A TAUBERIAN THEOREM FOR DISCRETE POWER SERIES METHODS

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ABSTRACT: A tauberian theorem from summability by a discrete power series method to ordinary convergence is proved.

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1. INTRODUCTION

Throughout this paper $\sum_{n=0}^{\infty} a_n$ is a series of real or complex numbers and $\{s_n\}$ represents its associated sequence of partial sums. The sequence $\{p_k\}$ is nonnegative with $p_0 > 0$ and satisfies $P_n := \sum_{k=0}^{n} p_k \to \infty$. Assume that the power series $p(x) := \sum_{k=0}^{\infty} p_k x^k$ has radius of convergence 1 and define $t_n := \frac{1}{P_n} \sum_{k=0}^{n} p_k s_k$ and $p_s(x) := \frac{1}{p(x)} \sum_{k=0}^{\infty} p_k s_k x^k$. The real sequence $\{\lambda_n\}$ satisfies $1 \leq \lambda_0 < \lambda_1 < \cdots \to \infty$ and we set $x_n = 1 - \frac{1}{\lambda_n}$.

Weighted mean and power series methods are defined as follows.

Definition 1.1 If $t_n \to s$ as $n \to \infty$ then we say that $\{s_n\}$ is limitable to s by the weighted mean method M_p and write $s_n \to s(M_p)$.

Definition 1.2 Suppose that $p_s(x)$ exists for each $x \in (0, 1)$. If $\lim_{x \to 1^-} p_s(x) = s$ then we say that $\{s_n\}$ is limitable to s by the power series method (P) and write $s_n \to s\ (P)$.

Weighted mean methods and power series methods have been studied extensively. It is known (see [3] or [4]) that the condition $P_n := \sum_{k=0}^{n} p_k \to \infty$ guarantees that both are regular. Moreover, (P) includes (M_p) in the sense that $s_n \to s(M_p)$ implies $s_n \to s(P)$ (see [4] or [5]).

If $p_k = 1$ for all k then the corresponding weighted mean and power series methods are the $(C, 1)$ method of Cesàro and ordinary Abel summability, (A), respectively. In the latter case, $p(x) = \dfrac{1}{1-x}$, and $p_s(x) = (1-x)\sum_{k=0}^{\infty} s_k x^k$.

Next we define the associated discrete methods.

Definition 1.3 We say that $\{s_n\}$ is limitable to s by the discrete weighted mean method, (M_{P_λ}), and write $s_n \to s(M_{P_\lambda})$ if $\tau_n := t_{[\lambda_n]} = \dfrac{1}{P_{[\lambda_n]}} \sum_{k=0}^{[\lambda_n]} p_k s_k \to s$ as $n \to \infty$ where $[\cdot]$ denotes the greatest integer function.

Definition 1.4 If $p_s(x_n)$ exists for all n and $\lim_{n\to\infty} p_s(x_n) = s$ then we say that s_n is limitable to s by the discrete power series method (P_λ) and write $s_n \to s(P_\lambda)$.

Discrete methods have been studied recently by Armitage and Maddox in [1], [2] and by the author in [6], [7].

Note that, trivially, (M_{P_λ}) includes (M_p) and (P_λ) includes (P). Consequently, (M_{P_λ}) and (P_λ) inherit regularity from the underlying weighted mean or power series method. Other abelian–type results have been given in [7]. Reverse implications only hold under additional hypotheses called "tauberian conditions". The main theorem of this paper is a tauberian result from (P_λ) summability to convergence. In general, one might expect to require tauberian conditions on both $\{\lambda_n\}$ and $\{p_n\}$. One might also expect tauberian theorems for the corresponding power series method (P) to suggest results for (P_λ). Our result is inspired by theorem 3 in [5].

2. THE MAIN THEOREM

Theorem 2.1 Suppose that

(i) $\frac{P_n}{p(x_n)} = \mathrm{O}(1)$ as $n \to \infty$,

(ii) $0 < p_k \leq M$ for all $k \geq 0$,

(iii) $\frac{\lambda_n}{P_n} = \mathrm{O}(1)$,

(iv) $\sum_{k=0}^{\infty} a_k = s(P_\lambda)$ and

(v) $a_k = \mathrm{o}(\frac{p_k}{P_k})$ as $k \to \infty$.

Then $\sum_{k=0}^{\infty} a_k = s$.

Proof. First, write

$$\begin{aligned} s_n - p_s(x_n) &= \frac{1}{p(x_n)} \left\{ \sum_{k=0}^{\infty} s_n p_k x_n^k - \sum_{k=0}^{\infty} s_k p_k x_n^k \right\} \\ &= \frac{1}{p(x_n)} \sum_{k=0}^{\infty} (s_n - s_k) p_k x_n^k \\ &= \frac{1}{p(x_n)} \left\{ \sum_{k=0}^{n-1} (s_n - s_k) p_k x_n^k + \sum_{k=n+1}^{\infty} (s_n - s_k) p_k x_n^k \right\} \\ &= I + J. \end{aligned}$$

It suffices to show that $I, J \to 0$ as $n \to \infty$.

