A first order divided difference

For a given function $f(x)$ and two distinct points x_0 and x_1, define

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

This is called the first order divided difference of $f(x)$.

By the Mean-value theorem,

$$f(x_1) - f(x_0) = f'(c)(x_1 - x_0)$$

for some c between x_0 and x_1. Thus

$$f[x_0, x_1] = f'(c)$$

and the divided difference is very much like the derivative, especially if x_0 and x_1 are quite close together. In fact,

$$f'\left(\frac{x_1 + x_0}{2}\right) \approx f[x_0, x_1]$$

is quite an accurate approximation of the derivative.

Second order divided difference

Given three distinct points x_0, x_1, and x_2, define

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

This is called the second order divided difference of $f(x)$.

We can show that $f[x_0, x_1, x_2] = \frac{1}{2} f''(c)$ for some c intermediate to x_0, x_1, and $_2$. In fact,

$$f''(x_1) \approx 2f[x_0, x_1, x_2]$$

in the case when nodes are evenly spaced, $x_1 - x_0 = x_2 - x_1$.

Example

Consider the table

x	1	1.1	1.2	1.3	1.4
$\cos x$	0.54030	0.45360	0.36236	0.26750	0.16997

Let $x_0 = 1$, $x_1 = 1.1$, and $x_2 = 1.2$. Then

$$f[x_0, x_1] = \frac{0.45360 - 0.54030}{1.1 - 1} = -0.86700$$

$$f[x_1, x_2] = \frac{0.36236 - 0.45360}{1.1 - 1} = -0.91240$$

$$\begin{aligned} f[x_0, x_1, x_2] &= \frac{f[x_1, x_2] - f[x_0, x1]}{x_2 - x_0} \\ &= \frac{-0.91240 - (-0.86700)}{1.2 - 1.0} = -0.22700 \end{aligned}$$

For comparison, $f'\left(\frac{x_1 + x_0}{2}\right) = -\sin(1.05) = -0.86742$

General divided differences

Given $n+1$ distinct points $x_0, \ldots, x_n$, with $n \geq 2$, define

$$f[x_0, \ldots, x_n] = \frac{f[x_1, \ldots, x_n] - f[x_0, \ldots, x_{n-1}]}{x_n - x_0}$$

This is a recursive definition of the n-th order divided difference of $f(x)$, using divided differences of order n. We can also relate this to the derivative as follows:

$$f[x_0, \ldots, x_n] = \frac{1}{n!} f^{(n)}(c)$$

for some c intermediate to the points $\{x_0, \ldots, x_n\}$.

Order of nodes

We see that

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(x_0) - f(x_1)}{x_0 - x_1} = f[x_1, x_0]$$

The order of x_0 and x_1 does not matter. We have for 2nd order divided difference:

$$\begin{aligned} f[x_0, x_1, x_2] &= \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} \\ &= \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)} \\ &\quad + \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)} \\ &\quad + \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)} \end{aligned}$$

Order of nodes

Using this formula, we can show that

$$f[x_0, x_1, x_2] = f[x_0, x_2, x_1]$$

Mathematically,

$$f[x_0, x_1, x_2] = f[x_{i_0}, x_{i_1}, x_{i_2}]$$

for any permutation (i_0, i_1, i_2) of $(0, 1, 2)$.

Newton's interpolating polynomial

Recall that the general interpolation problem: find a polynomial $P_n(x)$ of degree n for which

$$P_n(x_i) = y_i, \quad i = 0, 1, \dots, n$$

with given data points

$$(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n),$$

where $\{x_0, \dots, x_n\}$ distinct points.

Newton's interpolating polynomial

Let the data values be generated from a function $f(x)$:

$$y_i = f(x_i), \quad i = 0, 1, \ldots, n$$

Using divided differences $f[x_0, x_1], f[x_0, x_1, x_2], \ldots, f[x_0, \ldots, x_n]$ we can write the interpolation polynomials $P_1(x)$, $P_2(x), \ldots, P_n(x)$ in a way that is simple to compute.

$$\begin{aligned} P_1(x) &= f(x_0) + f[x_0, x_1](x - x_0) \\ P_2(x) &= f(x_0) + f[x_0, x_1](x - x_0) \\ &\quad + f[x_0, x_1, x_2](x - x_0)(x - x_1) \\ &= P_1(x) + f[x_0, x_1, x_2](x - x_0)(x - x_1) \end{aligned}$$

