Direct method

Remarks:

- Zeros may be created in the diagonal position.
- Rearrange the system to have the largest coefficient in the diagonal position.
- Where there are many equations, the round-off error may cause large effect.
- If the matrix is ill-conditioned, results are particularly sensitive to round-off error.

Direct method

LU decomposition method:

Any square matrix A can be expressed as

$$A = LU,$$

where L is the lower triangular matrix:

$$L = \begin{bmatrix} l_{11} & 0 & 0 & \cdot & \cdot & 0 \\ l_{21} & l_{22} & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ l_{n1} & l_{n2} & 0 & \cdot & \cdot & l_{nn} \end{bmatrix}$$

Direct method

and U is the upper triangular matrix:

$$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} & \cdot & \cdot & u_{1n} \\ 0 & u_{22} & u_{23} & \cdot & \cdot & u_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & u_{nn} \end{bmatrix}$$

Direct method

LU decomposition method:

A given linear system takes the following form using LU decomposition:

$$A\mathbf{x} = LU\mathbf{x} = \mathbf{b}.$$

Define the vector $\mathbf{z} = U\mathbf{x}$.

The system takes the form:

$$L\mathbf{z} = \mathbf{b}.$$

Direct method

LU decomposition method:

First, solve

$$L\mathbf{z} = \mathbf{b}$$

for $\mathbf{z}$.

Secondly, set

$$U\mathbf{x} = \mathbf{z}$$

and solve for $\mathbf{x}$.

Direct method

Example:

Develop an algorithm for LU factorization.

Solution

There are three algorithms available.

1. Doolittle's factorization.
   $l_{ii} = 1$ for $1 \leq i \leq n$.
2. Crout's factorization.
   $u_{ii} = 1$ for $1 \leq i \leq n$.
3. Cholesky's factorization.
   $U = L^T$ so that $l_{ii} = u_{ii}$ for $1 \leq i \leq n$.

Direct method

Crout's factorization.

The elements of L and U are determined by

$$u_{ii} = 1;\ i = 1, 2, \ldots, n.$$

$$l_{ij} = \left\{ a_{ij} - \sum_{k=1}^{j-1} l_{ik} u_{kj} \right\};\ i \geq j;\ i = 1, 2, \ldots, n$$

$$u_{ij} = \left\{ \frac{a_{ij} - \sum_{k=1}^{i-1} l_{ik} u_{kj}}{l_{ii}} \right\};\ i < j;\ j = 2, 3, \ldots, n.$$

Eigenvalues and eigenvectors

If A is a square matrix, then the roots λ's of $\det(A - \lambda I){=}0$ are eigenvalues of A. Further, if $x = [x_1, x_2, \ldots, x_n]^T \neq 0$ satisfies $(A - \lambda I)x = 0$, then x is an eigenvector.

Eigenvalues and eigenvectors

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With matlab

`[V,D]=eig(A)` produces a diagonal matrix `D` of eigenvalues and full matrix `V` whose columns are the corresponding eigenvectors so that `A*V=V*D`.

Eigenvalues and eigenvectors

The spectral radius of a matrix A is defined as

$$\rho(A) = \max |\lambda|,$$

where λ is an eigenvalue of A.

Norms

Vector norm:

l_p norm of the vector $x = [x_1, x_2, \ldots, x_n]^T$ is

$$\|x\|_p = \left\{ \sum_{i=1}^{n} x_i^2 \right\}^{1/p},$$

where $0 < p < \infty$, and the l_∞ norm is

$$\|x\|_\infty = \max_{1 \le i \le n} |x_i|$$

Norms

Exercise

For each $x \in \mathbb{R}^n$, show that $\|x\|_\infty \leq \|x\|_2 \leq \sqrt{n}\|x\|_\infty$

Matrix norm:

The norm of the matrix is a measure of the magnitude of the matrix, and is denoted by $\|A\|$.

