Interpolation

Interpolation is the process of estimating an intermediate value from a set of discrete or tabulated values.

Suppose we have the following tabulated values:

y	y_0	y_1	y_2??	y_3	y_4	y_5
x	x_0	x_1	x_2	x_3	x_4	x_5

How do we determine a missing value e.g. y_2?

Let us consider a polynomial $p(x)$ that fits to the tabulated values such that $y_i = p(x_i)$. Intermediate values of y can be estimated by evaluating this polynomial.

Interpolation

Example:

Develop an interpolation formula for the function $e^{0.5x}$ using the values at $x_0 = 0$ and $x_1 = 2$, and estimate the value of $e^{0.5x}$ at $x = 1$.

Solution:

Let $y = f(x) := e^{0.5x}$.

A simple approach would be to connect points $(x_0, f(x_0))$ and $(x_1, f(x_1))$ using a straight line $y = mx + c$.

Interpolation

We can evaluate y at $x = 1$, which is $y = m + c$ and this is the estimate of $e^{0.5x}$ at $x = 1$.

To find m and c, we use tabulated values to get the following:

$$\begin{aligned} y_0 &= mx_0 + c \\ y_1 &= mx_1 + c \end{aligned}$$

We solve this linear system using a method of our choice - such as - Gauss elimination, Jacobi etc.

Interpolation

Using Gauss elimination, we get

$$c = \frac{x_0 y_1 - x_1 y_0}{x_0 - x_1} \qquad m = \frac{y_0 - y_1}{x_0 - x_1}.$$

Therefore,

$$y = 1.0 + 0.8591x$$

is the interpolating polynomial.

Evaluating this polynomial at $x = 1$, we get $y = 1.8591$.

Using a calculator, we get the exact value is 1.6487. The error of this linear interpolation is 12.7626%.

What is the degree of the polynomial?

Interpolation: how to find the polynomial?

We can re-write the straight line such that

$$y = a_0 + a_1(x - x_0).$$

It is easy to see that $a_0 = y_0$ if $x = x_0$ and $a_1(x_1 - x_0) = y_1 - a_0$ if $x = x_1$ or $a_0 = 1.0$ and $a_1 = \frac{y_1 - y_0}{x_1 - x_0} = 0.8591$.

Similarly, we can write a polynomial of degree 2

$$y(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1)$$

and a polynomial of degree 3

$$y(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2)$$

Note that the number of tabulated data = the number of constants for each polynomial.

Interpolation

Find a general interpolating polynomial of degree n and discuss the limitations.

Let us suppose that we have a data set $\{(x_i, y_i)\}$ such that $y_i = f(x_i)$ for $i = 0, \ldots, n$.

Let the polynomial be of the form

$$f(x) = y = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} + a_n x^n.$$

To agree with the data set, the polynomial must pass through the point (x_i, y_i) for $i = 0, 1, \ldots, n$. Thus

$$y_i = a_0 + a_1 x_i + a_2 x_i^2 + \cdots + a_{n-1} x_i^{n-1} + a_n x_i^n.$$

Interpolation

These equations can be expressed in matrix form as

$$\mathbf{Ba} = \mathbf{y},$$

where the Vandermonde matrix is

$$\mathbf{B} = \begin{bmatrix} 1 & x_0 & x_0^2 & x_0^3 & \cdot & \cdot & \cdot & x_0^n \\ 1 & x_1 & x_1^2 & x_1^3 & \cdot & \cdot & \cdot & x_1^n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & x_n^3 & \cdot & \cdot & \cdot & x_n^n \end{bmatrix}$$

$$\mathbf{a} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

Interpolation

Home work exercise

The Vandermonde matrix is given by

$$\mathbf{B}\mathbf{a} = \mathbf{y},$$

where $\mathbf{B} = [x_i^j]$ for $i, j = 0, \ldots, n$, $\mathbf{a} = [a_0, \ldots, a_n]^T$, and $\mathbf{y} = [y_0, \ldots, y_n]^T$.

1. Show that
$$det(\mathbf{B}) = \Pi_{0 \leq j < i \leq n}(x_i - x_j)$$
2. Let $y_i = f(x_i)$ for $i = 0, \ldots, n$ represent $n + 1$ distinct data set. Show that there is a unique interpolating polynomial $p(x)$ of degree $\leq n$ such that $p(x_i) = y_i$.

Interpolation: remarks

- The system of equations $\mathbf{Ba} = \mathbf{y}$ can be solved using either a direct method (e.g. Gauss elimination) or an iterative method (e.g. Jacobi).
- A solution exists if and only if the rank of the augmented matrix $[\mathbf{B} \,|\, \mathbf{y}]$ is equal to that of the coefficient matrix $[\mathbf{B}]$.
- The solution will be unique if the rank is n. If rank $< n$, then one or more equations are redundant.
- However, note that the system is sensitive to the condition number of the matrix $\mathbf{B}$, and hence determining the polynomial such that $\mathbf{a} = (\mathbf{B})^{-1}\mathbf{y}$ is not an efficient approach.

