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Pancyclic PBD block-intersection graphs

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Abstract

A *block-intersection graph* of a design $\mathscr{D}$, denoted $G_{\mathscr{D}}$, is a graph derived from the blocks of the design $\mathscr{D}$, where the blocks are the vertices of the graph and there is an edge between two vertices if and only if the corresponding blocks have non-empty intersection. In this paper we examine the block-intersection graphs of PBD's and prove that they are pancyclic, that is, they have a cycle of every length from 3 to $|V(G_{\mathscr{D}})|$.

This paper is dedicated to my grandfather, Cyril Bull, who was always proud and supportive of me in everything I did. I know he would have been especially proud of me now.

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1 Introduction

Block-intersection graphs have been the focus of research for many years. In particular the block-intersection graphs of $BIBD(v, k, \lambda)$'s and $PBD(v, \mathcal{K}, \lambda)$'s have been researched in great detail. The discussion of block-intersection graphs began in 1987 when Ron Graham asked whether or not the block-intersection graph of a Steiner triple system has a Hamilton cycle [2]. Soon thereafter, Horák and Rosa proved this to be true [7]—a result which was also proved as a corollary by Alspach, Heinrich and Mohar [2]. Subsequently Alspach and Hare have proven that every $BIBD(v, k, 1)$ with $k \geq 3$ has an edge-pancyclic block-intersection graph [1]. Hare continued this research and showed that every $PBD(v, \mathcal{K}, 1)$ with $\min\{\mathcal{K}\} \geq 3$ has an edge-pancyclic block-intersection graph [5]. More recently, Horák, Pike and Raines proved that every $BIBD(v, 3, \lambda)$ with $v \geq 12$ has a Hamiltonian 1-block-intersection graph [6]. The most recent in this line of articles proved that every $BIBD(v, k, \lambda)$ with $\lambda \geq 2$ has a pancyclic block-intersection graph [8]. We modify the method used by Mamut, Pike and Raines to prove our result that the block-intersection graphs of PBD's with $\lambda \geq 2$ and $\max\{\mathcal{K}\} \leq 2\min\{\mathcal{K}\}$ are pancyclic.

To reach this conclusion we first present three lemmata. The first allows us to create cycles of any even length from 4 to 2α where α is the size of a maximum independent set. We use this in our main theorem, where we begin with a cycle of length 2α. The second lemma is an extension of the first. We prove in Lemma 2 that we can indeed get cycles of every odd length from 3 to $2\alpha - 1$. In our final lemma, we prove that we have paths of length at most 2 from the independent set used to create our cycle, to a vertex outside the cycle. Lemma 3 also tells us that the interior vertices of the 2-paths are all pairwise adjacent, a subtle, yet important result that is necessary in proving our main theorem. Using these results, our main theorem becomes a matter of cases based on which vertices in a set are adjacent.

Our motivation for proving this theorem is to follow the progression of proving results about block-intersection graphs. Most results are first proved for very simple cases such as triple systems, and are then generalised to BIBD's and finally PBD's. With this proof we finish another chapter in this progression. Related problems in this field are presented at the end of the paper.

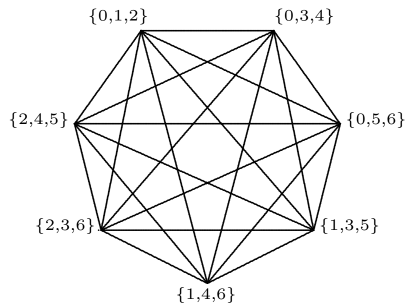
2 Preliminary Terminology

2.1 Combinatorial Design Theory

A *combinatorial design*, $\mathscr{D}$, is a pair $(V, \mathscr{B})$ where V is a finite collection and $\mathscr{B}$ is a finite collection of subsets of V where the subsets are selected with certain restrictions. A specific combinatorial design is a *balanced incomplete block design*, written $BIBD(v, k, \lambda)$ where $v = |V|$, k is the number of elements in each block and λ is the number of times each pair of elements occurs throughout the blocks. A more general design is a *pairwise balanced design*, written as $PBD(v, \mathcal{K}, \lambda)$. The restrictions on a $PBD(v, \mathcal{K}, \lambda)$ are similar to a $BIBD(v, k, \lambda)$ except that instead of each block having size k, the size of each block of a $PBD(v, \mathcal{K}, \lambda)$ is a member of the set $\mathcal{K}$.

