MATH 4341 – Combinatorial Design Theory Assignment #2
Winter 2020

Instructions

- Answer each question completely; justify your answers.
- This assignment is due at the start of class on Wednesday February 5th.

1. Prove that there is a $(7,7,4,4,2)$-BIBD and that it is unique up to isomorphism.

2. We define a *Latin square of order* n to be an $n \times n$ array in which each of the n^2 cells is filled with a symbol from a set S of cardinality n, such that each symbol occurs once in each row of the array and each symbol occurs once in each column of the array. Typically the set S is chosen to be $\{1,2,\ldots,n\}$. A Latin square is called symmetric if, for each i and j, the symbol in cell (i,j) is the same as the symbol in cell (j,i).

   (a) Construct an example of a Latin square of order 3 that is not symmetric.

   (b) Construct an example of a symmetric Latin square of order 3.

   (c) Construct an example of a Latin square of order 4 that is not symmetric.

   (d) Construct an example of a symmetric Latin square of order 4.

3. Suppose that A is a block of a (v,k,λ)-BIBD. For each $i \in \{0,1,\ldots,k\}$, let x_i denote the number of blocks other than A that intersect A in precisely i elements. Prove that $\sum_{i=0}^{k} ix_i = k(r-1)$ and also that $\sum_{i=0}^{k} i(i-1)x_i = k(k-1)(\lambda-1)$.

4. Suppose that a (v,k,λ)-BIBD is both a derived design and a residual design (of some other design(s)). Prove that $v = 2\lambda + 2$.

5. Let $(X,\mathcal{A})$ be a symmetric (v,k,λ)-BIBD. Prove that $k > \lambda$.

6. Let $(X,\mathcal{A})$ be a symmetric $(v,k,1)$-BIBD and let A_0 be any block of $\mathcal{A}$. Explain how the blocks of the residual design $Res(X,\mathcal{A},A_0)$ can be partitioned into resolution classes.

7. Another way of defining a projective plane is as a set X of points and a set $\mathcal{A}$ of subsets of X called lines such that the following three axioms hold:

   A1: Given any two points, there is exactly one line that contains both of them.

   A2: Given any two lines, there is exactly one point that is contained in both of them.

   A3: There is a subset of X consisting of 4 points, no three of which are collinear.

   Prove that under these axioms, every line of the design $(X,\mathcal{A})$ contains the same number of points.