MATH 2090.

Assignment 2.

1. Generally: $A(10) = P\left(1 + \frac{3.1}{2 \times 100}\right)^{2 \times 10}$

$A(10)$ is the value in 10 years time

i.e. $A(10) = 50{,}000$

P is the principal, or the 'present value' of the investment:—

So, $50000 = P.(1.0155)^{20}$

OR $P = \dfrac{50000}{(1.0155)^{20}}$

$\doteq 36759.656927$ (6 dec. places).

That is, I should invest \$36,759.66.

2. Note the definitions:

$$i_n = \frac{a(n) - a(n-1)}{a(n-1)}, \quad d_n = \frac{a(n-1) - a(n)}{a(n)}$$

Note also that $d_n = \dfrac{i_n}{1 + i_n}$

or $i_n = \dfrac{d_n}{1 - d_n}$

(a)

$$\begin{aligned}
L.H.S &= \frac{i_n}{1 + i_n}\left(1 + \tfrac{1}{2} i_n\right) \\
&= d_n\left(1 + \tfrac{1}{2} i_n\right) \\
&= d_n\left[1 + \tfrac{1}{2}\left(\frac{d_n}{1 - d_n}\right)\right] \\
&= d_n\left[1 + \frac{\frac{1}{2} d_n}{1 - d_n}\right] \\
&= d_n\left[\frac{1 - d_n + \frac{1}{2} d_n}{1 - d_n}\right] \\
&= \frac{d_n}{1 - d_n}\left(1 - \tfrac{1}{2} d_n\right) = R.H.S.
\end{aligned}$$

So the equation in (a) is proven.

(b) From definition, $i_k = \dfrac{a(k) - a(k-1)}{a(k-1)}$, we have

$$i_k\, a(k-1) = a(k) - a(k-1). \text{ So}$$

$$\sum_{k=1}^{n} i_k\, a(k-1) = \sum_{k=1}^{n} \left[a(k) - a(k-1)\right]$$

$$= \sum_{k=1}^{n} \left[a(k)\right] - \sum_{k=1}^{n} \left[a(k-1)\right]$$

But, $\displaystyle\sum_{k=1}^{n} \left[a(k-1)\right] = a(0) + \sum_{k=2}^{n} a(k-1)$

$$= a(0) + \sum_{j=1}^{n-1} a(j)$$

(where $k = j+1$)

$$\therefore \sum_{k=1}^{n} i_k\, a(k-1) = \sum_{k=1}^{n} a(k) - a(0) - \sum_{j=1}^{n-1} a(j)$$

$$= a(n) - a(0), \text{ as } \sum_{k=1}^{n-1} a(k) \equiv \sum_{j=1}^{n-1} a(j)$$

$$\therefore \underline{\sum_{k=1}^{n} i_k\, a(k-1) = a(n) - 1}, \text{ since } a(0) = 1.$$

As required.

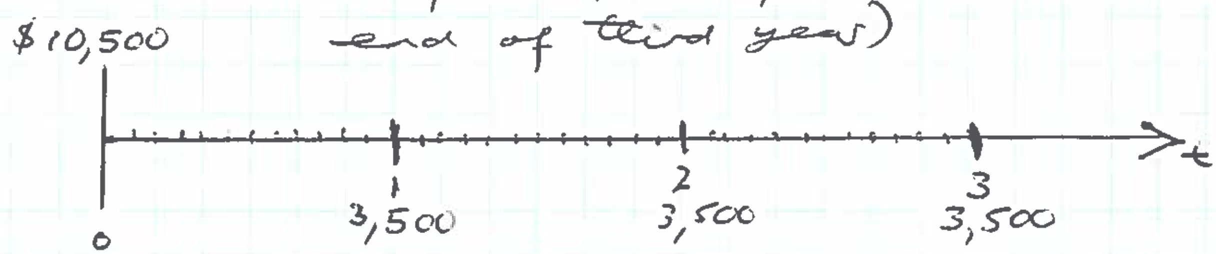
[Note: easy to see if you write terms out:

$$\sum_{k=1}^{n} \left[a(k) - a(k-1)\right] = \left[a(1) - a(0)\right] + \left[\cancel{a(2)} - \cancel{a(1)}\right] + \left(\cancel{a(3)} - \cancel{a(2)}\right]$$
$$+ \left[\cancel{a(4)} - \cancel{a(3)}\right] + \cdots + \left[a(n) - \cancel{a(n-1)}\right]]$$

