DISSERTATION

Titel der Dissertation

SYMMETRY METHODS IN THE ATMOSPHERIC SCIENCES

Verfasser

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angestrebter akademischer Grad

Doktor der Naturwissenschaften (Dr. rer. nat.)

Wien, im Oktober 2010

Studienkennzahl lt. Studienblatt:	A 091 415
Dissertationsgebiet lt. Studienblatt:	Meteorologie
Betreuer:	emer. O. Univ.-Prof. Dr. Michael Hantel

Abstract

Wide ranges of symmetry methods are applied to several differential equations arising in the atmospheric sciences. Lie point symmetries of the barotropic vorticity equation, the barotropic potential vorticity equation and the two-layer baroclinic model are computed. One- and two-dimensional inequivalent subalgebras of the respective maximal Lie invariance algebras are classified. Based on this classification, we determine various group-invariant solutions of the investigated differential equations. The physical relevance of these particular solutions is evaluated. Symmetries are used to find point transformations that map the barotropic potential vorticity equation on the β-plane and the barotropic vorticity equation on the rotating sphere to the respective equations in the inertial frame. Two refined techniques for the computation of the complete point symmetry group of differential equations are proposed within the framework of the direct method. The first technique is based on the invariance of megaideals of the maximal Lie invariance algebra under automorphisms generated by point symmetries. The second technique involves knowledge on the admissible transformations of classes of differential equations containing the given equation. It is shown how symmetries can be employed to determine closure schemes in the course of the parameterization problem. The methods we apply rest on techniques of direct and inverse group classifications. These methods are exemplified by parameterizing the eddy vorticity flux in the Reynolds averaged vorticity equation. This leads to several invariant parameterization schemes possessing different degrees of symmetry. The symmetries of the barotropic vorticity equation and the Saltzman convection equations are used to derive spectral finite-mode approximations. This is done using both Lie and discrete point symmetries as a criterion for the selection of Fourier modes. It is proved that the Lorenz–1960 model can be systematically re-derived with the aid of point symmetries of the vorticity equation. In a similar manner, it is demonstrated that the selection of modes for the Lorenz–1963 convection model is not compatible with the symmetries of the Saltzman equations. It is shown that the Hamiltonian and Nambu structures of the Lorenz–1963 model are not related to the Hamiltonian and Nambu forms of the Saltzman convection equations. A new six-component truncation of the convection equation is proposed. The selection of modes for this model is based on point symmetries of the convection equations. These modes are suitably scaled to allow the six-component model to be of Hamiltonian and Nambu forms analog to those of the original Saltzman equations.

Zusammenfassung

Zahlreiche Symmetriemethoden werden auf Differentialgleichungen der Atmosphärendynamik angewandt. Die Lie-Punktsymmetrien der barotropen Vorticitygleichung, der barotropen potentiellen Vorticitygleichung und des baroklinen Zweischichtmodels werden berechnet. Ein- und zweidimensionale inäquivalente Subalgebren der jeweiligen maximalen Lie-Invarianzalgebren werden klassifiziert und dazu verwendet, exakte Lösungen der jeweiligen Gleichungen zu bestimmen. Die physikalische Bedeutung dieser Lösungen wird untersucht und diskutiert. Mittels der Symmetrien der barotropen potentiellen Vorticitygleichung auf der β-Ebene und der barotropen Vorticitygleichung auf der rotierenden Kugel können Punkttransformationen gefunden werden, die beide Gleichungen in die jeweiligen Gleichungen im Inertialsystem transformieren. Zwei erweiterte Techniken zur Berechnung der gesamten Punktsymmetriegruppe von Differentialgleichungen werden vorgestellt, die im Rahmen der direkten Methode angewandt werden können. Die erste Technik basiert auf der Invarianz von Megaidealen der maximalen Lie-Invarianzalgebra unter von Punktsymmetrien erzeugten Automorphismen. Die zweite Technik verwendet Kenntnisse über *admissible transformations* von Klassen von Differentialgleichungen, die die untersuchte Gleichung enthalten. Weiters wird gezeigt wie Symmetrien dazu verwendet werden können, Schließungen im Zuge des Parameterisierungsproblems zu definieren. Für diesen Zweck werden Verfahren der direkten und inversen Gruppenklassifikation benützt. Als Beispiel werden verschiedene Parameterisierungen für den Eddy-Vorticityfluß in der Reynolds-gemittelten Vorticitygleichung konstruiert, die unterschiedliche Symmetrieeigenschaften besitzen. In einem weiteren Schritt werden die Symmetrien der barotropen Vorticitygleichung und der Saltzman'schen Konvektionsgleichungen dazu verwendet um spektrale, niedrigdimensionale Approximationen dieser Gleichungen zu erzeugen. Dazu werden Lie-Punkt- und diskrete Symmetrien als Kriterium zur Auswahl der Fouriermoden verwendet. Es wird bewiesen dass das Lorenz–1960 Modell systematisch unter Zuhilfenahme der Punktsymmetrien der Vorticitygleichung ableitbar ist. Auf ähnliche Weise wird demonstriert dass die Wahl der Moden des Lorenz–1963 Modells der thermischen Konvektion nicht mittels Symmetrien begründbar ist. Zudem wird gezeigt dass sowohl die Hamiltonsche als auch die Nambu Form des Lorenz–1963 Modells nicht mit der entsprechenden Hamiltonschen bzw. Nambu-Darstellung der Saltzman'schen Konvektionsgleichungen zusammenhängen. Aus diesem Grund wird ein sechskomponentiges Modell der Konvektionsgleichungen abgeleitet. Die Modenwahl dieses neuen Modells basiert vollständig auf Punktsymmetrien der Saltzman'schen Gleichungen. Durch geeignetes Skalieren dieser Moden ist es möglich eine Hamiltonsche bwz. Nambu-Darstellung dieses sechskomponentigen Modells zu finden, die der Hamilton- bzw. Nambuformulierung der kontinuierlichen Konvektionsgleichungen vollständig analog ist.

Contents

Chapter 1

Preface

1.1 General introduction

1.1.1 Motivation

Today's range of applicability of symmetry methods is enormous and it not easy to account appropriately even for the most recent directions in this field. Presently, symmetries play an important role in mathematics, chemistry, engineering and in almost all branches of theoretical physics, including classical mechanics, quantum mechanics and relativity.[1] One reason for the overall prominence of the concept of symmetry is its nativeness and its simplicity. Intuitively speaking, a symmetry is a *transformation* of an *object* leaving this object invariant. This is clearly such a general property that it can be recovered almost everywhere in nature and, correspondingly, in numerous areas of science and art. To be more specific, in the course of the thesis, our *objects* will be several differential equations of the atmospheric sciences and our *transformations* will be point transformations *preserving* these equations or *relating* them to each other.[2]

By definition, symmetries are attributes of their associated objects and thus in some sense provide an inverse way to characterize these objects. That is, by studying the transformations that leave an object invariant, we can already learn about the object itself. While this observation might appear to be somewhat trivial for symmetries of geometric objects, it formalizes in a rather nontrivial way in the field of symmetry analysis of differential equations. The most inspiring example of this finding stems from inverse group classification: Any differential equation can be represented as a function of the differential invariants of its admitted Lie symmetry group [118]. In other words, the knowledge of the symmetries of a differential equation (i.e. the *transformations*) suffice to determine the differential equation (i.e. the *object*) itself. Indeed, this is a main motivation for investigating symmetries of differential equations: They help to understand these equations, which is of inestimable value especially for all those differential equations, for which it is difficult to determine their general (or even only particular) solution(s) systematically.