2.2 Graph Theory

From any combinatorial design we can create a graph called a *block-intersection graph*. The vertices of a block-intersection graph are the blocks of the combinatorial design. To form the edges of the graph, we look at the pairwise intersections of the blocks. If two blocks have non-empty intersection then there is an edge between the two corresponding vertices. For example, for the $BIBD(7,3,1)$, our blocks are: $\{0,1,2\}$, $\{0,3,4\}$, $\{0,5,6\}$, $\{1,3,5\}$, $\{1,4,6\}$, $\{2,3,6\}$, $\{2,4,5\}$. Our corresponding block-intersection graph is:



A *cycle* of the graph G is a subgraph C_G with vertex set $V(C_G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(C_G) = \{(v_1, v_2), (v_2, v_3), \ldots, (v_{n-1}, v_n), (v_n, v_1)\}$. A cycle on n vertices is called an n-cycle. A graph with every cycle from length 3 to length $|V(G)|$ is called *pancyclic*.

For other terms used in this paper, please refer to [3].

3 Lemmata

We now turn our attention to proving the three lemmata needed for the proof of our main theorem. In Lemma 1 we show that given an independent set, A, of vertices with $|A| \geq 2$, we can find a cycle of length $2|A|$ in the block-intersection graph. We show this using an application of Hall's matching condition [4] which says in any bipartite graph, with bipartitions X and Y, if the size of the neighbour set of $Z \subseteq X$, $N(Z) \subseteq Y$, is greater than or equal to the size of Z for all $Z \subseteq X$ then there exists a matching from the neighbour set $N(Z)$ to the set Z that saturates every vertex in the set Z.

Lemma 1. *Let $\mathscr{D} = (V, \mathscr{B})$ be a $PBD(v, \mathcal{K}, \lambda)$ with $\lambda \geq 2$ and $\max\{\mathcal{K}\} \leq 2\min\{\mathcal{K}\}$. Let A be a set of independent vertices in $G_{\mathscr{D}}$ of size at least 2. Then $G_{\mathscr{D}}$ has a cycle of length $2|A|$ containing each vertex of A.*

Proof. We consider two cases, the first where $|A| = 2$ and the second where $|A| > 2$.

Assume $|A| = 2$ and let $A = \{a_1, a_2\}$. Let p_1 be a point in a_1 and let p_2 be a point in a_2. Clearly $p_2 \notin a_1$ and $p_1 \notin a_2$ since a_1 and a_2 are independent. Now, since $\lambda \geq 2$ there must exist at least two vertices in $G_{\mathscr{D}}$ that contain the pair (p_1, p_2); call them a_3 and a_4. Since a_1 and a_2 are both adjacent to a_3 and a_4, we have a cycle (a_1, a_3, a_2, a_4) of length 4, that is, whenever $|A| = 2$, we can find a cycle of length $2|A|$ containing both vertices of A.

Now assume $|A| > 2$. Let $A = \{a_1, a_2, \ldots, a_i\}$ be an independent set of vertices. Let $A_1 = \{(a_j, a_{j+1}) | 1 \leq j \leq i\}$ where for convenience we let $a_{i+1} = a_1$. Construct a bipartite graph B_1 having vertex set $A_1 \cup (\mathscr{B} - A)$, and in which a vertex (a_j, a_{j+1}) of A_1 is adjacent to a vertex $b \in (\mathscr{B} - A)$ if and only if $a_j \cap b \neq \emptyset$ and $a_{j+1} \cap b \neq \emptyset$. We now seek a matching in B_1 that saturates A_1, since each edge $\{(a_j, a_{j+1}), b\}$ of such a matching will correspond to a 2-path (a_j, b, a_{j+1}) in $G_{\mathscr{D}}$.

