \equiv 0$ (mod $2e$).*

Proof. Let $(U_0, \mathcal{A}_0)$ be a strongly equitably uniquely $(k-1)$-chromatic e-star system of order n_{k-1} with colour classes $C_1, \ldots, C_{k-1}$ such that $n_{k-1} \equiv 0$ (mod $2e$), $|C_1| =$

$\cdots = |C_{k-1}| = r > e$ where r is a positive integer and each vertex of C_s has colour s, for $1 \leqslant s \leqslant k-1$. For each $s \in \{1, \ldots, k-1\}$, let $C_s = \{c_1^s, \ldots, c_r^s\}$. For each $i \in \{1, \ldots, k\}$, let $U_i = U_0 \times \{i\}$ and $\mathcal{A}_i = \mathcal{A}_0 \times \{i\}$ where $\mathcal{A}_0 \times \{i\}$ denotes the set $\{S \times \{i\} \mid S \in \mathcal{A}_0\}$. So $(U_i, \mathcal{A}_i)$ has colour classes $C_1 \times \{i\}, \ldots, C_{k-1} \times \{i\}$. We construct a strongly equitable k-chromatic e-star system $(V, \mathcal{B})$ where $V = \bigcup_{i=1}^{k} U_i$. We need to decompose the edges between the subsets of V into e-stars such that the resulting e-star system $(V, \mathcal{B})$ is strongly equitably k-chromatic.

First, we decompose the edges between U_1 and U_2. For each $s \in \{1, \ldots, k-1\}$, let $D_s = \{c_1^s \times \{1\}, \ldots, c_e^s \times \{1\}\}$. Let $\mathcal{F}_1^1 = \bigcup_{j=1}^{r} \{\{c_j^1 \times \{2\}; D_1\}, \{c_j^1 \times \{2\}; D_2\}, \ldots, \{c_j^1 \times \{2\}; D_{k-1}\}\}$. Partition $U_1 \setminus \bigcup_{s=1}^{k-1} D_s$ into $m = \frac{n_{k-1} - e(k-1)}{e}$ sets $G_1, \ldots, G_m$ of cardinality e such that for each $\ell \in \{1, \ldots, m\}$, the vertices of G_ℓ do not all have the same colour. Since $(U_0, \mathcal{A}_0)$ is strongly equitably $(k-1)$-chromatic, such a partition exists. Let $\mathcal{F}_1^2 = \bigcup_{j=1}^{r} \{\{c_j^1 \times \{2\}; G_1\}, \{c_j^1 \times \{2\}; G_2\}, \ldots, \{c_j^1 \times \{2\}; G_m\}\}$ and $\mathcal{F}_1 = \mathcal{F}_1^1 \cup \mathcal{F}_1^2$. Then $\mathcal{F}_1$ is a decomposition of the edges between $C_1 \times \{2\}$ and U_1 into e-stars. In any putative $(k-1)$-colouring of the e-star system that we are building, the colouring of U_1 is unique since $(U_1, \mathcal{A}_1)$ is uniquely $(k-1)$-chromatic. The uniqueness of the $(k-1)$-colouring of $(U_1, \mathcal{A}_1)$ is such that all vertices of D_s $(1 \leqslant s \leqslant k-1)$ have colour s. But now some star of $\mathcal{F}_1^1$ is monochromatic if any vertex of $C_1 \times \{2\}$ has colour s for some $s \in \{1, \ldots, k-1\}$. Therefore the system that we are building is not $(k-1)$-colourable. By demonstrating a k-colouring, we establish that it is k-chromatic. Without loss of generality, colour each vertex of D_s with colour s and each vertex of $C_1 \times \{2\}$ with colour k.

Let $\mathcal{F}_2^1 = \bigcup_{j=1}^{r} \{\{c_j^2 \times \{2\}; D_1\}, \{c_j^2 \times \{2\}; D_3\}, \{c_j^2 \times \{2\}; D_4\}, \ldots, \{c_j^2 \times \{2\}; D_{k-1}\}\}$. Partition $U_1 \setminus \big(D_1 \cup (\bigcup_{s=3}^{k-1} D_s)\big)$ into $m = \frac{n_{k-1} - e(k-2)}{e}$ sets $G_1, \ldots, G_m$ such that for each $\ell \in \{1, \ldots, m\}$, the vertices of G_ℓ do not all have the same colour. Let $\mathcal{F}_2^2 =$

$\bigcup_{j=1}^{r}\big\{\{c_j^2\times\{2\};G_1\},\ldots,\{c_j^2\times\{2\};G_m\}\big\}$ and $\mathcal{F}_2=\mathcal{F}_2^1\cup\mathcal{F}_2^2$. Then $\mathcal{F}_2$ is a decomposition of the edges between $C_2\times\{2\}$ and U_1 into e-stars such that all the vertices in $C_2\times\{2\}$ are forced to have colour 2. For each $s\in\{3,\ldots,k-1\}$, decompose the edges between $C_s\times\{2\}$ and U_1 into a set of e-stars $\mathcal{F}_s$ in a similar manner such that all the vertices in $C_s\times\{2\}$ are forced to have colour s.

Let $\mathcal{U}_2^1=\bigcup_{s=1}^{k-1}\mathcal{F}_s$. Then $\mathcal{U}_2^1$ is a decomposition of the edges between U_1 and U_2 into e-stars forcing $(U_2,\mathcal{A}_2)$ to have colour classes $C_2^2,C_2^3,\ldots,C_2^k$ of equal size with no vertex of colour 1, where $C_2^2=C_2\times\{2\}$, $C_2^3=C_3\times\{2\}$, …, $C_2^k=C_1\times\{2\}$. For each $i\in\{3,\ldots,k\}$, decompose the edges between U_1 and U_i into a set of e-stars $\mathcal{U}_i^1$ in a similar manner forcing $(U_i,\mathcal{A}_i)$ to have colour classes $C_i^1,C_i^2,\ldots,C_i^{i-2},C_i^i,\ldots,C_i^k$ of equal size with no vertex of colour $i-1$, where $C_i^1=C_1\times\{i\}$, $C_i^2=C_2\times\{i\}$, …, $C_i^{i-2}=C_{i-2}\times\{i\}$, $C_i^i=C_i\times\{i\}$, …, $C_i^k=C_{i-1}\times\{i\}$.

Next, for each $j\in\{3,\ldots,k\}$, we decompose the edges between U_2 and U_j. Partition U_2 into $m=\frac{n_{k-1}}{e}$ sets $G_1,\ldots,G_m$ such that for each $\ell\in\{1,\ldots,m\}$, the vertices of G_ℓ do not all have the same colour. For each $j\in\{3,\ldots,k\}$, let $\mathcal{U}_j^2=\bigcup_{u\in U_j}\big\{\{u;G_1\},\{u;G_2\},\ldots,\{u;G_m\}\big\}$. Then $\mathcal{U}_j^2$ is a decomposition of the edges between U_2 and U_j into e-stars such that no e-star is monochromatic. For each $3\leqslant i<j\leqslant k$, decompose the edges between U_i and U_j into a set of e-stars $\mathcal{U}_j^i$ in a similar manner.

Let $\mathcal{B}=\big(\bigcup_{i=1}^{k}\mathcal{A}_i\big)\cup\big(\bigcup_{j=2}^{k}\mathcal{U}_j^1\big)\cup\big(\bigcup_{j=3}^{k}\mathcal{U}_j^2\big)\cup\ldots\cup\mathcal{U}_k^{k-1}$ and $C_1^1=C_1\times\{1\}$, $C_1^2=C_2\times\{1\}$, …, $C_1^{k-1}=C_{k-1}\times\{1\}$. Then $(V,\mathcal{B})$ is a strongly equitable k-chromatic e-star system of order $n_k=kn_{k-1}$ with colour classes $C_1^1\cup C_3^1\cup C_4^1\cup\cdots\cup C_k^1$, $C_1^2\cup C_2^2\cup C_4^2\cup\cdots\cup C_k^2$, …, $C_1^{k-1}\cup C_2^{k-1}\cup\cdots\cup C_{k-1}^{k-1}$, $C_2^k\cup C_3^k\cup\cdots\cup C_k^k$ where each colour class is of size $(k-1)r$. □

We now show how to construct a strongly equitable uniquely k-chromatic e-star system from a strongly equitable k-chromatic e-star system.

Theorem 3.3.4. *Let $k \geqslant 3$ and $e \geqslant 3$. If there exists a strongly equitably k-chromatic e-star system of order $n_0 \equiv 0 \pmod{2e}$, then there exists a uniquely k-chromatic e-star system of order n for some n such that $n \equiv 0 \pmod{2e}$.*

Proof. Let $(U_0, \mathcal{A}_0)$ be a strongly equitable k-chromatic e-star system of order $n_0 \equiv 0 \pmod{2e}$ with colour classes $C_1, \ldots, C_k$ such that each vertex of C_s has colour s, for $s = 1, \ldots, k$. For a positive integer ℓ that will be fixed later and for each $i \in \{1, \ldots, \ell\}$, let $U_i = U_0 \times \{i\}$ and $\mathcal{A}_i = \mathcal{A}_0 \times \{i\}$ where $\mathcal{A}_0 \times \{i\}$ denotes the set $\{S \times \{i\} \mid S \in \mathcal{A}_0\}$. So $(U_i, \mathcal{A}_i)$ has colour classes $C_1 \times \{i\}, \ldots, C_k \times \{i\}$. Let $U = \bigcup_{i=1}^{\ell} U_i$. Let $A^1 = \{a_1^1, \ldots, a_e^1\}, \ldots, A^k = \{a_1^k, \ldots, a_e^k\}$, $A_0^1 = \{(a_1)_0^1, \ldots, (a_e)_0^1\}$, $A = \bigcup_{i=1}^k A^i$ and $A' = A \cup A_0^1$. Similarly, let $F^1 = \{f_1^1, \ldots, f_e^1\}, \ldots, F^k = \{f_1^k, \ldots, f_e^k\}$, $F_0^1 = \{(f_1)_0^1, \ldots, (f_e)_0^1\}$, $F = \bigcup_{i=1}^k F^i$, and $F' = F \cup F_0^1$, $G^1 = \{g_1^1, \ldots, g_e^1\}, \ldots, G^k = \{g_1^k, \ldots, g_e^k\}$, $G_0^1 = \{(g_1)_0^1, \ldots, (g_e)_0^1\}$, $G = \bigcup_{i=1}^k G^i$, and $G' = G \cup G_0^1$, $H^1 = \{h_1^1, \ldots, h_e^1\}$, $\ldots, H^k = \{h_1^k, \ldots, h_e^k\}$, $H_0^1 = \{(h_1)_0^1, \ldots, (h_e)_0^1\}$, $H = \bigcup_{i=1}^k H^i$, and $H' = H \cup H_0^1$. We construct a uniquely k-chromatic e-star system $(\hat{V}, \hat{\mathcal{B}})$ where $\hat{V} = A \cup F \cup G \cup H \cup U$ if k is even and $\hat{V} = A' \cup F' \cup G' \cup H' \cup U$ if k is odd.

We need to decompose the edges between the subsets of $\hat{V}$ into e-stars such that the resulting e-star system $(\hat{V}, \hat{\mathcal{B}})$ is uniquely k-chromatic.

First, we decompose the edges between A and U into e-stars in a way such that no e-subset in A other than the e-subsets $A^1, \ldots, A^k$ are monochromatic in any putative k-colouring of $(\hat{V}, \hat{\mathcal{B}})$. Let $\mathcal{D}$ be the set of all e-subsets of A except $A^1, \ldots, A^k$. The number of e-subsets of the set A is $N = \binom{ke}{e}$ and $|\mathcal{D}| = N - k$. Let $a = k - 1$ and $|\mathcal{D}| = aq + r$ where q is a nonnegative integer and $0 \leqslant r < a$ is an integer. Partition set $\mathcal{D}$ into q sets $\mathbb{T}_1, \ldots, \mathbb{T}_q$ of size a of disjoint e-subsets and one set $\mathbb{T}_{q+1}$ of size r of disjoint e-subsets; such a partition is known to exist by Theorem 3.1.3. Let

$\mathbb{T}_i = \{T_1^i, \ldots, T_a^i\}$, $1 \leqslant i \leqslant q$, and $\mathbb{T}_{q+1} = \{T_1^{q+1}, \ldots, T_r^{q+1}\}$. Let $\tilde{A}_i = A \setminus \bigcup_{j=1}^{a} T_j^i$, $1 \leqslant i \leqslant q$ and $\tilde{A}_{q+1} = A \setminus \bigcup_{j=1}^{r} T_j^{q+1}$ if $r > 0$ and $\tilde{A}_{q+1} = \emptyset$ if $r = 0$. We now fix $\ell = q+1$ if $r > 0$ and $\ell = q$ if $r = 0$.

