F. 17

MATH2090 – Mathematics of Finance

Assignment 5

Name:

MUN Number:

Due Date: Monday, 23 October

1. Consider an annuity with n payments made at the end of each for n years with an annual interest rate of i. Suppose also that the payments form a geometric sequence,

$$P_k = Pr^{k-1}, \text{ for } k = 1, \ldots, n \text{ with } P \text{ and } r \text{ fixed constants.}$$

Show that the present value of the annuity (at $t = 0$) is given by,

$$P.V. = P\nu \left[\frac{1-(r\nu)^n}{1-r\nu} \right], \text{ where, as usual, } \nu = \frac{1}{1+i}.$$

What is the accumulated value, A.V., just after the last payment?

2. A court awards a settlement of \$1,500,000 in the following terms:

   - The money is to be invested with (minimum) annual interest rate of 2.7%;
   - the award is to be paid with the initial payment one year after the settlement date and the remaining payments made at the end of each subsequent year for 20 years; the payments are to increase in those subsequent years in line with an inflation rate of 2.1% per annum.

   (a) What is the amount of the first payment?

   (b) What is the amount of the last payment?

3. Jean takes a mortgage of \$167,000 with her bank at an interest rate of 4.6%, convertible semi-annually, and over a term of 25 years. Jean repays the mortgage with equal payments made at the end of each month.

   (a) What are Jean's monthly payments on her mortgage?

   (b) What is the outstanding principal on Jean's loan immediately after the last payment in the 20th year?

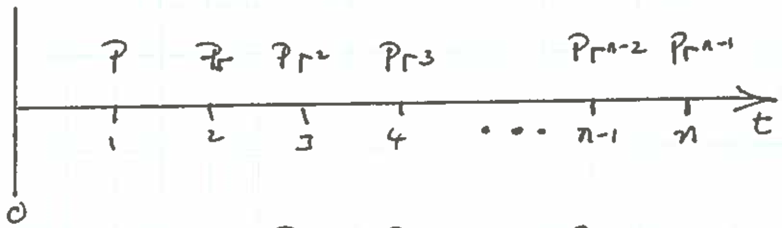
   (c) What portion of the last payment in the 20th year is interest and what portion is principal?

   (d) Jean refinances her loan, immediately after her last payment in the 20th year: Jean pays a lump sum of \$10,000, and then repays the remaining debt over the 10 years with an annual interest rate of 4.1%, and equal payments made at the end of each month. What are Jean's new monthly payments?

MATH2090 — MATH of FINANCE

Assignment 5 — Solutions.

1.



$$P.V. = \frac{P}{1+i} + \frac{Pr}{(1+i)^2} + \frac{Pr^2}{(1+i)^3} + \cdots + \frac{Pr^{n-1}}{(1+i)^n}$$

$$= P\left[v + rv^2 + r^2v^3 + \cdots + r^{n-1}v^n\right]$$

$$= Pv\left[1 + (rv) + (rv)^2 + \cdots + (rv)^{n-1}\right]$$

$$= Pv\left[\frac{1-(rv)^n}{1-(rv)}\right], \text{ sum geometric series.}$$

i.e. $P.V. = Pv\left[\frac{1-(rv)^n}{1-(rv)}\right]$, as required.

Now $A.V. = P.V.(1+i)^n$, i.e. n-compoundings of the P.V.

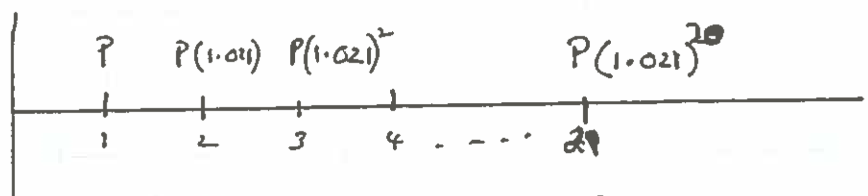
$$= Pv\left[\frac{1-(rv)^n}{1-(rv)}\right](1+i)^n$$

$$= Pv\left[\frac{(1+i)^n - r^n}{1-(rv)}\right], \text{ as } v(1+i) = 1.$$

i.e.

$$A.V. = Pv\left[\frac{(1+i)^n - r^n}{1-(rv)}\right]$$

2.