To establish such a matching in B_1, we need only to show that Hall's matching condition is satisfied; i.e. that $|S| \leq |N(S)|$ for each non-empty subset S of A_1. Let $S \subseteq A_1$ and let $\sigma = |S|$.

Denoting $\max\{\mathcal{K}\}$ by K and $\min\{\mathcal{K}\}$ by k observe that for each $j \in \{1, 2, \ldots, i\}$, the vertex (a_j, a_{j+1}) of S gives rise to at least k^2 pairs of points of the form (v_1, v_2) where $v_1 \in a_j$ and $v_2 \in a_{j+1}$ and that each pair of points occurs λ times among the blocks of $\mathscr{B} - A$. Any block of $\mathscr{B} - A$ that contains such a pair of points is a neighbour of (a_j, a_{j+1}) in B_1. The blocks of $N(S)$ collectively contain all $\sigma\lambda k^2$ such pairs of points that are formed from the σ vertices of S, and since each block holds at most $\binom{K}{2}$ pairs of points, it follows that:

$$|N(S)| \geq \frac{\sigma\lambda k^2}{\binom{K}{2}} = \frac{2\sigma\lambda k^2}{K(K-1)} \geq \frac{2\sigma\lambda k^2}{2k(2k-1)} = \frac{\sigma\lambda k}{2k-1} \geq \frac{2\sigma k}{2k-1} > \frac{2\sigma k}{2k} = \sigma = |S|$$

and so Hall's condition is satisfied.

Now let the edges $\{(a_j, a_{j+1}), b_j\}$, $j = 1, 2, \ldots, i$, comprise a matching in B_1 that saturates A_1. Then $(a_1, b_1, a_2, b_2, \ldots, a_i, b_i, a_1)$ is a $2|A|$-cycle in $G_{\mathscr{D}}$, containing each vertex in A. □

We have shown that $G_{\mathscr{D}}$ contains a cycle of length $2|A|$ for any independent set A. From this, we know $G_{\mathscr{D}}$ contains a cycle of every even length from 4 to 2α. We must now show that $G_{\mathscr{D}}$ contains a cycle of every odd length from 3 to $2\alpha - 1$. We do this by first showing that the replication number of a point contained in a vertex in a maximum independent set A is at least $\alpha + 1$. We then use Lemma 1 to form an even-length cycle of every length from 4 to $2\alpha - 2$ by taking a subset of vertices of a maximum independent set and showing that we can add an edge and a vertex to increase the cycle length by 1.

Lemma 2. *Let $\mathscr{D} = (V, \mathscr{B})$ be a PBD$(v, \mathcal{K}, \lambda)$ with $\lambda \geq 2$ and $\max\{\mathcal{K}\} \leq 2\min\{\mathcal{K}\}$. Then there exists a cycle of every length from 3 to 2α.*

Proof. First we consider the replication number, r, of the points contained in the vertices of $A = \{a_1, a_2, \ldots, a_\alpha\}$, a maximum independent set of vertices.