Partition U_0 into e-subsets $E_1, \ldots, E_m$, where $m = \frac{n_0}{e}$, such that each E_j has nonempty intersection with at least two colour classes, $1 \leqslant j \leqslant m$. For each $i \in \{1, 2, \ldots, \ell\}$ and for each $w \in \tilde{A}_i$, we decompose the edges between w and U_i into the set of e-stars $\mathcal{R}_w = \bigcup_{j=1}^{m} \big\{\{w; E_j \times \{i\}\}\big\}$. For each $i \in \{1, \ldots, q\}$, let $\mathcal{T}_i = \bigcup_{u \in U_i} \big\{\{u; T_1^i\}, \ldots, \{u; T_a^i\}\big\}$ and $\mathcal{T}_{q+1} = \bigcup_{u \in U_i} \big\{\{u; T_1^{q+1}\}, \ldots, \{u; T_r^{q+1}\}\big\}$. Then $\mathcal{P}_i = \mathcal{T}_i \cup \big(\bigcup_{w \in \tilde{A}_i} \mathcal{R}_w\big)$, $1 \leqslant i \leqslant \ell$, is a decomposition of the edges between A and U_i and $\bigcup_{i=1}^{\ell} \mathcal{P}_i$ is a decomposition of the edges between A and U into e-stars.

Recall that $(U_i, \mathcal{A}_i)$ is k-chromatic. So for any k-colouring of $(U_i, \mathcal{A}_i)$ there must be a vertex u_s^i of each colour s in U_i, $1 \leqslant s \leqslant k$. Note that for each $i \in \{1, \ldots, q\}$, $\{u_s^i; T_j^i\} \in \mathcal{T}_i$, $1 \leqslant j \leqslant a$, and $\{u_s^{q+1}; T_j^{q+1}\} \in \mathcal{T}_{q+1}$, $1 \leqslant j \leqslant r$, and so to ultimately obtain a valid k-colouring of $(\hat{V}, \hat{\mathcal{B}})$, the e vertices of T_j^i and T_j^{q+1} must not all have colour s, $1 \leqslant s \leqslant k$. If a colour s occurs on more than e vertices of A then there exists a T_j^i or T_j^{q+1} with only colour s and then $\{u_s^i; T_j^i\}$ or $\{u_s^i; T_j^{q+1}\}$ would be monochromatic. So in any valid k-colouring of $(\hat{V}, \hat{\mathcal{B}})$, each colour must occur on exactly e points of A. If the points of colour s are not all in A^1 or ... or A^k, then they are the leaves of a monochromatic e-star with centre u_s^i. Without loss of generality, we assign colour s to all e points of A^s, $1 \leqslant s \leqslant k$.

We now begin to decompose the edges between A and F into e-stars. For each $i \in \{1, \ldots, e\}$, let $\mathcal{K}_i^1 = \big\{\{f_i^1; A^2\}, \ldots, \{f_i^1; A^k\}\big\}$, $\mathcal{K}_i^2 = \big\{\{f_i^2; A^1\}, \{f_i^2; A^3\}, \ldots, \{f_i^2; A^k\}\big\}$, ..., $\mathcal{K}_i^k = \big\{\{f_i^k; A^1\}, \ldots, \{f_i^k; A^{k-1}\}\big\}$. Since each vertex of A^s has colour s, $2 \leqslant s \leqslant k$, then each vertex of F^1 must have colour 1 or else a monochromatic e-star would now exist. Likewise, each vertex of F^s must have colour s, $2 \leqslant s \leqslant k$.

Let $\mathcal{K}^1 = \bigcup_{i=1}^{e} \mathcal{K}_i^1, \ldots, \mathcal{K}^k = \bigcup_{i=1}^{e} \mathcal{K}_i^k$. We decompose the edges between A^i and F^i, $1 \leqslant i \leqslant k$, later. For each $j \in \{1, \ldots, k\}$, decompose the edges between A^j and $G^1 \cup \cdots \cup G^{j-1} \cup G^{j+1} \cup \cdots \cup G^k$ into a set of e-stars $\mathcal{L}_j$ in a similar manner, forcing the vertices of the subsets $G^1, \ldots, G^k$ to have colours $1, \ldots, k$ respectively. Let $\mathcal{L} = \bigcup_{j=1}^{k} \mathcal{L}_j$. We decompose the edges between A^i and G^i, $1 \leqslant i \leqslant k$, later.

Next, we decompose the edges between H and $F \cup G$. For each $i \in \{1, \ldots, e\}$, let $\mathcal{E}_i^1 = \big\{\{h_i^1; f_1^2, \ldots, f_e^2\}, \ldots, \{h_i^1; f_1^k, \ldots, f_e^k\}, \{h_i^1; f_1^1, \ldots, f_{e-1}^1, g_1^2\}, \{h_i^1; f_e^1, g_1^1, \ldots, g_{e-2}^1, g_2^2\}, \{h_i^1; g_{e-1}^1, g_e^1, g_3^2, \ldots, g_e^2\}, \{h_i^1; g_1^3, \ldots, g_e^3\}, \ldots, \{h_i^1; g_1^k, \ldots, g_e^k\}\big\}$ and $\mathcal{E}^1 = \bigcup_{i=1}^{e} \mathcal{E}_i^1$. Then $\mathcal{E}^1$ is a decomposition of the edges between H^1 and $F \cup G$ into e-stars such that all the vertices in H^1 are forced to have colour 1. For each $s \in \{2, \ldots, k\}$, decompose the edges between H^s and $F \cup G$ into a set of e-stars $\mathcal{E}^s$ in a similar manner, forcing every vertex in H^s to have unique colour s. Let $\mathcal{E} = \bigcup_{s=1}^{k} \mathcal{E}^s$.

Next, we decompose the edges between A^1 and $F^1 \cup G^1 \cup H$. For each $i \in \{1, \ldots, e\}$, let $\mathcal{M}_i^1 = \big\{\{a_i^1; f_1^1, \ldots, f_{e-1}^1, h_1^2\}, \{a_i^1; f_e^1, g_1^1, \ldots, g_{e-2}^1, h_2^2\}, \{a_i^1; g_{e-1}^1, g_e^1, h_3^2, \ldots, h_e^2\}, \{a_i^1; h_1^1, \ldots, h_{e-1}^1, h_1^3\}, \{a_i^1; h_e^1, h_2^3, \ldots, h_e^3\}, \{a_i^1; h_1^4, \ldots, h_e^4\}, \ldots, \{a_i^1; h_1^k, \ldots, h_e^k\}\big\}$. Then $\mathcal{M}^1 = \bigcup_{i=1}^{e} \mathcal{M}_i^1$ is a decomposition of the edges between A^1 and $F^1 \cup G^1 \cup H$ into e-stars, none of which are monochromatic. For each $j \in \{2, \ldots, k\}$, decompose the edges between A^j and $F^j \cup G^j \cup H$ into a set $\mathcal{M}^j$ of e-stars in a similar manner. Let $\mathcal{M} = \bigcup_{j=1}^{k} \mathcal{M}^j$.

Next, for each $1 \leqslant i \leqslant \ell$, we decompose the edges between $F \cup G$ and U_i into e-stars. Let $u_1^s, \ldots, u_e^s$ be the vertices of colour s, $1 \leqslant s \leqslant k$, in an equitable k-colouring of $(U_0, \mathcal{A}_0)$. For each $j \in \{1, \ldots, e\}$, let $\mathcal{N}_j^1 = \big\{\{u_j^1 \times \{i\}; f_1^2, \ldots, f_e^2\}, \ldots, \{u_j^1 \times \{i\}; f_1^k, \ldots, f_e^k\}, \{u_j^1 \times \{i\}; f_1^1, \ldots, f_{e-1}^1, g_1^2\}, \{u_j^1 \times \{i\}; f_e^1, g_1^1, \ldots, g_{e-2}^1, g_2^2\}, \{u_j^1 \times \{i\}; g_{e-1}^1, g_e^1, g_3^2, \ldots, g_e^2\}, \{u_j^1 \times \{i\}; g_1^3, \ldots, g_e^3\}, \ldots, \{u_j^1 \times \{i\}; g_1^k, \ldots, g_e^k\}\big\}$ and observe that $u_j^1 \times \{i\}$ is now forced to have colour 1. For each $j \in \{1, \ldots, e\}$ and each $s \in \{2, \ldots, k\}$, decompose the edges between $u_j^s \times \{i\}$ and $F \cup G$ into a set of e-stars

$\mathcal{N}_j^s$ in a similar manner so that $u_j^s \times \{i\}$ is forced to have colour s. Let $\mathcal{N}^s = \bigcup_{j=1}^{e} \mathcal{N}_j^s$ and $\mathcal{O}_i = \bigcup_{s=1}^{k} \mathcal{N}^s$. Then $\mathcal{O}_i$ is a decomposition of the edges between $F \cup G$ and U_i into e-stars which forces every vertex of U_i to have a unique colour. Let $\mathcal{O} = \bigcup_{i=1}^{\ell} \mathcal{O}_i$.

At this point every vertex of $\hat{V}$ is coloured, and the colouring is unique (up to a permutation of the colours). Moreover, the colour classes are equal in size and so the colouring is strongly equitable. All that remains is to complete the decomposition without introducing monochromatic e-stars.

For each $1 \leqslant i \leqslant e$, we begin to decompose the edges between F and G into e-stars. Let $\mathcal{Q}_i^1 = \big\{\{f_i^1; g_1^1, \ldots, g_{e-1}^1, g_1^2\}, \{f_i^1; g_e^1, g_2^2, \ldots, g_e^2\}\big\}, \{f_i^1; g_1^3, \ldots, g_e^3\}, \ldots, \{f_i^1; g_1^k, \ldots, g_e^k\}\big\}$. Then $\mathcal{Q}^1 = \bigcup_{i=1}^{e} \mathcal{Q}_i^1$ is a decomposition of the edges between F^1 and G into e-stars, none of which is monochromatic. For each $j \in \{2, \ldots, k\}$, decompose the edges between F^j and G into a set of e-stars $\mathcal{Q}^j$ in a similar manner. Let $\mathcal{Q} = \bigcup_{j=1}^{k} \mathcal{Q}^j$.

Next, for each $1 \leqslant i \leqslant \ell$, we decompose the edges between H and U_i into e-stars. Recall that $u_1^s, \ldots, u_e^s$ are the vertices of colour s, $1 \leqslant s \leqslant k$. Let $\mathcal{I}_j^1 = \big\{\{u_j^1 \times \{i\}; h_1^1, \ldots, h_{e-1}^1, h_1^2\}, \{u_j^1 \times \{i\}; h_e^1, h_2^2, \ldots, h_e^2\}, \{u_j^1 \times \{i\}; h_1^3, \ldots, h_e^3\}, \ldots, \{u_j^1 \times \{i\}; h_1^k, \ldots, h_e^k\}\big\}$, $1 \leqslant j \leqslant e$. For each $j \in \{1, \ldots, e\}$ and each $s \in \{2, \ldots, k\}$, decompose the edges between $u_j^s \times \{i\}$ and H into a set of e-stars $\mathcal{I}_j^s$ in a similar manner. Let $\mathcal{I}^s = \bigcup_{j=1}^{e} \mathcal{I}_j^s$ and $\mathcal{S}_i = \bigcup_{s=1}^{k} \mathcal{I}^s$. Then $\mathcal{S}_i$ is a decomposition of the edges between H and U_i into e-stars. Let $\mathcal{S} = \bigcup_{i=1}^{\ell} \mathcal{S}_i$. The edges between U_i and U_j, $1 \leqslant i < j \leqslant \ell$, are decomposed into a set $\mathcal{U}_j^i$ of e-stars in a similar manner.

Note that $|A| = |F| = |G| = |H| = ke$.