For I we have

$$\begin{aligned} |I| &= \frac{1}{p(x_n)} \left| \sum_{k=0}^{n-1} (s_n - s_k) p_k x_n^k \right| \\ &\leq \frac{1}{p(x_n)} \sum_{k=0}^{n-1} |s_n - s_k| p_k x_n^k \\ &\leq \frac{1}{p(x_n)} \sum_{k=0}^{n-1} |s_n - s_k| p_k \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{p(x_n)}\{|a_1+a_2+\cdots+a_n|p_0+|a_2+\cdots+a_n|p_1+\cdots+|a_n|p_{n-1}\}\\
&\leq \frac{1}{p(x_n)}\{|a_1|p_0+|a_2|(p_0+p_1)+\cdots+|a_n|(p_0+p_1+\cdots+p_{n-1})\}\\
&= \frac{P_n}{p(x_n)}\frac{1}{P_n}\sum_{k=1}^{n}|a_k|P_{k-1}\\
&= \frac{P_n}{p(x_n)}\frac{1}{P_n}\sum_{k=0}^{n}p_k\frac{|a_k|P_{k-1}}{p_k} \quad \text{where } P_{-1}=0.
\end{aligned}
$$

Now, $\dfrac{P_n}{p(x_n)} = \mathrm{O}(1)$ by condition (i). Moreover, since $\dfrac{|a_k|P_{k-1}}{p_k} \to 0$ and the weighted mean method M_p is regular, we have

$$
\frac{P_n}{p(x_n)}\frac{1}{P_n}\sum_{k=0}^{n}p_k\frac{|a_k|P_{k-1}}{p_k} \to 0.
$$

Assign $\epsilon > 0$ and consider J.

By condition (v) there is an n_0 such that $|a_k| \leq \dfrac{\epsilon p_k}{P_k}$ for all $k \geq n_0$.

Assume that $k > n \geq n_0$. Then

$$
\begin{aligned}
|s_k - s_n| &= |a_{n+1}+a_{n+2}+\cdots+a_k|\\
&\leq \epsilon\left\{\frac{p_{n+1}}{P_{n+1}}+\frac{p_{n+2}}{P_{n+2}}+\cdots+\frac{p_k}{P_k}\right\}\\
&=: \epsilon Q_k.
\end{aligned}
$$

Now,

$$
|J| \leq \frac{1}{p(x_n)}\sum_{k=n+1}^{\infty}\epsilon Q_k p_k x_n^k
$$

and

$$
\begin{aligned}
Q_k &\leq \frac{p_{n+1}}{P_n}+\frac{p_{n+2}}{P_n}+\cdots+\frac{p_k}{P_n}\\
&= \frac{P_k-P_n}{P_n}\\
&= \frac{P_k}{P_n}-1\\
&< \frac{P_k}{P_n}.
\end{aligned}
$$

Hence, using condition (ii),

$$\begin{aligned}
|J| &\leq \frac{\epsilon}{p(x_n)}\frac{1}{P_n}\sum_{k=n+1}^{\infty} P_k p_k x_n^k \\
&= \frac{\epsilon P_n}{p(x_n)}\frac{1}{{P_n}^2}\sum_{k=n+1}^{\infty} P_k p_k x_n^k \\
&\leq \frac{\epsilon M P_n}{p(x_n)}\frac{1}{{P_n}^2}\sum_{k=n+1}^{\infty} P_k x_n^k
\end{aligned}$$

and, by conditions (i), (ii) and (iii),

$$\begin{aligned}
|J| &\leq \frac{\epsilon M P_n}{p(x_n)}\frac{1}{{P_n}^2}\sum_{k=n+1}^{\infty} M(k+1) x_n^k \\
&= \frac{\epsilon M^2 P_n}{p(x_n)}\frac{1}{{P_n}^2}\sum_{k=n+1}^{\infty} (k+1) x_n^k \\
&\leq \frac{\epsilon M^2 P_n}{p(x_n)}\frac{1}{{P_n}^2}\sum_{k=0}^{\infty} (k+1) x_n^k \\
&= \frac{\epsilon M^2 P_n}{p(x_n)}\frac{{\lambda_n}^2}{{P_n}^2} \\
&\leq \epsilon M_1 \quad \text{for large } n \text{ and some constant } M_1.
\end{aligned}$$

This completes the proof.

As an illustration, in the special case of the discrete Abel method (A_λ), we get the following.

Corollary 2.2 If $na_n \to 0$ and there exist positive constants, γ_1 and γ_2, such that $\gamma_1 \leq \frac{\lambda_n}{n} \leq \gamma_2$ then $\sum_{k=0}^{\infty} a_k = s(A_\lambda)$ implies $\sum_{k=0}^{\infty} a_k = s$.

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