DISCRETE POWER SERIES METHODS

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1. Introduction

Throughout this paper $\sum_{n=0}^{\infty} a_n$ is a series of real or complex numbers and $\{s_n\}$ represents its associated sequence of partial sums. The sequence $\{p_k\}_0^{\infty}$ is nonnegative and satisfies $\sum_{k=n}^{\infty} p_k > 0$ for each n. In addition $p(x) := \sum_{k=0}^{\infty} p_k x^k$, $p_s(x) := \frac{1}{p(x)} \sum_{k=0}^{\infty} p_k s_k x^k$, and ρ_p denotes the radius of convergence of $p(x)$. The real sequence $\{\lambda_n\}$ satisfies $1 \leq \lambda_0 < \lambda_1 < \cdots \to \infty$.

Definition 1.1. *Suppose that $\rho_p > 0$ and $p_s(x)$ exists for each $x \in (0, \rho_p)$. If $\lim_{x \to \rho_p^-} p_s(x) = s$ then we say that $\{s_n\}$ is limitable to s by the power series method (P) and write $s_n \to s$ (P).*

When there can be no ambiguity about the underlying sequence $\{p_k\}$, we drop the subscript on the radius of convergence and use ρ instead of ρ_p.

Power series methods have been extensively studied. For example, see [6]. The basic regularity results, although available in [6], were concisely summarized by Borwein in [4] and his result is recalled here.

Theorem 1.1.

(1) *If $0 < \rho < \infty$ then (P) is regular if and only if $\sum_{k=0}^{\infty} p_k \rho^k = \infty$.*

(2) *If $\rho = \infty$ then (P) is regular.*

For ordinary Abel summability take $p_k = 1$ for all k. Then $p(x) = \frac{1}{1-x}$, $\rho = 1$, and $p_s(x) = (1-x) \sum_{k=0}^{\infty} s_k x^k$. For Borel summability

$p_k = \dfrac{1}{k!}$, $p(x) = e^x$, $\rho = \infty$, and $p_s(x) = e^{-x}\sum_{k=0}^{\infty}\dfrac{s_k}{k!}x^k$. These are two examples of continuous power series methods. Next we define the discrete types.

Set

$$x_n = \begin{cases} \rho - \frac{1}{\lambda_n} & \text{if } 0 < \rho < \infty, \\ \lambda_n & \text{if } \rho = \infty. \end{cases}$$

Definition 1.2. *If $p_s(x_n)$ exists for all n and $\lim_{n\to\infty} p_s(x_n) = s$ then we say that s_n is limitable to s by the discrete power series method (P_λ) and write $s_n \to s(P_\lambda)$.*

Discrete methods have been studied recently by Armitage and Maddox in [1] and [2]. Note that (P_λ) includes (P) in the sense that $s_n \to s(P)$ implies $s_n \to s(P_\lambda)$.

Theorem 1.2.

(1) *If $0 < \rho < \infty$ then (P_λ) is regular if and only if $\sum_{k=0}^{\infty} p_k\rho^k = \infty$.*

(2) *If $\rho = \infty$ then (P_λ) is regular.*

Proof. Part (2) and one direction of (1) follow immediately from theorem 1.1 and the observation preceeding this theorem. So, assume (P_λ) is regular and suppose, as we may, that for some fixed m, $p_m > 0$.

Now, $\lim_{n\to\infty} p(x_n) = \sum_{k=0}^{\infty} p_k\rho^k$ since, for any N,

$$\sum_{k=0}^{\infty} p_k\rho^k \geq \sum_{k=0}^{\infty} p_k x_n^k \geq \sum_{k=0}^{N} p_k x_n^k.$$

Hence

$$\sum_{k=0}^{\infty} p_k\rho^k \geq \lim_{n\to\infty} p(x_n) \geq \sum_{k=0}^{N} p_k\rho^k.$$

Define a sequence $\{s_k\}$ by

$$s_k = \begin{cases} 0 & \text{if } k \neq m \\ \frac{1}{p_m} & \text{if } k = m. \end{cases}$$

Then,

$$\begin{aligned} p_s(x_n) &= \frac{1}{p(x_n)} \sum_{k=0}^{\infty} p_k s_k x_n^k \\ &= \frac{x_n^m}{p(x_n)} \to 0 \text{ as } n \to \infty. \end{aligned}$$

Therefore $\lim_{n\to\infty} p(x_n) = \infty$ and, hence, $\sum_{k=0}^{\infty} p_k \rho^k = \infty$.

2. Abelian Results

Adopt the notation $E_\lambda = \{\lambda_n : n \geq 0\}$. We prove the following result which includes Theorem 1 of [2].

Theorem 2.1. *Suppose that* (P_λ) *is regular and that* $p_k > 0$ *for all* $k \geq 0$.

(1) $P_\lambda \subseteq P_\mu$ *if and only if* $E_\mu - E_\lambda$ *is finite.*
(2) $P_\mu = P_\lambda$ *if and only if* $E_\mu \Delta E_\lambda$ *is finite.*

Proof.
It suffices to prove part (1) for then (2) is immediate. Let $\{x_n\}$ be as defined in the previous section and let $\{y_n\}$ be the sequence that corresponds to $\{\mu_n\}$. That is,

$$y_n = \begin{cases} \rho - \frac{1}{\mu_n} & \text{if } 0 < \rho < \infty, \\ \mu_n & \text{if } \rho = \infty. \end{cases}$$

If $E_\mu - E_\lambda$ is finite then there exists an $N \in \boldsymbol{N}$ such that $\{y_n : n \geq N\} \subseteq \{x_n : n \in \boldsymbol{N}\}$. Set $y_n = x_{j_n}$ for $n \geq N$.

