MATH 2090 — Final Examination, Fall 2014

Solutions.

1. (a) $A(t) = 1500(1+it)$

So at $t=2$ $\quad A(2) = 1500(1+2i)$
$$= 1621$$
$$\therefore i = \tfrac{1}{2}\left[\frac{1621}{1500} - 1\right]$$
$$\doteqdot 0.04033$$
i.e. approximately 4.03%.

(b) $A(t) = 1500(1+i)^t$

At $t=2$, $\quad A(2) = 1500(1+i)^2$
$$= 1621$$
$$\therefore i = \sqrt{\frac{1621}{1500}} - 1$$
$$\doteqdot 0.039551$$
i.e. approximately 3.96%.

(c) $A(t) = 1500e^{\delta t}$

At $t=2$, $A(2) = 1500e^{2\delta}$
$$= 1621$$
$$\therefore e^{2\delta} = \frac{1621}{1500}$$
$$\therefore \delta = \tfrac{1}{2}\ln\left(\frac{1621}{1500}\right)$$
$$\doteqdot 0.038789$$
i.e. is approximately 3.88%.

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2. $1+i_1 = \dfrac{157500}{150000}$, $\quad 1+i_2 = \dfrac{137940}{157500-25000}$
$$= \frac{137940}{132500}$$

$$1+i_3 = \frac{157528.20}{152940}$$

Then time weighted rate of return, i_t, is given by
$$1+i_t = \frac{157500}{150000}\cdot\frac{137940}{132500}\cdot\frac{157528.20}{152940}$$

i.e. $1+i_t \doteqdot 1.1259027$

$\therefore \quad i_t = 0.1259027$

i.e. the time-weighted return is approximately 12.59%

(b) Let Dollar-weighted return be i (i.e. i is the effective annual interest rate)

The equation of value is

$$157528.20 = 150\,000(1+i)^{2\frac{1}{2}} - 25000(1+i)^2 + 15000(1+i)^1$$

Write $x = \sqrt{1+i}$, then $(1+i)^{2\frac{1}{2}} = x^5$, $(1+i)^2 = x^4$, $1+i = x^2$

So

$$\frac{157528.20}{150000} = x^5 - \frac{25000}{150000}x^4 + \frac{15000}{150000}x^2$$

i.e. $\quad x^5 - \frac{1}{6}x^4 + \frac{1}{10}x^2 - c = 0$

where $c = \dfrac{157228.20}{150000} \doteqdot 1.050188$

From the Scilab output (and using the fact that x is real and $x>1$) we have

$$x = 1.0245374$$

$\therefore \quad 1+i = x^2 = (1.0245374)^2 \doteqdot 1.049677$

i.e. $i \doteqdot 0.049677$

i.e. the return is approximately 4.97%.

3. (a) The annual rate is 4.2% we need this as an effective monthly rate i :- $(1+i)^{12} = 1+0.042$

so $i = (1.042)^{\frac{1}{12}} - 1$.

Let the monthly payments be X, then the total accumulated value is given by

$$1500\,000 = X\, s_{\overline{40\times12}|i} = X\frac{\left[(1.042)^{40} - 1\right]}{(1.042)^{\frac{1}{12}} - 1}$$

$$\therefore X = \frac{1500000\left[(1\cdot042)^{\frac{1}{12}}-1\right]}{(1\cdot042)^{40}-1}$$

$$\doteqdot 1231\cdot1008$$

i.e. monthly payments are $\underline{\$1,231\cdot10(08)}$

(b) The annuity, $1500000 = X_2\, a_{\overline{30}|j}$, $j = 0\cdot028$

Where j is the ~~monthly~~ annual rate of 2·8% and X_2 is annual payment. Then

$$X_2 = \frac{1500000\times0\cdot028}{1-(1\cdot025)^{-30}}$$

$$\doteqdot 74563\cdot735$$

i.e. Sue's annual annuity payments are approximately, $\underline{\$74,563\cdot74}$