Needless to say that symmetries are more than just a means to characterize differential equations. They can be used for several practical purposes as reviewed in the following sections. In fact, it is astonishing how many methods were developed and successfully applied in the field of

[1] Not to even mention the countless occurrences of symmetries in the fields of biology, fine arts and architecture.

[2] Whenever we subsequently write in general about *differential equations*, we indeed mean systems of either ordinary or partial differential equations.

symmetry analysis of differential equations during the past 50 years. It is the aim of the thesis to demonstrate the potential usefulness of some of these methods when applied to differential equations arising in the atmospheric sciences. More specifically, the thesis has three primary goals:

- Applying classical methods of symmetry analysis to several models of the atmospheric sciences.
- Systematically re-deriving a number of known results of dynamic meteorology using symmetry techniques.
- Opening new fields of applications for methods of symmetry analysis.

In addition, a few of the methods applied in the thesis might be regarded as an extension or simplification of techniques already existing.[3] Yet as the thesis is devoted to the practical usage of symmetries in the atmospheric sciences, we will focus more on the application of these methods in specific situations than on formulating them in a very rigorous manner.

In some way, this PhD thesis can be seen as a continuation of our master thesis, in which we made our first experiences with symmetry methods in dynamic meteorology. It is still the great success that the symmetry approach records in diverse mathematical and physical sciences that inspired us to apply it more thoroughly to typical models of geophysical fluid dynamics. It is hoped that the thesis will give an impression how fruitful these methods can be in this field.

1.1.2 Classical symmetry analysis

In the first part of the thesis, symmetries are mainly used to obtain exact solutions of selected differential equations of the atmospheric sciences. This is a classical usage of symmetries and is related to the original stimulation of Sophus Lie to develop the theory of continuous transformation groups at the end of the 19th century as a tool for the integration of ordinary differential equations [82]. Since we are solely dealing with partial differential equations, symmetries are used to determine exact solution by carrying out *group-invariant reduction*, which finally leads to group-invariant solutions. The method of group-invariant reduction rests on the possibility to introduce the invariants of a Lie symmetry (sub)group as new variables in the associated differential equation. It is then guaranteed that a symmetry (sub)group possessing r-dimensional orbits allows to reduce the number of independent variables $p \geq r$ by r. Any solution of the reduced differential equation gives rise to a particular solution of the initial differential equation after transforming back to the original variables of the initial equation. As this construction is classical, it can be found in virtually all textbooks on symmetry analysis, such as in [23, 62, 115, 149].[4]

Among exact solutions, Lie symmetries can also be used for several other purposes related to differential equations. Here we list a few important fields of applications that play a role in the present thesis. For further discussions, see the textbooks [22, 23, 62, 97, 115, 118].

Determining new solutions from known ones. Symmetries are transformations that map the set of solutions of a differential equation to itself. For this reason, a symmetry can be used to transform a given exact solution of a differential equation to another (possibly new) exact solution. Thus, by acting on the set of known solutions of a differential equation with

[3] We intentionally avoid the term *new method* in this place as it is used in almost an inflationary manner in symmetry analysis. Unfortunately, not all these “new“ methods are as new as they pretend to be.

[4] For an ordinary differential equation, symmetries can be used to lower the order of the equation and can thereby help to obtain its general solution, see also [23, 62, 115, 149].

symmetries of the same equation, series of exact solutions can be produced with very little effort. On the other hand, the property of a symmetry to map solutions to other solutions renders it necessary to classify the set of group-invariant solutions of a differential equation into dissimilar subsets, which cannot be related to each other by means of a symmetry. As equivalent subalgebras of the corresponding maximal Lie invariance algebra lead to similar subsets of invariant solutions, the classification of inequivalent subalgebras of the maximal Lie invariance algebras of the differential equations we investigate in the thesis is of striking importance.

Identification of related differential equations. Symmetries can be used to determine similar differential equations, which can be mapped to each other by point transformations. The criterion is that the symmetry groups of similar equations are similar with respect to the same point transformations. This necessary condition is used in the first part of the thesis to map the barotropic vorticity equation on the rotating sphere and the barotropic potential vorticity equation on the β-plane to the respective equations in the inertial frames. Moreover, this property is especially useful in the case where it is possible to find a linear differential equation associated with a nonlinear one, as it is then possible to linearize the nonlinear equation. A prominent example in fluid mechanics is the system of one-dimensional shallow-water equations, which can be transformed to a system of linear differential equations using a hodograph transformation [62].

Computation of discrete symmetries. Lie symmetries also provide a way to determine the complete point symmetry group of a differential equation, which includes both Lie and discrete point symmetries. The main idea is the following. Any point symmetry (continuous or discrete) of a differential equation induces an automorphism of the corresponding Lie invariance algebra. This condition in turn restricts the general form of a transformation that can be a point symmetry of a given differential equation. Plugging this restricted form of a general point transformation into the symmetry determining equations and factoring out the continuous symmetries allows to determine the group of discrete symmetries [61]. The outlined method can be further simplified by noting that there are certain subalgebras which are invariant under any transformation from the group of automorphisms of the given Lie algebra. By this method, which is discussed more thoroughly in the end of the first part of the thesis and exemplified with the barotropic vorticity equation on the β-plane and the baroclinic two-layer model, the complete group of point symmetries of a differential equation can be determined. A second approach that can simplify the computation of the complete point symmetry group of a differential equation uses information on the set of admissible transformation of a normalized class of differential equation containing the given differential equation. This method is also demonstrated for the vorticity equation on the β-plane using the set of admissible transformations of a class of generalized vorticity equations, which is derived in the second part of the thesis.

Computation of invariants and differential invariants. Invariants of a Lie group action play an important role in the construction of group-invariant or partially invariant solutions. The can be determined using the infinitesimal Lie symmetry generators, by solving a quasi-linear characteristic system of differential equations [118]. Another method is based on finite symmetry transformations and involves the moving frame method [40]. In the same way, differential invariants are obtainable as well. These are the invariants of the group action extended to derivatives of the dependent variables up to a fixed order. It was already noted above that any differential equation can be expressed in terms of the differential invariants of its symmetry group action. This way, the differential invariants allow to determine the most general class of differential equations admitting the chosen group as a symmetry group. This is of primary importance for various practical applications of symmetries, as virtually all theories of physics are based on symmetry principles (e.g. classical mechanics is based on the Galilei group and special relativity is based on the Poincaré group [131]). A more comprehensive discussion of differential invariants is presented in the second part of the thesis.