Let the first payment be P dollars, then the next payment is $P \times (1.021)$ as the inflation rate is 2.1%.
Confusing point: first payment then 20 more years so $n = 21$.

Now use question 1 :-
with $n = 21$

$$P.V. = Pv\left[\frac{1-(rv)^{21}}{1-rv}\right], \text{ with } r = 1.021 \text{ and } v = \frac{1}{1.027}$$

So Present Value equation gives

$$1500000 = \frac{P}{1.027}\left[\frac{1-\left(\frac{1.021}{1.027}\right)^{21}}{1-\left(\frac{1.021}{1.027}\right)}\right]$$

$$\therefore P = \frac{1500000 \times 1.027\left[1-\left(\frac{1.021}{1.027}\right)\right]}{\left[1-\left(\frac{1.021}{1.027}\right)^{21}\right]}$$

$$\doteqdot 77734.910020$$

i.e. the first payment is $\underline{\underline{\$77,734.91}}$

(b) The last payment is $P(1.021)^{20} \doteqdot 117796.108107$
i.e. last payment is $\underline{\underline{\$117,796.11}}$

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3. Convert the nominal semi annual rate to effective monthly rate j:

$$(1+j)^6 = 1 + \frac{4.6}{2 \times 100} = 1.023$$

i.e. $\underline{\underline{j = (1.023)^{\frac{1}{6}} - 1}}$

(a) Let original mortgage payments be X, then

$$167,000 = X\, a_{\overline{12 \times 25}|j}$$

$$\therefore X = 167,000\left[\frac{(1.023)^{\frac{1}{6}}-1}{1-(1.023)^{-\frac{300}{6}}}\right]$$

i.e. $$X = \frac{167000\left[(1.023)^{\frac{1}{6}}-1\right]}{1-(1.023)^{-50}}$$

$$\doteqdot 933.60648$$

i.e. Jean's monthly payments are \$933.60(648) //.

(b) The outstanding principle at the end of the 20th year, $p_{20\times12} = p_{240}$, is (by the prospective method)

$$p_{240} = X \cdot a_{\overline{12\times(25-20)}|}$$
$$= X \cdot a_{\overline{60}|}$$
$$= X \cdot \frac{\left[1-(1.023)^{-\left(\frac{60}{6}\right)}\right]}{(1.023)^{\frac{1}{6}}-1}$$
$$\doteqdot 933.60648 \cdot \frac{\left[1-(1.023)^{-10}\right]}{(1.023)^{\frac{1}{6}}-1}$$

$$\doteqdot 50009.096$$

i.e. The outstanding principal at the end of the 20th year is \$50,009.09(6) //.

(c) From lectures we know that the interest portion of the last payment in the 20th year (i.e. the $20\times12 = 240$th payment) is

$$i\,p_{239} = i\,X\,a_{\overline{12(26-20)}|}$$
$$= X(1-v^{72})$$
$$\doteqdot 933.60648\left[1-(1.023)^{-\frac{72}{6}}\right]$$

(4)

i.e. $\therefore i_{P_{239}} \doteqdot 933.60648\left[1-(1.023)^{-12}\right]$

$\doteqdot 222.95567$

i.e. the interest portion of the payment is

$\$222.95(567)$.

the principal portion is $933.60648 - 222.9556$

$= 710.65081$

i.e. the principal portion of the payment is

$\$710.65(081)$.

(d) By (b) the outstanding principal is $\$50,009.09(16)$; after pay $\$10,000$ this leaves an amount of $\$40,009.09(16)$ to be repaid at 4·1% (annually) with monthly payments over 10 years.

First we need the monthly interest rate i :–

$$(1+i)^{12} = 1 + \frac{4.1}{100} = 1.041.$$

$$\therefore 1+i = (1.041)^{\frac{1}{12}}, \quad i = (1.041)^{\frac{1}{12}} - 1.$$

The monthly payments, X, are given by:

$$40009.0916 = X \cdot a_{\overline{12\times10}|i}$$

$$\doteqdot X \cdot \frac{\left[1-(1.041)^{-\frac{12\times10}{12}}\right]}{(1.041)^{\frac{1}{12}}-1}$$

$$\therefore X \doteqdot 40009.0916 \times \frac{\left[(1.041)^{\frac{1}{12}}-1\right]}{1-(1.041)^{-10}} \doteqdot 405.546$$

$\therefore$ payments are $\$405.54(65)$ //.