Limited visibility Zombies and Survivors

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Abstract

Zombies and Survivors is a variant of Cops and Robbers where the zombies (cops) must always move closer to the survivor (robber). We survey the current state of knowledge about the game and provide a list of open questions. A graph is *zombie-win* if a single zombie can guarantee the capture of the survivor. It is shown that the strong product of zombie-win graphs is zombie-win. We introduce a new notion of *safe subgraphs*, which have applications to both Cops and Robbers and Zombies and Survivors. Safe subgraphs are used to prove that every cop-win graph (and zombie-win graph) with maximum degree 3 is chordal. A new limited visibility variant of Zombies and Survivors is introduced. We show that every *cycle-filling distance-2 dominating set* is a winning starting position for the zombies in the 2-visibility game. Finally, we use safe subgraphs to define *safe distance-k dominating sets*, which are used to obtain a lower bound in k-visibility Zombies and Survivors.

1 Introduction

The game of Cops and Robbers is a perfect-information two-player pursuit-evasion game played on a graph, first introduced by Quilliot [24] in 1978 and independently considered by Nowakowski and Winkler [19] in 1983. In the last forty years, there has been a great deal of research focused on this game and its many variants. The book by Bonato and Nowakowski [3] is an excellent introduction to the field.

In Cops and Robbers, one player controls a set of *cops* and the other player controls a single *robber*. To begin the game, each cop and robber chooses a vertex to occupy, with the cops choosing first. Play then alternates between cops and robbers, with the cops moving first. For the cops' move, each cop either moves to an adjacent vertex or stays at their current location. The robber's move is defined similarly. The cops win if, after some finite number of rounds, a cop occupies the same vertex as the robber. The robber wins if they can avoid capture indefinitely. We say a graph is *cop-win* if a single cop can force a win for the cop player.

More recently, a variant known as Zombies and Survivors has received a substantial amount of attention. This game was introduced by Fitzpatrick, Howell, Messinger and Pike [10] in 2016 and has been investigated further in [1, 8, 9, 16, 21]. Also see [2, 4, 23] for a probabilistic variant. The game is similar to Cops and Robbers, with a *zombie* in place of each cop and a *survivor* in place of the robber. While a cop can move to any adjacent vertex, a zombie must always move along a geodesic joining itself and the survivor. If there are multiple geodesics, a zombie is allowed to choose one. In this sense, the zombies do have some intelligence, although their available strategies are very limited. The survivor moves in exactly the same way as a robber. We say a graph is *zombie-win* if a single zombie can force a win for the zombie player. Every valid zombie strategy is clearly a valid cop strategy, so every zombie-win graph is cop-win.

Zombies and Survivors is notable for having rules which seem very similar to Cops and Robbers at first glance, but which lead to a game with surprisingly different properties. For example, while there is a well-known structural characterization of cop-win graphs [19], there is no similar result known for zombie-win graphs. Another example is given by the importance of the initial position of the zombies in Zombies and Survivors (see the graph G_5' in Figure 2), whereas the initial position of the cops in Cops and Robbers is irrelevant. This is evidence that Zombies and Survivors is a game of a fundamentally different nature.

In Section 2 we will first review necessary concepts from graph theory, then give a brief introduction to both Cops and Robbers and Zombies and Survivors. In Section 3 we introduce *safe subgraphs*, which generalize results from [3] and have applications to both Cops and Robbers and Zombies and Survivors. In Section 4, we will continue the tradition of the field by proposing yet another variant of Cops and Robbers known as *k-visibility Zombies and Survivors*. The idea of limiting the visibility of the cops was explored in [7], using a model where the cops do not know the robber's location at the beginning of the game. Our variant of Zombies and Survivors is based on a new perfect-information limited visibility model which is closely related to the theory of *distance-k dominating sets*. Finally, in Section 5 we give a list of open questions and discuss possible directions for future research.

2 Definitions and preliminary results

For standard graph theory notation, we follow that of [5]. Every graph in this paper is finite and nonempty. Given a graph G, denote its set of vertices by $V(G)$ and its set of edges by $E(G)$. If $X \subseteq V(G)$, then $G[X]$ refers to the subgraph of G induced by X. For a vertex $u \in V(G)$, the set $N_G(u) = \{v \in V(G) \mid uv \in E(G)\}$ is the *open neighbourhood of* u and $N_G[u] = N_G(u) \cup \{u\}$ is the *closed neighbourhood of* u. If the graph G is understood, we will simply write $N(u)$ and $N[u]$. For $X \subseteq V(G)$, the set $N[X] = \bigcup_{u \in X} N[u]$ is the *closed neighbourhood of* X. In the Cops and Robbers literature it is usually assumed that all graphs are *reflexive*, meaning that there is a loop at every vertex so that a vertex is adjacent to itself. We will adopt this convention, with the caveat that we only consider edges of the form uv with $u \neq v$ in the definition of $N(u)$. This is so that $u \notin N(u)$, allowing us to distinguish between $N(u)$ and $N[u]$.

We say that a vertex $u \in V(G)$ *dominates* a set of vertices X if u is adjacent to every vertex in $X \setminus \{u\}$. In this case, we also say that X is *dominated* by u. A vertex u is said to be a *universal vertex* if u dominates $V(G)$. A set of vertices X is a *dominating set of* G if every vertex in $V(G) \setminus X$ is adjacent to at least one vertex in X. The *domination number* $\gamma(G)$ is the size of a smallest dominating set of G.

The notation $(u_0, u_1, \ldots, u_k)$ refers to a walk starting on the vertex u_0 and ending on the vertex u_k. A path starting on u and ending on v will be referred to as a *u-v path*. A u-v path of minimum length is called a *u-v geodesic*. A *k-cycle* is a cycle with k vertices, which we will denote by $(u_0, u_1, \ldots, u_{k-1}, u_0)$. For vertices u and v in a graph G, define the *distance* $d_G(u, v)$ to be the length of any u-v geodesic if at least one u-v path exists, or ∞ otherwise. The function d_G satisfies the *triangle inequality* $d_G(u, v) \leq d_G(u, w) + d_G(w, v)$ for any $u, v, w \in V(G)$. If $u \in V(G)$, let $N_k[u] = \{v \in V(G) \mid d(u, v) \leq k\}$ be the *distance-k neighbourhood* of u. If $Z \subseteq V(G)$, let $N_k[Z] = \bigcup_{u \in Z} N_k[u]$ be the *distance-k neighbourhood of* Z.

There are a few simple graphs which we will encounter frequently. These include the *path*, *cycle* and *complete graph* on n vertices, denoted by P_n, C_n and K_n respectively. The *wheel graph* W_n is defined by adding a single universal vertex to the cycle C_n. In particular, $W_3 = K_4$. The *diamond graph* D has vertices $V(D) = \{a, b, c, d\}$ and edges $E(D) = \{ab, bc, cd, ad, ac\}$.

If H is an induced subgraph of G, then for all $u, v \in H$ with $d_H(u, v) = 2$, it follows that $d_G(u, v) = 2$, since $uv \notin E(G)$. An *isometric* subgraph H of G satisfies the stronger property that for all $u, v \in H$, $d_H(u, v) = d_G(u, v)$. Every isometric subgraph of a graph is an induced subgraph, but the converse is false in general. A family of examples is given by the graphs W_n for $n \geq 6$, each of which contains C_n as an induced but not isometric subgraph.

A mapping $f\colon V(H) \to V(G)$ is an *isometric embedding* if

$$d_G(f(u), f(v)) = d_H(u, v)$$

for any $u, v \in V(H)$. Any isometric embedding is necessarily an injective graph homomorphism (although homomorphisms of reflexive graphs are not injective in general). If there exists an isometric embedding $f\colon V(H) \to V(G)$, then H can be viewed as an isometric subgraph of G by identifying H with its image under f.

A graph G is *chordal* if G contains no induced cycles of length 4 or more. More generally, a graph G is *bridged* if G contains no isometric cycles of length 4 or more. Any tree is clearly chordal. The smallest chordal graph containing a 4-cycle is the diamond D. Chordal graphs are bridged, since isometric cycles must be induced. The wheel W_6 is bridged, but not chordal.

The *Cartesian product* of graphs G and H is denoted by $G \square H$, where $V(G \square H) = V(G) \times V(H)$ and $E(G \square H) = \{(x_1, y_1)(x_2, y_2) \mid x_1x_2 \in E(G)$ and $y_1 = y_2$, or $x_1 = x_2$ and $y_1y_2 \in E(H)\}$. The *strong product* of G and H is denoted by $G \boxtimes H$, where $V(G \boxtimes H) = V(G) \times V(H)$ and $E(G \boxtimes H) = E(G \square H) \cup \{(x_1, x_2)(y_1, y_2) \mid x_1x_2 \in E(G)$ and $y_1y_2 \in E(H)\}$.

A *robber-play* is a walk which records the vertices occupied by the robber in a particular game of Cops and Robbers. A *cop-play* is a walk which records the vertices occupied by a specific cop. If the robber is playing against a single cop, we will use the notation $(r_0, r_1, \ldots)$ and $(c_0, c_1, \ldots)$ for the robber-play and cop-play respectively. In this case, the game itself can be viewed as an interleaved sequence of vertices $(c_0, r_0, c_1, r_1, \ldots)$. In particular, c_0 is the cop's starting vertex and r_0 is the robber's starting vertex. Note that $r_{i+1} = r_i$ or $c_{i+1} = c_i$ for some $i \geq 0$ is possible. Furthermore, a robber-play is finite if and only if the cop wins. In this case, the robber-play is $(r_0, \ldots, r_k)$ and the cop-play is $(c_0, \ldots, c_k, c_{k+1})$ where $k \geq 0$ and $c_{k+1} = r_k$. In general, we say a play is *winning* if it leads to that player winning the game.

Similarly, a *survivor-play* in Zombies and Survivors is denoted by $(s_0, s_1, \ldots)$ and a *zombie-play* is denoted by $(z_0, z_1, \ldots)$. For a specific zombie, for all $i \geq 0$, z_{i+1} must be adjacent to z_i on some z_i-s_i geodesic $(z_i, z_{i+1}, \ldots, s_i)$. This implies that $z_{i+1} \neq z_i$ for all i. Note that the validity of a zombie-play depends on the survivor-play, since the set of vertices that the zombie is allowed to choose z_{i+1} from on turn i is determined by the survivor's current location s_i. This is not the case for Cops and Robbers, as the validity of any cop-play does not depend on the robber-play.

Define the *cop-number* of a graph G to be the minimum number of cops required to win on G. We write $c(G)$ for the cop-number of G, so that G is cop-win if and only if $c(G) = 1$. The *zombie-number* of a graph is defined similarly. We write $z(G)$ for the zombie-number of a graph, so that G is zombie-win if and only if $z(G) = 1$.

Theorem 2.1. *For any graph G, $c(G) \leq z(G)$.*

Proof. Every winning zombie-play is also a winning cop-play. In other words, if k zombies have a winning strategy, then k cops can win using the same strategy. □

It follows from Theorem 2.1 that every zombie-win graph is cop-win. However, the converse does not hold. The graph G_5 in Figure 1 has $c(G_5) = 1$ and $z(G_5) = 2$ (see [10] for a proof).

