AMAT3132 – Numerical Analysis, Winter 2010

Home work 3
(show all works)

Due Monday Feb 15, 2010 by 24:00 in the drop box#40
Full marks 30

Instruction:

- Please hand in the report into the box #40 located next to Math general office in HH.
- The first page of the report must include your student information, and a list of people/resources whom (if) you got help from. Examples of people whom you have discussed are your course instructor, your classmates, or any other people. Example of resources are books other than texts for this course, web sites etc.
- You are permitted to discuss each other or people outside the class. However, your matlab code and all results must be authored by yourself.
- Please submit your matlab code electronically using the “submit assignment” utility.
- You must have either one executable **.m** matlab script for the entire assignment - such as the template for this course or you must have one main code that should run and verify your results. If your code does not run, you will be penalized accordingly.
- To solve each problem, you will use functions and scripts. A function will implement the desired algorithm, and scripts will use the function and will display results.

1. [10] Let u be a given $n \times n$ array. We can map each element of u to a vector x of size N. The map depends on the size of the given array. In other words, there is a function $k = f(i, j, n)$ that determines the k-th index of x if the index (i, j) for an entry of the $n \times n$ array is known.

   Write a `MATLAB` function that takes (i, j, n) as input and returns k. Test your function so that you get a unique k for each pair (i, j). Take $n = 5$, and report in a table, showing values of k in one column and values of pairs (i, j) in other column.

   [Hint: The matlab function may contain only one or two lines of code (e.g. very easy).]

2. [10] Consider the matrix vector product Ax, where neither A nor x is know. The vector x is mapped to an array u of size $n \times n$. We have the following information to define Ax:

$$(Ax)_i := Au = 4u_{ij} - u_{i-1,j} - u_{i+1,j} - u_{ij-1} - u_{ij+1}$$

   for $i = 1 \ldots n$ and $j = 1 \ldots n$. Quantities $u_{0,j}$, $u_{n+1,j}$, $u_{i,0}$, $u_{i,n}$ etc are taken to be zero because such data are not available. **Determine** entries a_{ij} of the matrix A. What is the size of the matrix? Is this a blocked sparse matrix? Is the matrix diagonally dominant? Write down the matrix for $n = 3$ and $n = 6$.

3. [10] Consider the linear system $Ax = b(\nu)$, where the parameter ν perturbs the right hand side of the system and dictates effective resolution for a given tolerance.

   For $\nu = 5 \times 10^{-1}$, print the log-scale plot of the norm $||b - Ax||_2$ vs N, with the following information, where $N = n^2$ and $n = 8, 16, 32, 64, 128$.

   Write a matlab function that takes an $n \times n$ array u as input and returns the vector Ax as defined above, but using the follwing u.

   In order to verify your code, consider two series $\{x_i\}_{i=1}^n$ and $\{y_j\}_{j=1}^n$ such that $(x_1, y_1) = (-2, -2)$ and $(x_n, y_n) = (2, 2)$, and $x_{i+1} - x_i = y_{j+1} - y_j \ \forall i, j$. Use the `meshgrid` function of MATLAB to construct two $n \times n$ arrays x and y. Let $u(x, y) = \exp(-(x^2 + y^2)/\nu)$, which produces an $n \times n$ array and define a vector b that is mapped to the $n \times n$ array

$$-\left(\frac{x_n - x_1}{n - 1}\right)^2 \nabla \cdot (\nabla u)$$

   using $k = f(i, j, n)$. You can use `MATLAB` functions `gradient(u,dx,dy)` and `divergence(x,y,ux,uy)` to evaluate b.

These analytical questions are for practise only and can be solved as soon as corresponding materials are covered, do not hand in.

4. Let $f(x) = 2\sin(x) - e^x/4 - 1$ is zero for two values near $x = -5$.
   (a) Propose two initial intervals so that Bisection method would converge to the solution of $f(x) = 0$.
   (b) How many iterations are required to agree an error bound 10^{-6}?

5. Let $f(x) = (x - 0.3)(x - 0.5)$. Clear $f(x) = 0$ has two roots $x = 0.3$ and $x = 0.5$.
   (a) If you take an initial interval $[0.1, 0.6]$ for bisection method, which root do you expect to converge. Explain your answer.
   (b) Determine two good starting intervals so that bisection method converges to the exact root.
   (c) Let $[0, 0.49]$ be the starting interval. Which root will the bisection method converge to?

6. Let $f(x) = 2\sin(x) - e^x/4 - 1$ is zero for two values near $x = -5$.
   (a) Solve $f(x) = 0$ using regula falsi method with starting intervals $[-7, -5]$ and $[-5, -3]$.
   (b) Compare the number of iterations required to obtain solution of $f(x) = 0$ with a tolerance 10^{-5} using regula falsi method and secant method.
   (c) Explain why the secant method usually converges to reach a given tolerance faster than either bisection or regula falsi method.
   (d) Although secant method converges faster than regula falsi, explain when regula falsi is preferred.

7. Let $f(x) = 2\sin(x) - e^x/4 - 1$ is zero for two values near $x = -5$. Solve $f(x) = 0$ using Netwon's method with a tolerance 10^{-5}.

8. Let $f(x) = 4x^3 - 1 - \exp(x^2/2)$ has two zeros near $x = 1.0$ and $x = 3.0$.
   (a) If you begin the Newton's method at $x = 2$, which root is reached (do not use Newton's method)?
   (b) Use Newton's method starting at $x = 2$ with a tolerance 10^{-5}. How many iterations are required?
   (c) If you start Newton's method with $x = 3.5$, which root will be reached? Explain.

9. (a) Use Newton's method to derive an algorithm for finding the square root of N.

   (b) Use the above algorithm to show that

$$(A \cdot B)^{1/2} \approx \frac{A+B}{4} + \frac{A \cdot B}{A+B}.$$

10. Let $f(x) = (x-1)^2(x+1)$. Obviously, $f(x) = 0$ has roots at $x = +1$ and $x = -1$.

   (a) Start with values that differs from roots by 0.2, compare the number of iterations needed by Newton's method to compute solution with a tolerance 10^{-4}.

   (b) Explain the differences.