---

3. Note: 1. 5.6% is a nominal rate with 12 compoundings per year.

2. Payments are yearly (i.e. 12 month interval) (end of first, end of 2^{nd} and last one at end of third year)



Hence the amount owed after three years, including payment made at the end of that third year, is

$$10500 \times \left(1+\frac{5.6}{12\times100}\right)^{12\times3} - 3500\left(1+\frac{5.6}{12\times100}\right)^{12\times2}$$
$$- 3500\left(1+\frac{5.6}{12\times100}\right)^{12\times1}$$
$$- 3500\left(1+\frac{5.6}{12\times100}\right)^{12\times0}$$

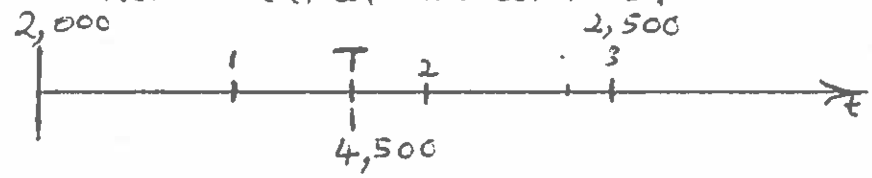
[i.e. compounded debt at end of 3 years <u>minus</u> compounding of each of the three payments]

Calculate to find the amount owed is

1301.096488

i.e. <u>(to nearest cent) $1,301.10</u> .

<u>4.</u> Suppose a single investment of $4,500 is made T years after Heather's initial investment.



Note: $0 < T < 3$ (Why?)

The annual interest rate is $i = \frac{3.9}{100} = 0.039$, so the equation of value at $t = 3$ years gives:-

$$4500 \times (1.039)^{3-T} = 2000\times(1.039)^3 + 2500\times(1.039)^0$$

i.e. $$4.5\times(1.039)^{3-T} = 2\times(1.039)^3 + 2.5$$

i.e. $$(1.039)^{3-T} = \frac{2.(1.039)^3 + 2.5}{4.5}$$

Take ln's of both sides :-

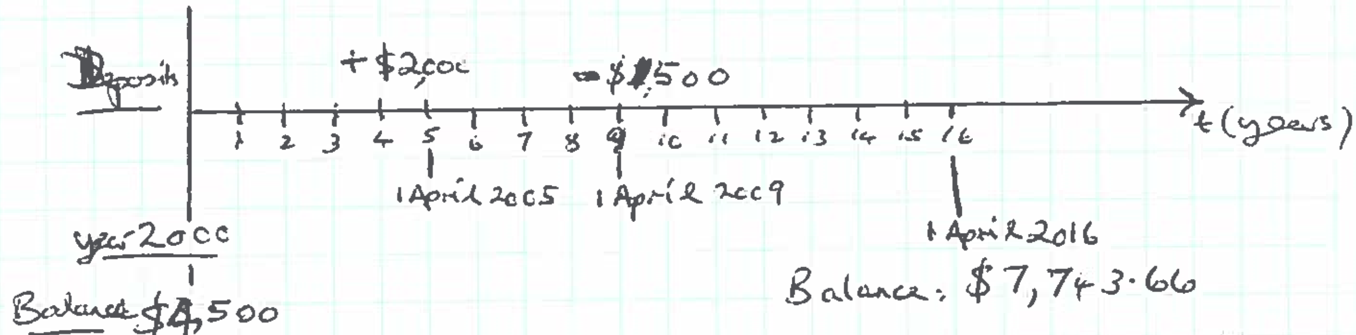
$$(3-T)\ln(1.039) = \ln\left[\frac{2.(1.039)^3+2.5}{4.5}\right]$$

$$\therefore \quad T = 3 - \frac{\ln\left[\frac{2.(1.039)^3+2.5}{4.5}\right]}{\ln(1.039)}$$

$$\doteqdot 1.623999$$

i.e. <u>One year and 228 days</u> (Nearest day)

5. We will count the 'financial years', i.e. 1 April to 31 March, as years (t) in our equations.



Suppose the effective annual interest rate is, i, then the equation of value is:—

$$7743{\cdot}66 = 4500(1+i)^{16} + 2000(1+i)^{11} - 1500(1+i)^{7}$$

Let $X = 1+i$, and divide through by 100:—

$$0 = 45X^{16} + 20X^{11} - 15X^{7} - 77{\cdot}4366$$

A polynomial equation which we must solve (numerically) for X — i.e. we must find the roots of the polynomial. Use MATLAB roots function to find the roots of this polynomial of degree 16:—

```
fx >> roots([45 0 0 0 0 20 0 0 0 -15 0 0 0 0 0 0 -77.4366])
```

MATLAB lists the 16 possible roots of which two (those with $0i$ part) which are real and one of them is greater than 1 (Note we want $X > 1$, as $X = 1+i$)

So we have $X \doteqdot 1{\cdot}026824$ (six dec. places)

$\therefore$, as $X = 1+i$, $i = 0{\cdot}026824$

i.e. the effective annual interest rate is

$2{\cdot}6824\%$.