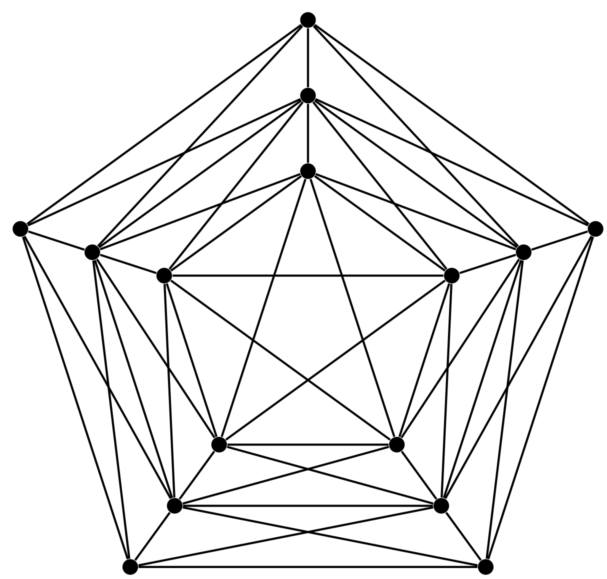

Figure 1: G_5.

It was shown in [9] that for any set $\{T_1, \ldots, T_n\}$ of n nontrivial trees, $z(\square_{i=1}^{n} T_i) = \lceil 2n/3 \rceil$. On the other hand, $c(\square_{i=1}^{n} T_i) = \lceil (n+1)/2 \rceil$ [18]. This implies that the difference between $c(G)$ and $z(G)$ can be arbitrarily large. More generally, for all $m \geq k \geq 1$, there exists a graph $Z_{k,m}$ such that $c(Z_{k,m}) = k$ and $z(Z_{k,m}) = m$ [21]. Also, for all $k \geq 2$, there exists a planar graph G_k such that $c(G_k) = 2$ and $z(G_k) \geq k$ [1].

Let $Z_0(G) \subseteq V(G)$ be the set of initial positions from which a single zombie can win. It follows that G is zombie-win if and only if $Z_0(G) \neq \emptyset$. A vertex v

is said to be *zombie-win* if $v \in Z_0(G)$. We say that G is *strongly zombie-win* if $Z_0(G) = V(G)$.

The graph G_5' in Figure 2 is obtained from G_5 by adding a vertex v and adding an edge from v to each of the five inner vertices in G_5. It was shown in [10] that $z(G_5') = 1$ and $Z_0(G_5') = \{v\}$. Hence G_5' is an example of a zombie-win graph that is not strongly zombie-win.

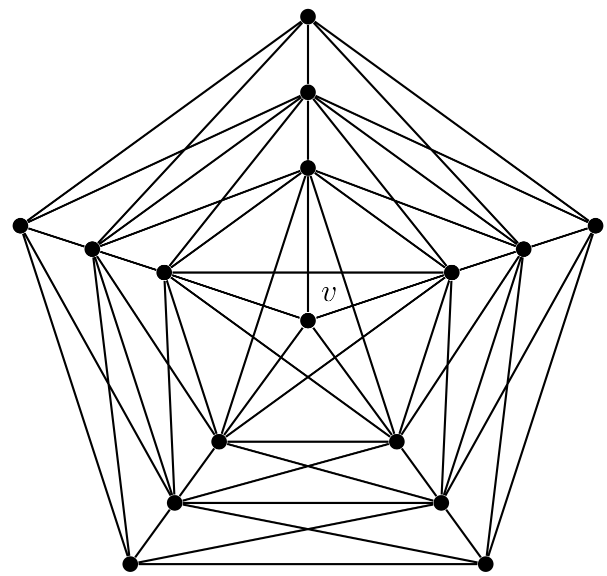

Figure 2: G_5'.

A large class of strongly zombie-win graphs is given by the following theorem.

Theorem 2.2. *(Corollary 7 of [10]) Any bridged graph is strongly zombie-win.*

Corollary 2.3. *Any chordal graph is strongly zombie-win.*

However, not all strongly zombie-win graphs are bridged. The smallest example is given by the wheel W_4. The fact that W_4 is strongly zombie-win is a special case of the following observation.

Theorem 2.4. *Any graph with a universal vertex is strongly zombie-win.*

Proof. Let G be a graph with a universal vertex u. Clearly, $u \in Z_0(G)$. Let $z_0 \in V(G) \setminus \{u\}$ and suppose a zombie starts on z_0. The survivor must start on a vertex $s_0 \notin N[z_0]$ to avoid losing immediately. Then (z_0, u, s_0) is a z_0-s_0 geodesic of length 2, so the zombie can move to u. Every vertex $s_1 \in N[s_0]$ is adjacent to u, so the survivor will be caught on the next turn. $\square$

An example of a strongly zombie-win graph that is not characterized by Theorem 2.2 or Theorem 2.4 is any graph of the form $P_n \boxtimes P_m$ with $n \geq 3$ and $m \geq 4$. We will show that a strong product of paths is strongly zombie-win in Theorem 3.3.

The most general known sufficient condition for a graph to be zombie-win is given by the following theorem, which generalizes Theorems 2.2 and 2.4.

Theorem 2.5. *(Theorem 6 of [10]) If there exists a breadth-first search of a graph G such that the associated spanning tree is also a cop-win spanning tree, then G is zombie-win.*

3 Safe subgraphs

Recall that a graph G is *robber-win* if it is not cop-win. Similarly, G is *survivor-win* if it is not zombie-win. To be more precise, G is survivor-win if and only if for every $z_0 \in V(G)$, there exists a survivor-play that wins against a single zombie starting on z_0. Note that every robber-win graph is survivor-win, but the converse is false in general. Any graph G with $c(G) = 1$ and $z(G) \geq 2$ is survivor-win but not robber-win. The graph G_5 from Figure 1 is an example. Also note that any disconnected graph is robber-win as the robber can simply choose to start in a different component than the cop.

For many robber-win (respectively survivor-win) graphs, the robber (respectively survivor) can win without using the entire graph. The concept of restricting the movement of the robber was first introduced in Clarke's doctoral thesis [6]. The related question of how to characterize optimal robber strategies was investigated in [20]. Continuing this line of inquiry, there are a number of interesting questions that arise when trying to find the smallest subgraph that the robber or survivor can use to win. In Section 4, we will apply ideas from this section to a limited visibility variant of Zombies and Survivors.

A subgraph H of G is *robber-safe with respect to v* if a robber can win against a single cop that starts on the vertex $v \in V(G)$, where the robber is restricted to moving along the vertices and edges of H. The cop player is allowed to use the entire graph G. If a subgraph H is robber-safe with respect to every $v \in V(G)$, we say that H is *robber-safe*. We define *survivor-safe with respect to v* and *survivor-safe* similarly, where the cop is replaced with a zombie. If H is robber-safe with respect to v then H is survivor-safe with respect to v, since if a robber can win against a cop starting on v while staying in the subgraph H, then a survivor can certainly win against a zombie starting on v while staying in H. In addition, if a subgraph H of G is robber-safe (respectively survivor-safe), then the entire graph G is robber-win (respectively survivor-win).

Given a graph G with a subgraph H, where H itself is a survivor-win graph, we would like to find sufficient conditions for H to be survivor-safe in G. In other words, we want to know when a winning survivor strategy in H can be adapted to the larger graph G, independently of the starting position of the zombie in G. First, it can be easily shown that H being an induced subgraph of G is not sufficient.

Theorem 3.1. *For every graph H, there exists a strongly zombie-win graph G which contains H as an induced subgraph.*

Proof. Construct a graph G from H by adding a new universal vertex u. Then G is strongly zombie-win as a consequence of Theorem 2.4, and H is an induced subgraph of G. □

Next, we will show that if a survivor-win graph H is an isometric subgraph of a graph G, then this is still not enough to conclude that H is survivor-safe in G.

Lemma 3.2. *Suppose G and H are connected graphs with $g, g' \in V(G)$ and $h, h' \in V(H)$. Let $a_0 = g$, $a_k = g'$, $b_0 = h$ and $b_\ell = h'$. Suppose $P = (a_0, \dots, a_k)$ is a g-g' geodesic in G of length k and $Q = (b_0, \dots, b_\ell)$ is a h-h' geodesic in H of length ℓ. Then*

$$R = \begin{cases} ((a_0, b_0), (a_1, b_1), \dots, (a_k, b_k)) & \text{if } k = \ell \\ ((a_0, b_0), (a_1, b_1), \dots, (a_k, b_k), (a_k, b_{k+1}), \dots, (a_k, b_\ell)) & \text{if } k < \ell \\ ((a_0, b_0), (a_1, b_1), \dots, (a_\ell, b_\ell), (a_{\ell+1}, b_\ell), \dots, (a_k, b_\ell)) & \text{if } k > \ell \end{cases}$$

is a (g, h)-(g', h') geodesic in $G \boxtimes H$ of length $\max\{k, \ell\}$.

Proof. This is the same argument used in the proof of Proposition 5.4 in [13]. □

Theorem 3.3. *If $G_1, \dots, G_n$ are zombie-win graphs, then*

$$G = \boxtimes_{i=1}^{n} G_i$$

is zombie-win. In particular, if $u_i \in Z_0(G_i)$ for each i, then $(u_1, \dots, u_n) \in Z_0(G)$.

Proof. If we can prove the theorem for $G = G_1 \boxtimes G_2$, the general case will follow by induction on n. Let $z_0 \in Z_0(G_1)$ and $w_0 \in Z_0(G_2)$. We will show that $(z_0, w_0) \in Z_0(G)$.

Suppose the survivor starts on vertex $(s_0, t_0) \in V(G)$. The zombie wins in G_1, so there exists a geodesic $P_1 = (z_0, a_1, \ldots, a_{k-1}, s_0)$ in G_1 such that the move z_0 to a_1 is part of a winning zombie-play in G_1. Similarly, the zombie wins in G_2, so there exists a geodesic $Q_1 = (w_0, b_1, \ldots, b_{\ell-1}, t_0)$ in G_2 such that the move w_0 to b_1 is part of a winning zombie-play in G_2.

Let R_1 be the geodesic in G obtained by applying Lemma 3.2 to P_1 and Q_1. Let $\pi_1 : G \to G_1$ and $\pi_2 : G \to G_2$ be the projection maps $\pi_1((u, v)) = u$ and $\pi_2((u, v)) = v$. The zombie's strategy is to move along R_1 from (z_0, w_0) to (a_1, b_1). By taking projections, this is equivalent to making the moves z_0 to a_1 in G_1 and w_0 to b_1 in G_2 simultaneously.

From here, the zombie can simply repeat this strategy. If P_i is a winning geodesic in G_1 and Q_i is a winning geodesic in G_2 at turn i, take the "product" geodesic R_i obtained from Lemma 3.2. Eventually, the zombie's projection will catch the survivor's projection in both G_1 and G_2.

If the survivor is caught in both factors on the same turn, the zombie immediately wins in G. Suppose the survivor is caught in one factor first, say G_1. After this point, if the survivor moves again in G_1, the zombie can move in G to recapture the survivor in G_1 while also moving along a winning geodesic in G_2. If the survivor does not move in G_1, the zombie can remain on the survivor's position in G_1 while also moving along a winning geodesic in G_2. In both cases, the zombie is using a geodesic obtained from Lemma 3.2. Thus, the survivor will eventually be caught in G. □

If v is a vertex of a connected graph G, recall that

$$e(v) = \max_{u \neq v} \{d_G(u, v)\}$$

is the *eccentricity* of v and

$$\mathrm{diam}(G) = \max_{v \in V(G)} \{e(v)\}$$

is the *diameter* of G.

