Iterative method

Let x^k denotes the approximate solution of $Ax = b$, then we can write

$$x^{k+1} = M^{-1}(b - Bx^k)$$

or

$$x^{k+1} = Gx^k + M^{-1}b.$$

x^k is called k-th iterate.

$G = -M^{-1}B$ is called the iteration matrix.

M is usually picked such that the resulting system is easy to solve.

Iterative method

Jacobi iteration

We can rewrite this process as

$$x^{k+1} = D^{-1}[b - (L + U + D - D)x^k],$$

which gives

$$x^{k+1} = D^{-1}[b - (L + U + D - D)x^k].$$

Therefore

$$x^{k+1} = x^k + D^{-1}\underbrace{[b - Ax^k]}_{r^k}.$$

Iterative method

Jacobi iteration

We have the Jacobi iteration matrix $-D^{-1}(L+U)$, where $A = L + D + U$.

Let $A = [a_{ij}]$ and $x = [x_j]$, where $i, j = 1, 2 \ldots, n$.

We can write

$$Ux = \sum_{j=i+1}^{n} a_{ij} x_j$$

and

$$Lx = \sum_{j=1}^{i-1} a_{ij} x_j.$$

Iterative method

Jacobi iteration

Therefore, the Jacobi iteration scheme can be written as

$$x_i^{k+1} = \frac{1}{a_{ii}}\left[b_i - \sum_{j=1}^{i-1} a_{ij}x_j^k - \sum_{j=i+1}^{n} a_{ij}x_j^k\right], \quad i = 1,2,\ldots,n;\ k = 1,2,\ldots,$$

$$x_i^{k+1} = \frac{1}{a_{ii}}\left[b_i - \sum_{j=1,j\neq i}^{n} a_{ij}x_j^k\right], \quad i = 1,2,\ldots,n;\ k = 1,2,\ldots,\ldots$$

Iterative method

Gauss-Seidel iteration:

Consider the scheme

$$Dx^{k+1} = b - Lx^k - Ux^k.$$

If we replace assumed values x_j^k with the approximated values x_j^{k+1} as soon as they are available, we get

$$Dx^{k+1} = b - Lx^{k+1} - Ux^k.$$

The Gauss-Seidel iteration takes the following form

$$x^{k+1} = (L + D)^{-1}[b - Ux^k].$$

Iterative method

Gauss-Seidel iteration:

This can also be stated as:

$$x_i^{k+1} = \frac{1}{a_{ii}} \left[b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{k+1} - \sum_{j=i+1}^{n} a_{ij} x_j^{k} \right], \quad i = 1, 2, \ldots, n; \; k = 1, 2, \ldots$$

Remark:

Jacobi method calculates all x_i^{k+1} using old values x_i^k.

Gauss-Seidel method uses x_i^{k+1} for calculating x_{i+1}^{k+1}

Iterative method

Convergence criterion iterative methods

The iterative process can be stopped if

$$\|b - Ax^{k+1}\| \leq \epsilon \text{ or } \|x^{k+1} - x^k\| \leq \epsilon$$

or if

$$\frac{\|b - Ax^{k+1}\|}{\|b\|} \leq \epsilon \text{ or } \frac{\|x^{k+1} - x^k\|}{\|x^k\|} \leq \epsilon$$

Iterative method

Convergence criterion:

The sufficient condition for convergence is the diagonally dominant property;

$$|a_{ii}| > \sum_{j=1, i\neq j}^{n} |a_{ij}|.$$

If the matrix A is diagonally dominant, both the Jacobi method and the Gauss-Seidel method converge irrespective of the initial values x^k.