Computation of conservation laws. Conservation laws have a significant relevance in the study of differential equations. They are useful to understand the behavior of the associated differential equations and provide essential information whether or not it may be possible to explicitly integrate these equations. Moreover, they can be used as a consistency test for numerical integration schemes [163]. Any conservation law of an ordinary differential equation is nothing but a first integral of this equation. The classical relation between symmetries and conservation laws was established by Emmy Noether [111]. More exactly, Noether's theorem relates (generalized) variational symmetries (i.e. symmetries of a variational problem) with conservation laws of the associated Euler–Lagrange equations. Methods for finding conservation laws of systems of differential equations that are not Euler–Lagrange equations of variational functionals were developed much later [3, 24, 160, 163], although the necessary theoretical background was well known [115]. The basic idea of the most effective approach is to solve the system that is adjoint to the system of determining equations for generalized symmetries. This yields the set of adjoint symmetries of the given differential equation. Any characteristic of every conservation law of a differential equation is an adjoint symmetry of the same equation but the inverse claim is not true. Characteristics should be separated from adjoint symmetries by satisfying additional conditions. This method also works for systems possessing a variational principle. In this case, adjoint symmetries and generalized symmetries coincide, which then leads back to the usual Noether case. For a more formal discussion and a description of other methods for the direct construction of conservation laws, see [130, 132, 163]. Conservation laws also play an outstanding role in the Hamiltonian formulation of physical models, which is a central matter in the third part of the thesis.

1.1.3 Symmetries and parameterization schemes

It is often the case that the differential equations modeling real world phenomena include one or more arbitrary parameters, thus representing *classes of differential equations* rather than a single equation. These parameters might be essential to fit the model to experimental data or to make it applicable to a wide range of situations. For such classes of differential equations, it can be the case that for different values of these parameters, the resulting differential equations possess different symmetry properties. Keeping in mind that equations of successful physical theories are characterized by wide symmetry groups, it might as well be beneficial to single out from a class of differential equations those elements that possess Lie invariance algebras of maximal dimensions. The complete and systematic description of symmetry properties of classes of differential equations is referred to as *group classification*. A discussion of the classical group classification problem can be found in the textbook [118] and in the recent paper [131] (see also the references therein).

The problem of group classification of classes of differential equations is hence a very natural one and it is essentially the issue of the second part of the present thesis. Namely, we use techniques of group classification to construct physical parameterization schemes that possess certain invariance properties. In general, the parameterization problem is certainly one of the most crucial problems that has to be treated appropriately in order to obtain valuable numerical weather and climate forecasts. Parameterizations are intimately linked to the discretization or averaging of a differential equation. In the course of discretization, it is necessary to choose a grid at whose vertices all the dependent variables of the differential equation to be discretized are defined. Obviously, by choosing a finite grid resolution (usually dictated by the limits of computer resources), all processes taking place at scales below those of the grid cannot be explicitly resolved by the model. Similarly, any measurement of real atmospheric data does not represent an instantaneous value at a single point but rather some mean value over a (suitable short) interval in time and a (suitable small) domain in space. To feed these measurements as initial or boundary conditions to a differential equation naturally also calls for an averaging of this equation, which in turn again introduces a splitting into resolved and unresolved scales. Unfortunately, as the governing equations of the atmospheric sciences are nonlinear, it is not reasonable to simply neglect the processes occurring at the unresolved scales. By nonlinear interaction of scales, the subgrid scale will, sooner or later, have an impact on the gridscale. If this interaction is not adequately taken into account, any result of a numerical weather or climate prediction will be spoilt. This is why it is essential to find a way to incorporate the effects of the subgrid scale processes on the resolved quantities. By definition, this is the issue of parameterization.

More mathematically, in the course of parameterization an expression of the unresolved quantities via the resolved ones is established, which we will call the *parameterization function*. By substituting this expression into the discretized (resp. averaged) differential equation, the latter is turned into a class of differential equation, whose tuple of arbitrary elements is precisely the parameterization function. There are several conditions a physical parameterization has to meet, such as correct dimensionality, tensorial properties and invariance under Galilean transformations, but all of these restrictions are not sufficient to completely determine the specific form of the parameterization function. It is the aim of the second part of the thesis to demonstrate that the Lie symmetries of a differential equation can be used to restrict the general form of the parameterization function. The main paradigm is to construct the parameterizations in a way such that the resulting equations possess different symmetry properties. For this reason, in the framework of the present thesis it is convenient to postulate the following assertion: *The (invariant) parameterization problem is a group classification problem.* That is, we solve the parameterization problem using well-known techniques of group classification. This program results in a list of invariant parameterization schemes for the models of geophysical fluid dynamics that could subsequently be tested with real data to assess, which of the parameterizations accounts best for the nature of the problem.

1.1.4 Symmetries, Nambu mechanics and finite-mode models

It was reported above that a classical way to employ symmetries is by carrying out group invariant reduction. This procedure finally results in a number of exact solutions of the original partial differential equation. In the third part of the thesis, we use symmetries to reduce the dynamic equations in another way, which in general does not result in new exact solutions of the given partial differential equations, but which may instead lead to some general insights in prevailing physical processes of atmospheric dynamics. Symmetries are used in the course of *spectral modeling*. Using the spectral method, the dependent variables of a system of partial differential equations are expanded into series with respect to orthogonal functional bases. In geophysical fluid dynamics, usually trigonometric functions or spherical harmonics are used as basis functions. This expansion is subsequently plugged into the dynamic equations. Multiplying the resulting relation with the basis functions, integrating over the given domain and using the orthogonality properties of the basis functions, the original partial differential equations are converted into a system of infinitely many ordinary differential equations that govern the evolution of the series expansion coefficients. To allow for a comprehensive mathematical and numerical treatment of the resulting problem, it is usually necessary to truncate this infinite system by dropping all but finitely many modes. At the same time, this leads to a negligence of various physical processes. See the textbook [77] for further details on spectral modeling.

Historically, the main motivation for low-dimensional models stems from the pre-supercomputer era, where it was difficult or even impossible to deal with the complete set of hydro-thermodynamical equations of the atmospheric sciences. It was argued that by comparing the results of such simplified model studies with the real state of the atmosphere, an understanding of the importance of the neglected processes can be gained [84]. In addition to the indisputable value of these simplified models for the atmosphere sciences, these studies also greatly stimulated the theory of dynamical systems, in particular following the pioneering work on chaotic dynamics by Edward N. Lorenz [85].

Generally speaking, there is no unique or natural criterion, which modes to retain and how to truncate the series expansion. Indeed, a key problem in spectral modeling is to assure that the resulting reduced model inherits some of the structural properties of the original system of partial differential equations. Choosing a poor truncation may lead to models violating conservation properties of the initial system, which in turn can give rise to unphysical solutions of the reduced model. In such a case, the aim of reduced model studies is at once rendered obsolete.

In the present thesis we restrict ourselves to the investigation of very low-dimensional model, i.e. models that only consist of up to six Fourier modes. The amount of physics in such heavily reduced models is obviously rather small and compared to the possibilities of today's numerical simulations it cannot be expected to gain new physical insights from these models. We nevertheless still consider the problem of structure-preserving low-dimensional models as important. For one reason, these models are well-suited for pedagogical purposes. There is a great amount of papers in the atmospheric sciences, in which very low-dimensional models are used to motivate issues of data assimilation, ensemble prediction or the parameterization problem. For these purposes, in turn, it is desirable to use reduced models that inherit some of the structure of the original system of equations, c.f. the discussion in Chapter 12. Moreover, the same problem of finding appropriate criteria for the truncation of a series expansion also arises in all present day's spectral numerical weather and climate prediction models. Although these models comprise of many more coefficients than the models discussed in this part, they can also suffer from similar deficiencies including violation of conservation properties, wrong turbulence spectra and wrong phase velocities of characteristic wave phenomena (such as gravity waves or Rossby waves), leading to inaccurate or incorrect forecasts (i.e. to "unphysical" solutions), especially in the course of long-term integrations. Owing to these reasons, any contribution to the solution of the truncation problem might be seen as an important preliminary stage on the route to more consistent atmospheric numerical models.