Lemma 3.4. *(Corollary 24 of [11]) If H is a connected graph with n vertices and* $\mathrm{diam}(G) = d$*, then there exist vertices $v_1, v_2, \ldots, v_{n-d} \in V(H)$ such that H can be isometrically embedded into the graph*

$$G = \mathop{\boxtimes}_{i=1}^{n-d} P_{e(v_i)+1}.$$

Also see Chapter 15 of [13] for a survey on isometric subgraphs of strong products.

Corollary 3.5. *For every connected graph H, there exists a strongly zombie-win graph G which contains H as an isometric subgraph.*

Proof. The graph G in Lemma 3.4 is strongly zombie-win as a consequence of Theorem 3.3, since any path is strongly zombie-win. □

If H is an isometric subgraph of G, then for all $u, v \in V(H)$, there is some u-v geodesic in G which is contained in H. However, there could be other u-v geodesics in G not contained in H which the zombie is allowed to choose. Therefore, H being an isometric subgraph of G is not enough to guarantee that the zombie will ever enter the subgraph H, or remain in H after entering it for the first time. It follows that even if H is survivor-win, the survivor will not necessarily be able to use their winning strategy in H against the zombie in G. On the other hand, this argument also shows that it is possible for a single zombie to lose in a strongly zombie-win graph by choosing the wrong geodesics.

To find survivor-safe subgraphs, we will need to try something different. One possibility would be to impose a stronger condition on H than being isometric. A subgraph H of a graph G is *convex* if for all $u, v \in V(H)$, every u-v geodesic in G is contained in H.

Theorem 3.6. *Once a zombie enters a convex subgraph H which contains the survivor, it can never leave H as long as the survivor remains in H.*

Proof. If H is a convex subgraph of G, then for all $u, v \in V(H)$, H contains all geodesics between u and v in G. All of the zombie's moves must be along geodesics, so after entering H, it will always move to a new vertex in H. □

Unfortunately, it seems to be difficult to determine a general condition for a convex survivor-win subgraph H of G to be survivor-safe in G. Even if the survivor wins in H, their winning strategy will still in general depend on the zombie's starting vertex in H. To use their H-winning strategy against a zombie in G, the survivor needs to be able to control where the zombie first enters H (if the zombie starts on some vertex $z_0 \in V(G) \setminus V(H)$), and this in turn depends on the way that H is embedded in G.

One way to deal with this problem is to only consider subgraphs that are survivor-safe with respect to a single vertex $z \in V(G)$. This is the approach we will take in Section 4.3. For now, let us instead try to find sufficient conditions for a subgraph to be robber-safe, recalling the fact that every robber-safe subgraph is survivor-safe.

The first method to find robber-safe subgraphs uses retracts. If H is an induced subgraph of a graph G, we say that H is a *retract* of G if there exists a homomorphism $f\colon G \to H$ such that $f|_H = \mathrm{id}_H$. It is critical that G is reflexive, since reflexive graphs admit more retracts than irreflexive graphs. Retracts are a powerful tool in the study of Cops and Robbers; see [3] for a number of applications.

For a subgraph H of G and $x \in V(H)$, recall that $N_H[x] = \{x\} \cup \{y \in V(H) \mid xy \in E(H)\}$. The proof of the following theorem is similar to the argument used to prove Theorem 1.9 in [3].

Theorem 3.7. *If a retract H of a graph G is robber-win, then H is a robber-safe subgraph of G.*

Proof. It suffices to consider a single cop. Let $f\colon G \to H$ be a retract. For any cop-play C in G, the image of C under f is a cop-play in H since f is a homomorphism. Call $f(C)$ the *shadow* S of the cop. The robber has a winning robber-play R against S in H, since we are assuming H is robber-win. We claim that R is also a winning robber-play against the cop-play C in G.

To see this, suppose for a contradiction that H is not robber-safe. Then there exists a cop-play C in G such that any robber-play in H will lose against C. Let R be a robber-play in H which wins against $S = f(C)$. The turn before the cop wins in G, the cop will be on a vertex $c_i \in V(G)$ and the robber will be on a vertex $r_i \in V(H)$ such that $N_H[r_i] \subseteq N_G[c_i]$. Applying the retract f to r_i and c_i, it follows that $N_H[r_i] \subseteq N_H[f(c_i)]$. But this means that the robber will be caught in the subgraph H by S the next turn. Hence the robber-play R loses against S in H, which is a contradiction. □

Figure 3 gives an example of a graph G with a robber-safe cycle C (the outer blue cycle) which is not a retract, since the vertex v cannot be mapped to a vertex in $V(C)$ by any retract $f\colon G \to C$. However, C is not the smallest robber-safe cycle in W. There are three robber-safe cycles of length four, and all of these cycles are retracts of G.

More generally, robber-safe cycles of minimum length are always retracts. Recall that the *girth* of a graph G, denoted by $g(G)$, is the minimum length of a cycle in G.

Theorem 3.8. *If C is a robber-safe cycle in G such that C has length equal to $g(G)$, then C is a retract of G.*

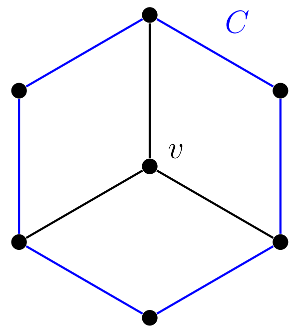

Figure 3: A graph G with a robber-safe subgraph C which is not a retract.

Proof. Proposition 2.51 of [15] says that if C is any cycle of minimum length in G, then C is a retract of G. □

If $g(G) = 3$, then G contains a triangle, but a triangle is never robber-safe. It follows that, for graphs containing a triangle, a robber-safe subgraph of minimum order cannot be a cycle of minimum length.

The following lemma gives another way to find robber-safe subgraphs. This is a generalization of Lemma 2.1 in [3].

Lemma 3.9. *Suppose H is a subgraph of a graph G such that for all $u \in V(H)$, for all $v \in V(G) \setminus \{u\}$, $N_H[u] \nsubseteq N_G[v]$. Then H is a robber-safe subgraph of G.*

Proof. Let $c_0 \in V(G)$. Note that no vertex $v \in V(G)$ can dominate $V(H)$, so there exists a vertex $r_0 \in V(H)$ such that $d_G(c_0, r_0) \geq 2$. From here, the robber has a simple winning strategy: they wait on vertex $r_i = r_0$ until $d_G(c_{i+1}, r_i) = 1$, which means that $r_i \in N_G[c_{i+1}]$. Then, the robber moves to a vertex $r_{i+1} \in V(H)$ such that $r_{i+1} \in N_H[r_i] \setminus N_G[c_{i+1}]$. Now $d_G(c_{i+1}, r_{i+1}) = 2$, and the same strategy can be repeated indefinitely. □

Lemma 3.10. *Suppose $C = (u_0, \ldots, u_k, u_0)$ is a cycle in G such that $k \geq 3$ and for all $i \in \{0, \ldots, k\}$, with subscript addition modulo $k+1$,*

(a) $u_{i-1}u_{i+1} \notin E(G)$, and

(b) for all $v \in V(G) \setminus \{u_{i-1}, u_i, u_{i+1}\}$, $\{u_{i-1}, u_i, u_{i+1}\} \nsubseteq N_G[v]$.

Then C is a robber-safe subgraph of G.

Proof. To apply Lemma 3.9, we need to show that for all $v \in V(G) \setminus \{u_i\}$, $\{u_{i-1}, u_i, u_{i+1}\} \nsubseteq N_G[v]$. Condition (a) ensures that $\{u_{i-1}, u_i, u_{i+1}\} \nsubseteq N_G[u_{i-1}]$ and $\{u_{i-1}, u_i, u_{i+1}\} \nsubseteq N_G[u_{i+1}]$. Condition (b) covers the case $v \notin \{u_{i-1}, u_{i+1}\}$. □

A graph G is said to be $(H_1, \ldots, H_k)$*-free* if none of the graphs $H_1, \ldots, H_k$ are an induced subgraph of G.

Theorem 3.11. *If a graph G has a cycle C such that $|V(C)| \geq 4$ and $G[N[V(C)]]$ is (D, K_4)-free, then $G[V(C)]$ is a robber-safe subgraph of G.*

Proof. Let $C = (u_0, \ldots, u_k, u_0)$ where $k \geq 3$. The main idea of the proof is that the (D, K_4)-free condition is equivalent to the condition that any two triangles in $G[N[V(C)]]$ are edge-disjoint.

Consider the subset of edges $F = \{u_i u_{i+2} \in E(G) \mid u_i \in V(C)\}$ with subscript addition modulo $k+1$.

Case 1: $F = \emptyset$.

In this case, we can show that the cycle C is a robber-safe subgraph of G using Lemma 3.10. Condition (a) is satisfied because $F = \emptyset$. For condition (b), suppose for a contradiction that for some u_i, there exists a vertex $v \in V(G) \setminus \{u_{i-1}, u_i, u_{i+1}\}$ which is adjacent to u_{i-1}, u_i and u_{i+1}. Then $(u_{i-1}, u_i, v, u_{i-1})$ and (u_i, u_{i+1}, v, u_i) are triangles in $G[N[V(C)]]$ sharing the edge $u_i v$, which is impossible. Thus, condition (b) is satisfied and it follows from Lemma 3.10 that C is robber-safe.

Case 2: $F \neq \emptyset$.

If $u_i u_{i+2} \in F$, then $u_{i-1} u_{i+1} \notin F$ and $u_{i+1} u_{i+3} \notin F$. To see this, if $u_{i-1} u_{i+1} \in F$, then $(u_{i-1}, u_i, u_{i+1}, u_{i-1})$ and $(u_i, u_{i+1}, u_{i+2}, u_i)$ are triangles in $G[V(C)]$ sharing the edge $u_i u_{i+1}$, which is a contradiction. Similarly, if $u_{i+1} u_{i+3} \in F$, then $(u_{i+1}, u_{i+2}, u_{i+3}, u_{i+1})$ and $(u_i, u_{i+1}, u_{i+2}, u_i)$ are triangles sharing the edge $u_{i+1} u_{i+2}$. Using this fact, we can write

$$F = \{u_{i_0} u_{i_0+2}, u_{i_1} u_{i_1+2}, \ldots, u_{i_j} u_{i_j+2}\} = \{e_0, \ldots, e_j\}$$

where $0 \leq j+1 \leq \lceil \frac{k}{2} \rceil$ and $0 \leq i_0 < i_0+2 \leq i_1 < i_1+2 \leq \cdots \leq i_j < i_j+2 \leq k$. In particular, $|i_j - i_{j+1}| \geq 2$ for all j.

Define a new cycle C' as follows: for every edge $u_i u_{i+2} \in F$, replace the subpath (u_i, u_{i+1}, u_{i+2}) of C with the path (u_i, u_{i+2}); see Figure 4 for an example. This results in a cycle $C' = (v_0, v_1, \ldots, v_t, v_0)$ such that $V(C') \subseteq V(C)$, $t = k - |F|$ and $F \subseteq E(C')$. We will show that C' is a robber-safe subgraph of G using Lemma 3.10.