Denote $\max\{\mathcal{K}\}$ by K and $\min\{\mathcal{K}\}$ by k. Any point, v_0, of a vertex $a_i \in A$, where $1 \leq i \leq \alpha$, occurs at least α times among the blocks of $\mathscr{B} - A$. To show this we note that there are at least $(\alpha - 1)k$ points in $(\bigcup_{j=1}^{\alpha} a_j) - a_i$, each of which must occur with v_0 λ times in the blocks of $\mathscr{B} - A$. Also, each block in $\mathscr{B} - A$ that contains the point v_0 can have at most $K - 1$ pairs of the form (v_0, p), where $p \in (\bigcup_{j=1}^{\alpha} a_j) - a_i$. Now noting that v_0 occurs exactly once in the blocks of A we have:

$$r \geq \frac{(\alpha-1)k\lambda}{K-1} + 1 \geq \frac{(\alpha-1)k\lambda}{2k-1} + 1 > \frac{(\alpha-1)k\lambda}{2k} + 1 \geq \frac{2k(\alpha-1)}{2k} + 1 = (\alpha - 1) + 1 = \alpha$$

Therefore we have $r > \alpha$, and since r is an integer, we can say that $r \geq \alpha + 1$ among the blocks of $\mathscr{B}$, and since v_0 occurs in exactly one block of A, it occurs in the blocks of $\mathscr{B} - A$

at least α times. Now we show that since the replication number of each point of A is at least $\alpha + 1$, we can create odd-length cycles of every length from 3 to $2\alpha - 1$.

We now assume that $\alpha \geq 2$ since if $\alpha = 1$ our block-intersection graph would be complete, and thus pancyclic.

Since $\alpha \geq 2$ and $r \geq \alpha + 1$ it is clear that we have a cycle of length 3, since every point of $\bigcup_{j=1}^{\alpha} a_j$ occurs in at least 3 blocks. To now show that there are cycles of every odd length from 5 to $2\alpha - 1$, we will construct cycles of every even length from 4 to $2\alpha - 2$ and to each cycle we will add one vertex and one edge, thus increasing the length of the cycle by one. To construct these even length cycles, we will use Lemma 1, and require that each set of independent vertices, S, be a subset of our maximum independent set A. Without loss of generality we will let $a_1 \in S$ be a vertex on a cycle of length $2|S|$, and let b_1 be one of its neighbours on the cycle. Since $a_1 \cap b_1 \neq \emptyset$, then they must share at least one point in common, say v_1. Now v_1 occurs at least $\alpha+1$ times in the blocks of $\mathscr{B}$ and at most α of these can be on the cycle, since we are looking at cycles of size at most $2\alpha - 2$ so at most $\alpha - 1$ of these vertices are independent with exactly one of our independent vertices containing the point v_0. It is now clear that there is at least one block outside the cycle that contains v_1; call it c_1. Now we can remove the edge $\{a_1, b_1\}$ and replace it with the 2-path (a_1, c_1, b_1) to increase the length of the cycle by one. $\square$

Now that we have cycles of every length from 3 to 2α, we must consider paths from these cycles to a vertex outside the cycle. This will allow us, in our proof of the main theorem, to increase the length of a cycle by 1. We use Hall's matching condition once more to prove the existence of paths of length at most 2. Some other properties are needed for our main theorem, and these too are discussed in this lemma.

Lemma 3. *Let $\mathscr{D} = (V, \mathscr{B})$ be a $PBD(v, \mathcal{K}, \lambda)$ with $\lambda \geq 2$ and $\max\{\mathcal{K}\} \leq 2\min\{\mathcal{K}\}$. Let A be a set of independent vertices in $G_{\mathscr{D}}$, and let $x \in (\mathscr{B} - A)$. Then there exists a set of $|A|$ internally disjoint paths of length at most 2 in $G_{\mathscr{D}}$ from x to A such that the interior vertices of the 2-paths are adjacent.*

Proof. Let $A = \{a_1, a_2, \ldots, a_i\}$ be a set of independent vertices and take $x \in (\mathscr{B} - A)$ with $v_0 \in x$. We consider two cases, first where $A \cup \{x\}$ is independent, and second where there are some vertices in A that are neighbours of x.