Fitting a polynomial to a data set

Suppose we have the following tabulated data. Let us estimate $f(3.0)$ by fitting a cubic polynomial to given data. Let $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ be a cubic polynomial.

x	$y = f(x)$
3.2	22.0
2.7	17.8
1.0	14.2
4.8	38.3
5.6	51.7

We get

$$\begin{array}{ll} x = 3.2 & a_0 + a_1(3.2) + a_2(3.2)^2 + a_3(3.2)^3 = 22.0 \\ x = 2.7 & a_0 + a_1(2.7) + a_2(2.7)^2 + a_3(2.7)^3 = 17.8 \\ x = 1.0 & a_0 + a_1(1.0) + a_2(1.0)^2 + a_3(1.0)^3 = 14.2 \\ x = 4.8 & a_0 + a_1(4.8) + a_2(4.8)^2 + a_3(4.8)^3 = 38.3 \end{array}$$

Solving these equations, we get $a_0 = 24.3499$, $a_1 = -16.1177$, $a_2 = 6.4952$, $a_3 = -0.5275$. Hence the polynomial is

$$24.3499 - 16.1177x + 6.4952x^2 - 0.5275x^3.$$

The derivation of Lagrange polynomial

The Lagrange interpolation does not require the solution of a linear system. Let us now derive a quadratic Lagrange interpolation formula.

Consider a quadratic polynomial that passes through three points (x_i, y_i), $i = 0, 1, 2$ and express this polynomial in the form

$$l(x) = a_0(x - x_1)(x - x_2) + a_1(x - x_0)(x - x_2) + a_2(x - x_0)(x - x_1),$$

where a_0, a_1, and a_2 are constants that have to be determined such that $l(x)$ passes through given data points (x_i, y_i), $i = 0, 1, 2$ such that $l(x_i) = y_i$.

The derivation of Lagrange polynomial

Therefore, we get

$$
\begin{aligned}
y_0 &= a_0(x_0 - x_1)(x_0 - x_2) \\
y_1 &= a_1(x_1 - x_0)(x_1 - x_2) \\
y_2 &= a_2(x_2 - x_0)(x_2 - x_1)
\end{aligned}
$$

Hence

$$
\begin{aligned}
a_0 &= \frac{y_0}{(x_0 - x_1)(x_0 - x_2)} \\
a_1 &= \frac{y_1}{(x_1 - x_0)(x_1 - x_2)} \\
a_2 &= \frac{y_2}{(x_2 - x_0)(x_2 - x_1)}
\end{aligned}
$$

Using values of a_0, a_1, and a_2 in $I(x)$, we get,

$$I(x) = y_0 \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + y_1 \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + y_2 \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

We can also write

$$I(x) = \sum_{i=0}^{2} y_i L_i(x),$$

where

$$L_i(x) = \prod_{j=0, j\neq i}^{2} \left[\frac{x-x_j}{x_i-x_j} \right], \quad i = 0, 1, 2$$

Example

Write out the Lagrangian polynomial from this table:

x	2.1	4.1	7.1
y	-12.4	7.3	10.1

Solution: Let

$$I(x) = \sum_{i=0}^{n} y_i L_i(x),$$

where

$$L_i(x) = \prod_{j=0, j\neq i}^{2} \left(\frac{x - x_j}{x_i - x_j} \right), \quad i = 0, 1, 2$$

Example

Therefore

$$
\begin{aligned}
L(x) &= -12.4\frac{(x-4.1)(x-7.1)}{(2.1-4.1)(2.1-7.1)} \\
&+ 7.3\frac{(x-2.1)(x-7.1)}{(4.1-2.1)(4.1-7.1)} \\
&\quad +10.1\frac{(x-2.1)(x-4.1)}{(7.1-2.1)(7.1-4.1)} \\
&= -1.7833x^2 + 20.9067x - 48.4395
\end{aligned}
$$

Example Verify that $L(x)$ reproduces the y values for each x values.

Solution We find that $L(2.1) = -12.4$

$L(4.1) = 7.3$

$L(7.1) = 10.1$

Example Using $L(x)$ estimate y at $x = 3$ and at $x = 8$.

Solution We find that $L(3) = -1.7695$ and

$L(8) = 4.6805$

Example

Write out the cubic Lagrangian polynomial from this table:

x	2	4	3	8
y	1	3	5	9

Solution: Let

$$I(x) = \sum_{i=0}^{n} y_i L_i(x),$$

where

$$L_i(x) = \prod_{j=0, j\neq i}^{2} \left(\frac{x - x_j}{x_i - x_j} \right), \quad i = 0, 1, 2$$

$$L(x) = 1\frac{(x-4)(x-3)(x-8)}{(2-4)(2-3)(2-8)}$$

$$+3\frac{(x-2)(x-3)(x-8)}{(4-2)(4-3)(4-8)} \qquad +5\frac{(x-2)(x-4)(x-8)}{(3-2)(3-4)(3-8)}$$

$(x-2)(x-4)(x-3)$ … 37 … 171 … 1051 … 199