Case 1. k is even. We let each of $(A, \mathcal{A}), (F, \mathcal{F}), (G, \mathcal{G})$, and $(H, \mathcal{H})$ be strongly equitably 2-chromatic e-star systems on the sets A, F, G, and H respectively. Let

$\hat{\mathcal{B}} = \mathcal{A}\cup\mathcal{F}\cup\mathcal{G}\cup\mathcal{H}\cup\big(\bigcup_{i=1}^{\ell}\mathcal{A}_i\big)\cup\big(\bigcup_{i=1}^{\ell}\mathcal{P}_i\big)\cup\big(\bigcup_{j=2}^{\ell}\mathcal{U}_j^1\big)\cup\big(\bigcup_{j=3}^{\ell}\mathcal{U}_j^2\big)\cup\ldots\cup\big(\mathcal{U}_\ell^{\ell-1}\big)\cup\big(\bigcup_{j=1}^{k}\mathcal{K}^j\big)\cup \mathcal{L}\cup\mathcal{E}\cup\mathcal{M}\cup\mathcal{O}\cup\mathcal{Q}\cup\mathcal{S}$. Then $(\hat{V},\hat{\mathcal{B}})$ is an e-star system of order $n = 4ke+\ell n_0$ which is strongly equitably uniquely k-chromatic.

Case 2. k is odd. For $j\in\{1,\ldots,e\}$, let $\mathcal{V}_j = \big\{\{(a_j)_0^1; f_1^2,\ldots,f_e^2\}, \{(a_j)_0^1; f_1^3,\ldots,f_e^3\}, \ldots, \{(a_j)_0^1; f_1^k,\ldots,f_e^k\}, \{(a_j)_0^1; f_1^1,\ldots,f_{e-1}^1, g_1^2\}, \{(a_j)_0^1; f_e^1, g_1^1,\ldots,g_{e-2}^1, g_2^2\}, \{(a_j)_0^1; g_{e-1}^1, g_e^1, g_3^2,\ldots,g_e^2\}, \{(a_j)_0^1; g_1^3,\ldots,g_e^3\}, \ldots,\{(a_j)_0^1; g_1^k,\ldots,g_e^k\}\big\} \cup \big\{\{(a_j)_0^1; h_1^1,\ldots,h_{e-1}^1, h_1^2\}, \{(a_j)_0^1; h_e^1, h_2^2,\ldots,h_{e-1}^2\}, \{(a_j)_0^1; h_1^3,\ldots,h_e^3\}, \ldots, \{(a_j)_0^1; h_1^k,\ldots,h_e^k\}\big\} \cup \bigcup_{i=1}^{\ell}\big(\big\{\{(a_j)_0^1; u_1^1\times\{i\}, u_2^1\times\{i\},\ldots,u_{e-1}^1\times\{i\}, u_1^2\times\{i\}\},\ \ldots,\ \{(a_j)_0^1; u_{(t-1)(e-1)+1}^1\times\{i\},\ldots,u_{t(e-1)}^1\times\{i\}, u_t^2\times\{i\}\},\ \{(a_j)_0^1; u_{t(e-1)+1}^1\times\{i\},\ldots,u_{|C_1|}^1\times\{i\}, u_{t+1}^2\times\{i\},\ldots,u_{t+m}^2\times\{i\}\}\big\}\big) \cup \bigcup_{i=1}^{\ell}\big(\big\{\{(a_j)_0^1; x_1^d\times\{i\},\ldots,x_e^d\times\{i\}\} \mid 1\leqslant d\leqslant \frac{n_0-(|C_1|+t+m)}{e}\big\}\big)$, where $t = \lfloor\frac{|C_1|}{e-1}\rfloor$ and $m = e-(|C_1|-t(e-1))$ are positive integers and the sets $\{x_1^d,\ldots,x_e^d\}$, $1\leqslant d\leqslant \frac{n_0-(|C_1|+t+m)}{e}$, form a partition of $U_i\setminus(C_1\times\{i\})\cup\big\{u_1^2\times\{i\}, u_2^2\times\{i\},\ldots,u_{t+m}^2\times\{i\}\big\}$. Let $\mathcal{V} = \bigcup_{j=1}^{e}\mathcal{V}_j$, then $\mathcal{V}$ is a decomposition of the edges between A_0^1 and $F\cup G\cup H\cup U$ into e-stars forcing the vertices of A_0^1 to have unique colour 1.

Decompose the edges between F_0^1 and $A'\cup G\cup H\cup U$ into set of e-stars $\mathcal{W}$ in a similar manner so that the vertices of F_0^1 are forced to have colour 1. Also, decompose the edges between G_0^1 and $A'\cup F'\cup H\cup U$ into set of e-stars $\mathcal{X}$ so that the vertices of G_0^1 are forced to have colour 1, and the edges between H_0^1 and $A'\cup F'\cup G'\cup U$ into set of e-stars $\mathcal{Y}$ so that the vertices of H_0^1 are forced to have colour 1.

Now, we decompose the edges of the complete graph on set A' into e-stars. We first take 2-chromatic e-star systems $Z_1 = (A^1\cup A^2, \mathcal{A}_2^1)$, $Z_2 = (A^3\cup A^4, \mathcal{A}_4^3)$, $\ldots$, $Z_{\frac{k+1}{2}} = (A^k\cup A_0^1, \mathcal{A}_1^k)$ of size $2e$. It is easy then to decompose the edges between Z_i and Z_j into set $\mathcal{Z}_j^i$ of e-stars $1\leqslant i<j\leqslant\frac{k+1}{2}$, such that no e-star is monochromatic. Then $(A',\mathcal{A}')$ where $\mathcal{A}' = \mathcal{A}_2^1\cup\mathcal{A}_4^3\cup\cdots\cup\mathcal{A}_1^k\cup\big(\bigcup_{j=2}^{\frac{k+1}{2}}\mathcal{Z}_j^1\big)\cup\big(\bigcup_{j=3}^{\frac{k+1}{2}}\mathcal{Z}_j^2\big)\cup\cdots\cup\mathcal{Z}_{\frac{k+1}{2}}^{\frac{k+1}{2}-1}$ is an e-star system which is uniquely k-chromatic as we already assign a unique colour s

to each vertex of A', for each $s \in \{1, 2, \ldots, k\}$. Decompose the edges of the complete graph on the sets F', G' and H' into set of e-stars $\mathcal{F}'$, $\mathcal{G}'$ and $\mathcal{H}'$ in a similar manner. Let $\hat{\mathcal{B}} = \mathcal{A}' \cup \mathcal{F}' \cup \mathcal{G}' \cup \mathcal{H}' \cup \big(\bigcup_{i=1}^{\ell} \mathcal{A}_i\big) \cup (\bigcup_{i=1}^{\ell} \mathcal{P}_i) \cup \big(\bigcup_{j=2}^{\ell} \mathcal{U}_j^1\big) \cup \big(\bigcup_{j=3}^{\ell} \mathcal{U}_j^2\big) \cup \ldots \cup \big(\mathcal{U}_\ell^{\ell-1}\big)$ $\cup\, (\bigcup_{j=1}^{k} \mathcal{K}^j) \cup \mathcal{L} \cup \mathcal{E} \cup \mathcal{M} \cup \mathcal{O} \cup \mathcal{Q} \cup \mathcal{S} \cup \mathcal{V} \cup \mathcal{W} \cup \mathcal{X} \cup \mathcal{Y}$. Then $(\hat{V}, \hat{\mathcal{B}})$ is an e-star system of order $n = 4e(k+1) + \ell n_0$ which is strongly equitably uniquely k-chromatic.

□

We now show how to construct a uniquely k-chromatic e-star system from a smaller uniquely k-chromatic e-star system.

Theorem 3.3.5. *For any $k \geqslant 3$, let $(V, \mathcal{B})$ be a uniquely k-chromatic e-star system of order n_0 constructed from Theorem 3.3.4 with colour classes $C_1, \ldots, C_k$. Then there exists a uniquely k-chromatic e-star system for all $n > n_0$ such that $n \equiv 0, 1$ (mod $2e$).*

Proof. For $s \in \{1, \ldots, k\}$, let $C_s = \{c_1^s, \ldots, c_{|C_s|}^s\}$. Since $n_0 \equiv 0 \pmod{2e}$, let $n_0 = 2et$, $t \geqslant 1$.

First, we construct a uniquely k-chromatic $S_e(2et+1)$, $(\hat{V}, \hat{\mathcal{B}})$, from $(V, \mathcal{B})$. Let $\hat{V} = V \cup \{2et+1\}$. Let $\mathcal{T}_1^{2et+1} = \big\{\{2et+1; c_1^2, \ldots, c_e^2\}, \{2et+1; c_1^3, \ldots, c_e^3\}, \ldots, \{2et+1; c_1^k, \ldots, c_e^k\}\big\}$ and $\mathcal{T}_2^{2et+1} = \big\{\{2et+1; c_1^1, \ldots, c_{e-1}^1, c_{e+1}^2\}, \{2et+1; c_e^1, \ldots, c_{2e-2}^1, c_{e+2}^2\}, \ldots, \{2et+1; c_{(r-1)(e-1)+1}^1, \ldots, c_{r(e-1)}^1, c_{e+r}^2\}, \{2et+1; c_{r(e-1)+1}^1, \ldots, c_{|C_1|}^1, c_{e+r+1}^2, \ldots, c_{e+r+r'}^2\}\big\}$ where $r = \lfloor \frac{|C_1|}{e-1} \rfloor$ and $r' = e - (|C_1| - r(e-1))$.

Partition the set $(C_2 \setminus \{c_1^2, \ldots, c_e^2, c_{e+1}^2, c_{e+2}^2, \ldots, c_{e+t+r}^2\}) \cup (C_3 \setminus \{c_1^3, \ldots, c_e^3\}) \cup \cdots \cup (C_k \setminus \{c_1^k, \ldots, c_e^k\})$ into a set $\mathbb{E}$ of m disjoint e-subsets $X_1, \ldots, X_m$ where $m = \frac{(|C_2|-(e+t+r)) + (|C_3|-e) + \cdots + (|C_k|-e)}{e}$. Let $\mathcal{T}_3^{2et+1} = \bigcup_{i=1}^{m} \big\{\{2et+1; X_i\}\big\}$.

Let $\hat{\mathcal{B}} = \mathcal{B} \cup \mathcal{T}_1^{2et+1} \cup \mathcal{T}_2^{2et+1} \cup \mathcal{T}_3^{2et+1}$. Then $(\hat{V}, \hat{\mathcal{B}})$ is a uniquely k-chromatic e-star system of order $2et+1$ with colour classes $C_1 \cup \{2et+1\}, C_2, \ldots, C_k$.

Next, we construct a uniquely k-chromatic $S_e(2et+2e)$, $(\mathring{V}, \mathring{\mathcal{B}})$, from $(\hat{V}, \hat{\mathcal{B}})$. Let $\mathring{V} = \hat{V} \cup V'$ where $V' = \{2et+2, 2et+3, \ldots, 2et+2e\}$ and let $v_0 \in C_1 \setminus \{c_1^1, \ldots, c_e^1\}$. Let $(V' \cup \{v_0\}, \mathcal{B}')$ be a strongly equitable 2-chromatic e-star system of order $2e$ constructed from Theorem 3.2.1. Let $\mathcal{T}_1^i = \bigcup_i \{\{i; c_1^2, \ldots, c_e^2\}, \{i; c_1^3, \ldots, c_e^3\}, \ldots, \{i; c_1^k, \ldots, c_e^k\}\}$, $i \in \{2et+3, 2et+5, \ldots, 2et+2e-1\}$ and $\mathcal{T}_1^j = \bigcup_j \{\{j; c_1^1, \ldots, c_e^1\}, \{j; c_1^3, \ldots, c_e^3\}, \ldots, \{j; c_1^k, \ldots, c_e^k\}\}$, $j \in \{2et+2, 2et+4, \ldots, 2et+2e\}$. For any $k \in \{2et+2, 2et+3, \ldots, 2et+2e\}$, decompose the remaining edges between vertex k and set $V \setminus \{v_0\}$ into sets of e-stars $\mathcal{T}_2^k$ and $\mathcal{T}_3^k$ in a manner similar to the construction of $\mathcal{T}_2^{2et+1}$ and $\mathcal{T}_3^{2et+1}$ in $(\hat{V}, \hat{\mathcal{B}})$ by labelling $2et+1$ with k.

Let $\mathring{\mathcal{B}} = \hat{\mathcal{B}} \cup \mathcal{B}' \cup \big(\bigcup_{k=2et+2}^{2et+2e} \mathcal{T}_1^k\big) \cup \big(\bigcup_{k=2et+2}^{2et+2e} \mathcal{T}_2^k\big) \cup \big(\bigcup_{k=2et+2}^{2et+2e} \mathcal{T}_3^k\big)$. Then $(\mathring{V}, \mathring{\mathcal{B}})$ is a uniquely k-chromatic e-star system of order $2et+2e$ with colour classes $C_1 \cup \{2et+3, 2et+5, \ldots, 2et+2e-1\}, C_2 \cup \{2et+2, 2et+4, \ldots, 2et+2e\}, C_3, \ldots, C_k$. Note that the result follows by applying the construction iteratively, with $(\mathring{V}, \mathring{\mathcal{B}})$ in place of $(V, \mathcal{B})$. □

We then conclude with the following corollary.