In the previous paragraphs we have discussed our motivation for addressing the problem of structure-preserving low-dimensional atmospheric models. Now it remains to specify *what* structures we regard as essential to be preserved. As the thesis is concerned with symmetries, it is natural to attempt preserving symmetry properties in low-dimensional modeling. More precisely, we use point symmetries of the original partial differential equations and investigate their implications on the series expansion coefficients. This allows to place restrictions on the modes that must be included in the truncated expansion.

A second structure we intend to preserve is related to conservation laws. It was mentioned in Section 1.1.2 that conservation laws play an important role in the theory of differential equations. As the equations of the atmospheric sciences often possess a large number of conservation laws it is sensible to propose reduced models that inherit as many conservation laws as possible. It happens that several of these conservation laws are associated with the specific Hamiltonian formulation of the governing field equations. Preserving the Hamiltonian form (in particular the *Poisson bracket*) in the course of the derivation of reduced models is therefore a primary concern in the present thesis.

A Hamiltonian formulation of an atmospheric low-dimensional model always guarantees energy conservation due to antisymmetry of the Poisson bracket. On the other hand, the equations of ideal hydro-thermodynamics also have conserved quantities related to integrals over functions of the vorticity vector, such as the vorticity moments in two-dimensional fluid mechanics or the helicity in three-dimensional fluid mechanics. In the Hamiltonian formulation of the equations of fluid mechanics, these conserved quantities do not explicitly appear in the Poisson bracket formulation in the same way as the energy does. The occurrence of multiple conserved quantities inspired Yōichirō Nambu [104] to propose an extension of the Hamiltonian formalism that allows to incorporate other conserved quantities besides the energy in the representation of the system. This is done by defining multi-linear, totally antisymmetric *Nambu brackets*. Using the Nambu bracket, energy and other conserved quantities enter the representation of a system on an equal level. While originally proposed for discrete systems, the idea of Nambu was passed over to ideal fluid mechanics. For various systems it was found that it is possible to extend the bilinear Poisson brackets to trilinear Nambu-like brackets. Although a theoretical foundation of this trilinear Nambu field formulation is vastly lacking, for the application to numerical modeling it is only the antisymmetry of a Nambu bracket that counts. Saving the trilinear Nambu bracket in the course of spectral or finite-difference discretization in turn guarantees two conserved quantities (usually the energy and one vorticity quantity) to be numerically preserved. The inclusion of only one additional conserved quantity may appear to be rather restrictive, especially in view of the infinite number of conserved quantities in two-dimensional ideal fluid mechanics. The main problem indeed is that there exists so far no universally applicable procedure how to construct discrete approximations of Hamiltonian field equations, c.f. the discussion in Section 12.2. On the other hand, for very low-dimensional models as considered in the third part of the thesis, preserving two conserved quantities is already quite enough and goes beyond various finite-mode models in the literature. This is why we find it appropriate to derive low-dimensional models that inherit a Nambu representation similar to those of the original field equations.

It should be kept in mind that Hamiltonian mechanics has been an increasingly wide field in mathematical physics since several decades. It plays a key role in various areas including classical and quantum mechanics, fluid mechanics, stability theory and perturbation analysis. Having said that, it is important to remark that for the present purpose we only need the Hamiltonian or Nambu field representation of the governing equations. That is, we are *not* concerned of what can be concluded from these formulations apart from the guarantee of preserving distinct conservative properties. Detailed information on both the theoretical foundation as well as on applications of discrete Hamiltonian mechanics and Hamiltonian field equations can be found in several textbooks, including [9, 10, 90, 115, 142]. In addition, all the necessary material concerning Hamiltonian mechanics and the Nambu representation for the thesis is presented in the introductory sections of the first and third papers in the third part (Chapters 10 and 12).

1.2 Structure of the thesis

The thesis is divided into three main parts, to which we refer as: (i) *Classical symmetry analysis*, (ii) *Symmetry preserving parameterization schemes* and (iii) *Symmetries, Nambu mechanics and finite-mode models*. Each part is composed of one or more papers that are either already published or presently under review. In the latter case, preprints of the papers are made available using the preprint net arXiv. For the sake of brevity, in some of the papers it was necessary to present results concerning the classification of inequivalent subalgebras without given detailed calculations. Two examples of these calculations can be found in the end of the first part (Chapter 7).

The first part of the thesis is based on the following papers:

- Bihlo, A. and R.O. Popovych, 2009. Symmetry analysis of barotropic potential vorticity equation. *Commun. Theor. Phys.*, **52** (4), 697–700.
- Bihlo, A. and R.O. Popovych, 2009. Lie symmetries and exact solutions of the barotropic vorticity equation. *J. Math. Phys.* **50** (12), 123102, 12 pp.
- Bihlo, A. and R.O. Popovych, 2010. Point symmetry group of the barotropic vorticity equation. *Submitted to the Proceedings of the Fifth workshop "Group Analysis of Differential Equations & Integrable Systems"*, arXiv:1009.1523v1, 13 pp.
- Bihlo, A. and R.O. Popovych, 2010. Lie symmetry analysis and exact solutions of the quasi-geostrophic two-layer problem, arXiv:1010.1542v1, 23 pp.

The first paper is a direct comment to two papers [59, 152] that appeared in the journal *Communications in Theoretical Physics*. In both of these papers, the barotropic potential vorticity equation was investigated using the classical Lie symmetry method. While in the first paper a basis element of the maximal Lie invariance algebra is missing, in both of the papers group-invariant reductions are carried out in a non-systematic fashion. The systematic investigation of group-invariant reduction is based on the computation of an optimal list of inequivalent subalgebras, which was considered in none of these papers. In our paper, we present the optimal lists of one- and two-dimensional subalgebras and discuss reductions of the barotropic potential vorticity equation based on this list. This allows to greatly simplify the construction of invariant solutions compared to those presented in [59, 152]. Another key result of this paper is that it is possible to set $\beta = 0$ in the barotropic potential vorticity equation by means of a point transformation. This means that by studying reductions using Lie symmetries, it is possible to consider the reduced case $\beta = 0$ initially and finally obtain a solution of the equation with $\beta \neq 0$ by applying the point transformation relating these two cases. This once more simplifies the construction of particular solutions of the barotropic potential vorticity equation.