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4. (a) First need the monthly rate, i; $(1+i)^{12} = 1+\frac{6\cdot2}{100}$

i.e. $i = (1\cdot062)^{\frac{1}{12}}-1$

Equation of present value

$$245000 = X\, a_{\overline{12\times25}|i} \quad ; X \text{ monthly payments}$$

$$a_{\overline{12\times25}|i} = \frac{1-(1\cdot062)^{-12\times25}}{(1\cdot062)^{\frac{1}{12}}-1}$$

so

$$X = \frac{245,000\left[(1\cdot062)^{\frac{1}{12}}-1\right]}{1-(1\cdot062)^{-12\times25}}$$

$$\doteqdot 1583\cdot1086$$

i.e. Edwardo's monthly payments are $\underline{\$1,583\cdot10(86)}$

(b) By prospective method the outstanding principal

$$P_{25,20} = X\, a_{\overline{12\times(25-20)}|}$$

$$= 1583\cdot1086\times\left[\frac{1-(1\cdot062)^{-5}}{(1\cdot062)^{\frac{1}{12}}-1}\right]$$

$$\doteqdot 81827\cdot151$$

i.e. outstanding principal is $81,827.15(1)

(c) The interest in the first payment of the 21st year is $i\,P_{25,20} \doteq 81827.151 \times \left[(1.062)^{\frac{1}{12}}-1\right]$

$$\doteq 411.21516$$

i.e. $411.21(516)

The remaining portion, $X - i\,P_{25,20}$, goes toward the principal $X - i\,P_{25,20} \doteq 1171.8934$

i.e. $1,171.89(34).

5. (a) Annual interest on loan is

$$24000 \times \frac{5.7}{100} = 1368$$

i.e. $1,368.

(b) Sinking fund:- X dollars per month for 10 years with monthly rate $i = (1.039)^{\frac{1}{12}} - 1$, annual rate is 3.9%

Then,

$$24000 = X\, s_{\overline{12\times10}|i}$$

$$= X\left[\frac{(1.039)^{10}-1}{(1.039)^{\frac{1}{12}}-1}\right]$$

$$\therefore\ X \doteq 164.4369$$

i.e. monthly sinking fund payments are $164.43(69).

(c) Outlay is:-

$$12 \times 164.4369 + 1368 = 3341.2428$$

i.e. $3,341.24(28)

For the ~~[illegible]~~ equivalent amortized loan we have 10 annual payments of $3,341.24(28) to repay the \$24,000 loan ~~so~~

$$24000 = 3341.2425 \times a_{\overline{10}|j}\ ,\ j \text{ the}$$

annual interest rate for the loan.

$$\therefore \quad a_{\overline{10}|j} = \frac{1-(1+j)^{-10}}{j}$$

$$= \frac{24000}{3341.2425}$$

$$\text{ie.} \quad \frac{1-(1+j)^{-10}}{j} = \frac{24000}{3341.2425}$$

$$\text{ie.} \quad \frac{1-x^{-10}}{x-1} = \frac{24000}{3341.2425}$$

$$\text{or} \quad \frac{x-1}{1-x^{-10}} = \frac{3341.2425}{24000}$$

$$\text{ie.} \quad \frac{x^{11}-x^{10}}{x^{10}-1} = \alpha, \text{ where } \alpha = \frac{3341.2425}{24000} \doteqdot 0.1392184$$

$$\text{ie.} \quad x^{11} - (1+\alpha)x^{10} + \alpha = 0,$$

[Can then use SciLab to get $X = 1.065176$
so interest rate is 6.52%.]

6. $$\text{Price} = 1000 \times \frac{6.4}{2 \times 100}\, a_{\overline{2 \times 20}|i} + (1+i)^{-2\times 20} \times 1000$$

where i is the effective ~~rate~~ 6 monthly rate :- $i = 0.041$

$$\therefore \quad \text{Price} = 32 \times \frac{[1-(1.041)^{-2\times 20}]}{0.041} + (1.041)^{-2\times 20} \times 1000$$

$$\doteqdot 824.4855$$

ie. bond price is \$824.48(55)

Check Coupon < yield bond at discount
Yes!

6. ctd

(b) Book value end of 10th year:–

$$B_{2\times10} = 32 \times \frac{\left[1-(1.041)^{-2(20-10)}\right]}{0.041} + (1.041)^{-2(20-10)} \times 1000$$

$$\doteqdot 878.76303$$

i.e. \$878.76 (2e3)

(c)

	Fr Coupon	iBk Interest	iBk − Fr Prin. Adjustment	Bk Book Value
End of 10th year:	...	...	...	878.7630
	32	36.029284	4.02984	882.79231

7. yield, 3% < 3.2 % coupon

So bond at premium price decreases to face value. So at earliest date $25\frac{1}{2}$ year (end)

$$\text{Price} = 10000 \times 0.032\, a_{\overline{51|}i=0.03} + (1+i)^{-51} \times 10000$$

$$\doteqdot 10519.02$$

i.e. \$10,519.02.

