# MATH 4341 – Combinatorial Design Theory    Assignment #1
# Winter 2018

**Instructions**

- Answer each question completely; justify your answers.
- This assignment is due at 17:00 on Tuesday January 23rd in Assignment Box #32.

1. (a) What are the necessary conditions for the existence of a $(v, 4, 1)$-BIBD?
   (b) What is the smallest order $v$ for which the existence of a nontrivial $(v, 4, 1)$-BIBD cannot be excluded?

2. (a) Prove that every (6,3,2)-BIBD is simple.
   (b) Prove that all (6,3,2)-BIBDs are isomorphic.

3. (a) Consider the set $S = \{1, 2, 3, 4, 5, 6\}$. Find a partition of $S$ into subsets of size three, such that each subset is either of the form $\{x, y, z\}$ such that $x + y = z$ or of the form $\{x, y, z\}$ such that $x + y + z = 13$.
   (b) For each set $A = \{x, y, z\}$ from part (a), where $x < y < z$, let $A'$ be the set $\{0, x, x + y\}$. Let $\sigma$ be the permutation $(0\ 1\ 2 \cdots 12)$. For each set $A'$, list the sets $\sigma^i(A')$ for $i = 0, 1, 2, \dots, 12$.
   (c) Taken as blocks, what type of design do the sets from part (b) form?

4. Let $A_0$ be a block in a $(v, k, 1)$-BIBD, say $(X, \mathcal{A})$.
   (a) Find a formula for the number of blocks $A \in \mathcal{A}$ such that $|A \cap A_0| = 1$.
   (b) Use this formula to show that $b \geqslant k(r - 1) + 1$ if a $(v, k, 1)$-BIBD exists.
   (c) Using the facts that $vr = bk$ and $v = r(k - 1) + 1$, deduce that $(r - k)(r - 1)(k - 1) \geqslant 0$ and hence $r \geqslant k$, which implies Fisher's Inequality.

5. Let $A_0$ be a block in a $(v, k, 1)$-BIBD, say $(X, \mathcal{A})$. Let $x \in X \setminus A_0$ and show that there are at least $k$ blocks that contain $x$ and intersect $A_0$. From this, deduce that $r \geqslant k$, which implies Fisher's Inequality.

6. We define a *Latin square of order n* to be an $n \times n$ array in which each of the $n^2$ cells is filled with a symbol from a set $S$ of cardinality $n$, such that each symbol occurs once in each row of the array and each symbol occurs once in each column of the array. Typically the set $S$ is chosen to be $\{1, 2, \dots, n\}$. A Latin square is called symmetric if, for each $i$ and $j$, the symbol in cell $(i, j)$ is the same as the symbol in cell $(j, i)$.
   (a) Construct an example of a Latin square of order 3 that is not symmetric.
   (b) Construct an example of a symmetric Latin square of order 3.
   (c) Construct an example of a Latin square of order 4 that is not symmetric.
   (d) Construct an example of a symmetric Latin square of order 4.