Corollary 3.3.2. *Let $e \geqslant 3$ and $k \geqslant 3$. There exists some integer n_0 where $n_0 \equiv 0$ (mod $2e$) such that for all admissible $n \geqslant n_0$ where $n \equiv 0, 1$ (mod $2e$), there exists a uniquely k-chromatic e-star system of order n.*

Proof. By Corollary 3.3.1, there exists a uniquely 2-chromatic e-star system of order n for all sufficiently large n such that $n \equiv 0, 1 \pmod{2e}$. For $k \geqslant 3$, apply Theorem 3.3.3, Theorem 3.3.4, and Theorem 3.3.5 to recursively construct a uniquely k-chromatic e-star system of order n for all sufficiently large n such that $n \equiv 0, 1 \pmod{2e}$. □

Chapter 4

Colourings of path systems

In this chapter, we investigate k-colourings of path systems. We first construct some small systems that are used as ingredients to construct larger systems. We then observe that there exists a k-chromatic P_m system for any $k \geqslant 2$ and $m \geqslant 4$ where m is even. Next, we prove that there exists an equitably 2-chromatic P_4 system for each admissible order n. Finally, we show that for any integer $k \geqslant 3$, there exists some integer n_k such that for all admissible $n \geqslant n_k$, there exists a k-chromatic P_4 system of order n.

4.1 Some small explicit solutions

In this section, we first construct (strongly) equitably 2-chromatic P_4 systems of order 4, 6, and 7. We then present decompositions of some complete bipartite graphs and other decompositions of complete bipartite graphs plus some set of edges into P_4 paths. These decompositions are used as ingredients to construct larger systems in the next section.

Lemma 4.1.1. *There exist strongly equitably 2-chromatic P_4 systems of order 4 and*

6, and an equitably 2-chromatic P_4 system of order 7.

Proof. We first construct a strongly equitably 2-chromatic P_4 system of order 4. Let $\hat{V} = \{v_1, v_2, v_3, v_4\}$, $\hat{\mathcal{B}} = \{(v_3, v_1, v_2, v_4),\ (v_1, v_4, v_3, v_2)\}$, $\hat{R} = \{v_1, v_3\}$, and $\hat{Y} = \{v_2, v_4\}$. Then $(\hat{V}, \hat{\mathcal{B}})$ is a $P_4(4)$ for which $\hat{R}$ and $\hat{Y}$ constitute a strongly equitable 2-colouring.

Next, we construct a strongly equitably 2-chromatic P_4 system of order 6. Let $\tilde{V} = \{v_1, v_2, v_3, v_4, v_5, v_6\}$, $\tilde{\mathcal{B}} = \{(v_6, v_1, v_2, v_5), (v_6, v_2, v_3, v_5), (v_2, v_4, v_1, v_3), (v_4, v_3, v_6, v_5), (v_1, v_5, v_4, v_6)\}$, $\tilde{R} = \{v_1, v_3, v_5\}$, and $\tilde{Y} = \{v_2, v_4, v_6\}$. Then $(\tilde{V}, \tilde{\mathcal{B}})$ is a $P_4(6)$ for which $\tilde{R}$ and $\tilde{Y}$ constitute a strongly equitable 2-colouring.

Finally, we construct an equitably 2-chromatic P_4 system of order 7. Let $V' = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$, $\mathcal{B}' = \{(v_2, v_7, v_4, v_3), (v_7, v_3, v_6, v_5), (v_1, v_7, v_6, v_2), (v_2, v_4, v_1, v_3), (v_1, v_5, v_4, v_6), (v_2, v_3, v_5, v_7), (v_6, v_1, v_2, v_5)\}$, $R' = \{v_1, v_3, v_5, v_7\}$, and $Y' = \{v_2, v_4, v_6\}$. Then $(V', \mathcal{B}')$ is a $P_4(7)$ for which R' and Y' constitute an equitable 2-colouring. □

Lemma 4.1.2. *There exists a strongly equitably 2-chromatic decomposition of $K_{3,3}$ into P_4 paths.*

Proof. Let $V = \{v_1, v_2, v_3\} \cup \{w_1, w_2, w_3\}$ be the set of vertices, $\mathcal{B} = \{(v_2, w_1, v_1, w_3), (v_1, w_2, v_3, w_1),\ (v_3, w_3, v_2, w_2)\}$, $R = \{v_1, v_3, w_2\}$, and $Y = \{v_2, w_1, w_3\}$. Then $\mathcal{B}$ is a decomposition of $K_{3,3}$ into P_4 paths for which R and Y constitute a strongly equitable 2-colouring. □

Lemma 4.1.3. *There exists an equitably 2-chromatic decomposition of $K_{4,3}$ into P_4 paths.*

Proof. Let $V = \{v_1, v_2, v_3, v_4\} \cup \{w_1, w_2, w_3\}$ be the set of vertices, $\mathcal{B} = \{(v_4, w_3, v_1, w_1), (v_2, w_1, v_4, w_2), (v_1, w_2, v_3, w_1), (v_3, w_3, v_2, w_2)\}$, $R = \{v_1, v_3, w_1, w_3\}$, and $Y = \{v_2, v_4, w_2\}$. Then $\mathcal{B}$ is a decomposition of $K_{4,3}$ into P_4 paths for which R and Y constitute an equitable 2-colouring. □

Lemma 4.1.4. *There exists a strongly equitably 2-chromatic decomposition of $K_{7,3}$ into P_4 paths.*

Proof. Let $V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\} \cup \{w_1, w_2, w_3\}$ be the set of vertices, $\mathcal{B} = \{(v_1, w_1, v_7, w_3), (v_7, w_2, v_2, w_1), (v_2, w_3, v_3, w_1), (v_3, w_2, v_4, w_1), (v_4, w_3, v_5, w_1), (v_5, w_2, v_6, w_1), (v_6, w_3, v_1, w_2)\}$, $R = \{v_1, v_3, v_5, v_7, w_2\}$, and $Y = \{v_2, v_4, v_6, w_1, w_3\}$. Then $\mathcal{B}$ is a decomposition of $K_{4,3}$ into P_4 paths for which R and Y constitute a strongly equitable 2-colouring. □

Lemma 4.1.5. *There exists a strongly equitably 2-chromatic decomposition of $K_{6,2}$ into P_4 paths.*

Proof. Let $V = \{v_1, v_2, v_3, v_4, v_5, v_6\} \cup \{w_1, w_2\}$ be the set of vertices, $\mathcal{B} = \{(v_2, w_1, v_3, w_2), (v_2, w_2, v_4, w_1), (v_5, w_1, v_6, w_2), (v_5, w_2, v_1, w_1)\}$, $R = \{v_1, v_3, v_5, w_1\}$, and $Y = \{v_2, v_4, v_6, w_2\}$. Then $\mathcal{B}$ is a decomposition of $K_{6,2}$ into P_4 paths for which R and Y constitute a strongly equitable 2-colouring. □

Lemma 4.1.6. *There exists an equitably 2-chromatic decomposition of $K_{6,3}$ into P_4 paths.*

Proof. Let $V = \{v_1, v_2, v_3, v_4, v_5, v_6\} \cup \{w_1, w_2, w_3\}$ be the set of vertices, $\mathcal{B} = \{(v_1, w_1, v_2, w_2), (v_2, w_3, v_3, w_1), (v_3, w_2, v_4, w_1), (v_4, w_3, v_5, w_1), (v_5, w_2, v_6, w_1), (v_6, w_3, v_1, w_2)\}$, $R = \{v_1, v_3, v_5, w_1, w_3\}$, and $Y = \{v_2, v_4, v_6, w_2\}$. Then $\mathcal{B}$ is a decomposition of $K_{6,3}$ into P_4 paths for which R and Y constitute an equitable 2-colouring. □

Lemma 4.1.7. *There exists a strongly equitably 2-chromatic decomposition of $K_{6,4}$ into P_4 paths.*

Proof. Let $V = \{v_1, v_2, v_3, v_4, v_5, v_6\} \cup \{w_1, w_2, w_3, w_4\}$ be the set of vertices, $\mathcal{B} = \{(v_1, w_1, v_2, w_2), (v_1, w_3, v_2, w_4), (v_1, w_2, v_3, w_1), (v_1, w_4, v_3, w_3), (v_4, w_1, v_5, w_2), (v_4, w_3, v_5, w_4),$

$(v_4, w_2, v_6, w_1), (v_4, w_4, v_6, w_3)\}$, $R = \{v_1, v_3, v_5, w_1, w_3\}$, and $Y = \{v_2, v_4, v_6, w_2, w_4\}$. Then $\mathcal{B}$ is a decomposition of $K_{6,4}$ into P_4 paths for which R and Y constitute a strongly equitable 2-colouring. □

Lemma 4.1.8. *There exists an equitably 2-chromatic decomposition of $K_{6,5}$ into P_4 paths.*

Proof. Let $V = \{v_1, v_2, v_3, v_4, v_5, v_6\} \cup \{w_1, w_2, w_3, w_4, w_5\}$ be the set of vertices, $\mathcal{B} = \{(v_1, w_1, v_2, w_2),\ (v_2, w_3, v_3, w_1),\ (v_3, w_2, v_4, w_1),\ (v_4, w_3, v_5, w_1), (v_5, w_2, v_6, w_1), (v_6, w_3, v_1, w_2), (v_2, w_4, v_3, w_5), (v_2, w_5, v_4, w_4), (v_5, w_4, v_6, w_5), (v_5, w_5, v_1, w_4)\}$, $R = \{v_1, v_3, v_5, w_1, w_3, w_5\}$, and $Y = \{v_2, v_4, v_6, w_2, w_4\}$. Then $\mathcal{B}$ is a decomposition of $K_{6,5}$ into P_4 paths for which R and Y constitute an equitable 2-colouring. □

Lemma 4.1.9. *There exists a strongly equitably 2-chromatic decomposition of $K_{6,6}$ into P_4 paths.*

Proof. Let $V = \{v_1, v_2, v_3, v_4, v_5, v_6\} \cup \{w_1, w_2, w_3, w_4, w_5, w_6\}$ be the set of vertices, $\mathcal{B} = \{(v_1, w_1, v_2, w_2),\ (v_2, w_3, v_3, w_1),\ (v_3, w_2, v_4, w_1),\ (v_4, w_3, v_5, w_1),\ (v_5, w_2, v_6, w_1), (v_6, w_3, v_1, w_2), (v_1, w_4, v_2, w_5), (v_2, w_6, v_3, w_4), (v_3, w_5, v_4, w_4), (v_4, w_6, v_5, w_4), (v_5, w_5, v_6, w_4), (v_6, w_6, v_1, w_5)\}$, $R = \{v_1, v_3, v_5, w_1, w_3, w_5\}$, and $Y = \{v_2, v_4, v_6, w_2, w_4, w_6\}$. Then $\mathcal{B}$ is a decomposition of $K_{6,6}$ into P_4 paths for which R and Y constitute a strongly equitable 2-colouring. □

Lemma 4.1.10. *There exists an equitably 2-chromatic decomposition of $K_{6,7}$ into P_4 paths.*

Proof. Let $V = \{v_1, v_2, v_3, v_4, v_5, v_6\} \cup \{w_1, w_2, w_3, w_4, w_5, w_6, w_7\}$ be the set of vertices, $\mathcal{B} = \{(v_1, w_1, v_2, w_2), (v_2, w_3, v_3, w_1), (v_3, w_2, v_4, w_1), (v_4, w_3, v_5, w_1), (v_5, w_2, v_6, w_1), (v_6, w_3, v_1, w_2), (v_1, w_4, v_2, w_5), (v_1, w_6, v_2, w_7), (v_1, w_5, v_3, w_4), (v_1, w_7, v_3, w_6), (v_4, w_4, v_5, w_5), (v_4, w_6, v_5, w_7),\ (v_4, w_5, v_6, w_4), (v_4, w_7, v_6, w_6)\}$, $R = \{v_1, v_3, v_5, w_1, w_3, w_5, w_7\}$, and

$Y = \{v_2, v_4, v_6, w_2, w_4, w_6\}$. Then $\mathcal{B}$ is a decomposition of $K_{6,7}$ into P_4 paths for which R and Y constitute an equitable 2-colouring. □

