MATH 4341 – Combinatorial Design Theory Winter 2018 — Assignment #3

Instructions

- Answer each question completely; justify your answers.
- This assignment is due at 17:00 on Thursday February 15th in Assignment Box #32.

1. Suppose $(X, \mathcal{A})$ is a STS(v) on point set $X = \{x_1, \ldots, x_v\}$ and let $\mathcal{F}_1, \ldots, \mathcal{F}_v$ be the v 1-factors of a 1-factorisation of K_{v+1}, where the vertex set Y of this K_{v+1} is disjoint from X. Consider the design with $X \cup Y$ as its point set and $\mathcal{B} = \mathcal{A} \cup \bigcup_{i=1}^{v} \{B \cup \{x_i\} : B \in \mathcal{F}_i\}$ as its block set.
   (a) Explain why $(X \cup Y, \mathcal{B})$ is a STS.
   (b) Use this construction to build a STS(15).
2. (a) Construct a Latin square L of order $m = 6$.
   (b) Let $C_{i,j}$ denote the cell whose location is row i column j of L, and let $L_{i,j}$ denote the symbol contained in cell $C_{i,j}$. Let $X = \{C_{i,j} : 1 \leqslant i \leqslant 6, 1 \leqslant j \leqslant 6\}$. For each cell $C_{i,j}$ define $B_{i,j} = \big(\{C_{s,j} : 1 \leqslant s \leqslant 6\} \cup \{C_{i,t} : 1 \leqslant t \leqslant 6\} \cup \{C_{s,t} : L_{s,t} = L_{i,j}, 1 \leqslant s \leqslant 6, 1 \leqslant t \leqslant 6\}\big) \setminus \{C_{i,j}\}$. Let $\mathcal{B} = \{B_{i,j} : 1 \leqslant i \leqslant 6, 1 \leqslant j \leqslant 6\}$.
      i. List the elements of $B_{1,2}$.
      ii. Verify that $(X, \mathcal{B})$ is a symmetric BIBD.
   (c) When the construction above is adapted for any Latin square of order $m \geqslant 2$, does it yield a symmetric BIBD? If yes, then express v, k and λ in terms of m.
3. A *biplane of order* n is a symmetric (v, k, λ)-BIBD for which $n = k - \lambda$ and $\lambda = 2$. Non-trivial biplanes of order n are currently known to exist only for $n \in \{1, 2, 3, 4, 7, 9, 11\}$.
   (a) Find a biplane of order 1.
   (b) Prove that no biplane of order 5 exists.
   (c) Prove that no biplane of order 8 exists.
   (d) Prove that no biplane of order 10 exists.
   (e) Show that the Bruck-Ryser-Chowla theorem does not preclude the existence of a biplane of order 14.
4. For which of the following parameters does the Bruck-Ryser-Chowla theorem preclude the existence of a (v, k, λ)-BIBD?
   (a) $(25, 9, 3)$
   (b) $(34, 12, 4)$
   (c) $(45, 12, 3)$
   (d) $(103, 18, 3)$
5. Find a $(15, 7, 3)$-difference set in $(\mathbb{Z}_{15}, +)$.
6. Verify that $\{0, 1, 7, 19, 23, 44, 47, 49\}$ is a $(57, 8, 1)$-difference set in $(\mathbb{Z}_{57}, +)$.
7. Find a $(31, 15, 7)$-difference set in $(\mathbb{Z}_{31}, +)$.