MATH 2090 — Assignment 4

1. Two parts to question:
   - First: Calculate the accumulated value of Jane's retirement savings.
   - Second: use the first part to calculate the value of the pension.

Accumulated value of the savings at the time of the last deposit is

$$7500\, s_{\overline{25}|} = 7500 \frac{[(1+i)^{25}-1]}{i}, \quad i = \frac{3.01}{100}$$

$$\doteq 273{,}803.780753$$

This now becomes the present value of the (pension) annuity with annual (end of year payments) and interest rate $i = \frac{2.04}{100}$

$$273{,}803.780753 \doteq X\, a_{\overline{20}|}, \quad \text{then as } a_{\overline{20}|} = \frac{1-(1.0204)^{-20}}{0.0204}$$

, we have

$$X \doteq 273{,}803.780753 \times \frac{(0.0204)}{1-(1.0204)^{-20}}$$

$$= 16{,}809.647537$$

So Jane's annual pension payment will be (nearest cent) $\underline{\$16{,}809.65}$.

2. (a) Remember: Interest period **must** be the same as the payment period! So we need a monthly effective interest rate j:–

$$\underbrace{(1+j)^3}_{\text{3 months one quarter year.}} = 1 + \underbrace{\frac{6.1}{4 \times 100}}_{\text{Nominal, } i^{(4)}\text{, interest of } 6.1\%\text{, i.e. } \frac{6.1}{4}\% \text{ per quarter}}$$

i.e. $1+j = (1+0.01525)^{\frac{1}{3}}$

Jack makes 12×5 monthly repayments so the Present Value equation is

$$8000 = X\, a_{\overline{60|}j} \quad , X \text{ dollars the monthly payment}$$

$$\therefore \quad X = \frac{8000}{a_{\overline{60|}j}} = 8000 \times \frac{\left[(1.01525)^{\frac{1}{3}} - 1\right]}{1 - (1.01525)^{\frac{-60}{3}}}$$

$$\doteqdot 154.920153$$

i.e. Jack's monthly repayments are <u>$154.92</u> (nearest cent)

(b) We first need to calculate Ron's effective monthly interest rate j:-

$$(1+j)^{12} = 1 + \frac{6.2}{100}$$

12 month to give one year

annual interest rate of 6.2%.

$$\therefore \quad 1+j = (1.062)^{\frac{1}{12}}$$

If Ron's repayments are X dollars a month for the $12 \times 5 = 60$ months then

$$8000 = X\, a_{\overline{60|}i} \quad \text{so}$$

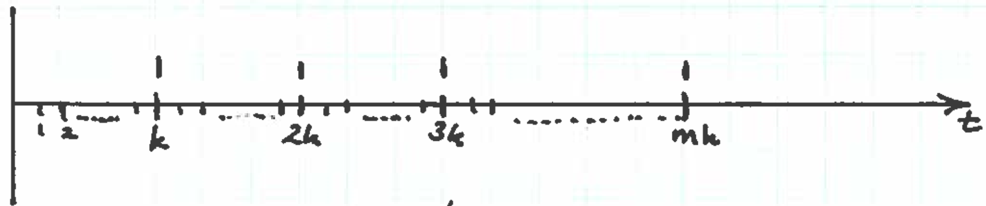
$$X = 8000 \times \frac{\left[(1.062)^{\frac{1}{12}} - 1\right]}{\left[1 - (1.062)^{\frac{-60}{12}}\right]}$$

$$\doteqdot 154.775874$$

i.e. Ron's monthly repayments are <u>$154.78</u> (nearest cent)

So Ron is pay 14¢ less per month than Jack! Over the 60 payments this is a difference of $8.66 to the nearest cent.

Q3.



Payments are every k years, m times; so we need the effective interest rate for k years, call it j then for the annual rate i we have

$$(*) \quad 1+j = (1+i)^k, \quad k \text{ comoundings of annual gives the } k\text{-yearly rate}$$

So the <u>interest rate</u> j is taken of the same period as the payments — that is every k years.

(a) The present value is

$$a_{\overline{m}|j} = \frac{1-(1+j)^{-m}}{j}$$

$$= \frac{1-(1+i)^{-mk}}{(1+i)^k-1}, \text{ using } (*).$$

$$= \left[\frac{i}{(1+i)^k-1}\right] \cdot \left[\frac{1-(1+i)^{-mk}}{i}\right]$$

$$= \frac{a_{\overline{mk}|}}{s_{\overline{k}|}} \text{ ; as required, because}$$

$$a_{\overline{mk}|} = \frac{1-(1+i)^{-mk}}{i} \text{ and } s_{\overline{k}|} = \frac{(1+i)^k-1}{i}$$

(b) Accumulated Value just after last payment is

$$s_{\overline{m}|j} = \frac{(1+j)^m-1}{j}$$

$$= \frac{(1+i)^{mk}-1}{(1+i)^k-1} = \left[\frac{i}{(1+i)^k-1}\right]\left[\frac{(1+i)^{mk}-1}{i}\right]$$

$$= \frac{s_{\overline{mk}|}}{s_{\overline{k}|}}$$

$$\text{as} \quad s_{\overline{mk}|} = \frac{(1+i)^{mk}-1}{i} \text{ and } s_{\overline{k}|} = \frac{(1+i)^k-1}{i}$$

i.e. the require accumulated value is $\dfrac{s_{\overline{mk|}}}{s_{\overline{k|}}}$

Q.4. In general $a_{\overline{\infty|}} = \frac{1}{i}$, i annual rate.

So the minim needed is $3000\, a_{\overline{\infty|}} = \dfrac{3000}{(2.6/100)}$

$\doteq 115384.615385$

i.e. The benefactor must invest \$115,384.62.

Q.5. We will do the calculation to find j the effective monthly interest rate as the \$600 payments are monthly. The equation of value for the accumulated value is

$$1\,500\,000 = 600\, s_{\overline{12\times 40|}\, j} = 600 \times \frac{[(1+j)^{12\times 40} - 1]}{j}$$

Put $X = 1+j$ so we have, after dividing by 600,

$$\frac{X^{480} - 1}{X - 1} = 2500 \quad \text{, rearrange:}$$

$$X^{480} - 2500X + (2500 - 1) = 0$$

i.e. $X^{480} - 2500X + 2{,}499 = 0$.

Use MATLAB to find the root of this degree 480 polynomial, note that we require the solution for which $X > 1$.

In MATLAB:

```
>> a = zeros(1,481)
>> a(1,1)=1
>> a(1,480) = -2500
>> a(1,481)= 2499
>> roots(a)
```

From the list of possible values for X given by MATLAB there is only <u>one</u> with $X > 1$:

$$X \doteq 1.00568833809918 + 0.i$$

This gives the monthly rate $j = X - 1$, we require the annual rate i given by $1 + i = (1+j)^{12} = X^{12}$

Using MATLAB with X as above:

```
>>    X^12 - 1
ans =
      0.0704366476597076
```

So the effective annual interest rate is $i = 0.0704$ (to 4 dec. places), i.e. 7.04%.