Lemma 4.1.11. *Let (V_1, V_2) be a bipartition of the set of vertices of the complete bipartite graph $K_{6,3}$ where $|V_1| = 6$ and $|V_2| = 3$. Let E_1 be the set of edges of $K_{6,3}$ and E_2 be the set of edges of the complete graph on V_2. Then the subgraph induced by $E_1 \cup E_2$ has a decomposition into P_4 paths which is equitably 2-chromatic.*

Proof. Let $V_1 = \{v_1, v_2, v_3, v_4, v_5, v_6\}$, $V_2 = \{w_1, w_2, w_3\}$, and $\mathcal{B} = \big\{(v_1, w_1, w_2, w_3),$ $(w_2, v_2, w_1, w_3), (v_2, w_3, v_3, w_1)$, $(v_3, w_2, v_4, w_1), (v_4, w_3, v_5, w_1), (v_5, w_2, v_6, w_1), (v_6, w_3, v_1, w_2)\big\}$, $R = \{v_1, v_3, v_5, w_1, w_3\}$, and $Y = \{v_2, v_4, v_6, w_2\}$. Then $\mathcal{B}$ is a decomposition of $E_1 \cup E_2$ into P_4 paths for which R and Y constitute an equitable 2-colouring. □

Lemma 4.1.12. *Let (V_1, V_2) be a bipartition of the set of vertices of the complete bipartite graph $K_{7,2}$ where $|V_1| = 7$ and $|V_2| = 2$. Let E_1 be the set of edges of $K_{7,2}$ and E_2 be the set of edges of the complete graph on V_2. Then the subgraph induced by $E_1 \cup E_2$ has a decomposition into P_4 paths which is equitably 2-chromatic.*

Proof. Let $V_1 = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$, $V_2 = \{w_1, w_2\}$, and $\mathcal{B} = \big\{(v_2, w_1, v_3, w_2),$ $(v_2, w_2, w_1, v_4), (v_4, w_2, v_5, w_1), (v_6, w_1, v_7, w_2), (v_6, w_2, v_1, w_1)\big\}$, $R = \{v_1, v_3, v_5, v_7, w_1\}$, and $Y = \{v_2, v_4, v_6, w_2\}$. Then $\mathcal{B}$ is a decomposition of $E_1 \cup E_2$ into P_4 paths for which R and Y constitute an equitable 2-colouring. □

Lemma 4.1.13. *Let (V_1, V_2) be a bipartition of the set of vertices of the complete bipartite graph $K_{7,3}$ where $|V_1| = 7$ and $|V_2| = 3$. Let E_1 be the set of edges of $K_{7,3}$ and E_2 be the set of edges of the complete graph on V_2. Then the subgraph induced by $E_1 \cup E_2$ has a decomposition into P_4 paths which is strongly equitably 2-chromatic.*

Proof. Let $V_1 = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$, $V_2 = \{w_1, w_2, w_3\}$, and $\mathcal{B} = \big\{(v_4, w_1, v_5, w_3),$ $(v_5, w_2, v_4, w_3), (v_6, w_1, v_7, w_3), (v_7, w_2, v_6, w_3), (v_1, w_3, v_3, w_1), (v_2, w_1, v_1, w_2), (v_3, w_2, w_3, v_2),$

$(v_2, w_2, w_1, w_3)\big\}$, $R = \{v_1, v_3, v_5, v_7, w_1\}$, and $Y = \{v_2, v_4, v_6, w_2, w_3\}$. Then $\mathcal{B}$ is a decomposition of $E_1 \cup E_2$ into P_4 paths for which R and Y constitute a strongly equitable 2-colouring. □

Lemma 4.1.14. *Let (V_1, V_2) be a bipartition of the set of vertices of the complete bipartite graph $K_{4,2}$ where $|V_1| = 4$ and $|V_2| = 2$. Let E_1 be the set of edges of $K_{4,2}$ and E_2 be the set of edges of the complete graph on V_2. Then the subgraph induced by $E_1 \cup E_2$ has a decomposition into P_4 paths which is strongly equitably 2-chromatic.*

Proof. Let $V_1 = \{v_1, v_2, v_3, v_4\}$, $V_2 = \{w_1, w_2\}$, and $\mathcal{B} = \big\{(v_1, w_1, w_2, v_4), (v_4, w_1, v_2, w_2), (v_1, w_2, v_3, w_1)\big\}$, $R = \{v_1, v_3, w_1\}$, and $Y = \{v_2, v_4, w_2\}$. Then $\mathcal{B}$ is a decomposition of $E_1 \cup E_2$ into P_4 paths for which R and Y constitute a strongly equitable 2-colouring. □

Lemma 4.1.15. *Let (V_1, V_2) be a bipartition of the set of vertices of the complete bipartite graph $K_{4,5}$ where $|V_1| = 4$ and $|V_2| = 5$. Let E_1 be the set of edges of $K_{4,5}$ and E_2 be the set of edges of the complete graph on V_2. Then the subgraph induced by $E_1 \cup E_2$ has a decomposition into P_4 paths which is equitably 2-chromatic.*

Proof. Let $V_1 = \{v_1, v_2, v_3, v_4\}$, $V_2 = \{w_1, w_2, w_3, w_4, w_5\}$, and $\mathcal{B} = \big\{(v_4, w_3, v_1, w_1), (v_2, w_1, v_4, w_2), (v_1, w_2, v_3, w_1), (v_3, w_3, v_2, w_2), (v_1, w_4, w_5, v_4), (v_4, w_4, v_2, w_5), (v_1, w_5, v_3, w_4), (w_4, w_1, w_2, w_3), (w_4, w_2, w_5, w_3), (w_4, w_3, w_1, w_5)\big\}$, $R = \{v_1, v_3, w_1, w_3, w_5\}$, and $Y = \{v_2, v_4, w_2, w_4\}$. Then $\mathcal{B}$ is a decomposition of $E_1 \cup E_2$ into P_4 paths for which R and Y constitute an equitable 2-colouring. □

4.2 k-colourings of path systems

In this section, we first observe that there exists a k-chromatic P_m system for any $k \geqslant 2$ and $m \geqslant 4$ where m is even. Next, we prove that there exists an equitably

2-chromatic P_4 system for each admissible order n. We then show that for any integer $k \geqslant 3$, there exists some integer n_k such that for all admissible $n \geqslant n_k$, there exists a k-chromatic P_4 system of order n.

4.2.1 k-colourings of P_m systems for m even

In this subsection, we prove that there exists a k-chromatic P_m system for any $k \geqslant 2$ and $m \geqslant 4$ where m is even. Recall that for any $m \geqslant 2$, necessary and sufficient conditions for the existence of a P_m system of order n are given in the following theorem from [34].

Theorem 4.2.1. *[34] Let $m \geqslant 2$. There exists a P_m system of order n if and only if*

- *$n = 1$ or $n \geqslant m$; and*
- *$n(n-1) \equiv 0$ (mod $2m-2$).*

To prove the main result of this subsection, we begin with the following theorem from [22].

Theorem 4.2.2. *[22] Let k, n and λ be positive integers such that $k \geqslant 2$, $n \geqslant 3$ and $(k, n) \neq (2, 3)$. Then there is an integer $N(k, n, \lambda)$ such that there exists a k-chromatic BIBD(v, n, λ) for all admissible integers $v \geqslant N(k, n, \lambda)$.*

We now show that any complete graph K_n, where n is even, has a decomposition into Hamilton paths.

Lemma 4.2.1. *For any even n, the complete graph K_n has a decomposition into Hamilton paths.*

Proof. The complete graph K_{n+1} has a Hamilton cycle decomposition as demonstrated in the 19th century by Walecki [27]. Remove one vertex from K_{n+1} to get a decomposition of K_n into Hamilton paths. This lemma also follows from Theorem 4.2.1. □

We then obtain the main result of this subsection.

Corollary 4.2.1. *For any $m \geqslant 4$, m even, and $k \geqslant 2$, there exists a k-chromatic P_m system.*

Proof. By Theorem 4.2.2, for any $k \geqslant 2$, $t \geqslant 2$, and any sufficiently large admissible integer v, there exists a weakly k-chromatic BIBD$(v, 2t, 1)$. By Lemma 4.2.1, the complete graph K_{2t} has a decomposition into Hamilton paths. Let $m = 2t$. Therefore, there exists a k-chromatic P_m system for any $k \geqslant 2$ and even $m \geqslant 4$. □

Note that a BIBD$(v, 4, 1)$ exists if and only if $v \equiv 1, 4 \pmod{12}$. Therefore, by Theorem 4.2.2, for any $k \geqslant 2$, there exist k-chromatic P_4 systems of order $v \equiv 1, 4 \pmod{12}$ for all sufficiently large v. In the next two subsections, we show that for any $k \geqslant 2$, there exist k-chromatic P_4 systems for all sufficiently large admissible v.

4.2.2 2-colourings of $\mathbf{P_4}$ systems

In this subsection, we prove that for each admissible order n, there exists an equitably 2-chromatic P_4 system of order n. Recall that a P_4 system of order n exists if and only if $n \equiv 0, 1, 3, 4 \pmod 6$ by Theorem 4.2.1. To prove the main result of this subsection, we first show the following lemmas.

Lemma 4.2.2. *Let $t \geqslant 1$. There exist a strongly equitably 2-chromatic P_4 system of order $6t$ and an equitably 2-chromatic P_4 system of order $6t + 1$.*

Proof. For $t = 1$ refer to Lemma 4.1.1. For each $t > 1$, we first construct an equitably 2-chromatic P_4 system $P_4(6t)$, $(V, \mathcal{B})$. Let $V = \{1, \ldots, 6t\}$ be the set of points. Partition the set of points V into t subsets $A_1 = \{1, 2, 3, 4, 5, 6\}, \ldots, A_t =$

$\{6t-5, 6t-4, 6t-3, 6t-2, 6t-1, 6t\}$.

Note that by Lemma 4.1.1 and Lemma 4.1.9, there exist an equitably 2-chromatic P_4 system of order six and a decomposition of $K_{6,6}$ into P_4 paths which is equitably 2-chromatic. So, for each $1 \leqslant i \leqslant t$ and $1 \leqslant k < \ell \leqslant t$, let $(A_i, \mathcal{A}_i)$ be an equitably 2-chromatic P_4 system of order six similar to that constructed from Lemma 4.1.1, and let $\mathcal{A}_{k,\ell}$ be an equitably 2-chromatic decomposition of the edges between A_k and A_ℓ into P_4 paths which is constructed in a similar manner to Lemma 4.1.9.

Let $\mathcal{B} = (\bigcup_{i=1}^{t} \mathcal{A}_i) \cup \big(\bigcup_{k=1}^{t-1} (\bigcup_{\ell=k+1}^{t} \mathcal{A}_{k,\ell})\big)$, $R = \{1, 3, \ldots, 6t-1\}$, and $Y = \{2, 4, \ldots, 6t\}$. Then $(V, \mathcal{B})$ is a $P_4(6t)$ for which R and Y constitute a strongly equitable 2-colouring.

Next, we construct an equitably 2-chromatic P_4 system $P_4(6t+1)$, $(\hat{V}, \hat{\mathcal{B}})$. Let $\hat{V} = \{1, \ldots, 6t+1\}$ be the set of points. Partition the set of points $\hat{V}$ into t subsets $\hat{A}_1 = \{1, 2, 3, 4, 5, 6, 7\}, \hat{A}_2 = \{8, 9, 10, 11, 12, 13\}, \ldots, \hat{A}_t = \{6t-4, 6t-3, 6t-2, 6t-1, 6t, 6t+1\}$.

Note that by Lemma 4.1.1, Lemma 4.1.9, and Lemma 4.1.10 there exist equitably 2-chromatic P_4 systems of order six and seven and decompositions of $K_{6,6}$ and $K_{6,7}$ into P_4 paths which are equitably 2-chromatic. So, let $(\hat{A}_1, \hat{\mathcal{A}}_1)$ be an equitably 2-chromatic P_4 system of order seven, and $(\hat{A}_i, \hat{\mathcal{A}}_i)$ for $2 \leqslant i \leqslant t$, be an equitably 2-chromatic P_4 system of order six similar to that constructed from Lemma 4.1.1. Also, for each $1 \leqslant k < \ell \leqslant t$, let $\hat{\mathcal{A}}_{k,\ell}$ be an equitably 2-chromatic decomposition of the edges between $\hat{A}_k$ and $\hat{A}_\ell$ into P_4 paths which is constructed in a similar manner to Lemma 4.1.9 and Lemma 4.1.10.