The second paper provides the link to our master thesis. In the master thesis, Lie symmetries were used to derive group-invariant solutions of the barotropic vorticity equation on the β-plane. Group-invariant reductions were carried out using inequivalent one-dimensional subalgebras of the maximal Lie invariance algebra of the vorticity equation. This enabled reductions to $(1+1)$-dimensional equations. Reductions to ordinary differential equations by means of two-dimensional subalgebras were not considered in the master thesis and hence are the natural starting point of the present work. In the second step, we extend the investigation of the barotropic vorticity equation to the spherical case. Different exact solutions are constructed. In the master thesis, the famous Rossby wave solution was obtained as a group-invariant solution. In the present work, it is shown that also the Rossby–Haurwitz wave solution on the sphere can be realized as a group-invariant solution. Moreover, a number of partially invariant solutions are computed for the vorticity equation on the β-plane, using the natural splitting of this equation into a diagnostic equation for the definition of the vorticity in terms of the stream function and the prognostic equation governing the evolution of the vorticity.

The third paper is devoted to the investigation of the complete point symmetry group of the barotropic vorticity equation on the β-plane. This involves both Lie and discrete point symmetries. Two different techniques are introduced that allow an a priori simplification of the calculations that are necessary in order to determine the complete point symmetry group. The first technique explicitly uses knowledge about the maximal Lie invariance algebra of the given differential equation. It is different from the method proposed in [61] in that it only uses minimal information on the automorphism group rather than computing the complete automorphism group. This minimal information is encoded in the set of megaideals of the maximal Lie invariance algebra. The second technique uses information about the set of admissible transformations of a normalized class of differential equations. For such classes, the point symmetry group of any equation belonging to this class is contained in the projection of the associated equivalence group to the space of independent and dependent variables. After either of these techniques was applied, the usual direct method for the computation of symmetries must be used to finally obtain the complete point symmetry group of the studied differential equation.

Finally, in the fourth paper, the classical Lie problem is solved for the baroclinic two-layer equations. Again, Lie symmetries are computed and the optimal lists of one- and two-dimensional inequivalent subalgebras are determined. Lie reductions are carried out on the basis of this classification. For the reductions in one variable, the resulting differential equations are again investigated for their symmetry properties. In cases where the reduced equations admit hidden symmetries, i.e. symmetries that are not induced by Lie symmetries of the original differential equations, we also carry out Lie reductions to ordinary differential equations using these hidden symmetries. This allows us to extend the set of exact solutions of the two-layer equations that is obtainable by Lie methods. The physical significance of these exact solutions is discussed. It is found that (baroclinic) Rossby waves provide an important class of invariant solutions of the two-layer model. Moreover, also the complete point symmetry group of the two-layer equations is computed, which enables the identification of discrete symmetries. For this aim, we use the first technique described in the third paper of the thesis. We find three independent discrete mirror symmetries. An interpretation in terms of Lie symmetries is given for the method of reduction of linear equations by a generalized ansatz. This interpretation is based on consideration of multiple copies of the initial equation. The method is exemplified with a linear submodel of the two-layer equations.

The second part of the thesis consists of the paper:

- R.O. Popovych and A. Bihlo, 2010. Symmetry preserving parameterization schemes. arXiv:1010.3010v1, 30 pp.

In this paper, we present general paradigms for the construction of parameterization schemes that exhibit symmetry properties. The main idea is to use techniques of inverse and direct group classification to determine possible closure assumptions for subgrid scale terms arising in averaged nonlinear differential equations. Using inverse group classification, differential invariants of subgroups of the maximal Lie invariance group of the differential equation under consideration are determined. By constructing the basis of differential invariants and the associated operators of invariant differentiation, an exhaustive description of differential invariants of arbitrary order is made available. These differential invariants can be assembled to parameterizations schemes, yielding invariant representations of the subgrid scale terms. Using direct group classification, first a general ansatz for the parameterizations has to be chosen. This yields a class of differential equations for which the equivalence algebra is determined. By investigating extensions of the kernel of Lie invariance algebras of the given class of differential equations induced by inequivalent subalgebras of the equivalence algebra, different ansatzes for the unknown subgrid scale terms can be adopted. In addition, the set of admissible transformations for the chosen class may be computed. For classes possessing the normalization property, it is guaranteed that the parameterizations obtained by extensions induced by the equivalence algebra exhaust the set of all possible invariant parameterizations. The resulting closure schemes have different symmetry properties and are still general enough to allow the incorporation of other desirable physical attributes. Both of these general techniques are illustrated by parameterizing the eddy vorticity flux in the Reynolds averaged barotropic vorticity equation. Distinct parameterization schemes possessing different degrees of invariance are constructed and discussed. Although several of these invariant parameterization schemes are obviously unphysical, we are also able to recover ansatzes that are well-known in the atmospheric sciences.

The third part of the thesis is based on the following papers:

- Bihlo, A., 2008. Rayleigh–Bénard Convection as a Nambu–metriplectic problem. *J. Phys. A*, **41** (29), 292001, 6 pp.
- Bihlo, A. and R.O. Popovych, 2010. Symmetry justification of Lorenz' maximum simplification. *Nonlin. Dyn.*, **61** (1), 101–107.
- Bihlo, A. and J. Staufer, 2010. Minimal atmospheric finite-mode models preserving symmetry and generalized Hamiltonian structures, arXiv:0909.1957v3, 18 pp.

In the first paper it is shown that the Saltzman equations governing Rayleigh–Bénard convection can be cast into Nambu bracket form. The starting point of the discussion is the noncanonical Hamiltonian formulation of these equations. The Casimir functionals of the noncanonical Poisson bracket are presented. There is an infinite number of Casimir functionals that split into two different classes. A representative of the first class of Casimir functionals can be used to extend the Poisson bracket to a twofold antisymmetric Nambu tri-bracket. Using this bracket, the conservative part of the convection equations is completely determined. The difference between the Hamiltonian and a representative of the second class of Casimir functionals is used to define the generalized free energy as introduced in [100]. Using this generalized free energy it is possible to define a symmetric two-bracket, which allows to formulate the dissipative part of the convection equations. The sum of the antisymmetric tri-bracket and the symmetric two-bracket defines the Nambu–metriplectic bracket. The dissipative Saltzman equations can then be rewritten in a completely geometric manner using this Nambu–metriplectic bracket.

The second paper is devoted to a study of the Lorenz–1960 model. It was the aim of Lorenz [84] to derive the maximum simplification of the atmospheric equations in a way that the resulting system still accounts for nonlinear model interactions. He started with a Fourier expansion of the barotropic vorticity equation on the torus and neglected all but three Fourier modes. In the course of his derivations he made two observations that allow one to constrain the number of modes necessary in the final model. We find that these two observations can be interpreted from the symmetry point of view. This motivates us to re-derive the Lorenz–1960 model by a selection of modes that is based on point symmetries of the vorticity equation. We start with the truncation of the vorticity equation to an eight component finite-mode model, obtained by restricting the Fourier modes to take only the values $-1, 0, 1$, thereby setting to zero the mean value. We use induced subgroups of the symmetry group of the vorticity equation in the space of Fourier components to derive several low-dimensional models, possessing five, four or three components. It is demonstrated that the three-component Lorenz–1960 model is really the major nontrivial simplification compatible with point symmetries of the vorticity equation.