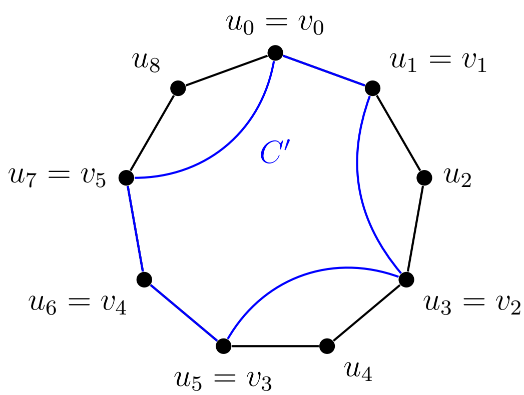

Figure 4: An example with $k = 8$, $|F| = 3$ and $t = 5$.

Condition (a) in Lemma 3.10 is equivalent to the condition that $v_i v_{i+2} \notin E(G)$ for all i. Suppose for a contradiction that $v_i v_{i+2} \in E(G)$ for some i. There are four cases to consider. If $\{v_i, v_{i+1}, v_{i+2}\} = \{u_{i'}, u_{i'+1}, u_{i'+2}\}$ for some index i', then $v_i v_{i+2} = u_{i'} u_{i'+2}$. Hence $v_i v_{i+2} \in F \setminus E(C')$, which contradicts $F \subseteq E(C')$. If $\{v_i, v_{i+1}, v_{i+2}\} = \{u_{i'}, u_{i'+1}, u_{i'+3}\}$, then $(v_i, v_{i+1}, v_{i+2}, v_i)$ and $(v_{i+1}, u_{i'+2}, v_{i+2}, v_{i+1})$ are triangles sharing the edge $v_{i+1}v_{i+2}$. If $\{v_i, v_{i+1}, v_{i+2}\} = \{u_{i'}, u_{i'+2}, u_{i'+3}\}$, then $(v_i, v_{i+1}, v_{i+2}, v_i)$ and $(v_i, u_{i'+1}, v_{i+1}, v_i)$ are triangles sharing the edge $v_i v_{i+1}$. If $\{v_i, v_{i+1}, v_{i+2}\} = \{u_{i'}, u_{i'+2}, u_{i'+4}\}$, then similarly $(v_i, v_{i+1}, v_{i+2}, v_i)$ and $(v_i, u_{i'+1}, v_{i+1}, v_i)$ are triangles sharing the edge $v_i v_{i+1}$. All four cases are impossible, so condition (a) holds.

To show that condition (b) holds for C', we can use the same argument that we used for C in the case $F = \emptyset$. This is possible because the subgraph $G[N[V(C')]]$ of $G[N[V(C)]]$ is also (D, K_4)-free.

Finally, Lemma 3.10 implies that C' is a robber-safe subgraph of G. □

We can do a little better than Lemma 3.9.

Lemma 3.12. *Suppose H is a subgraph of a graph G such that*

(a) $V(H)$ is not dominated by any $v \in V(G)$, and

(b) for all $u \in V(H)$, for all $v \in V(G)$ such that $d_G(u, v) = 2$, for all $w \in N_G(u) \cap N_G(v)$, $N_H[u] \nsubseteq N_G[w]$.

Then H is a robber-safe subgraph of G.

Proof. Let $c_0 \in V(G)$. Condition (a) guarantees that c_0 does not dominate $V(H)$, so there exists a vertex $r_0 \in V(H)$ such that $d_G(c_0, r_0) \geq 2$. The robber's strategy is to wait until $d_G(c_{i+1}, r_i) = 2$, then choose $r_{i+1} = r_i$.

Now, for all $c_{i+2} \in N_G[c_{i+1}]$, there exists a vertex $r_{i+2} \in N_H[r_{i+1}]$ such that $d_G(c_{i+2}, r_{i+2}) = 2$. This strategy can be repeated indefinitely. $\square$

Figure 5 gives two graphs where Lemma 3.12 can be used to show that the blue cycle is robber-safe. Note that in both of these examples, Lemma 3.10 is not applicable.

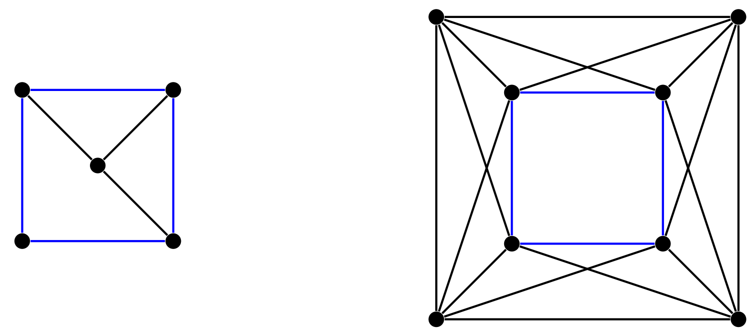

Figure 5: Two examples illustrating Lemma 3.12.

There are also examples of robber-safe subgraphs that do not meet the hypotheses of Lemma 3.12. A simple example is given by the graph G in Figure 6. Observe that $d_G(u, v) = 2$, $w \in N_G(u) \cap N_G(v)$ and $N_C[u] \subseteq N_G[w]$, so Lemma 3.12 cannot be applied. Nevertheless, it is clear that C is a robber-safe cycle.

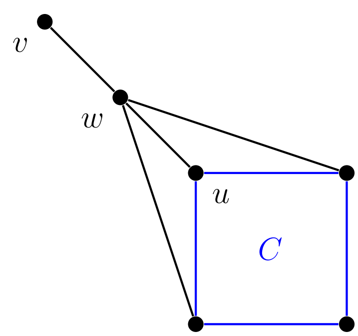

Figure 6: An example where Lemma 3.12 cannot be applied.

For connected graphs with maximum degree 3, the concepts of robber-win and survivor-win are equivalent. In fact, we can show that these graphs must contain a particularly simple safe cycle.

Theorem 3.13. *Suppose G is connected and $\Delta(G) \leq 3$. Then the following are equivalent:*

(a) G is robber-win.

(b) G is survivor-win.

(c) G contains a cycle $C = (u_0, \ldots, u_k, u_0)$ such that $k \geq 3$ and for all $i \in \{0, \ldots, k\}$, $u_{i-1}u_{i+1} \notin E(G)$, with subscript addition modulo $k + 1$.

Proof. $(a) \implies (b)$: This is the contrapositive of the fact that zombie-win graphs are cop-win.
$(b) \implies (c)$: Suppose that all cycles C of length at least 4 in G have the property that $u_{i-1}u_{i+1} \in E(G)$ for some $u_i \in V(C)$. Then in particular, G has no induced cycles of length at least 4, so G is chordal. It follows from Corollary 2.3 that G is zombie-win.
$(c) \implies (a)$: We will show that the cycle C is robber-safe in G using Lemma 3.12. First, it needs to be shown that no vertex $v \in V(G)$ dominates $V(C)$. The two cases are $v \in V(C)$ and $v \notin V(C)$. If $v = u_i \in V(C)$, then $u_iu_{i+2} \notin E(G)$ implies that v cannot dominate $V(C)$. If $v \notin V(C)$, then v cannot dominate $V(C)$ since C has at least 4 vertices and $\Delta(G) \leq 3$.

Let $u_i \in V(C)$ and $v \in V(G)$ such that $d_G(u_i, v) = 2$. To apply Lemma 3.12, we need to show that for all $w \in N_G(v)$, $\{u_{i-1}, u_i, u_{i+1}\} \nsubseteq N_G[w]$.

Case 1: $w \notin \{u_{i-1}, u_{i+1}\}$.

Suppose for a contradiction that $\{u_{i-1}, u_i, u_{i+1}\} \subseteq N_G[w]$. It follows that $\{v, u_{i-1}, u_i, u_{i+1}\} \subseteq N_G(w)$ since $w \notin \{v, u_{i-1}, u_i, u_{i+1}\}$. But then $\deg(w) \geq 4$, which is impossible.

Case 2: $w \in \{u_{i-1}, u_{i+1}\}$.

First suppose that $w = u_{i-1}$. We assumed that $u_{i-1}u_{i+1} \notin E(G)$, which implies that $u_{i+1} \notin N_G[w]$. If $w = u_{i+1}$, a similar argument shows that $u_{i-1} \notin N_G[w]$. In either case, $\{u_{i-1}, u_i, u_{i+1}\} \nsubseteq N_G[w]$.

It follows from Lemma 3.12 that C is robber-safe in G. Thus G is robber-win. □

Corollary 3.14. *Suppose $\Delta(G) \leq 3$. Then the following are equivalent:*

(a) G is cop-win.

(b) G is zombie-win.

(c) G is chordal.

Proof. The implication $(c) \implies (b)$ follows from Corollary 2.3 and the implication $(b) \implies (a)$ follows from Theorem 2.1. To prove that $(a) \implies (c)$, use the contrapositive of $(c) \implies (a)$ in Theorem 3.13. □

Theorem 3.13 does not generalize to graphs G with $\Delta(G) \geq 4$. The smallest counterexample is given by the wheel W_4. If we also require that the cycle C is not dominated by any single vertex, a counterexample is given by the graph G in Figure 7. The centre 4-cycle is induced and is not dominated by any single vertex, but G is strongly zombie-win, so this cycle cannot be survivor-safe.

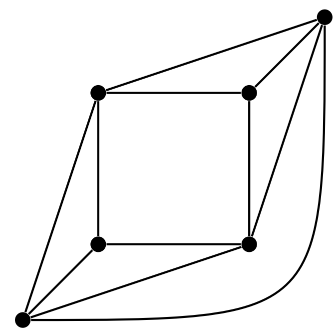

Figure 7: A strongly zombie-win graph G with an induced 4-cycle.

We say a graph G is *strongly robber-win* if G is connected, G has no universal vertices and for every $u, v \in V(G)$ with $d(u,v) \geq 2$, if a cop starts on v and a robber starts on u, then the robber wins. A *strongly survivor-win* graph is defined similarly, but with a zombie and survivor in place of a cop and robber. We can show that these graph classes are equal using Lemma 3.12.

Theorem 3.15. *For any graph G, the following are equivalent:*

(a) G is strongly robber-win.

(b) G is strongly survivor-win.

(c) G is connected, G has no universal vertices and for all $u, v \in V(G)$ such that $d(u,v) = 2$, for all $w \in N_G(u) \cap N_G(v)$, $N_G[u] \not\subseteq N_G[w]$.

Proof. $(a) \implies (b)$: A survivor can win whenever a robber can.
$(b) \implies (c)$: Let $z_0 \in V(G)$. Then for all $s_0 \in V(G)$ such that $d(z_0, s_0) = 2$ (there is at least one such vertex s_0 since G is connected and has no universal vertices), the survivor must be able to win. Thus $N_G[s_0] \not\subseteq N_G[w]$ for any $w \in N_G[z_0]$, which is what we wanted to prove.
$(c) \implies (a)$: Apply Lemma 3.12 with $H = G$. □

Any cycle graph C_n with $n \geq 4$ is strongly robber-win. The graph G in Figure 6 is robber-win but not strongly robber-win.

4 k-Visibility Zombies and Survivors

4.1 Rules and basic results

We will now study a variant of the game of Zombies and Survivors we call *k-visibility Zombies and Survivors.* The game starts similarly to Zombies and Survivors on a graph G, with the zombie player placing a single zombie on every vertex of a set $Z \subseteq V(G)$, followed by the survivor choosing a vertex $s_0 \in V(G) \setminus Z$. After this, the turns alternate between the zombie player and the survivor player moving on the vertices of G. The zombies win if, after a finite number of turns, some zombie occupies the same vertex as the survivor. If the survivor can evade capture indefinitely, then the survivor wins.

