PMAT 4282 – Cryptography Assignment #4
Winter 2009

Instructions

- Answer each question completely; justify your answers.
- This assignment is due at: 3:00 pm on Thursday February 19th.

1. By $\mathbb{Z}_n^*$ we denote the set $\{a \in \mathbb{Z}_n \,|\, \mathrm{GCD}(a,n) = 1\}$. A *quadratic residue* modulo n is any element x of $\mathbb{Z}_n^*$ that is a square (i.e., $x = y^2$ for some $y \in \mathbb{Z}_n^*$). What are the quadratic residues for

   (a) $\mathbb{Z}_{23}^*$
   (b) $\mathbb{Z}_{26}^*$
   (c) $\mathbb{Z}_{27}^*$

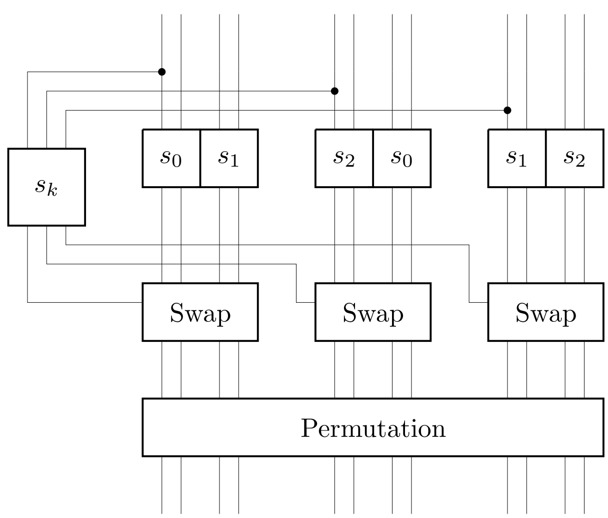
   In a few weeks we will see in class how to determine whether a given element $x \in \mathbb{Z}_p^*$ is a quadratic residue, where p is a prime. For instance, we'll learn how to answer the question: Is 789 a quadradic residue in $\mathbb{Z}_{5683}^*$?

2. Below is a schematic diagram for the function $f : \{0,1\}^{12} \to \{0,1\}^{12}$, which is used in each round of computation of an NDS-like cryptosystem in which $n = 12$ and $r = 16$.

   The specifications of this cryptosystem are such that s_0 is the identity function, s_1 is the complement function, s_2 swaps the first and second bits, and the permutation in the final step of f simply reverses the order of the 12 bits.

   You have gained access to an implementation of the encryption algorithm for this cryptosystem, using the key s_k that Alice and Bob have as their secret. This implementation is online on the course website.

   (a) How many possible choices are there for the key s_k?
   (b) Perform a chosen plaintext attack on the cryptosystem, and thereby determine s_k.



A *group* $(\mathcal{G}, \cdot)$ consists of a set $\mathcal{G}$ along with a binary operation $\cdot$, such that:

- $\mathcal{G}$ is closed under $\cdot$
- $\cdot$ is associative (i.e., $a \cdot (b \cdot c) = (a \cdot b) \cdot c$)
- there exists an identity with respect to $\cdot$
  (i.e., there exists an element $e \in \mathcal{G}$ such that $a \cdot e = e \cdot a = a$ for all $a \in \mathcal{G}$)
- each element has an inverse
  (i.e., for each $a \in \mathcal{G}$ there exists an element $b \in \mathcal{G}$ such that $a \cdot b = b \cdot a = e$)

Note that we often refer to the group as $\mathcal{G}$ rather than $(\mathcal{G}, \cdot)$. Notice also that $\cdot$ need not be commutative; a group in which $\cdot$ is commutative is called an *Abelian* group.

The notation a^n will be used to denote $\underbrace{a \cdot a \cdot a \cdots a}_{n\ a\text{'s}}$. The *order* of an element $a \in \mathcal{G}$ is the smallest positive integer t such that $a^t = e$. If a has order $|\mathcal{G}|$ then a is a *generator* of $\mathcal{G}$.

3. Find all generators of each of the following groups:
   (a) $\mathbb{Z}_9^*$
   (b) $\mathbb{Z}_{15}^*$
   (c) $\mathbb{Z}_{17}^*$
   (d) $\mathbb{Z}_{25}^*$

4. Prove the following statement:
   If the order of $a \in \mathbb{Z}_n^*$ is t and $a^s \equiv 1 \pmod{n}$, then t divides s.

5. (a) Suppose that α is a generator of $\mathbb{Z}_n^*$. Prove that α^k is also a generator of $\mathbb{Z}_n^*$ if and only if $\mathrm{GCD}(k, \phi(n)) = 1$, where ϕ is Euler's totient function.
   (b) Provided that $\mathbb{Z}_n^*$ has at least one generator, then how many generators does it have?
   (c) When p is prime, $\mathbb{Z}_p^*$ is known to have a generator. How many generators are there in:
      i. $\mathbb{Z}_{19}^*$
      ii. $\mathbb{Z}_{31}^*$
      iii. $\mathbb{Z}_{181}^*$
      iv. $\mathbb{Z}_{257}^*$
      v. $\mathbb{Z}_{2^t+1}^*$, where $(2^t + 1)$ is prime
      vi. $\mathbb{Z}_{2t+1}^*$, where t and $(2t + 1)$ are prime

6. Algorithm 4.9 on page 162 of the Handbook of Applied Cryptography, is as follows:

   Input: A multiplicative finite group $\mathcal{G}$ of order n, an element $a \in \mathcal{G}$, and the prime factorisation $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$.
   Output: The order t of a.

   1 Set $t \leftarrow n$.
   2 For $i = 1, 2, \cdots, k$ do each of the following:
      2.1 Set $t \leftarrow \dfrac{t}{p_i^{e_i}}$.

2.2 Compute $b \leftarrow a^t$.

2.3 While b is not the multiplicative identity do the following: compute $b \leftarrow b^{p_i}$ and set $t \leftarrow tp_i$.

3 Return t.

Use this algorithm to determine the order of each of the following elements:

(a) 5 in $\mathbb{Z}_7^*$

(b) 12 in $\mathbb{Z}_{25}^*$

(c) 3 in $\mathbb{Z}_{61}^*$

Which of these elements are generators?

7. Solve for x (i.e., find the smallest non-negative integer solution):

(a) $5^x \equiv 4 \pmod{37}$

(b) $6^x \equiv 16 \pmod{41}$

(c) $13^x \equiv 12 \pmod{197}$

(d) $55^x \equiv 444 \pmod{569}$