In the third paper, we investigate the relation between the Lorenz–1963 model and point symmetries and the Nambu–metriplectic form of the Saltzman convection equations. It is shown that the discrete Nambu form of the conservative part of the Lorenz–1963 model is not related to the tri-bracket derived in the first paper of this part. Moreover, the selection of modes for this model cannot be motivated using the same technique successfully applied to the Lorenz–1960 model. For this reason, we propose a new low-dimensional system that extends the Lorenz–1963 model by three additional modes. The selection of modes of this extended model is based on point symmetries of the convection equations. It also inherits the appropriate tri-bracket structure from the convection equations. The symmetric bracket for the dissipative six-component model is formulated using a metric tensor. Hence it is shown that the extended Lorenz–1963 model has a discrete Nambu–metriplectic representation and thus an analog geometric form as the original Saltzman equations.

Part I

Classical symmetry analysis

Chapter 2

Symmetry analysis of barotropic potential vorticity equation

Abstract Recently F. Huang [Commun. Theor. Phys. **42** (2004) 903] and X. Tang and P.K. Shukla [Commun. Theor. Phys. **49** (2008) 229] investigated symmetry properties of the barotropic potential vorticity equation without forcing and dissipation on the beta-plane. This equation is governed by two dimensionless parameters, F and β, representing the ratio of the characteristic length scale to the Rossby radius of deformation and the variation of earth' angular rotation, respectively. In the present paper it is shown that in the case $F \neq 0$ there exists a well-defined point transformation to set $\beta = 0$. The classification of one- and two-dimensional Lie subalgebras of the Lie symmetry algebra of the potential vorticity equation is given for the parameter combination $F \neq 0$ and $\beta = 0$. Based upon this classification, distinct classes of group-invariant solutions is obtained and extended to the case $\beta \neq 0$.

2.1 Introduction

There is a long history in dynamic meteorology to use simplified models of the complete set of hydro-thermodynamical equations to gain insides in the different processes characterising the various structures and pattern occurring in the atmosphere. One of the most classical models in atmospheric science is the barotropic (potential) vorticity equation. It has been successfully used both for theoretical considerations [29, 139] and practical numerical weather predictions [30] since it is capable of describing some prominent features of mid-latitude weather phenomena such as the well-known Rossby waves and blocking regimes. In nondimensional form it reads [122]

$$\frac{\partial \zeta}{\partial t} - F\frac{\partial \psi}{\partial t} + J(\psi, \zeta) + \beta\frac{\partial \psi}{\partial x} = 0, \quad \zeta = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}. \tag{2.1}$$

Here ψ is the stream function, ζ the vorticity,

$$J(a, b) = \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial a}{\partial y}\frac{\partial b}{\partial x}$$

is the Jacobian and F and β represent the ratio of the characteristic length scale to the Rossby radius of deformation and a parameter describing the variation of earth' angular rotation, respectively.

2.2 The Lie symmetries

The symmetries of (2.1) have first been investigated in [21] and were applied to construct new solutions from known ones. Later, they have been studied by [59] and [152], but without reference to the group classification problem. This problem arises since different values of F and β lead to different symmetry properties of (2.1), which in turn characterise different physical properties of model (2.1). However, as shown in this paper, there are only three essential combinations of the values of these two parameters, given by $F = 0$, $\beta = 0$; $F = 0$, $\beta \neq 0$ and $F \neq 0$. The first combination leads to the usual vorticity form of Euler's equation which is the issue of e.g. [6]. This combination is of particular interest, since it gives rise to a new symmetry [13] (a so-called potential symmetry) which is not present in the velocity form of Euler's equation. The second combination of parameters was discussed in [15, 63]. It is the usual barotropic vorticity equation. The parameter β can be set to 1 by scaling and/or changing signs of variables. Note that for both combinations the associated Lie invariance algebras are infinite dimensional.

Now we show that if $F \neq 0$, we can always set $\beta = 0$. (Then the nonvanishing parameter F can be scaled to ± 1.) For this purpose, we recompute the symmetries of (2.1) for the case $F \neq 0$ and β arbitrary. This is done upon using the computer algebra packages MuLie [51] and DESOLV [28]. For this combination, equation (2.1) admits the six-dimensional Lie symmetry algebra $\mathfrak{a}_\beta$ generated by the operators

$$\begin{aligned} &\mathcal{D} = t\partial_t - \frac{\beta}{F}t\partial_x - \left(\psi - \frac{\beta}{F}y\right)\partial_\psi, \quad \mathbf{v}_r = -y\partial_x + \left(x + \frac{\beta}{F}t\right)\left(\partial_y + \frac{\beta}{F}\partial_\psi\right), \\ &\mathbf{v}_t = \partial_t, \quad \mathbf{v}_x = \partial_x, \quad \mathbf{v}_y = \partial_y, \quad \mathbf{v}_\psi = \partial_\psi. \end{aligned} \tag{2.2}$$

This algebra is not singular in β and consequently it also includes the case $\beta = 0$. Moreover, computing the symmetries for the case $\beta = 0$ explicitly, we obtain the same algebra (2.2) with $\beta = 0$. Hence

$$\begin{aligned} &\mathcal{D} = t\partial_t - \psi\partial_\psi, \quad \mathbf{v}_r = -y\partial_x + x\partial_y, \\ &\mathbf{v}_t = \partial_t, \quad \mathbf{v}_x = \partial_x, \quad \mathbf{v}_y = \partial_y, \quad \mathbf{v}_\psi = \partial_\psi, \end{aligned} \tag{2.3}$$

are the generators of the Lie symmetry algebra $\mathfrak{a}_0$. The physical importance of these generators is the following: $\mathcal{D}$ generates simultaneous scaling in t and ψ, $\mathbf{v}_r$ is the rotation operator in the (x, y)-plane, $\mathbf{v}_t$, $\mathbf{v}_x$, $\mathbf{v}_y$ and $\mathbf{v}_\psi$ are the infinitesimal generators of translations in t, x, y and ψ, respectively. The nonzero commutation relations between basis elements (2.3) are exhausted by

$$[\mathbf{v}_t, \mathcal{D}] = \mathbf{v}_t, \quad [\mathbf{v}_\psi, \mathcal{D}] = -\mathbf{v}_\psi, \quad [\mathbf{v}_x, \mathbf{v}_r] = \mathbf{v}_y, \quad [\mathbf{v}_y, \mathbf{v}_r] = -\mathbf{v}_x.$$

Therefore, the algebra $\mathfrak{a}_0$ has a simple structure. It is a solvable Lie algebra and can be represented as the direct sum $\mathfrak{g}_{3.4}^{-1} \oplus \mathfrak{e}(2)$, where $\mathfrak{e}(2) = \langle \mathbf{v}_x, \mathbf{v}_y, \mathbf{v}_r \rangle$ is the Euclidean algebra in the (x, y)-plane and $\mathfrak{g}_{3.4}^{-1} = \langle \mathbf{v}_t, \mathbf{v}_\psi, \mathcal{D} \rangle$ is a three-dimensional almost Abelian Lie algebra from Mubarakzyanov's classification of low dimensional Lie algebras [103].