To explain how the k-visibility variant differs, suppose the survivor is currently occupying vertex s_i. During the zombie player's turn, if there is a zombie occupying vertex z_i and if $s_i \in N_k[z_i]$, then the zombie on z_i must move closer to the survivor. If $s \notin N_k[z_i]$, then the zombie on z_i cannot move. In other words, if the survivor is within distance k of z_i, then the zombie on z_i follows the rules of the regular Zombies and Survivors game. If not, the zombie on z_i remains inactive until the survivor enters $N_k[z_i]$.

We say $Z \subseteq V(G)$ is *k-visibility zombie-win* if the zombies starting on Z win k-visibility Zombies and Survivors. If this is not the case, then the survivor wins against zombies starting on Z, so we say Z is *k-visibility survivor-win.* If there is any vertex $v \in V(G)$ such that $v \notin N_k[Z]$, then the survivor can win by staying on v indefinitely, since all of the zombies will remain inactive. Therefore, if Z is k-visibility zombie-win, then $V(G) \setminus Z \subseteq N_k[Z]$. A set Z satisfying this condition is said to be a *distance-k dominating set* of G, or just a *k-dominating set* for short. See [14] for an introduction to distance domination in graphs.

Let $\gamma_k(G)$ be the *k-domination number* of G, the size of any smallest k-dominating set of G, and let $z_k(G)$ be the *k-visibility zombie number* of G, the size of any smallest k-visibility zombie-win set of G. The next theorem follows from the previous discussion.

Theorem 4.1. *For any graph G and any $k \geq 1$, $z_k(G) \geq \gamma_k(G)$.*

An important note is that exactly one zombie is placed on each vertex in Z. This is different from Zombies and Survivors, where we do allow more than one zombie to start on the same vertex. Zombies are still allowed to share vertices with other zombies at any point after the first turn of the game. The main reason for this restriction is that it allows us to identify the set Z with the starting position of the zombies, so that there is a 1-1 correspondence between k-dominating sets and nontrivial zombie starting positions. It follows that $z_k(G)$ is always the size of a set of vertices, similarly to $\gamma_k(G)$. In Section 4.3 we will see an example where removing this restriction can simplify the zombies' winning strategy. Unless stated otherwise, we will always assume this restriction holds.

There are two other items to keep in mind. The first is that once a zombie is activated, it remains active for the remainder of the game. This is because the survivor can never leave a zombie's vision radius after entering it. Second, both players have perfect information, which is not the case for the limited visibility model in [7]. In particular, the zombies can communicate and are permitted to use strategies that require coordination. For example, the zombies may need to choose a specific combination of geodesics to force the survivor to move into the vision radius of an inactive zombie.

For $k = 1$, the 1-visibility zombie-win sets can be completely characterized. First, note that a 1-dominating set of G is the same as a dominating set of G, so $\gamma_1(G) = \gamma(G)$.

Theorem 4.2. *For any graph G, $z_1(G) = \gamma(G)$.*

Proof. It suffices to show that $Z \subseteq V(G)$ is a dominating set of G if and only if Z is 1-visibility zombie-win. First, suppose that Z is a dominating set. No matter where the survivor starts, they will be caught by the zombies' next move in the 1-visibility game. Hence Z is 1-visibility zombie-win. Conversely, if Z is 1-visibility zombie-win, then it must be a dominating set as was already argued. □

For $k \geq 2$, it is possible for the inequality $\gamma_k(G) \leq z_k(G)$ to be strict. The smallest example is given by the cycle $G = C_4$, which has $\gamma_2(C_4) = 1$ and $z_2(C_4) = 2$. We can show that equality is attained in the case where G is strongly zombie-win (in the full visibility game).

Lemma 4.3. *Let G be a zombie-win graph and let $k \geq 1$. Suppose G has a minimum k-dominating set Z such that $Z \subseteq Z_0(G)$. Then $z_k(G) = \gamma_k(G)$.*

Proof. It suffices to show that Z is a k-visibility zombie-win set. This is clear since no matter where the survivor starts, they will attract the attention of at least one zombie, and this zombie will catch them since $Z \subseteq Z_0(G)$. □

Theorem 4.4. *If G is strongly zombie-win and $k \geq 1$, then $z_k(G) = \gamma_k(G)$.*

Proof. If G is strongly zombie-win, then $Z_0(G) = V(G)$. It follows that any minimum k-dominating set Z satisfies $Z \subseteq Z_0(G)$, so we can apply Lemma 4.3. □

To see which player wins in the case $k = 1$, it suffices to check whether or not Z is a dominating set. In the case $k = 2$, if Z is 2-visibility zombie-win then Z is a 2-dominating set, but the converse does not hold in general. Our next goal is to determine a sufficient condition for Z to be 2-visibility zombie-win.

4.2 2-visibility Zombies and Survivors

To help understand the 2-visibility game, let us start with an example. Consider the graph G_7 in Figure 8. Let $Z = \{z_1, \ldots, z_7\}$ and let $W = \{w_1, w_2\}$. It is straightforward to check that Z is the unique minimum 2-dominating set of G_7, so $\gamma_2(G_7) = 7$. Also, $Z \cup W$ is a minimum 2-visibility zombie-win set, so $z_2(G_7) = 9$. The minimality of $Z \cup W$ follows from the fact that any minimum 2-visibility zombie-win set Z_2 of G must contain Z, but $|Z_2 \setminus Z| \leq 1$ is not sufficient for the zombies to win.

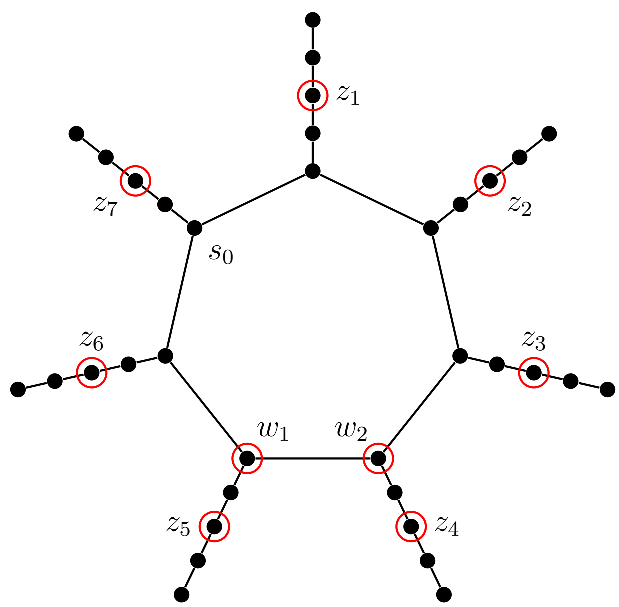

Figure 8: A graph G_7 with $\gamma_2(G_7) = 7$ and $z_2(G_7) = 9$.

The set W has an interesting property: if Zombies and Survivors is played on G_7 with two zombies starting on W, then the survivor can win by starting on s_0 and moving clockwise, even though $z(G_7) = 2$. This is because both zombies must move along the shorter arc of the induced 7-cycle, so two zombies cannot win from this starting position. However, this strategy does not work against 2-visibility zombies. The survivor must start on some vertex $u \notin N[W]$ to avoid losing immediately. Then either $u \notin N_2[w_1]$ or $u \notin N_2[w_2]$, so one of w_1 or w_2 will remain inactive. In fact, if W' is any set of two vertices from the centre 7-cycle, then $Z \cup W'$ is 2-visibility zombie-win. This is the key property of the 2-visibility game that will be used to obtain an upper bound on $z_2(G)$ in Theorem 4.7.

As a first step towards Theorem 4.7, suppose a survivor is playing 2-visibility Zombies and Survivors on a graph G against zombies placed on a 2-dominating set $Z \subseteq V(G)$. Starting from s_0, a survivor-play is a walk $S = (s_0, s_1, s_2, \ldots)$ on the vertices of G. Then S is a winning survivor-play if and only if S is an infinite walk. Assuming S is winning, there are a few observations we can make.

When the survivor selects s_0 at the start of the game, there will be at least one vertex $z_0 \in Z$ such that $d(z_0, s_0) \leq 2$, since Z is a 2-dominating set. If $d(z_0, s_0) = 1$, then the survivor will be caught immediately, so we can assume that $d(z_0, s_0) = 2$. The zombie starting on z_0 will then move to a vertex z_1 so that $d(z_1, s_0) = 1$. If the survivor chooses to stay on vertex s_0, then they will be caught by this zombie's next move, so $s_1 \neq s_0$.

In general, $s_{i+1} \neq s_i$ for all $i \geq 0$. Furthermore, for all $i \geq 0$, after the survivor moves to s_i, there will be at least one zombie that moves to a vertex $z_{i+1} \in N(s_i)$. The survivor then moves to some $s_{i+1} \in N(s_i)$ such that $s_{i+1} \neq z_{i+1}$. In fact, $s_{i+1} \notin N[z_{i+1}]$ and so $d(s_{i+1}, z_{i+1}) = 2$, since otherwise the survivor will be caught next turn. Now the zombie on z_{i+1} can move to the vertex s_i that was previously occupied by the survivor. If the zombies do this every turn, then $s_{i+2} \neq s_i$ for all $i \geq 0$. We will call this strategy the *survivor-trailing strategy*, since at least one zombie always moves to the survivor's previous position.

Motivated by the example in Figure 8, we introduce the following definition: a subset Z of $V(G)$ is said to be *cycle-filling* if, for every cycle C in G with length at least 4, $|V(C) \cap Z| \geq 2$. Let $\gamma_2^c(G)$ be the *cycle-filling 2-domination number* of G, the size of a minimum cycle-filling 2-dominating set of G.

Clearly, $\gamma_2(G) \leq \gamma_2^c(G)$ for any graph G.

Theorem 4.5. *If T is a tree, then $\gamma_2(T) = z_2(T) = \gamma_2^c(T)$.*

Proof. Trees are strongly zombie-win by Theorem 2.2, so $\gamma_2(T) = z_2(T)$ follows from Theorem 4.4. For the other equality, it is clear that $\gamma_2(T) \leq \gamma_2^c(T)$. A tree has no cycles, so every 2-dominating set is trivially a cycle-filling 2-dominating set. Thus $\gamma_2^c(T) \leq \gamma_2(T)$. □

Lemma 4.6. *If a graph G has components $H_1, \ldots, H_t$, then $z_2(G) = \sum_{i=1}^{t} z_2(H_i)$ and $\gamma_2^c(G) = \sum_{i=1}^{t} \gamma_2^c(H_i)$.*

Proof. First, observe that placing $z_2(H_i)$ zombies on each component is sufficient for the zombies to win, so $z_2(G) \le \sum_{i=1}^{t} z_2(H_i)$. Next, suppose that fewer than $\sum_{i=1}^{t} z_2(H_i)$ zombies are placed on G. By the pigeonhole principle, there is a component H_j with fewer than $z_2(H_j)$ zombies, so the survivor can win by starting in H_j. Hence $z_2(G) \ge \sum_{i=1}^{t} z_2(H_i)$. The second equality is straightforward to prove. □

Theorem 4.7. *For any graph G, $z_2(G) \le \gamma_2^{\mathsf{c}}(G)$.*

Proof. It suffices to prove the theorem for connected graphs G. To see this, suppose G has components $G_1, \ldots, G_t$ and that Theorem 4.7 holds for each G_i. Applying Lemma 4.6,

$$z_2(G) = \sum_{i=1}^{t} z_2(G_i) \le \sum_{i=1}^{t} \gamma_2^{\mathsf{c}}(G_i) = \gamma_2^{\mathsf{c}}(G).$$

Let G be a connected graph and let Z be any cycle-filling 2-dominating set of G. It suffices to show that Z is 2-visibility zombie-win, where the zombies use the survivor-trailing strategy. Suppose for a contradiction that the survivor has a winning play $S = (s_0, s_1, s_2, \ldots)$ for some $s_0 \in V(G) \setminus Z$.