Norms

Matrix norm:

$$\|A\|_1 = \max_{1\le j\le n} \sum_{i=1}^{n} |a_{ij}| = \text{maximum column sum}$$

$$\|A\|_\infty = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}| = \text{maximum row sum}$$

$$\|A\|_f = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2 \right)^{1/2} \quad \text{(Frobenius norm)}$$

Norms

L_2 norm of a matrix:

The spectral radius of a matrix A is defined as

$$\rho(A) = \max_{1 \le i \le n} |\lambda_i|,$$

where λ_i's are eigenvalues of A. This is also known as the L_2 norm of a matrix.

Norms

Properties of matrix norm:

1. $\|A\| \geq 0$ and $\|A\| = 0$ if and only if $A = 0$.
2. $\|kA\| = |k|\|A\|$.
3. $\|A + B\| \leq \|A\| + \|B\|$.
4. $\|AB\| \leq \|A\|\|B\|$.

Sensitivity of linear system

What are ill conditioned systems $A\mathbf{x} = \mathbf{b}$.

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For an ill-conditioned system, the matrix A is nearly singular.

Sensitivity of linear system

What are ill conditioned systems $A\mathbf{x} = \mathbf{b}$.

A system is called ill-conditioned if small change in $\mathbf{b}$ causes a large change in $\mathbf{x}$.

For an ill-conditioned system, the matrix A is nearly singular.

The degree of ill-conditioning of a matrix is measured by its condition number.

Sensitivity of linear system

Problem: Express upper and lower bounds for relative error using the condition number of the coefficient matrix.

Solution:

Define the error $e = \mathbf{x} - \tilde{x}$.

Sensitivity of linear system

Hence, $\mathbf{e} = A^{-1}\mathbf{r}$.

Taking norms: $\|\mathbf{e}\| \leq \|A^{-1}\|\|\mathbf{r}\|$.

Sensitivity of linear system

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Taking norms: $\|\mathbf{e}\| \leq \|A^{-1}\|\|\mathbf{r}\|$.

Taking the norm of $\mathbf{r} = A\mathbf{e}$, we get $\|\mathbf{r}\| \leq \|A\|\|\mathbf{e}\|$.

Therefore,

$$\frac{\|\mathbf{r}\|}{\|A\|} \leq \|\mathbf{e}\| \leq \|A^{-1}\| \, \|\mathbf{r}\|.$$

Sensitivity of linear system

Taking norm of $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} = A^{-1}\mathbf{b}$, we get

$$\frac{\|\mathbf{b}\|}{\|A\|} \leq \|\mathbf{x}\| \leq \|A^{-1}\| \, \|\mathbf{b}\|.$$

Sensitivity of linear system

Taking norm of $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} = A^{-1}\mathbf{b}$, we get

$$\frac{\|\mathbf{b}\|}{\|A\|} \leq \|\mathbf{x}\| \leq \|A^{-1}\| \, \|\mathbf{b}\|.$$

Hence,

$$\frac{1}{cond(A)} \frac{\|\mathbf{r}\|}{\|\mathbf{b}\|} \leq \frac{\|\mathbf{e}\|}{\|\mathbf{x}\|} \leq cond(A) \frac{\|\mathbf{r}\|}{\|\mathbf{b}\|}.$$

This is the desired relationship. (Here, $cond(A) = ||A|| \cdot ||A^{-1}||$ or $C(A) = ||A|| \cdot ||A^{-1}||$.

Sensitivity of linear system

Relative error can be as large as the relative residual multiplied by the condition number.

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Relative error can also be as small as the relative residual divided by the condition number.

If condition number is large, relative residual gives little information about the accuracy.

If the condition number is near unity, the relative residual is a good measure of the relative error.

Sensitivity of linear system

Problem:

Show that the error of the solution relative to the norm of the computed solution can be as large as the relative error in the coefficients of A multiplied by the condition number of A.

Proof:

Want to show that

$$\frac{\|\mathbf{x} - \tilde{x}\|}{\|\tilde{\mathbf{x}}\|} \leq C(A)\frac{\|E\|}{\|A\|}.$$

Sensitivity of linear system

Let $\tilde{A} = A + E$ be the perturbed coefficient matrix. We desire to know how large is $\mathbf{x} - \tilde{x}$, where $\mathbf{x}$ is the solution of $A\mathbf{x} = \mathbf{b}$ and $\tilde{x}$ is the solution of $\tilde{A}\tilde{x} = \mathbf{b}$.