Suppose that $s_n \to s(P_\lambda)$. Then $\lim_{n\to\infty} p_s(x_n) = s$. But we then have $\lim_{n\to\infty} p_s(y_n) = \lim_{n\to\infty} p_s(x_{j_n}) = s$.

For the reverse implication suppose, by way of contradiction, that $P_\lambda \subseteq P_\mu$ but that $E_\mu - E_\lambda$ is infinite. Then there exists a subsequence μ' of μ, say $\{\mu_{n_k}\}$ such that $E_{\mu'} \cap E_\lambda = \emptyset$. Construct, as we may, a continuous function $\phi : (-\infty, \infty) \to [0, \infty)$ such that $\phi(\lambda_k) = 0$, and $\phi(\mu_{n_k}) = p(y_{n_k})$.

Suppose first that $0 < \rho < \infty$. By a theorem of Carleman (see [5] or [3]) there exists an entire function g such that

$$|g(x) - \phi(x)| < \frac{1}{1+|x|} \text{ for all } x \in \boldsymbol{R}.$$

Set $f(z) = g(\frac{1}{\rho - z})$. Then f is analytic for $|z| < \rho$ so there exists a sequence $\{s_k\}$ such that $f(z) = \sum_{k=0}^{\infty} s_k z^k$ for $|z| < \rho$. Set $b_k = \frac{s_k}{p_k}$ for all $k \geq 0$. Then, for the sequence $\{b_k\}$,

$$\begin{aligned} |p_s(x_n)| &= \frac{1}{p(x_n)} |\sum_{k=0}^{\infty} p_k b_k x_n^k| \\ &= \frac{1}{p(x_n)} |\sum_{k=0}^{\infty} s_k x_n^k| \\ &= \frac{1}{p(x_n)} |f(x_n)| \\ &= \frac{1}{p(x_n)} |g(\lambda_n)| \\ &= \frac{1}{p(x_n)} |g(\lambda_n) - \phi(\lambda_n)| \\ &< \frac{1}{p(x_n)} \frac{1}{\lambda_n} \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

Hence $b_n \to 0(P_\lambda)$.

On the other hand,

$$
\begin{aligned}
|p_s(y_{n_k})| &= \frac{1}{p(y_{n_k})}|\sum_{j=0}^{\infty} p_j b_j y_{n_k}^j| \\
&= \frac{1}{p(y_{n_k})}|\sum_{j=0}^{\infty} s_j y_{n_k}^j| \\
&= \frac{1}{p(y_{n_k})}|f(y_{n_k})| \\
&= \frac{1}{p(y_{n_k})}|g(\mu_{n_k})| \\
&\geq \frac{1}{p(y_{n_k})}\{|\phi(\mu_{n_k})| - |g(\mu_{n_k}) - \phi(\mu_{n_k}|\} \\
&\geq 1 - \frac{1}{p(y_{n_k})\mu_{n_k}} \to 1 \text{ as } k \to \infty.
\end{aligned}
$$

Therefore $b_k \not\to 0(P_\mu)$. This is a contradiction.

Now suppose $\rho = \infty$. In this case, $g(z) = \sum_{k=0}^{\infty} s_k z^k$. With $\{b_k\}$ as above we get,

$$
\begin{aligned}
|p_s(x_n)| &= \frac{1}{p(x_n)}|\sum_{k=0}^{\infty} p_k b_k x_n^k| \\
&= \frac{1}{p(x_n)}|g(x_n)| \\
&= \frac{1}{p(\lambda_n)}|g(\lambda_n) - \phi(\lambda_n)| \\
&< \frac{1}{p(\lambda_n)}\frac{1}{\lambda_n} \\
&\to 0 \text{ as } n \to \infty.
\end{aligned}
$$

Hence $b_k \to 0(P_\lambda)$.

But,

$$\begin{aligned} |p_s(y_{n_k})| &= \frac{1}{p(y_{n_k})}|\sum_{j=0}^{\infty} p_j b_j y_{n_k}^j| \\ &= \frac{1}{p(y_{n_k})}|g(y_{n_k})| \\ &= \frac{1}{p(\mu_{n_k})}|g(\mu_{n_k})| \\ &\geq \frac{1}{p(\mu_{n_k})}\{|\phi(\mu_{n_k})| - |g(\mu_{n_k}) - \phi(\mu_{n_k}|\} \\ &\geq 1 - \frac{1}{p(\mu_{n_k})\mu_{n_k}} \to 1 \text{ as } k \to \infty. \end{aligned}$$

Therefore, as previously, $b_k \not\to 0(P_\mu)$. This completes the proof.

Corollary 2.2. *If* (P_λ) *is regular and* $p_k > 0$ *for all* $k \geq 0$, *then* $(P) \subset (P_\lambda)$.

Proof.
Set $\mu_n = \dfrac{\lambda_n + \lambda_{n+1}}{2}$ for $n \geq 0$. Then $E_\lambda \cap E_\mu = \emptyset$. Hence we cannot have $(P_\lambda) \subseteq (P_\mu)$. Therefore there exists a sequence $\{s_k\}$ such that $s_k \to s(P_\lambda)$ but $\{s_k\}$ is not limitable (P_μ). But we always have $(P) \subseteq (P_\mu)$. Therefore $s_k \to s(P_\lambda)$ but $s_k \not\to s(P)$.

References

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