If G is a tree, $z_2(G) \le \gamma_2^{\mathsf{c}}(G)$ follows from Theorem 4.5. Therefore, we can assume G is not a tree.

We claim that for all $i \ge 0$ and $j \in \{1, 2, 3\}$, $s_{i+j} \ne s_i$. For $j \in \{1, 2\}$, this was already shown in the discussion preceding Theorem 4.5. The remaining case is $j = 3$. Suppose for a contradiction that $s_i = s_{i+3}$ for some i. Then $(s_i, s_{i+1}, s_{i+2}, s_i)$ is a cycle of length 3 in S. After the survivor moves to s_{i+2}, the trailing zombie will move to s_{i+1}. But then $s_{i+3} = s_i \in N(s_{i+1})$, and the survivor will be caught when this zombie moves to s_i.

Define the set $Z' = \{z \in Z :$ there exists a cycle C in G such that $z \in V(C)\} \subseteq Z$. Our goal is to show that $V(S) \cap Z' = \emptyset$. In other words, if we record all the vertices in some cycle of G that are occupied by a zombie at the beginning of the game, then the survivor cannot occupy any of these vertices during the course of the game.

Suppose that $V(S) \cap Z' \ne \emptyset$. Then there is a first instance t_0 such that $s_{t_0} \in V(S) \cap Z'$. Consider the subwalk $S_0 = (s_0, s_1, \ldots, s_{t_0})$ of S. We already know that for all $s_i \in V(S_0)$, if $j \in \{1, 2, 3\}$ and $s_{i+j} \in V(S_0)$, then $s_{i+j} \ne s_i$. In fact, S_0 must be a path. To prove this, it remains to show that S_0 contains no cycles of length at least 4. If this was not true, letting $C = (s_i, s_{i+1}, \ldots, s_{i+j} = s_i)$ be a cycle in S_0 with $i \ge 0$ and $j \ge 4$, C would contain at least two distinct vertices with zombies (at the beginning of the game), as a consequence of the cycle-filling property of Z. At least one of these zombies must be on some $s_{i'}$ with $i' < t_0$, contradicting the minimality of t_0.

The problem the survivor faces is that they will be passing through the vertex s_{t_0}, but this vertex is occupied by a zombie at the beginning of the game. To accomplish this, the survivor needs to "lure" the zombie off of s_{t_0} before occupying it for the first time. To be more precise, let t_1 be the first instance such that $s_{t_1} \in V(S) \cap N_2[s_{t_0}]$. The assumption was that the survivor wins, so $d(s_{t_1}, s_{t_0}) = 2$. Let $P_1 = (s_{t_1}, u_1, s_{t_0})$ be the geodesic of length 2 in G that the zombie on s_{t_0} moves along after the survivor enters s_{t_1}, where $u_1 \in V(G)$.

Our first observation is that $t_1 \leq t_0 - 2$. This is because $t_1 \leq t_0$ using the minimality of t_1, and $d(s_{t_1}, s_{t_0}) = 2$ implies that $t_1 \neq t_0$ and $t_1 \neq t_0 - 1$. It follows that $(s_{t_1}, \ldots, s_{t_0})$ is a subpath of S_0.

Next, we will show that $u_1 \notin V(S_0)$. From the definition of P_1, $u_1 \neq s_{t_0}$ and $u_1 \neq s_{t_1}$. If $u_1 = s_i$ for some $i < t_1$, then i would be a smaller index than t_1 with $s_i \in V(S) \cap N_2[s_{t_0}]$, which contradicts the definition of t_1. Suppose $u_1 = s_i$ for some i with $t_1 < i < t_0$. If $u_1 = s_{t_1+1}$, then when the survivor enters s_{t_1}, the zombie on s_{t_0} will move to u_1. From this position, the survivor cannot avoid this zombie since S_0 is a path, and so they will be caught by the zombies' next move, which is a contradiction. The last remaining possibility is $u_1 = s_i$ for $t_1 + 2 \leq i < t_0$. This would imply that $(s_{t_1}, \ldots, u_1, s_{t_1})$ is a cycle with vertices in S_0, implying the existence of two more zombies on $V(S_0) \cap Z'$, contradicting the minimality of t_0. Therefore, either $u_1 \in V(G) \setminus V(S)$ or $u_1 \in \{s_{t_0+1}, s_{t_0+2}, \ldots\}$.

We can now conclude that $C_1 = (s_{t_1}, \ldots, s_{t_0}, u_1, s_{t_1})$ is a cycle in G of length at least 4, where we have concatenated the subpath $(s_{t_1}, \ldots, s_{t_0})$ of S_0 and the path P_1. The cycle-filling condition implies there must be at least two zombies on this cycle at the beginning of the game. We already know there must be one zombie on s_{t_0}. The other one must be on u_1, since every other vertex of C_1 is in $V(S_0) \setminus \{s_{t_0}\}$, which is zombie-free. Hence $u_1 \in Z$. Now let t_2 be the first instance such that $s_{t_2} \in V(S) \cap N_2[u_1]$. Let $P_2 = (s_{t_2}, u_2, u_1)$ be the geodesic of length 2 the zombie on u_1 moves along after the survivor enters s_{t_2}, where $u_2 \in V(G)$.

Our next goal is to show that $C_2 = (s_{t_2}, \ldots, s_{t_1}, u_1, u_2, s_{t_2})$ is a cycle of length at least 4. First, $t_2 \leq t_1 - 1$, using the fact that $d(s_{t_1}, u_1) = 1$. Next, $u_1 \neq s_{t_2}$ and $u_1 \neq s_{t_1}$, using the fact that $u_1 \notin V(S_0)$. It remains to show that $u_2 \notin \{s_{t_2}, s_{t_1}, u_1\}$. We have $u_2 \neq s_{t_2}$ and $u_2 \neq u_1$ from the definition of P_2. Suppose for a contradiction that $u_2 = s_{t_1}$. If $t_2 \leq t_1 - 2$, then $(s_{t_2}, \ldots, s_{t_1}, s_{t_2})$ would be a cycle with vertices in S_0, which is impossible as we have already seen. If $t_2 = t_1 - 1$, then the zombie on u_1 will move to $s_{t_1} = u_2$ when the survivor enters s_{t_2}. This will result in the survivor's capture by the zombies' next move, using the fact that S_0 is a path. Thus, all of the vertices $s_{t_2}, s_{t_1}, u_1, u_2$ are distinct. We can now conclude that $u_2 \in Z$ and $u_2 \notin V(S_0)$.

By iterating the previous argument, we can conclude that for all $k \geq 2$, there is a cycle $C_k = (s_{t_k}, \ldots, s_{t_{k-1}}, u_{k-1}, u_k, s_{t_k})$ with length at least 4 such that $\{u_{k-1}, u_k\} \subseteq Z$, where $u_k \notin \{u_1, \ldots, u_{k-1}\}$ and $t_k \leq t_{k-1} - 1$. But then for some k we will have $t_k < 0$, which is impossible. Thus, $V(S) \cap Z' = \emptyset$.

To conclude the proof, $V(S) \cap Z' = \emptyset$ implies that S does not contain any cycles of length 4 or more. This is because a cycle of length 4 or more contained in S would contain two vertices with zombies at the beginning of the game by the cycle-filling condition, and both of these vertices would be elements of $V(S) \cap Z'$. Hence $s_{i+j} \neq s_i$ for all $i \geq 0$ and $j \geq 4$. Using the previous result for $j \in \{1, 2, 3\}$, it follows that $s_{i+j} \neq s_i$ for all $i \geq 0$ and $j \geq 1$. But then every vertex of the infinite walk S is distinct, which is impossible on a finite graph. Therefore, the survivor will be caught after a finite number of turns. □

While the bound $z_2(G) \leq \gamma_2^{\mathsf{c}}(G)$ holds for any graph G, the ratio $\frac{\gamma_2^{\mathsf{c}}(G)}{z_2(G)}$ can be arbitrarily large.

Theorem 4.8. *For all $k \geq 2$, there exists a connected graph G_k such that $\gamma_2(G_k) = 2 = z_2(G_k)$ and $\gamma_2^{\mathsf{c}}(G_k) = 2k$.*

Proof. Construct the graph G_k as follows: take k disjoint copies of C_4, each with vertices $\{u_{i,1}, u_{i,2}, u_{i,3}, u_{i,4}\}$ for $1 \leq i \leq k$, then add $2(k-1)$ edges

$$u_{2,1}u_{1,1},\ u_{2,2}u_{1,2},\ u_{3,1}u_{1,1},\ u_{3,2}u_{1,2},\ \ldots,\ u_{k,1}u_{1,1},\ u_{k,2}u_{1,2}.$$

Let $Z_k = \{u_{1,1}, u_{1,2}\}$ and let $Z_k^{\mathsf{c}} = \{u_{1,1}, u_{1,2}, u_{2,1}, u_{2,2}, \ldots, u_{k,1}, u_{k,2}\}$. We claim that Z_k is both a minimum 2-dominating set and a minimum 2-visibility zombie-win set of G_k, while Z_k^{c} is a minimum cycle-filling 2-dominating set of G_k.

It is straightforward to verify that $\gamma_2(G_k) = 2$, so $z_2(G_k) \geq 2$. To see that Z_k is 2-visibility zombie-win, suppose $s_0 = u_{i,j}$ for $1 \leq i \leq k$ and $1 \leq j \leq 4$. If $i = 1$ and $j \in \{1, 2\}$, then the survivor will be caught by the zombies' next move. If $i \geq 2$ and $j = 3$, then the zombie on $u_{1,2}$ will move to $u_{i,2}$ and the zombie on $u_{1,1}$ will remain inactive. From here, the survivor's only option is to move to $u_{i,4} \in N_2(u_{1,1})$, activating the second zombie. The two zombies can now move to trap the survivor on the 4-cycle $H_i = (u_{i,1}, u_{i,2}, u_{i,3}, u_{i,4}, u_{i,1})$. A similar argument works if $i \geq 2$ and $j = 4$.