It is straightforward to show that the Lie algebra (2.2) maps to the Lie algebra (2.3) under the transformation given by

$$\tilde{t} = t, \quad \tilde{x} = x + \frac{\beta}{F}t, \quad \tilde{y} = y, \quad \tilde{\psi} = \psi - \frac{\beta}{F}y. \tag{2.4}$$

This transformation also maps (2.1) to the same equation with $\beta = 0$. That is, (2.4) is an equivalence transformation for the class of equations of the form (2.1) with $F \neq 0$. Hence the

β-term can be neglected under symmetry analysis. Every solution of the equation with $\beta = 0$ can be extended to a solution with $\beta \neq 0$ by means of the transformation (2.4).

We also note that the maximal (infinite dimensional) Lie symmetry algebras in the cases $F = 0$, $\beta = 0$ and $F = 0$, $\beta \neq 0$ are neither isomorphic to each other nor isomorphic to the (finite dimensional) algebra $\mathfrak{a}_0$. Consequently, it is not possible to find point transformations that relate the corresponding PDEs to each other. All the transformations used for the reductions of the parameters F and β belong to the equivalence group of class (2.1). The group classification list for class (2.1) is therefore exhausted by the three inequivalent cases $F = 0$, $\beta = 0$; $F = 0$, $\beta = 1$ and $F = \pm 1$, $\beta = 0$.

2.3 Classification of subalgebras

Classification of subgroups of Lie symmetry groups of differential equations is an essential part in the study of these equations. This is since classification allows for an efficient computation of group-invariant solutions, without the possibility of an occurrence of equivalent solutions. Classifying subgroups may further lead to the construction of simple ansätze for the corresponding equivalence classes of reduced differential equations. Thereby, the classification also provides an important step for further investigations of properties of these reduced equations.

The classification of subgroups of symmetry groups is usually done by the classification of the associated Lie subalgebras with respect to the adjoint representation [115, 118]. The potential vorticity equation (2.1) is a (2+1) model and thus Lie reductions up to an ordinary differential equations require the classification of one- and two-dimensional subalgebras. An exhaustive classification of subalgebras exists only for Lie algebras up to dimension four [121]. Thus we need to classify subalgebras of $\mathfrak{a}_0$. This problem is not difficult because the algebra $\mathfrak{a}_0$ has a simple solvable structure.

The adjoint representation of a Lie group on it's Lie algebra is given as the Lie series

$$\mathbf{w}(\varepsilon) = \mathrm{Ad}(e^{\varepsilon \mathbf{v}})\mathbf{w}_0 := \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!}\{\mathbf{v}^n, \mathbf{w}_0\}, \tag{2.5}$$

with $\{\cdot,\cdot\}$ being defined recursively:

$$\{\mathbf{v}^0, \mathbf{w}_0\} := \mathbf{w}_0, \quad \{\mathbf{v}^n, \mathbf{w}_0\} := (-1)^n[\mathbf{v}, \{\mathbf{v}^{n-1}, \mathbf{w}_0\}].$$

Alternatively, the adjoint representation can also be calculated by integrating the initial value problem

$$\frac{\mathrm{d}\mathbf{w}(\varepsilon)}{\mathrm{d}\varepsilon} = [\mathbf{w}(\varepsilon), \mathbf{v}], \quad \mathbf{w}(0) = \mathbf{w}_0.$$

For basis elements (2.3) of the algebra $\mathfrak{a}_0$ we obtain the following nontrivial adjoint actions

$$\begin{aligned}
&\mathrm{Ad}(e^{\varepsilon \mathbf{v}_t})\mathcal{D} = \mathcal{D} - \varepsilon \mathbf{v}_t, && \mathrm{Ad}(e^{\varepsilon \mathbf{v}_\psi})\mathcal{D} = \mathcal{D} + \varepsilon \mathbf{v}_\psi,\\
&\mathrm{Ad}(e^{\varepsilon \mathcal{D}})\mathbf{v}_t = e^{\varepsilon}\mathbf{v}_t, && \mathrm{Ad}(e^{\varepsilon \mathcal{D}})\mathbf{v}_\psi = e^{-\varepsilon}\mathbf{v}_\psi,\\
&\mathrm{Ad}(e^{\varepsilon \mathbf{v}_x})\mathbf{v}_r = \mathbf{v}_r - \varepsilon \mathbf{v}_y, && \mathrm{Ad}(e^{\varepsilon \mathbf{v}_y})\mathbf{v}_r = \mathbf{v}_r + \varepsilon \mathbf{v}_x,\\
&\mathrm{Ad}(e^{\varepsilon \mathbf{v}_r})\mathbf{v}_x = \mathbf{v}_x \cos\varepsilon + \mathbf{v}_y \sin\varepsilon, && \mathrm{Ad}(e^{\varepsilon \mathbf{v}_r})\mathbf{v}_y = -\mathbf{v}_x \sin\varepsilon + \mathbf{v}_y \cos\varepsilon.
\end{aligned}$$

2.3.1 One-dimensional subalgebras

The classification of one-dimensional subalgebras of the whole symmetry algebra (2.3) is done by an inductive approach [115]: We start with the most general infinitesimal generator,

$$\mathbf{v} = a_1\mathcal{D} + a_2\mathbf{v}_r + a_3\mathbf{v}_t + a_4\mathbf{v}_x + a_5\mathbf{v}_y + a_6\mathbf{v}_\psi,$$

and simplify it as much as possible by means of adjoint actions. Depending on the respective values of the coefficients a_i, $i = 1, \ldots, 6$, we find the following list of inequivalent one-dimensional subalgebras of (2.3):

$$\begin{aligned}
&1.\ a_1 \neq 0, a_2 \neq 0: \\
&\qquad \langle \mathcal{D} + a\mathbf{v}_r \rangle. \\
&2.\ a_1 \neq 0, a_2 = 0, (a_4, a_5) \neq (0,0): \\
&\qquad \langle \mathcal{D} + a\mathbf{v}_x \rangle. \\
&3.\ a_1 = 0, a_2 \neq 0, a_3 \neq 0: \\
&\qquad \langle \mathbf{v}_r \pm \mathbf{v}_t + a\mathbf{v}_\psi \rangle. \\
&4.\ a_1 = a_3 = 0, a_2 \neq 0: \\
&\qquad \langle \mathbf{v}_r + c\mathbf{v}_\psi \rangle. \\
&5.\ a_1 = a_2 = 0, a_3 \neq 0, (a_4, a_5) \neq (0,0): \\
&\qquad \langle \mathbf{v}_t + a\mathbf{v}_x + c\mathbf{v}_\psi \rangle. \\
&6.\ a_1 = a_2 = a_3 = 0, (a_4, a_5) \neq (0,0): \\
&\qquad \langle \mathbf{v}_x + c\mathbf{v}_\psi \rangle. \\
&7.\ a_1 = a_2 = a_3 = a_4 = a_5 = 0: \\
&\qquad \langle \mathbf{v}_\psi \rangle.
\end{aligned} \tag{2.6}$$

where $c \in \{-1, 0, 1\}$ and $a, a_i \in \mathbb{R}$. In case 5 we can additionally set $a \in \{-1, 0, 1\}$ if $c = 0$.