8. (a) $V(t) = 100\,S(t) + 10\,B(t)$

So, $V(0) = 100 \times 25 + 10 \times 100$
$= 3500$

$$V(1) = \begin{cases} 100 \times 30 + 10\times(1+0.042)\times100, & \text{prob } 0.3 \\ 100 \times 15 + 10\times(1.042)\times100, & \text{prob } 0.7 \end{cases}$$

$$= \begin{cases} 4{,}042, & \text{prob } 0.3 \\ 2{,}542, & \text{prob } 0.7 \end{cases}$$

(b)

$$i_v = \frac{V(1)-V(0)}{V(0)} = \begin{cases} \frac{542}{3{,}500}, & \text{prob } 0.3 \\ -\frac{958}{3500}, & \text{prob } 0.7 \end{cases} = \begin{cases} 0.1549, & \text{prob } .3 \\ -0.2737, & \text{prob } .7 \end{cases}$$

(c) $E(i_v) = 0.154851 \times 0.3 + (-0.2737143) \times 0.7$

$= -0.1451429$

i.e. expect a loss of 14.51%.

Risk as measured by standard deviation:-

$$\sigma_{i_v} = \sqrt{\{(0.154851 + 0.1451429)^2 \times 0.3 + (-0.2737143 + 0.1451429)^2 \times 0.7\}}$$

$\doteqdot 0.1963933$

i.e. approximately 19.64%.

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9. (a) Call option strike price \$25.

$$C(0) = \left(\frac{30-25}{30-15}\right) \times \left[25 - \frac{15}{1.04^2}\right]$$

$\doteqdot 3.53487$

i.e. cost of call option is \$3.53 (487).

(b) $$C(1) = \begin{cases} 30-25, & \text{prob. } 0.3 \\ 0, & \text{prob } 0.7 \end{cases}$$

Yield $$i_c = \frac{C(1) - C(0)}{C(0)} = \begin{cases} \frac{5 - 3.53487}{3.53487}, & 0.3 \\ -1, & \text{prob. } 0.7. \end{cases}$$

$$= \begin{cases} 0.41448, & 0.3 \\ -1, & 0.7 \end{cases}$$

(c) Expected value, $E(i_v) = 0.41448 \times 0.3 - 1 \times 0.7$

$\doteqdot -0.57566$

i.e. expected loss of 57.57%

Risk: $\sigma_{K_V} = \sqrt{\{(0.41448 + 0.57566)^2 \times 0.3 + (-1 + 0.57566)^2 \times 0.7\}}$

$\doteqdot 0.64820$

i.e. risk is at 64.820 %.

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10. (a) $P(0) = \left(\frac{25-15}{30-15}\right)\left[\frac{30}{1.042} - 25\right] \doteqdot 2.5271913$

Put option price is (approx) \$2.53.

(b) $V(t) = x\,S(t) + \frac{(1000 - 25x)}{P(0)} P(t)$

Profit / Loss :- $V(1) - V(0) = x\,S(1) + \frac{(1000 - 25x)P(1)}{P(0)} - 1000$

as $P(1) = \begin{cases} 0, & \text{prob. } 0.3 \\ 25 - 15 = 10, & \text{prob } 0.7 \end{cases}$

We have

$$V(1) - V(0) = \begin{cases} 30x - 1000, & \text{prob } 0.3 \\ 15x + \frac{(1000 - 25x)}{P(0)} \times 10, & \text{prob } 0.7 \end{cases}$$

No loss :- $30x - 1000 \geq 0$. take $x = \frac{100}{3}$

then $V(1) - V(0) = \begin{cases} 0, & \text{prob } 0.3 \\ 159.4930, & \text{prob } 0.7. \end{cases}$

Optimal portfolio is $\frac{100}{3}$ shares and $\frac{1000 - 25x}{P(0)} \doteqdot 65.950$ put options.