SAMPLE FINAL

MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

Mathematics 3000 **FINAL EXAMINATION** December, 2006

ANSWER ALL QUESTIONS TIME: 2 HOURS

MARKS

12 1. Give a precise mathematical definition of each of the following.

(a) $\{x_n\}_{n=1}^{\infty}$ is a *Cauchy sequence.*

(b) The function $f: I \mapsto \mathbb{R}$ is *continuous* at $c \in I$, an open interval.

(c) The function f is *uniformly continuous* on the interval I.

12 2. State (**NO PROOF**) each of the following.

(a) The **Bolzano–Weierstrass Theorem.**

(b) The **Sequential Criterion for Continuity.**

(c) The **Intermediate Value Theorem.**

(d) The **Uniform Continuity Theorem.**

6 3. Answer each of the following and include a **brief** explanation.

(a) Give an example of a function $f : [0,\ 1] \mapsto \mathbb{R}$ that is discontinuous at every x.

(b) Give an example of a continuous function on an interval I that is not uniformly continuous on I.

30 4. Prove each of the following using the appropriate "$\epsilon - N$" or "$\epsilon - \delta$" definition.

(a) $\displaystyle\lim_{n\to\infty} \frac{3n^2 - 2}{n^2 + 1} = 3.$

(b) $f(x) = \dfrac{x^2 + 2}{x}$ is continuous at $x = 1$.

(c) $f(x) = \dfrac{1}{1 + x^2}$ is uniformly continuous on $(1, \infty)$.

15 5. Let $x_1 = 1$ and $x_{n+1} = \sqrt{5 + 4x_n}$ for $n \geq 1$.

(a) Show that $x_n < x_{n+1}$ for $n \geq 1$.

(b) Prove that $1 \leq x_n < 5$ for $n \geq 1$.

(c) Find $\lim_{n \to \infty} x_n$. Justify.

12 6. Express S as one or more intervals where $S = \{x \in \mathbb{R} : \frac{1 - 2x}{x^2 - 3} < -3\}$. Find $\sup S$ and $\inf S$ by inspection. Show all work.

8 7. Using the Intermediate Value Theorem, show that the equation

$$\frac{x^4 + 2x^2 + 5}{x - 1} + \frac{x^6 + 2x^4 + 6}{x} = 0$$

has a solution in the open interval $(0, 1)$.

5 8. Suppose that $f \colon (a, b) \mapsto \mathbb{R}$ is uniformly continuous. If $|f(x)| \geq k > 0$ for all $x \in (a, b)$ prove that $\frac{1}{f(x)}$ is uniformly continuous on (a, b).

5 9. **BONUS QUESTION.** Suppose that $f \colon [a, b] \mapsto \mathbb{R}$ is continuous.

Prove **EXACTLY ONE** of the following.

(a) $f(x)$ is bounded on $[a, b]$.

(b) $f(x)$ is uniformly continuous on $[a, b]$.