The cycles H_i for $1 \leq i \leq k$ are pairwise vertex-disjoint in G_k, so $\gamma_2^{\mathsf{c}}(G) \geq 2k$. The set Z_k^{c} is cycle-filling since every cycle C in G_k contains both $u_{i,1}$

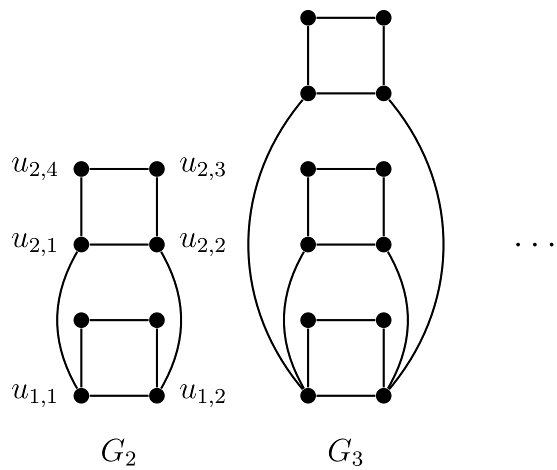

Figure 9: The first two graphs G_2 and G_3 defined in Theorem 4.8.

and $u_{i,2}$ for some i. It is also clear that Z_k^{c} is a 2-dominating set. Hence $\gamma_2^{\mathsf{c}}(G) = 2k = |Z_k^{\mathsf{c}}|$. □

4.3 Safe dominating sets

Let Z be a k-dominating set of a connected graph G. For each $z \in Z$, define the subgraph $H_z = G[V(G) \setminus N_k[Z \setminus \{z\}]]$ of G. We say Z is *safe* if, for some $z \in Z$, H_z is survivor-safe (in the full visibility game) with respect to z. If Z is not safe, we say it is *unsafe*.

Theorem 4.9. *If Z is a safe k-dominating set of a graph G, then Z is k-visibility survivor-win.*

Proof. The subgraph H_z is survivor-safe with respect to z, so the survivor has a strategy to evade the zombie starting on z indefinitely while staying in the subgraph H_z. Furthermore, $V(H_z) \cap N_k[w] = \emptyset$ for every $w \in Z \setminus \{z\}$, so every other zombie will remain inactive. It follows that the survivor's strategy to evade the zombie starting on z is also a winning strategy for the k-visibility game in G. Thus Z is a k-visibility survivor-win set. □

Note that if the survivor starts on a vertex $s_0 \in V(H_z)$, then $s_0 \in N_k[z]$ since Z is a k-dominating set of G. It follows that the survivor will immediately attract the attention the zombie starting on z. Also note that it is possible to have $z \notin V(H_z)$. This can happen if there exists some $w \in Z \setminus \{z\}$ such that $z \in N_k[w]$.

Let $\gamma_k^{\mathsf{u}}(G)$ be the size of a minimum unsafe k-dominating set of G. We are now in a position to improve the trivial lower bound on $z_k(G)$ from Theorem 4.1.

Corollary 4.10. *If G is a graph and $k \geq 1$, then $\gamma_k(G) \leq \gamma_k^{\mathsf{u}}(G) \leq z_k(G)$.*

Proof. The inequality $\gamma_k(G) \leq \gamma_k^{\mathsf{u}}(G)$ is obvious. It remains to show that $\gamma_k^{\mathsf{u}}(G) \leq z_k(G)$. This follows from Theorem 4.9, since every k-visibility zombie-win set $Z \subseteq V(G)$ is an unsafe k-dominating set. □

We say that a graph G is *strongly 2-zombie-win* if $z(G) = 2$ and two zombies can win Zombies and Survivors starting on any $u, v \in V(G)$. Note that $u = v$ is permitted.

Theorem 4.11. *If G is strongly 2-zombie-win, then for all $k \geq 1$, $z_k(G) = \gamma_k^{\mathsf{u}}(G)$.*

Proof. It suffices to show that if G is strongly 2-zombie-win, then any unsafe k-dominating set Z is k-visibility zombie-win. If we can show this, then $z_k(G) \leq \gamma_k^{\mathsf{u}}(G)$, and consequently $z_k(G) = \gamma_k^{\mathsf{u}}(G)$.

Let Z be an unsafe k-dominating set of G. Suppose for a contradiction that the survivor has a winning strategy against zombies starting in Z. If the survivor starts in $N_k[z] \cap N_k[w]$ for some $z, w \in Z$, then they will be caught by the zombies starting on z and w since G is strongly 2-zombie-win. Therefore, they must start in H_z for some $z \in Z$. If the survivor remains in H_z, they will lose since H_z is unsafe. It follows that the survivor must eventually leave H_z, at which point they will enter $N_k[w]$ for some $w \neq z$. From this point, they will lose since two zombies are now active. Thus Z is k-visibility zombie-win. □

To conclude this paper, we will use safe k-dominating sets to study the k-visibility game on Cartesian products of graphs.

Lemma 4.12. *If G and H are any two graphs such that $|V(G)| \geq 2$ and $|V(H)| \geq 2$, then $c(G \,\square\, H) \geq 2$.*

Proof. If G or H is disconnected, then $G \,\square\, H$ is disconnected (Corollary 5.3 in [13]), so $c(G \,\square\, H) \geq 2$. Otherwise, choose any edge $gg' \in E(G)$ and any edge $hh' \in E(H)$ such that $g \neq g'$ and $h \neq h'$. This gives us a cycle

$$C = ((g,h),(g',h),(g',h'),(g,h'),(g,h))$$

in $G \,\square\, H$. We will use Lemma 3.10 to show that C is robber-safe. Condition (a) of Lemma 3.10 is satisfied since $(g,h)(g',h') \notin E(G \,\square\, H)$ and

$(g',h)(g,h') \notin E(G \square H)$. To see that condition (b) holds, suppose for a contradiction that $(u,v) \notin V(C)$ is adjacent to three consecutive vertices of C, say (g,h), (g',h) and (g',h'). Then either $u = g'$ or $v = h$, but not both. If $u = g'$, then $(g',v)(g,h) \in E(G \square H)$, so $g = g'$. If $v = h$, then $(u,h)(g',h') \in E(G \square H)$, so $h = h'$. Both cases give a contradiction, so condition (b) holds. It follows that the cycle C is robber-safe in $G \square H$. $\square$

Theorem 4.13. *If G and H are strongly zombie-win graphs such that $|V(G)| \geq 2$ and $|V(H)| \geq 2$, then $G \square H$ is strongly 2-zombie-win.*

Proof. This is a special case of Theorem 2 from [16], which states that for any graphs G and H, $z(G \square H) \leq z(G) + z(H)$ (Proposition 5.1 of [13] gives a proof of Lemma 1 of [16]). It follows from [16] that if G and H are both zombie-win graphs and $u \in Z_0(G)$ and $v \in Z_0(H)$, then for any $x \in V(G)$ and $y \in V(H)$, two zombies can win starting from the vertices (u,y) and (x,v) in $G \square H$. We are assuming G and H are both strongly zombie-win, meaning that $Z_0(G) = V(G)$ and $Z_0(H) = V(H)$, so two zombies can win starting from any two vertices in $G \square H$. In particular, $z(G \square H) \leq 2$. Applying Lemma 4.12, $z(G \square H) \geq c(G \square H) \geq 2$. Thus $G \square H$ is strongly 2-zombie-win. $\square$

Corollary 4.14. *If G and H are strongly zombie-win graphs, then for all $k \geq 1$, $z_k(G \square H) = \gamma_k^u(G \square H)$.*

Proof. This follows from Theorem 4.11 and Theorem 4.13. $\square$

In the special case where G and H are both trees, we can say even more about $z_k(G \square H)$.

Given a graph G, $u \in V(G)$ and $k \geq 1$, let $S_k(u) = \{v \in V(G) \mid d(u,v) = k\}$. Recall that a set of vertices $X \subseteq V(G)$ is *independent* if there are no edges $uv \in E(G)$ such that $u \in X$, $v \in X$ and $u \neq v$.

Lemma 4.15. *If G is a bipartite graph, then for all $u \in V(G)$ and all $k \geq 1$, $S_k(u)$ is an independent set.*

Proof. Let (A,B) be a bipartition of G. Let $u \in V(G)$ and suppose without loss of generality that $u \in A$. Then $S_1(u) \subseteq B$, so $S_1(u)$ is independent since B is independent. Next, $S_2(u) \subseteq A$, so $S_2(u)$ is independent since A is independent. By induction on k, $S_k(u)$ is independent for all $k \geq 1$. $\square$

It is worth mentioning that the converse of Lemma 4.15 also holds, although we will not need to use this fact.

Recall that the *chromatic number* $\chi(G)$ of a graph G is the smallest number k such that $V(G)$ can be partitioned into k independent sets. It follows that a graph G is bipartite if and only if $\chi(G) \leq 2$.

Lemma 4.16. *If G and H are both bipartite, then $G \square H$ is bipartite.*

Proof. Theorem 26.1 of [13] states that for any two graphs G and H, $\chi(G \square H) = \max\{\chi(G), \chi(H)\}$. The graphs G and H are both bipartite, so $\chi(G) \leq 2$ and $\chi(H) \leq 2$ and therefore $\chi(G \square H) \leq 2$. □

Theorem 4.17. *If T_1 and T_2 are any two trees and $k \geq 1$, then $z_k(T_1 \square T_2) \leq 2\gamma_k(T_1 \square T_2)$.*

Proof. Let $G = T_1 \square T_2$ and let Z be any minimum k-dominating set of G. We will construct a new k-dominating set Z' such that $|Z'| \leq 2|Z| = 2\gamma_k(G)$ and Z' is k-visibility zombie-win. This is sufficient to conclude that $z_k(G) \leq |Z'| \leq 2\gamma_k(G)$.

Define Z' according to the following algorithm: Let $Z' := Z$. While there exists some $z \in Z'$ such that $N(z) \cap Z' = \emptyset$, choose any $w \in N(z)$ and set $Z' := Z' \cup \{w\}$. This is possible because G is connected, so $N(z) \neq \emptyset$ for all $z \in V(G)$. When this algorithm terminates, Z' has the property that every $z \in Z'$ is adjacent to at least one other vertex $w \in Z'$. At most $|Z|$ vertices need to be added, so $|Z'| \leq 2|Z|$.

Trees are strongly zombie-win, so G is strongly 2-zombie-win by Theorem 4.13. Then by Corollary 4.14 (and the proof of Theorem 4.11), if we can show that Z' is unsafe, it will follow that Z' is k-visibility zombie-win. To do this, we need to show that for all $z \in Z'$, the subgraph H_z is unsafe with respect to z. Let $w \in N(z) \cap Z'$. Then $V(H_z) \subseteq N_k[z] \setminus N_k[w]$. Furthermore, $N_k[z] \setminus N_k[w] \subseteq S_k(z)$, since if $u \in N_k[z] \setminus N_k[w]$ and $d(u, z) < k$, then $d(u, w) \leq d(u, z) + 1 \leq k$ by the triangle inequality, which is a contradiction. The graph G is bipartite by Lemma 4.16, so Lemma 4.15 implies that $S_k(z)$ is an independent set. Hence $V(H_z)$ is independent. This means that there are no edges in the subgraph H_z, so the survivor clearly cannot evade the zombie starting on z while remaining in H_z. □

Figure 10 illustrates the idea behind the proof of Theorem 4.17. The sets $N_3[z]$ and $N_3[w]$ are outlined by the red diamonds. The blue circled vertices lie in the set $N_3[z] \setminus N_3[w]$, which is independent since there are no diagonal edges in $P_8 \square P_7$.

