Fixed point iteration

Suppose that we can re-arrange $f(x)$ such that

$$f(x) = x - g(x).$$

Then, we construct the following scheme:

$$x_{n+1} = g(x_n), \quad n = 0, 1, 2, \ldots.$$

The above scheme is known as fixed point iteration.

Fixed point iteration

Theorem: Derive the rate of convergence for the fixed point iteration method.

We have $x_{n+1} = g(x_n)$. Let x^* be the true solution. Then, error after $n+1$ iteration is given by

$$e_{n+1} := x^* - x_{n+1} = g(x^*) - g(x_n).$$

This can be expressed as

$$e_{n+1} = \frac{g(x^*) - g(x_n)}{x^* - x_n}(x^* - x_n).$$

Fixed point iteration

Using mean value theorem, there exists a number ξ_n in the interval $[x^*, x_n]$ such that

$$g'(\xi_n) = \frac{g(x^*) - g(x_n)}{x^* - x_n}.$$

This implies that

$$|e_{n+1}| \leq |g'(\xi_n)| \cdot |e_n|.$$

The magnitude of $g'(\xi_n)$ tells us whether e_n is a decreasing sequence or not.

Fixed point iteration

If $|g'(\xi_n)| < 1$ then $|e_{n+1}| \leq |e_n|$.

Therefore, fixed-point method converges linearly if initial guess is taken from an interval, where $|g'(\xi_n)| < 1$.