MATH 2320 – Discrete Mathematics Assignment #6
Winter 2020

Instructions

- Answer each question completely; justify your answers.
- This assignment is due at 17:00 on Thursday March 19th in Assignment Box #43.

1. Let $S = \{1, 2, 3, 4\}$ and $T = \{a, b, c, d\}$. Define functions $f : S \to T$ and $g : S \to S$ such that $f = \{(1, d), (2, b), (3, d), (4, a)\}$ and $g = \{(1, 3), (2, 2), (3, 1), (4, 4)\}$.
   (a) Either find $f \circ f$ or explain why it does not exist.
   (b) Either find $f \circ g$ or explain why it does not exist.
   (c) Either find $g \circ f$ or explain why it does not exist.
   (d) Either find $g \circ g$ or explain why it does not exist.
   (e) Is f surjective? Is f injective? Find f^{-1} if it exists.
   (f) Is g surjective? Is g injective? Find g^{-1} if it exists.
2. Let $A = \{x \in \mathbb{R} \mid x \neq \frac{1}{3}\}$ and define $f : A \to \mathbb{R}$ by $f(x) = \frac{x}{3x-1}$.
   (a) Show that f is injective.
   (b) Is f is surjective?
   (c) What is the range of f?
   (d) Let D be the range of f and define $g : A \to D$ such that $g : x \mapsto f(x)$. What is g^{-1}?
3. Let $A = \mathbb{R} \setminus \{0, 1\}$. Define functions $f : A \to A$, $g : A \to A$ and $h : A \to A$ by $f(x) = \frac{1}{x}$, $g(x) = \frac{x-1}{x}$ and $h(x) = \frac{1}{1-x}$.
   (a) For each function:
      i. Determine whether the function is bijective.
      ii. If it is bijective, find its inverse.
   (b) What is $f \circ f$?
   (c) What is $f \circ g$?
   (d) What is $f \circ h$?
   (e) What is $f \circ g \circ h$?
4. Let $f : A \to B$ and $g : B \to C$ be functions. Prove that if $g \circ f$ is surjective and g is injective then f is surjective.
5. Show that there is a bijection between the sets $(-20, 5)$ and $(2, 3)$.
6. Define $f : (-1, 1) \to \mathbb{R}$ such that $f(x) = \frac{x}{1-x^2}$.
   (a) Prove that f is bijective.
   (b) Prove that there is a one-to-one correspondence between $(-1, 1)$ and $(0, 1)$.
   (c) Deduce that $\mathbb{R}$ is uncountable.

(over)

7. Prove that each of the following sets is countable by listing its elements in a systematic way with a first element, second element, etc. List at least the first ten elements of each set.

   (a) All integral powers of 2 (*i.e.*, every number of the form 2^n where n is an integer).

   (b) Those natural numbers that leave a remainder of 1 when divided by 3.

   (c) $\mathbb{N} \times \{0, 1, 2\}$

8. Determine, with justification, whether each of the following sets is finite, countably infinite, or uncountable.

   (a) $\{x \in \mathbb{N} \mid 1 < x < 2\}$

   (b) $\{x \in \mathbb{Q} \mid 1 < x < 2\}$

   (c) $\{x \in \mathbb{R} \mid 1 < x < 2\}$

   (d) $\{a + bi \in \mathbb{C} \mid a, b \in \mathbb{N}\}$

   (e) $\{(a, b) \in \mathbb{Q}^2 \mid a + b = 1\}$

   (f) $\{(a, b) \in \mathbb{R}^2 \mid b = \sqrt{1 - a^2}\}$