If we allow more than one zombie to start on the same vertex, Theorem 4.17 becomes much easier to prove: simply place another zombie on each vertex in a minimum k-dominating set Z and apply Theorem 4.13. In fact, none of the results on safe subgraphs are necessary! This strategy still uses $2|Z|$ zombies in the worst case, but the number of distinct vertices is only $|Z|$. While in some sense this is a more efficient zombie strategy since it requires

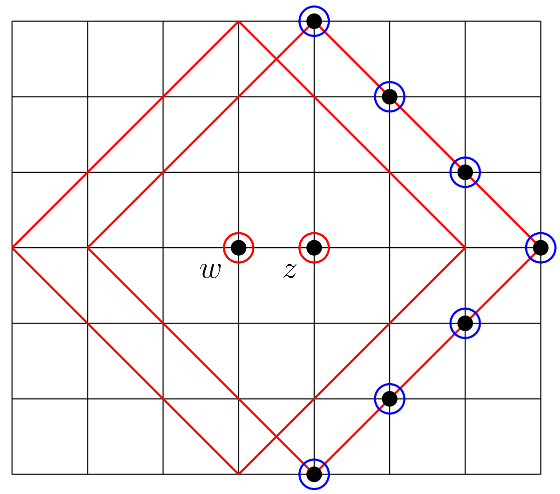

Figure 10: $N_3[z] \setminus N_3[w]$ in the graph $P_8 \,\square\, P_7$.

fewer distinct vertices, it is not clear if relaxing the rules in this way can ever reduce $z_k(T_1 \,\square\, T_2)$.

Theorem 4.17 can be used to obtain an upper bound on the k-visibility zombie number of the Cartesian product of paths, where we are again requiring that at most one zombie starts on any vertex.

Corollary 4.18. *If* $k \geq 1$ *and* $\min\{m, n\} > 2(2k^2 + 2k + 1)$, *then*

$$z_k(P_m \,\square\, P_n) \leq 2 \left\lfloor \frac{(m + 2k)(n + 2k)}{2k^2 + 2k + 1} \right\rfloor - 8.$$

Proof. Use Theorem 4.17 and the bound $\gamma_k(P_m \,\square\, P_n) \leq \left\lfloor \frac{(m+2k)(n+2k)}{2k^2+2k+1} \right\rfloor - 4$ for $k \geq 1$ and $\min\{m, n\} > 2(2k^2 + 2k + 1)$ from [12]. $\square$

Fix $k \geq 2$. Then

$$\limsup_{n \to \infty} \frac{z_k(P_n \,\square\, P_n)}{n^2} \leq \frac{2}{2k^2 + 2k + 1}.$$

For example, the fraction of vertices needed for the zombies to win the 2-visibility game on $P_n \,\square\, P_n$ for large n is at most $\frac{2}{13}$. If we allow two zombies to start on each vertex, then this fraction becomes $\frac{1}{13}$. For $k = 2$ we also have the cycle-filling bound from Theorem 4.7, but the new upper bound on $z_2(P_m \,\square\, P_n)$ from Corollary 4.18 seems to be much better in practice.

Finally, we give a theorem which generalizes the argument used in the proof of Theorem 4.17.

Theorem 4.19. *Let $k \geq 1$ and let Z be a k-dominating set of a connected graph G. If the subgraph H_z is a forest for every $z \in Z$, then Z is unsafe.*

Proof. We will show that every subgraph H_z is unsafe with respect to z. Let $s_0 \in V(H_z)$ and let $z_0 = z$. The survivor is restricted to moving in H_z, so they will never be able to leave the component of H_z containing s_0. Hence without loss of generality, we can assume H_z is a tree. The sequence $(d(s_i, z_i))_{i=0}^{\infty}$ is non-increasing for any survivor-play $S = (s_0, s_1, \ldots)$ and any zombie-play $(z_0, z_1, \ldots)$. Thus, it is enough to show that, no matter what the survivor does, there is a zombie strategy such that $d(s_{i+1}, z_{i+1}) \leq d(s_i, z_i) - 1$ for some $i \geq 0$, and that this zombie strategy can be repeated every time $d(s_i, z_i)$ decreases. Eventually, $d(s_j, z_j) = 0$ for some j, and the survivor will be caught.

Suppose for a contradiction that the survivor can maintain $d(s_k, z_k) = d(s_0, z_0)$ for all $k \geq 1$. Then the survivor wins, so S is an infinite walk on H_z. Fix some $i \geq 0$. Let $d = d(s_0, z_0)$ and let $H = H_z$. After the survivor moves to some $s_i \in V(H)$, the zombie moves to a vertex $z_{i+1} \in V(G)$ which lies on a s_i-z_i geodesic of the form $P = (s_i, \ldots, w, z_{i+1}, z_i)$. The path P has length d, since $d(s_i, z_i) = d$. By assumption, there must exist some $s_{i+1} \in N_H(s_i)$ such that $d(s_{i+1}, z_{i+1}) = d$. Observe that $s_{i+1} \neq s_i$, since $s_{i+1} = s_i$ would reduce the distance between the zombie and the survivor. The zombie can now move from z_{i+1} to $z_{i+2} = w$. This is because $(s_{i+1}, s_i, \ldots, w, z_{i+1})$ is an s_{i+1}-z_{i+1} path of length d, and is therefore an s_{i+1}-z_{i+1} geodesic. It follows that $s_{i+2} \neq s_i$, since $d(s_i, z_{i+2}) = d - 2$. But then S is an infinite walk on a tree such that for all $i \geq 0$, $s_{i+1} \neq s_i$ and $s_{i+2} \neq s_i$, which is impossible. □

5 Open questions

Despite our best efforts, Zombies and Survivors is still poorly understood. There are many basic questions about the game which are easy to state, but which seem to be difficult to answer using familiar tools from Cops and Robbers. We have compiled a list of open questions about Zombies and Survivors, organized into Sections 5.1 through 5.5. In Sections 5.6 and 5.7, we pose some questions about safe subgraphs and k-visibility Zombies and Survivors, respectively. Our hope is that by collecting these questions in one place, we will be able to provide a helpful resource for future research.

5.1 Characterizing zombie-win graphs

One of the most fascinating open questions about Zombies and Survivors is the following.

Question 5.1. *Is there a structural characterization of the class of zombie-win graphs? What about the class of strongly zombie-win graphs?*

Corollary 3.14 characterizes zombie-win graphs with maximum degree three, but this only answers Question 5.1 in a very special case.

A natural place to start is the converse of Theorem 2.5. Every zombie-win graph G is cop-win and every cop-win graph has at least one cop-win spanning tree (see [10]), but that does not necessarily mean that this cop-win spanning tree coincides with a spanning tree which results from a breadth-first search of G.

Question 5.2. *If G is a zombie-win graph and $z \in Z_0(G)$, is the spanning tree of G which results from a breadth-first search with z as the root necessarily a cop-win spanning tree?*

Motivated by Theorem 3.6, it would also be worth investigating the relationship between the structure of the convex subgraphs of G and Zombies and Survivors. The books [13] and [22] may contain some useful ideas.

5.2 New classes of zombie-win graphs

Question 5.3. *If H is a retract of G, is $z(H) \leq z(G)$? In particular, is the class of zombie-win graphs closed under retracts?*

Instead of considering retracts of all zombie-win graphs, we could start with the special case of retracts of strong products of paths. These turn out to be equivalent to *Helly graphs* [13, 15].

Question 5.4. *Are Helly graphs zombie-win?*

It was proven that *visibility graphs* are cop-win in [17].

Question 5.5. *Are visibility graphs zombie-win?*

Question 5.6. *Is the line graph of a zombie-win graph zombie-win?*

5.3 Zombie number of strong products

Theorem 2.8 of [3] leads to two natural questions about the zombie number of strong products of graphs.

Question 5.7. *Let G and H be graphs with $z(G) \geq 2$ or $z(H) \geq 2$. Is $z(G \boxtimes H) \leq z(G) + z(H) - 1$?*

Question 5.8. *If $z(G) = 1$ and $z(H) = k$, is $z(G \boxtimes H) = k$?*

5.4 Identifying zombie-win graphs at a vertex

Let G and H be zombie-win graphs and let $u \in Z_0(G)$ and $v \in Z_0(H)$. Define $G_u * H_v$ to be the graph obtained by identifying the vertices u and v. The main question we want to answer is the following:

Question 5.9. *Is $G_u * H_v$ zombie-win? In particular, is $u = v$ a zombie-win vertex in $G_u * H_v$?*

The answer to Question 5.9 is positive if G and H are both bridged, since then $G_u * H_v$ is bridged for all $u \in V(G)$ and all $v \in V(H)$. Also, if u is universal in G and v is universal in H, then $u = v$ is clearly universal in $G_u * H_v$.

A possible way to approach Question 5.9 would be to show that the survivor can never occupy a vertex that has been previously occupied by the zombie. This is true for the cop-win strategy and is referred to as the *no-backtrack property* [3, pg.34].

Question 5.10. *Does every zombie-win graph have the no-backtrack property? That is, is there always a winning zombie-play such that the survivor can never occupy any vertex that has been previously occupied by the zombie?*

5.5 Adding leaves and subdividing edges

The next two questions were posed in [1].

Question 5.11. *For every graph G, and for a graph G' obtained from G by successively adding vertices of degree 1, does it always hold that $z(G') = z(G)$?*

Question 5.12. *For any graph G, is there an integer k such that, for G'_k the graph obtained from G by subdividing all edges k times then adding the original edges, $z(G'_k) \geq z(G) + 1$?*

5.6 Safe subgraphs

Question 5.13. *Let H be a minimum robber-safe subgraph of G. Is H necessarily a retract of G? Furthermore, does H need to be a cycle?*

Recall that robber-safe subgraphs were only defined for a single cop, in which case a cycle is probably sufficient. If the definition is generalized to allow more than one cop, it seems unlikely that a cycle will be sufficient.

Question 5.14. *When exactly does a graph G with $c(G) \geq 2$ have a robber-safe cycle? Furthermore, can we give a characterization of the cycles which are robber-safe?*

Theorem 3.13 answers Question 5.14 for graphs with maximum degree three.

5.7 Limited visibility

Question 5.15. *If we apply our limited visibility model to Cops and Robbers, how does this differ from the model given in [7]? That is, consider cops that remain inactive until the robber first moves within distance k, after which point they can move like a normal cop.*

Question 5.16. *Can allowing more than one zombie to start on the same vertex ever reduce $z_k(G)$?*

Question 5.17. *Is it true that $z(G) \leq z_k(G)$ for all graphs G?*

If $k \geq \operatorname{diam}(G)$ and we also allow more than one zombie to start on the same vertex, then the k-visibility game becomes equivalent to the original game, which implies that $z(G) = z_k(G)$ (where we are defining $z_k(G)$ for the relaxed rules).

The upper bound given in Theorem 4.7 seems to be quite poor for most graphs. It seems likely that this bound can be tightened as follows:

Question 5.18. *Is Theorem 4.7 still true if the definition of cycle-filling 2-dominating sets is relaxed to only require two vertices on every induced cycle of length at least 4? Or even just isometric cycles?*

It also seems likely that the argument used to prove Theorem 4.7 can be generalized to all $k \geq 2$.

Question 5.19. *Can Theorem 4.7 be generalized to all $k \geq 2$? In particular, is $\min\{k, \lfloor \ell/2 \rfloor\}$ zombies on every cycle of length $\ell \geq 4$ good enough?*

Question 5.20. *If G is a graph such that $z(G)$ zombies can win from any starting position, is $z_k(G) \leq z(G)\gamma_k(G)$ for all $k \geq 1$?*

It might be necessary to require that $\delta(G) \geq z(G) - 1$ in Question 5.20 so that $z(G) - 1$ zombies can be placed around every vertex in a minimum k-dominating set. Also observe that if multiple zombies are allowed to start on the same vertex, then this question is easy to answer, since we can just place $z(G)$ zombies on every vertex in a minimum k-dominating set Z.

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