Assume $A \cup \{x\}$ is independent. Construct the bipartite graph B_2 having vertex set $A \cup (\mathscr{B} - (A \cup \{x\}))$, in which a vertex $a_j \in A$ is adjacent to $b \in \mathscr{B} - (A \cup \{x\})$ if and only if $a_j \cap b \neq \emptyset$ and $v_0 \in b$. It now suffices to show that in B_2 there exists a matching that saturates A, since each edge $\{a_j, b\}$ of such a matching in B_2 will correspond to a 2-path (a_j, b, x) in $G_{\mathscr{D}}$.

Denote $\max\{\mathcal{K}\}$ by K and $\min\{\mathcal{K}\}$ by k. Since v_0 does not occur in any blocks of A, then among the blocks of $\mathscr{B} - A$, each point $v_1 \in a_j$ must be paired with v_0 exactly λ times. Note that since each such pair of points contains the point v_0, then each block of $\mathscr{B} - A$ can contain at most $K - 1$ such pairs.

Let $S \subseteq A$ and let $\sigma = |S|$. Then the vertices of S give rise to at least $\sigma\lambda k$ pairs of points that each contain the point v_0 and which must be contained in blocks of $N(S)$. It follows that:

$$|N(S)| \geq \frac{\sigma\lambda k}{K - 1} \geq \frac{\sigma\lambda k}{2k - 1} > \frac{\sigma\lambda k}{2k} \geq \frac{2k\sigma}{2k} = \sigma = |S|$$

Since Hall's condition is satisfied, B_2 has a matching that saturates A, and so when $A \cup \{x\}$ is independent there are internally disjoint 2-paths from x to every point in A such that the interior vertices are adjacent.

Now assume that $A \cup \{x\}$ is not independent, that is, there exists at least one vertex, $a_j \in A$, such that there is an edge between a_j and x. Without loss of generality let $\{a_1, a_2, \ldots, a_\mu\}$ be the set of vertices in A that have 2-paths to x and let $\{a_{\mu+1}, a_{\mu+2}, \ldots, a_i\}$ be the set of vertices in A that are adjacent to x. Let $A' = \{a_1, a_2, \ldots, a_\mu\}$. Now $A' \cup \{x\}$ is a set of independent vertices, and from our first case we know that there are internally disjoint 2-paths from x to every point in A' such that the interior vertices are adjacent. □

4 Main Theorem

We now look at the main theorem in which we first construct a cycle of size 2α, twice the size of a maximum independent set (Lemma 1), and then using paths from our cycle to a vertex x outside the cycle (Lemma 3), we show that we can indeed extend the length of the cycle by 1.

Theorem. *Let $\mathscr{D} = (V, \mathscr{B})$ be a $PBD(v, \mathcal{K}, \lambda)$ with $\lambda \geq 2$ and $\max\{\mathcal{K}\} \leq 2\min\{\mathcal{K}\}$. Then $G_{\mathscr{D}}$ is pancyclic.*

Proof. First observe that from Lemma 1 and Lemma 2 we know that we have cycles of every length from 3 to 2α. We now consider the cycles longer than 2α.

Let $A = \{a_1, a_2, \ldots, a_\alpha\}$ be a maximum set of independent vertices in $G_{\mathscr{D}}$, and suppose that C is a cycle in $G_{\mathscr{D}}$ that contains each vertex of A. If C is not a Hamilton cycle, then we can proceed to construct a cycle C' having one edge more than C, such that C' contains each vertex of A. By initially using Lemma 1 to select such a cycle C of length 2α, and then iterating this process, we establish that $G_{\mathscr{D}}$ is pancyclic.

