MATH2090 – Mathematics of Finance

Instructor: Dr. C. Radford

Final Examination, FALL 2014

This is a two hour examination.
You may attempt all questions. The mark value of each question is indicated.
The total mark value of this test is 100.

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YOU MUST SHOW ADEQUATE WORKING WITH YOUR ANSWERS.

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FORMULA SHEET ATTACHED.

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1. A bank account had a balance of \$1,500 on January 1, 2012 and \$1,621 on January 1, 2014; there were no other deposits/withdrawals during this period.

   (a) Assuming simple interest what is the annual interest rate?

   (b) What is the annual effective compound interest rate for the account?

   (c) Assuming continuous compounding find the force of interest for this account.

   [7 Marks]

2. An investment account with *Beaver Investments* shows the following activity:

   – January 1, 2011, a balance of \$150,000.
   – June 30, 2011, a balance of \$157,500.
   – July 1, 2011, a withdrawal of \$25,000.
   – June 30, 2012, a balance of \$137,940.
   – July 1, 2012, a deposit of \$15,000.
   – June 30, 2013 a balance of \$157,528.20.

   (a) Find the *time-weighted* rate of return for the entire period covered in the account summary above.

   (b) First show that the equation for the *dollar-weighted* rate of return (i.e. the effective annual compound interest rate, i) for the entire period can be written as,

$$X^5 - \frac{1}{6}X^4 + \frac{1}{10}X^2 - c = 0, \text{ where } X = \sqrt{1+i} \text{ and } c \approx 1.050188$$

[Question 2 continued over page.]

Next, using SciLab we have

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-->roots([1 -1/6 0 1/10 0 -1.050188])
ans =
- 0.7917680 + 0.5749525i
- 0.7917680 - 0.5749525i
0.3628327 + 0.9689806i
0.3628327 - 0.9689806i
1.0245374
```

use this information to find the dollar-weighted rate of return, i.
[Recall: The 'i' in the SciLab output is $\sqrt{-1}$.] [12 Marks]

3. Sue wishes to save \$1,500,000 for retirement over the next 40 years of her working life, she will do this by putting equal monthly payments into a savings account with an annual interest rate of 4.2%. At retirement Sue will use the \$1,500,000 to buy an annuity, with an annual interest rate of 2.8%, which is to make 30 equal annual payments.

   (a) Find the value of the monthly payments Sue must make during her 40 years of work.

   (b) What is the annual payment Sue will receive for 30 years in the retirement?

   [8 Marks]

4. Eduardo takes out a home mortgage of \$245,000 with his bank at an annual interest rate of 6.2%, over a term of 25 years. Eduardo repays the mortgage with equal payments made at the end of each month.

   (a) What are Eduardo's monthly payments on his mortgage?

   (b) What is the outstanding principal on Eduardo's loan after the last payment of the 20th year?

   (c) What portion of the first payment of the 21st year is interest and what portion is principal?

   [12 Marks]

5. Consider a loan of \$24,000 at an annual interest rate of 5.7% where interest is paid annually and a sinking fund is established to repay the principal; the sinking fund is to be built by equal payments made at the end of each month for 10 years. If the sinking fund account has an effective annual interest rate of 3.9% find the following.

   (a) The annual interest payment.

   (b) The monthly sinking fund payment.

   (c) The annual outlay for the loan. Hence show that the effective annual interest rate, i, for the equivalent amortized loan satisfies the equation

$$X^{11} - (1+\alpha)X^{10} + \alpha = 0, \text{ where } \alpha \approx 0.1392184 \text{ and } X = 1 + i.$$

   [10 Marks]

6. Consider a 20 year par-valued bond with a face value of \$1,000, a coupon of 6.4% convertible semi-annually and a semi-annual yield rate of 4.1%.

[Question 6 continued over page.]

(a) What is the price of this bond?

(b) What is the book value of the bond at the end of its 10th year?

(c) Give the first line of a bond amortization schedule for the 11th year, *i.e.* the line for the mid-year coupon.

[11 Marks]

7. Consider a 30 year, par-valued, *callable* bond with a face value of \$10,000 and a coupon of 6.4%, convertible semi-annually. If the bond is redeemable at any coupon date after the payment of the last coupon in the 25th year. What is the minimum price an investor should pay to guarantee a semi-annual yield of 3%? [6 Marks]

8. Consider a market with just shares and bonds. Suppose the bond yield rate is 4.2% and the share and bond prices are, in the usual notation,

$$B(0) = 100,\ S(0) = 25,\ \text{and } S(1) = \begin{cases} 30, & \text{with probability } 0.3; \\ 15, & \text{with probability } 0.7. \end{cases}$$

For an investor holding a portfolio of (shares, bonds)$= (100, 10)$ answer the following questions.

(a) Write down the value function, $V(t)$, for the portfolio. Hence calculate values for $V(0)$ and $V(1)$.

(b) Calculate the possible yields, i_V, for the portfolio.

(c) Calculate the expected yield and the associated risk (standard deviation of i_V) for this portfolio.

[10 Marks]

9. Consider the market in Question 8, i.e. $B(0)$, bond yield rate (i_B), $S(0)$ and $S(1)$ as in Question 8.

(a) What is the price of a call option with strike price $P_C = 25$ dollars?

(b) Find the expected yield of the call options, *i.e.* $E(i_C)$, and the associated risk.

[10 Marks]

10. Consider the market in Question 8, i.e. $B(0)$, bond yield rate (i_B), $S(0)$ and $S(1)$ as in Question 8.

(a) What is the price of a put option with strike price $P_P = 25$ dollars?

(b) Let us suppose that our investor has \$1,000 to spend on shares, $S(0)$, and put options, $P(0)$. Construct the optimal portfolio implementing the 'no loss" strategy with initial outlay of \$1,000 worth of shares and put options.
[Hint: Let the investor buy x shares then she has $(1000 - xS(0))$ dollars to spend on put options.]

[14 Marks]

Formula Sheet

$$\nu = \frac{1}{1+i}, \quad a_{\overline{n}|} = \frac{1-\nu^n}{i}, \quad s_{\overline{n}|} = (1+i)^n a_{\overline{n}|}.$$

Prospective method, $p_k = X a_{\overline{n-k}|}$

$$P = Fra_{\overline{n}|} + C\nu^n$$

$$B_k = Fra_{\overline{n-k}|} + C\nu^{n-k}, \quad B_k = B_{k-1} + (iB_{k-1} - Fr)$$

$$C(0) = \left(\frac{S^u - P_C}{S^u - S^d}\right)\left[S(0) - \frac{S^d}{1+i_B}\right], \quad C(t) = \left(\frac{S^u - P_C}{S^u - S^d}\right)\left[S(t) - S^d\frac{B(t)}{B(1)}\right]$$

$$P(0) = \left(\frac{P_p - S^d}{S^u - S^d}\right)\left[\frac{S^u}{1+i_B} - S(0)\right], \quad P(t) = \left(\frac{P_p - S^d}{S^u - S^d}\right)\left[S^u\frac{B(t)}{B(1)} - S(t)\right]$$