Newton's interpolating polynomial

For the case of general n-th degree polynomial, we have

$$\begin{aligned} P_n(x) &= f(x_0) + f[x_0, x_1](x - x_0) \\ &\quad + f[x_0, x_1, x_2](x - x_0)(x - x_1) \\ &\quad + \dots \\ &\quad + f[x_0, \dots, x_n](x - x_0)\dots(x - x_{n-1}) \end{aligned}$$

Therefore

$$P_n(x) = P_{n-1}(x) + f[x_0, \dots, x_n](x - x_0)\dots(x - x_{n-1})$$

in which $P_{n-1}(x)$ interpolates $f(x)$ at points in $\{x_0, \dots, x_{n-1}\}$

Example

Consider the table

x	1	1.1	1.2	1.3	1.4
$\cos x$	0.54030	0.45360	0.36236	0.26750	0.16997

Define

$$D^k f(x_i) = f[x_i, \ldots, x_{i+k}]$$

i	x_i	$f(x_i)$	$D^1 f(x_i)$	$D^2 f(x_i)$	$D^3 f(x_i)$	$D^4 f(x_i)$
0	1.0	0.54030	−0.8670	−0.2270	0.1533	0.0125
1	1.1	0.45360	−0.9124	−0.1810	0.1583	
2	1.2	0.36236	−0.9486	−0.1335		
3	1.3	0.26750	−0.9753			
4	1.4	0.16997				

These were computed using the recursive definition

$$f[x_0, \ldots, x_n] = \frac{f[x_1, \ldots, x_n] - f[x_0, \ldots, x_{n-1}]}{x_n - x_0}$$

Then

$$\begin{aligned} P_1(x) &= 0.5403 - 0.8670(x-1) \\ P_2(x) &= P_1(x) - 0.2270(x-1)(x-1.1) \\ P_3(x) &= P_2(x) + 0.1533(x-1)(x-1.1)(x-1.2) \\ P_4(x) &= P_3(x) + 0.0125(x-1)(x-1.1)(x-1.2)(x-1.3) \end{aligned}$$

We can now estimate cos(1.05) using various order polynomials and the results are tabulated below:

n	1	2	3	4
$P_n(1.05)$	0.49695	0.49752	0.49758	0.49757
Error	6.20×10^{-4}	5.0×10^{-5}	-1.0×10^{-5}	0.0

Error of interpolation

Let $P_n(x)$ be a n-th degree interpolating polynomial that agree with $f(x)$ on $n+1$ distinct points. Let the error $E(x)$ be defined as

$$E(x) = f(x) - P_n(x)$$

Since $E(x_i) = 0$ for $i = 0, \ldots, n$, we can write

$$E(x) = (x - x_0)(x - x_1)\ldots(x - x_n)g(x),$$

where the function $g(x)$ has to be determined.

We see that

$$f(x) - P_n(x) - (x - x_0)(x - x_1)\ldots(x - x_n)g(x) = 0$$

Error of interpolation

Let us introduce a new function

$$W(t) = f(t) - P_n(t) - (t - x_0)(t - x_1)\dots(t - x_n)g(x)$$

Note that $W(t) = 0$ at $t = x, x_0, x_1, \dots, x_n$.

Therefore, $W(t)$ has $n + 2$ zeros. We assume that $W(t)$ is continuous and differentiable and use mean value theorem. Therefore, $W'(c) = 0$ if c is a number intermediate in $\{x_0, x_1, \dots, x_n, x\}$.

Error of interpolation

Using mean value theorem repeatedly, we get

$$W^{n+1}(c) = 0 = f^{(n+1)}(c) - (n+1)!g(x)$$

for c intermediate in $\{x_0, x_n, x\}$.

Thus

$$g(x) = \frac{f^{(n+1)}(c)}{(n+1)!}$$

Hence

$$E(x) = (x - x_0)(x - x_1)\ldots(x - x_n)\frac{f^{(n+1)}(c)}{(n+1)!}$$

Error of interpolation

Example: If the function $f(x) = \sin x$ is approximated by a polynomial of degree 9 that interpolates $f(x)$ at ten points in the interval $[0, 1]$, estimate the interpolation error.