Newton's method

Let x_0 be an approximation to the solution of $f(x) = 0$ and h be the error such that $f(x_0 + h) = 0$.

We aim to approximate h and set $x_1 = x_0 + h$.

Taylor's series expansion gives:

$$f(x_0 + h) = f(x_0) + hf'(x_0) + \text{ h.o.t..}$$

Then

$$f(x_0) + hf'(x_0) \approx 0$$

Newton's method

We get

$$h \approx -\frac{f(x_0)}{f'(x_0)},$$

We set

$$x_1 = x_0 + h = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Repeating the process, we write

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Convergence of the Newton's method

The Newton's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

can be treated as a fixed point iteration

$$x_{n+1} = g(x_n)$$

if we define

$$g(x_n) = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Convergence of the Newton's method

Let

$$g(x) = x - \frac{f(x)}{f'(x)}.$$

Therefore

$$g'(x) = \frac{f(x)f''(x)}{[f'(x)]^2}.$$

Let x^* be the true solution such that $f(x^*) = 0$. We can show that

$$g'(x^*) = \frac{f(x^*)f''(x^*)}{[f'(x^*)]^2} = 0.$$

Convergence of the Newton's method

The Taylor series expansion of $g(x_n)$ implies

$$g(x_n) = g(x^*) + \cancel{g'(x^*)(x^* - x_n)} + g''(\xi)(x^* - x_n)^2/2$$

Let $e_n = x^* - x_n$ be the error. Therefore, error after n-th iteration is

$$e_{n+1} = x^* - x_{n+1} = g(x^*) - g(x_n) = -g''(\xi)(e_n)^2/2$$

$$|e_{n+1}| \leq |g''(\xi)|e_n|^2.$$

The rate of convergence for Newton's method is quadratic.

Examples

Example: Let $f(x) = 4x^3 - 1 - e^{x^2/2}$

Q. Find intervals for two roots of $f(x)$.

We calculate $f(0) = -2$, $f(1) = 1.3513$, $f(2) = 23.611$, $f(3) = 16.983$, $f(4) = -2726$.

We expect one root in the interval $[0, 1]$ and another root in the interval $[3, 4]$

Examples

Q. Use Newton's method to find the root in $[0, 1]$

Solution: $f(x) = 4x^3 - 1 - e^{x^2/2}$

$f'(x) = 12x^2 - xe^{x^2/2}$.

Let $x_0 = 0.5$. Then

$$f(x_0) = -1.6331,\ f'(x_0) = 2.4334,\ h = -\frac{f(x_0)}{f'(x_0)} = 0.6711$$

Therefore $x_1 := x_0 + h = 1.1711$.

$$f(x_1) = 3.4397,\ f'(x_1) = 14.1335,\ h = -0.2433$$

Therefore $x_2 := x_1 + h = 0.9277$.

$$f(x_2) = 0.6563,\ f'(x_2) = 8.9021,\ h = -0.0073$$

Therefore $x_3 := x_2 + h = 0.8540$.

Examples

Q. Since the convergence of Newton's method depends on the starting solution, can you propose a better starting solution.

Solution: From the previous calculation, we see that the desired root is in the interval $[0, 1]$.

Using Bisection method, we can assume that the desired root is $(0 + 1)/2 = 0.5$.

We see that $f(0.5) \cdot f(1) < 0$.

We can pick an initial guess in the interval $[0.5, 1]$.

Using bisection method $x_0 = (0.5 + 1)/2 = 0.75$ would be a better choice for starting solution.

Examples

Q. Can you find the error bound for the improved initial guess in the interval $[0, 1]$.

Solution: For the improved initial guess, we have used 2 steps of bisection method.

Therefore, the error bound for the improved initial guess is

$$(b - a)/2^n = \frac{1}{2^2} = 0.25.$$

Examples

Q. We see that 3 iterations are necessary for Newton's method to get an error $\mathcal{O}(10^{-1})$. How many iterations will require to get similar accuracy if the Bisection method was used?

Solution: We solve $2^{-n}(b-a) = 10^{-2}$.

$$n = \frac{\ln(b-a) + 2\ln(5)}{\ln(2)} + 2 \sim 7$$

About 7 or more iterations are required, if the Bisection method were used. For this example, the Newton's method improves the speed by a factor of about $2-3$ ($\sim 7/3$) compared to the Bisection method.

Example

- Let $f(x) = 2\sin(x) - e^x/4 - 1$ is zero for two values near $x = -5$.

  1. Propose two initial intervals so that Bisection method would converge to the solution of $f(x) = 0$.

  2. How many iterations are required to agree an error bound 10^{-6}?

Example

Solution:

$f(-7) = -2.3142$, $f(-5) = 0.9162$, and $f(-3) = -1.2947$

We must have one solution in $[-7, -5]$ and the other in $[-5, -3]$.

To agree an error bound 10^{-6}, we must solve

$$\frac{b-a}{2^n} = 10^{-6}$$

Example

$$\ln(b-a) - n\ln 2 = -6\ln(10)$$

$$n\ln 2 = \ln(b-a) + 6\ln 5 + 6\ln 2$$

$$n = \frac{\ln(b-a) + 6\ln 5}{\ln 2} + 6$$

For the interval $[-5, -3]$ we need about 21 iterations.

For the interval $[-7, -5]$ we also need about 21 iterations.

Example

- Let $f(x) = (x - 0.3)(x - 0.5)$. Clearly $f(x) = 0$ has two roots $x = 0.3$ and $x = 0.5$.
  1. If you take an initial interval $[0.1, 0.6]$ for bisection method, which root do you expect to converge. Explain your answer.
  2. Determine two good starting intervals so that bisection method converges to the exact root.
  3. Let $[0, 0.49]$ be the starting interval. Which root will the bisection method converge to?

Solution

The interval $[0.1, 0.6]$ is not suitable for bisection method because $f(x)$ does not change sign in this interval.

By inspection, $f(0.2) = 0.03$, $f(0.4) = -0.01$, and $f(0.6) = 0.03$. So $[0.2, 0.4]$ and $[0.4, 0.6]$ are good choices.

The starting interval $[0, 0.49]$ is expected to converge to $x^* = 0.3$