Let $\hat{\mathcal{B}} = (\bigcup_{i=1}^{t} \hat{\mathcal{A}}_i) \cup \big(\bigcup_{k=1}^{t-1} (\bigcup_{\ell=k+1}^{t} \hat{\mathcal{A}}_{k,\ell})\big)$, $\hat{R} = \{1, 3, \ldots, 6t+1\}$, and $\hat{Y} = \{2, 4, \ldots, 6t\}$. Then $(\hat{V}, \hat{\mathcal{B}})$ is a $P_4(6t+1)$ for which $\hat{R}$ and $\hat{Y}$ constitute an equitable 2-colouring. □

Lemma 4.2.3. *Let $t \geqslant 1$. There exist an equitably 2-chromatic P_4 system of order*

$6t+3$ *and a strongly equitably 2-chromatic* P_4 *system of order* $6t+4$.

Proof. By Lemma 4.2.2, there exists a strongly equitably 2-chromatic $P_4(6t)$, $(V, \mathcal{B})$, where $V = \{1, \ldots, 6t\}$ is the set of points and $\mathcal{B}$ is the set of blocks. Let $R = \{1, 3, \ldots, 6t-1\}$ and $Y = \{2, 4, \ldots, 6t\}$ constitute a strongly equitable 2-colouring of $(V, \mathcal{B})$. Partition the set of points V into t subsets $A_1 = \{1, 2, 3, 4, 5, 6\}, \ldots, A_t = \{6t-5, 6t-4, 6t-3, 6t-2, 6t-1, 6t\}$.

First, we construct an equitably 2-chromatic $P_4(6t+3)$, $(\hat{V}, \hat{\mathcal{B}})$, from $(V, \mathcal{B})$. Let $\hat{V} = V \cup \{6t+1, 6t+2, 6t+3\}$. Decompose the edges between A_1 and $\{6t+1, 6t+2, 6t+3\}$ along with the edges $\{6t+1, 6t+2\}, \{6t+2, 6t+3\}, \{6t+3, 6t+1\}$ into the set of P_4 paths $\hat{\mathcal{A}}_1$ in a manner similar to Lemma 4.1.11. Also, for each $2 \leqslant i \leqslant t$, decompose the edges between A_i and $\{6t+1, 6t+2, 6t+3\}$ into a set $\hat{\mathcal{A}}_i$ of P_4 paths in a manner similar to Lemma 4.1.6.

Let $\hat{\mathcal{B}} = \mathcal{B} \cup (\bigcup_{i=1}^{t} \hat{\mathcal{A}}_i)$, $\hat{R} = \{1, 3, \ldots, 6t+3\}$, and $\hat{Y} = \{2, 4, \ldots, 6t+2\}$. Then $(\hat{V}, \hat{\mathcal{B}})$ is a $P_4(6t+3)$ for which $\hat{R}$ and $\hat{Y}$ constitute an equitable 2-colouring.

Next, we construct a strongly equitably 2-chromatic $P_4(6t+4)$, $(\tilde{V}, \tilde{\mathcal{B}})$, from $(V, \mathcal{B})$. Let $\tilde{V} = V \cup \{6t+1, 6t+2, 6t+3, 6t+4\}$. Decompose the edges between A_1 and $\{6t+1, 6t+2, 6t+3, 6t+4\}$ along with the edges $\{6t+1, 6t+2\}, \{6t+2, 6t+3\}, \{6t+3, 6t+4\}, \{6t+4, 6t+1\}, \{6t+1, 6t+3\}, \{6t+2, 6t+4\}$ into the set of P_4 paths $\tilde{\mathcal{A}}_1 = \big\{(1, 6t+1, 6t+2, 2), (1, 6t+3, 6t+2, 6t+4), (6t+4, 2, 6t+3, 6t+1), (2, 6t+1, 6t+4, 6t+3), (1, 6t+2, 3, 6t+1), (1, 6t+4, 3, 6t+3), (4, 6t+1, 5, 6t+2), (4, 6t+3, 5, 6t+4), (4, 6t+2, 6, 6t+1), (4, 6t+4, 6, 6t+3)\big\}$. Also, for each $2 \leqslant i \leqslant t$, decompose the edges between A_i and $\{6t+1, 6t+2, 6t+3, 6t+4\}$ into a set $\tilde{\mathcal{A}}_i$ of P_4 paths in a manner similar to Lemma 4.1.7.

Let $\tilde{\mathcal{B}} = \mathcal{B} \cup (\bigcup_{i=1}^{t} \tilde{\mathcal{A}}_i)$, $\tilde{R} = \{1, 3, \ldots, 6t+3\}$, and $\tilde{Y} = \{2, 4, \ldots, 6t+4\}$. Then $(\tilde{V}, \tilde{\mathcal{B}})$ is a $P_4(6t+4)$ for which $\tilde{R}$ and $\tilde{Y}$ constitute a strongly equitable 2-colouring. $\square$

We then obtain the following theorem.

Theorem 4.2.3. *For each admissible order n, there exists an equitably 2-chromatic P_4 system of order n.*

Proof. For $n \leqslant 7$ apply Lemma 4.1.1. Otherwise apply Lemma 4.2.2 and Lemma 4.2.3. $\square$

4.2.3 k-colourings of P_4 systems for $k \geqslant 3$

In this subsection, we prove that for any integer $k \geqslant 3$, there exists some integer n_k such that for all admissible $n \geqslant n_k$, there exists a k-chromatic P_4 system of order n. To prove the main result of this subsection, we first show the following lemmas.

Lemma 4.2.4. *Let $t \geqslant 1$ and $k \geqslant 3$. If there exists a k-chromatic P_4 system of order $6t$, then there exist k-chromatic P_4 systems of order $6t+3$, $6t+4$, $6t+6$, and $6t+7$.*

Proof. Suppose that there exists a k-chromatic $P_4(6t)$, $(V, \mathcal{B})$, where $V = \{1, \ldots, 6t\}$ is the set of points and $\mathcal{B}$ is the set of blocks. Let $C_1, C_2, \ldots, C_k$ be the colour classes of a k-colouring of $(V, \mathcal{B})$. Partition the set of points V into t subsets $A_1 = \{1, 2, 3, 4, 5, 6\}, \ldots, A_t = \{6t-5, 6t-4, 6t-3, 6t-2, 6t-1, 6t\}$.

First, we construct a k-chromatic $P_4(6t+3)$, $(V_1, \mathcal{B}_1)$, from $(V, \mathcal{B})$. Let $V_1 = V \cup U_1$ where $U_1 = \{6t+1, 6t+2, 6t+3\}$. Let $w_1 = 6t+1$, $w_2 = 6t+2$, and $w_3 = 6t+3$. Decompose the edges between A_1 and U_1 along with the edges $\{6t+1, 6t+2\}, \{6t+2, 6t+3\}, \{6t+3, 6t+1\}$ into a set $\mathcal{A}_1$ of P_4 paths in a manner similar to Lemma 4.1.11. For each $2 \leqslant i \leqslant t$, decompose the edges between A_i and U_1 into a set $\mathcal{A}_i$ of P_4 paths in a manner similar to the 2-chromatic decomposition of $K_{6,3}$ into P_4 paths constructed from Lemma 4.1.6. Let $\mathcal{B}_1 = \mathcal{B} \cup (\bigcup_{i=1}^{t} \mathcal{A}_i)$. Then $(V_1, \mathcal{B}_1)$ is a $P_4(6t+3)$ for which $C_1 \cup \{6t+1\}, C_2 \cup \{6t+2\}, C_3 \cup \{6t+3\}, C_4, \ldots, C_k$ constitute a k-colouring.

Next, we construct a k-chromatic $P_4(6t+4)$, $(V_2, \mathcal{B}_2)$, from $(V, \mathcal{B})$. Let $V_2 = V \cup U_2$ where $U_2 = \{6t+1, 6t+2, 6t+3, 6t+4\}$. Decompose the edges of the complete graph on U_2 into a set $\mathcal{A}$ of P_4 paths in a manner similar to the 2-chromatic decomposition of K_4 into P_4 paths constructed from Lemma 4.1.1 where $v_1 = 6t+1$, $v_2 = 6t+2$, $v_3 = 6t+3$, and $v_4 = 6t+4$. For each $1 \leqslant i \leqslant t$, decompose the edges between A_i and U_2 into a set $\mathcal{A}_i$ of P_4 paths in a manner similar to the 2-chromatic decomposition of $K_{6,4}$ into P_4 paths constructed from Lemma 4.1.7 where $w_1 = 6t+1$, $w_2 = 6t+2$, $w_3 = 6t+3$, and $w_4 = 6t+4$. Let $\mathcal{B}_2 = \mathcal{B} \cup \mathcal{A} \cup (\bigcup_{i=1}^{t} \mathcal{A}_i)$. Then $(V_2, \mathcal{B}_2)$ is a $P_4(6t+4)$ for which $C_1 \cup \{6t+1, 6t+3\}, C_2 \cup \{6t+2, 6t+4\}, C_3, \ldots, C_k$ constitute a k-colouring.

Next, we construct a k-chromatic $P_4(6t+6)$, $(V_3, \mathcal{B}_3)$, from $(V, \mathcal{B})$. Let $V_3 = V \cup U_3$ where $U_3 = \{6t+1, 6t+2, 6t+3, 6t+4, 6t+5, 6t+6\}$. Let $v_1 = 6t+1$, $v_2 = 6t+2$, $v_3 = 6t+3$, $v_4 = 6t+4$, $v_5 = 6t+5$, and $v_6 = 6t+6$. Decompose the edges of the complete graph on U_3 into a set $\mathcal{U}$ of P_4 paths in a manner similar to the 2-chromatic decomposition of K_6 into P_4 paths constructed from Lemma 4.1.1. For each $1 \leqslant i \leqslant t$, decompose the edges between A_i and U_3 into a set $\mathcal{A}_i$ of P_4 paths in a manner similar to the 2-chromatic decomposition of $K_{6,6}$ into P_4 paths constructed from Lemma 4.1.9. Let $\mathcal{B}_3 = \mathcal{B} \cup \mathcal{U} \cup (\bigcup_{i=1}^{t} \mathcal{A}_i)$. Then $(V_3, \mathcal{B}_3)$ is a $P_4(6t+6)$ for which $C_1 \cup \{6t+1, 6t+3, 6t+5\}, C_2 \cup \{6t+2, 6t+4, 6t+6\}, C_3, \ldots, C_k$ constitute a k-colouring.

Finally, we construct a k-chromatic $P_4(6t+7)$, $(V_4, \mathcal{B}_4)$, from $(V, \mathcal{B})$. Let $V_4 = V \cup U_4$ where $U_4 = \{6t+1, 6t+2, 6t+3, 6t+4, 6t+5, 6t+6, 6t+7\}$. Decompose the edges of the complete graph on U_4 into a set $\mathcal{U}$ of P_4 paths in a manner similar to the 2-chromatic decomposition of K_7 into P_4 paths constructed from Lemma 4.1.1 where $v_1 = 6t+1$, $v_2 = 6t+2$, $v_3 = 6t+3$, $v_4 = 6t+4$, $v_5 = 6t+5$, $v_6 = 6t+6$, and $v_7 = 6t+7$. For each $1 \leqslant i \leqslant t$, decompose the edges between A_i and U_4 into a set $\mathcal{A}_i$ of P_4 paths in a manner similar to the 2-chromatic decomposition of $K_{6,7}$ into P_4 paths constructed from Lemma 4.1.10 where $w_1 = 6t+1$, $w_2 = 6t+2$, $w_3 = 6t+3$, $w_4 = 6t+4$, $w_5 = 6t+5$, $w_6 = 6t+6$, and $w_7 = 6t+7$. Let $\mathcal{B}_4 = \mathcal{B} \cup \mathcal{U} \cup (\bigcup_{i=1}^{t} \mathcal{A}_i)$. Then $(V_4, \mathcal{B}_4)$ is a $P_4(6t+7)$ for which $C_1 \cup \{6t+1, 6t+5, 6t+7\}, C_2 \cup \{6t+2, 6t+4, 6t+6\}, C_3 \cup \{6t+3\}, \ldots, C_k$ constitute a k-colouring. □

Lemma 4.2.5. *Let $t \geqslant 1$ and $k \geqslant 3$. If there exists a k-chromatic P_4 system of order $6t+1$, then there exist k-chromatic P_4 systems of order $6t+3$, $6t+4$, $6t+6$, and $6t+7$.*

Proof. Suppose that there exists a k-chromatic $P_4(6t+1)$, $(V, \mathcal{B})$, where $V = \{1, \ldots, 6t+1\}$ is the set of points and $\mathcal{B}$ is the set of blocks. Let $C_1, C_2, \ldots, C_k$ be the colour classes of a k-colouring of $(V, \mathcal{B})$. Partition the set of points V into t subsets $A_1 = \{1, 2, 3, 4, 5, 6\}, \ldots, A_{t-1} = \{6t-11, 6t-10, 6t-9, 6t-8, 6t-7, 6t-6\}, A_t = \{6t-5, 6t-4, 6t-3, 6t-2, 6t-1, 6t, 6t+1\}$.