So, given such a cycle C that is not a Hamilton cycle and which contains each vertex of A, let $x \in \mathscr{B}$ be a vertex not contained in the cycle C. Using Lemma 3, we obtain a set of α paths of length at most 2 such that each pair of paths has only the vertex x in common and such that the interior vertices of the 2-paths are pairwise adjacent. Since some of our paths are 1-paths and some of them are 2-paths, without loss of generality we can label the vertices in A which have 2-paths to x by $\{a_1, a_2, \ldots, a_\mu\}$ and label the vertices in A which have 1-paths to x by $\{a_{\mu+1}, a_{\mu+2}, \ldots, a_\alpha\}$. Now label the b's that are the middle vertices in the 2-paths from x to $\{a_1, a_2, \ldots, a_\mu\}$ by $\{b_1, b_2, \ldots, b_\mu\}$. Some of these b's are in the cycle C and some of them are not. Now, we can assume without loss of generality that the vertices $\{b_1, b_2, \ldots, b_\omega\}$ are on the cycle and the vertices $\{b_{\omega+1}, b_{\omega+2}, \ldots, b_\mu\}$ are not on the cycle. Also, we can assume without loss of generality that the 2-paths from the vertices $\{a_1, a_2, \ldots, a_\omega\}$ to x are the 2-paths whose interior vertices are on the cycle C, and the 2-paths from the vertices $\{a_{\omega+1}, a_{\omega+2}, \ldots, a_\mu\}$ to x are the 2-paths whose interior vertices are not on the cycle C.

Fix an orientation of the cycle C and, referring to this orientation, let z^+ denote the vertex of C subsequent to the vertex z of C, and let z^{+2} denote the vertex of C subsequent to the vertex z^+. Now consider the set $S = \{x, b_1^+, b_2^+, \ldots, b_\omega^+, a_{\omega+1}^{+2}, a_{\omega+2}^{+2}, \ldots, a_\mu^{+2}, a_{\mu+1}^+, a_{\mu+2}^+, \ldots, a_\alpha^+\}$ of vertices in $G_{\mathscr{D}}$. Clearly $b_i^+ \neq b_j^+$ whenever $i \neq j$, $a_i^+ \neq a_j^+$ whenever $i \neq j$ and $a_i^{+2} \neq a_j^{+2}$

whenever $i \neq j$. Also $b_i^+ \neq a_k^+$ and $a_j^{+2} \neq a_k^+$ whenever $i \in \{1, 2, \ldots, \omega\}$, $j \in \{1, 2, \ldots, \alpha\}$ and $k \in \{1, 2, \ldots, \alpha\}$. And for each $i \in \{1, 2, \ldots, \alpha\}$, $x \neq b_i^+$, $x \neq a_i^+$ and $x \neq a_i^{+2}$.

If $b_i^+ = a_j^{+2}$ for some $i \in \{1, 2, \ldots, \omega\}$ and some $j \in \{\omega+1, \omega+2, \ldots, \mu\}$, then we can form the cycle C' by removing from C the edge $\{a_j, b_i\}$ and replacing it with the 2-path (a_j, b_j, b_i).

Otherwise, we may now assume that $|S| = \alpha + 1$. Hence S cannot be a set of independent vertices and so there must exist an edge between some pair of vertices of S. Nine cases result:

(1) If x and b_i^+ are adjacent, where $1 \leq i \leq \omega$, then we construct C' by removing the edge $\{b_i, b_i^+\}$ from C and replacing it with the 2-path (b_i, x, b_i^+).

(2) If x and a_i^{+2} are adjacent, where $\omega + 1 \leq i \leq \mu$, then we construct C' by removing the 2-path (a_i, a_i^+, a_i^{+2}) from C and replacing it with the 3-path (a_i, b_i, x, a_i^{+2}).

(3) If x and a_i^+ are adjacent, where $\mu + 1 \leq i \leq \alpha$, then we construct C' by removing the edge $\{a_i, a_i^+\}$ from C and replacing it with the 2-path (a_i, x, a_i^+).

(4) If b_i^+ and b_j^+ are adjacent, where $1 \leq i \leq \omega$ and $1 \leq j \leq \omega$, then we construct C' by removing the edges $\{b_i, b_i^+\}$ and $\{b_j, b_j^+\}$ from C and inserting the edge $\{b_i^+, b_j^+\}$ as well as the 2-path (b_i, x, b_j).