Sensitivity of linear system

Let $\tilde{A} = A + E$ be the perturbed coefficient matrix. We desire to know how large is $\mathbf{x} - \tilde{x}$, where $\mathbf{x}$ is the solution of $A\mathbf{x} = \mathbf{b}$ and $\tilde{x}$ is the solution of $\tilde{A}\tilde{x} = \mathbf{b}$.

Since $A\mathbf{x} = \mathbf{b}$ and $\tilde{A}\tilde{x} = \mathbf{b}$, we get

$$\mathbf{x} = A^{-1}(\tilde{A}\tilde{x}) = \tilde{x} + A^{-1}E\tilde{x}.$$

Taking norm of $\mathbf{x} - \tilde{\mathbf{x}}$ we conclude the proof.

Sensitivity of linear system

Exercise

Suppose that $\tilde{x}$ is an approximation to the solution of $Ax = b$, A is a non-singular matrix, and r is the residual vector for $\tilde{x}$. Then, show that

(i) $\|x - \tilde{x}\| \leq \|r\| \cdot \|A^{-1}\|$

(ii) $\dfrac{\|x - \tilde{x}\|}{\|x\|} \leq \|A\| \cdot \|A^{-1}\| \dfrac{\|r\|}{\|b\|}$.

Iterative method

Can we think for an alternate method for solving $Ax = b$?

The solution requires the inversion of the coefficient matrix:

$$x = A^{-1}b.$$

In `Matlab`, we use

```
x = A\b
```

Iterative method

Let us consider the system:

$$\begin{aligned} 6x - 2y + z &= 11 \\ -2x + 7y + 2z &= 5 \\ x + 2Y - 5z &= -1 \end{aligned}$$

We have a 3×3 matrix

$$A = \begin{bmatrix} 6 & -2 & 1 \\ -2 & 7 & 2 \\ 1 & 2 & -5 \end{bmatrix}$$

Iterative method

How about any of the following matrices:

$$L = \begin{bmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ 1 & 2 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -5 \end{bmatrix}, \quad U = \begin{bmatrix} 0 & -2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Each of these matrices are easy to invert.

None of these matrices are obtained by an appropriate elimination process.

$Dx = b$ is easier to solve than $Ax = b$.

Iterative method

Let us go back to the example and solve each equation for one variable.

$$
\begin{aligned}
6x - 2y + z &= 11 \\
-2x + 7y + 2z &= 5 \\
x + 2y - 5z &= -1
\end{aligned}
$$

Iterative method

Let us go back to the example and solve each equation for one variable.

$$
\begin{aligned}
x &= \frac{1}{6}(11 + 2y - z) \\
y &= \frac{1}{7}(5 + 2x - 2z) \\
z &= \frac{1}{-5}(-1 - x - 2y)
\end{aligned}
$$

Iterative method

If we assume

$$x = 0, y = 0, z = 0$$

an approximate solution, then we get

$$x = 11/6, y = 5/7, z = 1/5.$$

We can switch between assumed solution and obtained solution to end up with an approximate solution iteratively.

Iterative method

Jacobi iteration

Let us consider the decomposition

$$A = L + D + U,$$

where

D is the diagonal elements of A,

L is the strictly lower triangular elements of A, and

U is the strictly upper triangular elements of A.

Iterative method

Jacobi iteration

We can re-write the system $Ax = b$ as

$$Dx = b - (L + U)x.$$

Since D is a diagonal matrix and it is easy to find D^{-1}, we construct the Jacobi method:

$$x^{k+1} = D^{-1}[b - (L + U)x^k].$$

Iterative method

Let $A = M + B$. A linear system $Ax = b$ can be written as

$$Mx = b - Bx,$$

where M is a non-singular matrix that is easy to invert.

Therefore, one writes

$$x = M^{-1}(b - Bx).$$