First, we construct a k-chromatic $P_4(6t+3)$, $(V_1, \mathcal{B}_1)$, from $(V, \mathcal{B})$. Let $V_1 = V \cup U_1$ where $U_1 = \{6t+2, 6t+3\}$. Decompose the edges between A_t and U_1 along with the edge $\{6t+2, 6t+3\}$ into a set $\mathcal{A}_t$ of P_4 paths in a manner similar to Lemma 4.1.12. For each $1 \leqslant i \leqslant t-1$, decompose the edges between A_i and U_1 into a set $\mathcal{A}_i$ of P_4 paths in a manner similar to the 2-chromatic decomposition of $K_{6,2}$ into P_4 paths constructed from Lemma 4.1.5. Let $\mathcal{B}_1 = \mathcal{B} \cup (\bigcup_{i=1}^{t} \mathcal{A}_i)$. Then $(V_1, \mathcal{B}_1)$ is a $P_4(6t+3)$ for which $C_1 \cup \{6t+2\}, C_2 \cup \{6t+3\}, C_3, \ldots, C_k$ constitute a k-colouring.

Next, we construct a k-chromatic $P_4(6t+4)$, $(V_2, \mathcal{B}_2)$, from $(V, \mathcal{B})$. Let $V_2 = V \cup U_2$ where $U_2 = \{6t+2, 6t+3, 6t+4\}$. Let $w_1 = 6t+2$, $w_2 = 6t+3$, and $w_3 = 6t+4$. Decompose the edges between A_t and U_2 along with the edges $\{6t+2, 6t+3\}, \{6t+3, 6t+4\}, \{6t+4, 6t+2\}$ into a set $\mathcal{A}_t$ of P_4 paths in a manner similar to Lemma 4.1.13.

For each $1 \leqslant i \leqslant t-1$, decompose the edges between A_i and U_2 into a set $\mathcal{A}_i$ of P_4 paths in a manner similar to the 2-chromatic decomposition of $K_{6,3}$ into P_4 paths constructed from Lemma 4.1.6. Let $\mathcal{B}_2 = \mathcal{B} \cup (\bigcup_{i=1}^{t} \mathcal{A}_i)$. Then $(V_2, \mathcal{B}_2)$ is a $P_4(6t+4)$ for which $C_1 \cup \{6t+2\}, C_2 \cup \{6t+3\}, C_3 \cup \{6t+4\}, C_4 \ldots, C_k$ constitute a k-colouring.

Next, we construct a k-chromatic $P_4(6t+6)$, $(V_3, \mathcal{B}_3)$, from $(V, \mathcal{B})$. Let $V_3 = V \cup U_3$ where $U_3 = F_1 \cup F_2$, $F_1 = \{6t+2, 6t+3\}$ and $F_2 = \{6t+4, 6t+5, 6t+6\}$. Decompose the edges between A_t and F_1 along with the edge $\{6t+2, 6t+3\}$ into a set $\mathcal{A}_t$ of P_4 paths in a manner similar to Lemma 4.1.12. For each $1 \leqslant i \leqslant t-1$, decompose the edges between A_i and F_1 into a set $\mathcal{A}_i$ of P_4 paths in a manner similar to the 2-chromatic decomposition of $K_{6,2}$ into P_4 paths constructed from Lemma 4.1.5. Let $w_1 = 6t+4, w_2 = 6t+5$, and $w_3 = 6t+6$. Decompose the edges between A_t and F_2 along with the edges $\{6t+4, 6t+5\}, \{6t+5, 6t+6\}, \{6t+6, 6t+4\}$ into a set $\mathcal{A}'_t$ of P_4 paths in a manner similar to Lemma 4.1.13. For each $1 \leqslant i \leqslant t-1$, decompose the edges between A_i and F_2 into a set $\mathcal{A}'_i$ of P_4 paths in a manner similar to the 2-chromatic decomposition of $K_{6,3}$ into P_4 paths constructed from Lemma 4.1.6. Decompose the edges between F_1 and F_2 into a set of P_4 paths $\mathcal{F} = \big\{(6t+4, 6t+2, 6t+5, 6t+3), (6t+4, 6t+3, 6t+6, 6t+2)\big\}$. Let $\mathcal{B}_3 = \mathcal{B} \cup (\bigcup_{i=1}^{t} \mathcal{A}_i) \cup (\bigcup_{i=1}^{t} \mathcal{A}'_i) \cup \mathcal{F}$. Then $(V_3, \mathcal{B}_3)$ is a $P_4(6t+6)$ for which $C_1 \cup \{6t+2, 6t+5\}, C_2 \cup \{6t+3, 6t+6\}, C_3 \cup \{6t+4\}, C_4, \ldots, C_k$ constitute a k-colouring.

Finally, we construct a k-chromatic $P_4(6t+7)$, $(V_4, \mathcal{B}_4)$, from $(V, \mathcal{B})$. Let $V_4 = V \cup U_4$ where $U_4 = \{6t+2, 6t+3, 6t+4, 6t+5, 6t+6, 6t+7\}$. Let $v_1 = 6t+2$, $v_2 = 6t+3$, $v_3 = 6t+4$, $v_4 = 6t+5$, $v_5 = 6t+6$, and $v_6 = 6t+7$. Decompose the edges of the complete graph on U_4 into a set $\mathcal{U}$ of P_4 paths in a manner similar to the 2-chromatic decomposition of K_6 into P_4 paths constructed from Lemma 4.1.1. For each $1 \leqslant i \leqslant t-1$, decompose the edges between A_i and U_4 into a set $\mathcal{A}_i$ of P_4

paths in a manner similar to the 2-chromatic decomposition of $K_{6,6}$ into P_4 paths constructed from Lemma 4.1.9. Decompose the edges between A_t and U_4 into a set $\mathcal{A}_t$ of P_4 paths in a manner similar to the 2-chromatic decomposition of $K_{6,7}$ into P_4 paths constructed from Lemma 4.1.10. Let $\mathcal{B}_4 = \mathcal{B} \cup \mathcal{U} \cup (\bigcup_{i=1}^{t} \mathcal{A}_i)$. Then $(V_4, \mathcal{B}_4)$ is a $P_4(6t+7)$ for which $C_1 \cup \{6t+2, 6t+6\}, C_2 \cup \{6t+3, 6t+5\}, C_3 \cup \{6t+4, 6t+7\}, C_4 \ldots, C_k$ constitute a k-colouring. □

Lemma 4.2.6. *Let $t \geqslant 1$ and $k \geqslant 3$. If there exists a k-chromatic P_4 system of order $6t+3$, then there exist k-chromatic P_4 systems of order $6t+6$, $6t+7$, $6t+9$, and $6t+10$.*

Proof. Suppose that there exists a k-chromatic $P_4(6t+3)$, $(V, \mathcal{B})$, where $V = \{1, \ldots, 6t+3\}$ is the set of points and $\mathcal{B}$ is the set of blocks. Let $C_1, C_2, \ldots, C_k$ be the colour classes of a k-colouring of $(V, \mathcal{B})$. Partition the set of points V into $t+1$ subsets $A_1 = \{1, 2, 3, 4, 5, 6\}, \ldots, A_t = \{6t-5, 6t-4, 6t-3, 6t-2, 6t-1, 6t\}$, and $A_{t+1} = \{6t+1, 6t+2, 6t+3\}$.

First, we construct a k-chromatic $P_4(6t+6)$, $(V_1, \mathcal{B}_1)$, from $(V, \mathcal{B})$. Let $V_1 = V \cup U_1$ where $U_1 = \{6t+4, 6t+5, 6t+6\}$. Let $w_1 = 6t+4$, $w_2 = 6t+5$, and $w_3 = 6t+6$. Decompose the edges between A_1 and U_1 along with the edges of the complete graph on U_1 into a set $\mathcal{A}_1$ of P_4 paths in a manner similar to Lemma 4.1.11. For each $2 \leqslant i \leqslant t$, decompose the edges between A_i and U_1 into a set $\mathcal{A}_i$ of P_4 paths in a manner similar to the 2-chromatic decomposition of $K_{6,3}$ into P_4 paths constructed from Lemma 4.1.6. Also, decompose the edges between A_{t+1} and U_1 into a set $\mathcal{A}_{t+1}$ of P_4 paths in a manner similar to the 2-chromatic decomposition of $K_{3,3}$ into P_4 paths constructed from Lemma 4.1.2. Let $\mathcal{B}_1 = \mathcal{B} \cup (\bigcup_{i=1}^{t+1} \mathcal{A}_i)$. Then $(V_1, \mathcal{B}_1)$ is a $P_4(6t+6)$ for which $C_1 \cup \{6t+4\}, C_2 \cup \{6t+5\}, C_3 \cup \{6t+6\}, C_4, \ldots, C_k$ constitute a k-colouring.

Next, we construct a k-chromatic $P_4(6t+7)$, $(V_2, \mathcal{B}_2)$, from $(V, \mathcal{B})$. Let $V_2 = V \cup U_2$

where $U_2 = \{6t+4, 6t+5, 6t+6, 6t+7\}$. Decompose the edges of the complete graph on U_2 into a set $\mathcal{U}$ of P_4 paths in a manner similar to the equitable 2-chromatic decomposition of K_4 into P_4 paths constructed from Lemma 4.1.1 where $v_1 = 6t+4$, $v_2 = 6t+5$, $v_3 = 6t+6$, and $v_4 = 6t+7$. For each $1 \leqslant i \leqslant t$, decompose the edges between A_i and U_2 into a set $\mathcal{A}_i$ of P_4 paths in a manner similar to the 2-chromatic decomposition of $K_{6,4}$ into P_4 paths constructed from Lemma 4.1.7 where $w_1 = 6t+4$, $w_2 = 6t+6$, $w_3 = 6t+5$, and $w_4 = 6t+7$. Also, decompose the edges between A_{t+1} and U_2 into a set $\mathcal{A}_{t+1}$ of P_4 paths in a manner similar to the 2-chromatic decomposition of $K_{4,3}$ into P_4 paths constructed from Lemma 4.1.3 where $v_1 = 6t+4, v_2 = 6t+5, v_3 = 6t+6$, and $v_4 = 6t+7$. Let $\mathcal{B}_2 = \mathcal{B} \cup \mathcal{U} \cup (\bigcup_{i=1}^{t+1} \mathcal{A}_i)$. Then $(V_2, \mathcal{B}_2)$ is a $P_4(6t+7)$ for which $C_1 \cup \{6t+4, 6t+5\}, C_2 \cup \{6t+6, 6t+7\}, C_3, \ldots, C_k$ constitute a k-colouring.

Next, we construct a k-chromatic $P_4(6t+9)$, $(V_3, \mathcal{B}_3)$, from $(V, \mathcal{B})$. Let $V_3 = V \cup U_3$ where $U_3 = \{6t+4, 6t+5, 6t+6, 6t+7, 6t+8, 6t+9\}$. Let $v_1 = 6t+4$, $v_2 = 6t+5$, $v_3 = 6t+6$, $v_4 = 6t+7$, $v_5 = 6t+8$, and $v_6 = 6t+9$. Decompose the edges of the complete graph on U_3 into a set $\mathcal{U}$ of P_4 paths in a manner similar to the equitable 2-chromatic decomposition of K_6 into P_4 paths constructed from Lemma 4.1.1. For each $1 \leqslant i \leqslant t$, decompose the edges between A_i and U_3 into a set $\mathcal{A}_i$ of P_4 paths in a manner similar to the 2-chromatic decomposition of $K_{6,6}$ into P_4 paths constructed from Lemma 4.1.9. Also, decompose the edges between A_{t+1} and U_3 into a set $\mathcal{A}_{t+1}$ of P_4 paths in a manner similar to the 2-chromatic decomposition of $K_{6,3}$ into P_4 paths constructed from Lemma 4.1.6. Let $\mathcal{B}_3 = \mathcal{B} \cup \mathcal{U} \cup (\bigcup_{i=1}^{t+1} \mathcal{A}_i)$. Then $(V_3, \mathcal{B}_3)$ is a $P_4(6t+9)$ for which $C_1 \cup \{6t+4, 6t+6, 6t+8\}, C_2 \cup \{6t+5, 6t+7, 6t+9\}, C_3, \ldots, C_k$ constitute a k-colouring.