(5) If b_i^+ and a_j^{+2} are adjacent, where $1 \leq i \leq \omega$ and $\omega + 1 \leq j \leq \mu$, then we construct C' by removing the edge $\{b_i, b_i^+\}$ and the 2-path (a_j, a_j^+, a_j^{+2}) from C and inserting the edge $\{b_i^+, a_j^{+2}\}$ and the 3-path (b_i, x, b_j, a_j).

(6) If b_i^+ and a_j^+ are adjacent, where $1 \leq i \leq \omega$ and $\mu + 1 \leq j \leq \alpha$, then we construct C' by removing the edges $\{b_i, b_i^+\}$ and $\{a_j, a_j^+\}$ from C and inserting the edge $\{b_i^+, a_j^+\}$ and the 2-path (b_i, x, a_j).

(7) If a_i^{+2} and a_j^{+2} are adjacent, where $\omega + 1 \leq i \leq \mu$ and $\omega + 1 \leq j \leq \mu$, then we construct C' by removing the 2-paths (a_i, a_i^+, a_i^{+2}) and (a_j, a_j^+, a_j^{+2}) from C and inserting the edge $\{a_i^{+2}, a_j^{+2}\}$ and the 4-path (a_i, b_i, x, b_j, a_j).

(8) If a_i^{+2} and a_j^+ are adjacent, where $\omega + 1 \leq i \leq \mu$ and $\mu + 1 \leq j \leq \alpha$, then we construct C' by removing the edge $\{a_j, a_j^+\}$ and the 2-path (a_i, a_i^+, a_i^{+2}) from C and inserting the edge $\{a_i^{+2}, a_j^+\}$ and the 3-path (a_i, b_i, x, a_j).

(9) If a_i^+ and a_j^+ are adjacent, where $\mu + 1 \leq i \leq \alpha$ and $\mu + 1 \leq j \leq \alpha$, then we construct C' by removing the edges $\{a_i, a_i^+\}$ and $\{a_j, a_j^+\}$ and inserting the edge $\{a_i^+, a_j^+\}$ and the 2-path (a_i, x, a_j).

Note that in each case (even in the cases in which we remove a 2-path from C), each vertex of A is contained within the new cycle C'. □

5 Related Problems and Future Research

We have now proven that all PBD's with $\max\{\mathcal{K}\} \leq 2\min\{\mathcal{K}\}$ and $\lambda \geq 2$ are pancyclic. From here, our research can turn to at least a couple of other problems. The first is a more generalised version of our theorem, and the second is a much stronger result relating to edge-pancyclicity of the block-intersection graph of every $PBD(v, \mathcal{K}, \lambda)$ with $\max\{\mathcal{K}\} \leq 2\min\{\mathcal{K}\}$. We state these problems as conjectures.

Conjecture 1. *Let* $\mathscr{D} = (V, \mathscr{B})$ *be a* $PBD(v, \mathcal{K}, \lambda)$ *with* $\lambda \geq 2$ *and* $\max\{\mathcal{K}\} \leq \lambda \min\{\mathcal{K}\}$. *Then* $G_{\mathscr{D}}$ *is pancyclic.*

Conjecture 2. *Let* $\mathscr{D} = (V, \mathscr{B})$ *be a* $PBD(v, \mathcal{K}, \lambda)$ *with* $\lambda \geq 2$ *and* $\max\{\mathcal{K}\} \leq 2 \min\{\mathcal{K}\}$. *Then* $G_{\mathscr{D}}$ *is edge-pancyclic.*

It is worthwhile to note that to prove Conjecture 1 it needs only to be shown that $G_{\mathscr{D}}$ has a cycle of length $2|A|$ for every independent set A in $G_{\mathscr{D}}$, $|A| \geq 2$, since we can easily modify the proofs of Lemma 2 and Lemma 3 to hold for $\max\{\mathcal{K}\} \leq \lambda \min\{\mathcal{K}\}$.

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