MATH 2320 – Discrete Mathematics **Assignment #5**
Winter 2021

Instructions

- Answer each question completely; justify your answers.
- This assignment is due at 23:59 (Newfoundland time) on Tuesday March 2nd.
- Submit your assignment via the D2L shell for the course.

1. Let $S = \{1, 2, 3, 4\}$ and $T = \{a, b, c, d\}$. Define functions $f : S \to T$ and $g : S \to S$ such that $f = \{(1, a), (2, b), (3, d), (4, c)\}$ and $g = \{(1, 3), (2, 3), (3, 2), (4, 4)\}$.
   (a) Either find $f \circ f$ or explain why it does not exist.
   (b) Either find $f \circ g$ or explain why it does not exist.
   (c) Either find $g \circ f$ or explain why it does not exist.
   (d) Either find $g \circ g$ or explain why it does not exist.
   (e) Is f surjective? Is f injective? Find f^{-1} if it exists.
   (f) Is g surjective? Is g injective? Find g^{-1} if it exists.

2. Let $A = \{x \in \mathbb{R} \mid x \neq 2\}$ and define $f : A \to \mathbb{R}$ by $f(x) = \frac{12x}{3x-6}$.
   (a) Show that f is injective.
   (b) Is f is surjective?
   (c) What is the range of f?
   (d) Let D be the range of f and define $g : A \to D$ such that $g : x \mapsto f(x)$. What is g^{-1}?

3. Let $f : A \to B$ and $g : B \to C$ be functions. Prove that if $g \circ f$ is injective then f is injective.

4. Let $A = \mathbb{R} \setminus \{0, 1\}$. Define functions $f : A \to A$, $g : A \to A$ and $h : A \to A$ by $f(x) = \frac{1}{x}$, $g(x) = \frac{x-1}{x}$ and $h(x) = \frac{1}{1-x}$.
   (a) For each function:
      i. Determine whether the function is bijective.
      ii. If it is bijective, find its inverse.
   (b) What is $f \circ f$?
   (c) What is $f \circ g$?
   (d) What is $f \circ h$?
   (e) What is $f \circ g \circ h$?

5. Let $f : A \to B$ and $g : B \to C$ be functions. Prove that if $g \circ f$ is surjective and g is injective then f is surjective.

6. Define $f : (-1, 1) \to \mathbb{R}$ such that $f(x) = \frac{x}{1-x^2}$.

   (a) Prove that f is bijective.

   (b) Prove that there is a one-to-one correspondence between $(-1, 1)$ and $(0, 1)$.

   (c) Use (a) and (b) to deduce that $\mathbb{R}$ is uncountable.

7. Prove that each of the following sets is countable by listing its elements in a systematic way with a first element, second element, etc. List at least the first ten elements of each set.

   (a) All integral powers of 2 (*i.e.*, every number of the form 2^n where n is an integer).

   (b) $\mathbb{N} \times \{0, 1, 2\}$

8. Determine, with justification, whether each of the following sets is finite, countably infinite, or uncountable.

   (a) $\{x \in \mathbb{N} \mid -1 < x < 1\}$

   (b) $\{x \in \mathbb{Z} \mid -1 < x < 1\}$

   (c) $\{x \in \mathbb{Q} \mid -1 < x < 1\}$

   (d) $\{x \in \mathbb{R} \mid -1 < x < 1\}$

   (e) $\{a + bi \in \mathbb{C} \mid a, b \in \mathbb{N}\}$

   (f) $\{(a, b) \in \mathbb{Q}^2 \mid a - b = 3\}$

   (g) $\{(a, b) \in \mathbb{R}^2 \mid b = \sqrt{1 - a^2}\}$

9. Explain why $|\mathbb{Z}^2| = \aleph_0$