Finally, we construct a k-chromatic $P_4(6t+10)$, $(V_4, \mathcal{B}_4)$, from $(V, \mathcal{B})$. Let $V_4 = V \cup U_4$

where $U_4 = \{6t+4, 6t+5, 6t+6, 6t+7, 6t+8, 6t+9, 6t+10\}$. Decompose the edges of the complete graph on U_4 into a set $\mathcal{U}$ of P_4 paths in a manner similar to the equitable 2-chromatic decomposition of K_7 into P_4 paths constructed from Lemma 4.1.1 where $v_1 = 6t+4$, $v_2 = 6t+5$, $v_3 = 6t+6$, $v_4 = 6t+7$, $v_5 = 6t+8$, $v_6 = 6t+9$, and $v_7 = 6t+10$. For each $1 \leqslant i \leqslant t$, decompose the edges between A_i and U_4 into a set $\mathcal{A}_i$ of P_4 paths in a manner similar to the 2-chromatic decomposition of $K_{6,7}$ into P_4 paths constructed from Lemma 4.1.10 where $w_1 = 6t+4$, $w_2 = 6t+5$, $w_3 = 6t+6$, $w_4 = 6t+7$, $w_5 = 6t+8$, $w_6 = 6t+9$, and $w_7 = 6t+10$. Also, decompose the edges between A_{t+1} and U_4 into a set $\mathcal{A}_{t+1}$ of P_4 paths in a manner similar to the 2-chromatic decomposition of $K_{7,3}$ into P_4 paths constructed from Lemma 4.1.4 where $v_1 = 6t+4$, $v_2 = 6t+5$, $v_3 = 6t+6$, $v_4 = 6t+7$, $v_5 = 6t+8$, $v_6 = 6t+9$, and $v_7 = 6t+10$. Let $\mathcal{B}_4 = \mathcal{B} \cup \mathcal{U} \cup (\bigcup_{i=1}^{t+1} \mathcal{A}_i)$. Then $(V_4, \mathcal{B}_4)$ is a $P_4(6t+10)$ for which $C_1 \cup \{6t+4, 6t+8\}, C_2 \cup \{6t+5, 6t+7, 6t+9\}, C_3 \cup \{6t+6, 6t+10\}, C_4, \ldots, C_k$ constitute a k-colouring. □

Lemma 4.2.7. *Let $t \geqslant 1$ and $k \geqslant 3$. If there exists a k-chromatic P_4 system of order $6t+4$, then there exist k-chromatic P_4 systems of order $6t+6$, $6t+7$, $6t+9$, and $6t+10$.*

Proof. Suppose that there exists a k-chromatic $P_4(6t+4)$, $(V, \mathcal{B})$, where $V = \{1, \ldots, 6t+4\}$ is the set of points and $\mathcal{B}$ is the set of blocks. Let $C_1, C_2, \ldots, C_k$ be the colour classes of a k-colouring of $(V, \mathcal{B})$. Partition the set of points V into $t+1$ subsets $A_1 = \{1, 2, 3, 4, 5, 6\}, \ldots, A_t = \{6t-5, 6t-4, 6t-3, 6t-2, 6t-1, 6t\}$, and $A_{t+1} = \{6t+1, 6t+2, 6t+3, 6t+4\}$.

First, we construct a k-chromatic $P_4(6t+6)$, $(V_1, \mathcal{B}_1)$, from $(V, \mathcal{B})$. Let $V_1 = V \cup U_1$ where $U_1 = \{6t+5, 6t+6\}$. For each $1 \leqslant i \leqslant t$, decompose the edges between A_i and U_1 into a set $\mathcal{A}_i$ of P_4 paths in a manner similar to the 2-chromatic decomposition of $K_{6,2}$ into P_4 paths constructed from Lemma 4.1.5. Decompose the edges between A_{t+1} and U_1 along with the edges of the complete graph on U_1 into a set $\mathcal{A}_{t+1}$ of P_4 paths in a manner similar to Lemma 4.1.14. Let $\mathcal{B}_1 = \mathcal{B} \cup (\bigcup_{i=1}^{t+1} \mathcal{A}_i)$. Then $(V_1, \mathcal{B}_1)$ is a $P_4(6t+6)$ for which $C_1 \cup \{6t+5\}, C_2 \cup \{6t+6\}, C_3, \ldots, C_k$ constitute a k-colouring.

Next, we construct a k-chromatic $P_4(6t+7)$, $(V_2, \mathcal{B}_2)$, from $(V, \mathcal{B})$. Let $V_2 = V \cup U_2$ where $U_2 = \{6t+5, 6t+6, 6t+7\}$. Let $w_1 = 6t+5$, $w_2 = 6t+6$, and $w_3 = 6t+7$. Decompose the edges between A_1 and U_2 along with the edges of the complete graph on U_2 into a set $\mathcal{A}_1$ of P_4 paths in a manner similar to Lemma 4.1.11. For each $2 \leqslant i \leqslant t$, decompose the edges between A_i and U_2 into a set $\mathcal{A}_i$ of P_4 paths in a manner similar to the 2-chromatic decomposition of $K_{6,3}$ into P_4 paths constructed from Lemma 4.1.6. Also, decompose the edges between A_{t+1} and U_2 into a set $\mathcal{A}_{t+1}$ of P_4 paths in a manner similar to the 2-chromatic decomposition of $K_{4,3}$ into P_4 paths constructed from Lemma 4.1.3. Let $\mathcal{B}_2 = \mathcal{B} \cup (\bigcup_{i=1}^{t+1} \mathcal{A}_i)$. Then $(V_2, \mathcal{B}_2)$ is a $P_4(6t+7)$ for which $C_1 \cup \{6t+5\}, C_2 \cup \{6t+6\}, C_3 \cup \{6t+7\}, C_4, \ldots, C_k$ constitute a k-colouring.

Next, we construct a k-chromatic $P_4(6t+9)$, $(V_3, \mathcal{B}_3)$, from $(V, \mathcal{B})$. Let $V_3 = V \cup U_3$ where $U_3 = \{6t+5, 6t+6, 6t+7, 6t+8, 6t+9\}$. Let $w_1 = 6t+5$, $w_2 = 6t+6$, $w_3 = 6t+7$, $w_4 = 6t+8$, and $w_5 = 6t+9$. For each $1 \leqslant i \leqslant t$, decompose the edges between A_i and U_3 into a set $\mathcal{A}_i$ of P_4 paths in a manner similar to the 2-chromatic decomposition of $K_{6,5}$ into P_4 paths constructed from Lemma 4.1.8. Decompose the edges between A_{t+1} and U_3 along with the edges of the complete graph on U_3 into a set $\mathcal{A}_{t+1}$ of P_4 paths in a manner similar to Lemma 4.1.15. Let $\mathcal{B}_3 = \mathcal{B} \cup (\bigcup_{i=1}^{t+1} \mathcal{A}_i)$. Then $(V_3, \mathcal{B}_3)$ is a $P_4(6t+9)$ for which $C_1 \cup \{6t+5, 6t+8\}, C_2 \cup \{6t+6, 6t+9\}, C_3 \cup \{6t+7\}, C_4, \ldots, C_k$ constitute a k-colouring.

Finally, we construct a k-chromatic $P_4(6t+10)$, $(V_4, \mathcal{B}_4)$, from $(V, \mathcal{B})$. Let $V_4 = V \cup U_4$ where $U_4 = \{6t+5, 6t+6, 6t+7, 6t+8, 6t+9, 6t+10\}$. Let $v_1 = 6t+5$, $v_2 = 6t+6$, $v_3 =$

$6t+7$, $v_4 = 6t+8$, $v_5 = 6t+9$, and $v_6 = 6t+10$. Decompose the edges of the complete graph on U_4 into a set $\mathcal{U}$ of P_4 paths in a manner similar to the equitable 2-chromatic decomposition of K_6 into P_4 paths constructed from Lemma 4.1.1. For each $1 \leqslant i \leqslant t$, decompose the edges between A_i and U_4 into a set $\mathcal{A}_i$ of P_4 paths in a manner similar to the 2-chromatic decomposition of $K_{6,6}$ into P_4 paths constructed from Lemma 4.1.9. Also, decompose the edges between A_{t+1} and U_4 into a set $\mathcal{A}_{t+1}$ of P_4 paths in a manner similar to the 2-chromatic decomposition of $K_{4,6}$ into P_4 paths constructed from Lemma 4.1.7. Let $\mathcal{B}_4 = \mathcal{B} \cup \mathcal{U} \cup (\bigcup_{i=1}^{t+1} \mathcal{A}_i)$. Then $(V_4, \mathcal{B}_4)$ is a $P_4(6t+10)$ for which $C_1 \cup \{6t+5, 6t+9\}, C_2 \cup \{6t+6, 6t+8\}, C_3 \cup \{6t+7, 6t+10\}, C_4, \ldots, C_k$ constitute a k-colouring. □

We then obtain the following theorem.

Theorem 4.2.4. *For any integer $k \geqslant 3$, there exists some integer n_k such that for all admissible $n \geqslant n_k$, there exists a k-chromatic P_4 system of order n.*

Proof. By Corollary 4.2.1, for any integer $k \geqslant 3$, there exists a k-chromatic P_4 system of some order n_k. Apply Lemma 4.2.4, Lemma 4.2.5, Lemma 4.2.6, and Lemma 4.2.7 to construct a k-chromatic P_4 system of order n for all admissible $n \geqslant n_k$. □

Chapter 5

Conclusion and future work

In this thesis, we studied the chromatic index of block intersection graphs of Steiner triple systems. We observed that every STS(v) for which $v \equiv 3$ or $7 \pmod{12}$ has an overfull block intersection graph and hence is of Class 2. We then conjectured that whenever a Steiner triple system has a block intersection graph with an even number vertices, the graph is Class 1. We proved this to be true for Kirkman triple systems and cyclic Steiner triple systems of order $v \equiv 9 \pmod{12}$. We also proved that the conjecture holds for cyclic Steiner triple systems of order $v \equiv 1 \pmod{12}$ for which $\frac{\varphi(v)}{v-1} \geqslant \frac{2}{3}$, where φ is Euler's totient function. It is a future project to prove the conjecture for the remaining cyclic STS(v) with $v \equiv 1 \pmod{12}$ and also for non-cyclic STS(v).

It is well known that STS block intersection graphs belong to the family of strongly regular graphs [31]. Specifically, a graph is said to be *strongly regular* if it is regular, each pair of adjacent vertices has a constant number λ of common neighbours, and each pair of nonadjacent vertices has a constant number μ of common neighbours. Recently, Cioabă, Guo and Haemers in [12] determined the chromatic index of many strongly regular graphs. All investigated connected strongly regular graphs of even order are of Class 1, and they conjectured that this is the case for all connected strongly regular graphs of even order. It is an open problem to prove that all strongly regular graphs of even order are of Class 1.

We also studied colourings of star systems and path systems. We showed that for any integer $k \geqslant 2$, there exists a k-chromatic 3-star system of order n for all sufficiently large admissible n. We then generalised this result to e-star systems for any $e \geqslant 3$. We showed that for all $k \geqslant 2$ and $e \geqslant 3$, there exists a k-chromatic e-star system of order n for all sufficiently large n such that $n \equiv 0{,}1 \pmod{2e}$. We also proved that for all $k \geqslant 2$ and $e \geqslant 3$, there exists a uniquely k-chromatic e-star system of order n for all sufficiently large n such that $n \equiv 0{,}1 \pmod{2e}$. We then proved some analogous results for path systems. We observed that there exists a k-chromatic P_m system for any $k \geqslant 2$ and $m \geqslant 4$ where m is even. We proved that there exists an equitably 2-chromatic P_4 system for each admissible order n. Finally, we showed that for any integer $k \geqslant 3$, there exists a k-chromatic P_4 system of order n for all sufficiently large admissible n. In this thesis, we have not included colourings of P_m systems for any $m \geqslant 5$. We plan to prove some analogous results for P_m systems for any $m \geqslant 5$. We also plan to study unique colourings of path systems. We wish to find a uniquely k-chromatic P_m system for any $k \geqslant 2$ and $m \geqslant 4$.

In 2003, Forbes [17] showed that for every admissible $v \geqslant 25$, there exists a 3-balanced Steiner triple system with a unique 3-colouring and also a Steiner triple system which has a unique, nonequitable 3-colouring. It is an open problem to investigate unique k-colourings of Steiner triple systems for any $k \geqslant 4$. Also, it is a future project to study unique k-colourings of BIBDs with block sizes greater than three for any $k \geqslant 3$.

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