2.3.2 Two-dimensional subalgebras

The classification procedure of two-dimensional subalgebras follows in essential the same way as the one-dimensional case. The two most general linearly independent infinitesimal generators

$$\mathbf{v}^1 = a_1^1\mathcal{D} + a_2^1\mathbf{v}_r + a_3^1\mathbf{v}_t + a_4^1\mathbf{v}_x + a_5^1\mathbf{v}_y + a_6^1\mathbf{v}_\psi,$$
$$\mathbf{v}^2 = a_1^2\mathcal{D} + a_2^2\mathbf{v}_r + a_3^2\mathbf{v}_t + a_4^2\mathbf{v}_x + a_5^2\mathbf{v}_y + a_6^2\mathbf{v}_\psi,$$

are simultaneously subjected to the adjoint actions and nonsingular linear combining under some assumptions on the coefficients a_i^j, $i = 1, \ldots, 6$, $j = 1, 2$. Moreover, the required closure property of the subalgebra $\langle \mathbf{v}^1, \mathbf{v}^2 \rangle$ with respect to the Lie bracket (i.e. $[\mathbf{v}^1, \mathbf{v}^2] \in \langle \mathbf{v}^1, \mathbf{v}^2 \rangle$) eventually places further restrictions on the coefficients. By applying this technique, we find a set of inequivalent two-dimensional subalgebras of (2.3). For reason of brevity, we only list the subalgebras without the corresponding conditions on the respective coefficients a_i^j:

$$\begin{gathered}
\langle \mathcal{D}, \mathbf{v}_r \rangle, \quad \langle \mathcal{D} + a\mathbf{v}_r, \mathbf{v}_t \rangle, \quad \langle \mathcal{D} + a\mathbf{v}_x, \mathbf{v}_t \rangle, \quad \langle \mathcal{D} + a\mathbf{v}_x, \mathbf{v}_y \rangle, \\
\langle \mathcal{D} + a\mathbf{v}_r, \mathbf{v}_\psi \rangle, \quad \langle \mathcal{D} + a\mathbf{v}_x, \mathbf{v}_\psi \rangle, \quad \langle \mathbf{v}_r + c\mathbf{v}_\psi, \mathbf{v}_t + b\mathbf{v}_\psi \rangle, \quad \langle \mathbf{v}_r + c\mathbf{v}_t, \mathbf{v}_\psi \rangle, \\
\langle \mathbf{v}_t + a\mathbf{v}_x + c\mathbf{v}_\psi, \mathbf{v}_y + b\mathbf{v}_\psi \rangle, \quad \langle \mathbf{v}_t + a\mathbf{v}_x, \mathbf{v}_\psi \rangle, \quad \langle \mathbf{v}_x + c\mathbf{v}_\psi, \mathbf{v}_y + b\mathbf{v}_\psi \rangle, \quad \langle \mathbf{v}_x, \mathbf{v}_\psi \rangle,
\end{gathered} \tag{2.7}$$

where $c \in \{-1, 0, 1\}$ and $a, b \in \mathbb{R}$. Moreover, in the case $c = 0$ we can scale the coefficient b to obtain $b \in \{-1, 0, 1\}$. Additionally, if $c = b = 0$ in the ninth subalgebra then we can set $a \in \{-1, 0, 1\}$.

2.4 Group-invariant solutions

Selected group-invariant solutions of (2.1) have been studied in [59] but without reference to classes of inequivalent subalgebras and without noting that it is possible to set $\beta = 0$. Consequently, some of the ansätze presented in [59], which lead to a reduction of the number of independent variables of (2.1), are overly intricate. More precisely, they could be realised by means of considering reduction using one of the inequivalent subalgebras (2.6) or (2.7) and subsequently acting on the resulting invariant solutions by finite symmetry transformations.

We investigate potentially interesting group-invariant reductions upon using ansätze that are based on the above classification of subalgebras. If possible, we relate them to the solutions given in [59].

2.4.1 Reductions with one-dimensional subalgebras

Here we give the complete list of reduced equations of (2.1) with the parameters $F \neq 0$, $\beta = 0$ based on the classification of inequivalent subalgebras (2.6). In what follows v and w are functions of p and q.

1. $\langle \mathcal{D} + a\mathbf{v}_r \rangle$. Suitable invariants of this subalgebra for reduction are

$$p = x\cos(a\ln t) + y\sin(a\ln t), \quad q = -x\sin(a\ln t) + y\cos(a\ln t), \quad v = t\psi.$$

Using them as new variables, the vorticity equation (2.1) is reduced to:

$$w - a(qw_p - pw_q) - F(v - a(qv_p - pv_q)) + v_q w_p - v_p w_q = 0, \quad w = v_{pp} + v_{qq}.$$

2. $\langle \mathcal{D} + a\mathbf{v}_x \rangle$. Invariants of this subalgebra are

$$p = x - a\ln t, \quad q = y, \quad v = t\psi,$$

which reduce (2.1) to

$$w + aw_p - F(v + av_p) - v_p w_q + v_q w_p = 0, \quad w = v_{pp} + v_{qq}.$$

3. $\langle \mathbf{v}_r \pm \mathbf{v}_t + a\mathbf{v}_\psi \rangle$. Defining $\varepsilon = \pm 1$, we find the following invariants:

$$p = x\cos\varepsilon t + y\sin\varepsilon t, \quad q = -x\sin\varepsilon t + y\cos\varepsilon t, \quad v = \psi - \varepsilon a t.$$

In these variables, (2.1) reads

$$\varepsilon(qw_p - pw_q) - \varepsilon F(qv_p - pv_q + a) + v_p w_q - v_q w_p = 0, \quad w = v_{pp} + v_{qq}.$$

4. $\langle \mathbf{v}_r + c\mathbf{v}_\psi \rangle$. Here we have the invariants

$$p = \sqrt{x^2 + y^2}, \quad q = t, \quad v = \psi + c\arctan\frac{x}{y}.$$

This gives the same ansatz as was used in [59] (Case 2) upon additionally applying transformation (2.4). The corresponding reduced equation reads

$$w_q - Fv_q - \frac{c}{p}w_p = 0, \quad w = v_{pp} + \frac{1}{p}v_p.$$

5. $\langle \mathbf{v}_t + a\mathbf{v}_x + c\mathbf{v}_\psi \rangle$. Invariants of this subalgebra are

$$p = x - at, \quad q = y, \quad v = \psi - ct.$$

This is an ansatz for a traveling wave solution in x-direction, which includes the well-known Rossby waves. In [59] a similar ansatz was combined with a traveling wave ansatz also in y-direction (Case 3). However, as was indicated above, the additional consideration of waves in y-direction is not necessary at this stage of reduction. The equation corresponding to the above ansatz is

$$aw_p - F(av_p - c) - v_p w_q + v_q w_p = 0, \quad w = v_{pp} + v_{qq}.$$

6. $\langle \mathbf{v}_x + c\mathbf{v}_\psi \rangle$. A suitable ansatz for the invariants of this subalgebra is provided by

$$p = x, \quad q = y, \quad v = \psi - ct.$$

In this case (2.1) is reduced to

$$Fc - v_p w_q + v_q w_p = 0, \quad w = v_{pp} + v_{qq}.$$

7. $\langle \mathbf{v}_\psi \rangle$. It is not possible to make an ansatz for ψ for this subalgebra in the framework of the classical Lie approach. Hence, no reduction can be achieved through the gauging operator $\mathbf{v}_\psi$.

2.4.2 Reductions with two-dimensional subalgebras

To give also an example for a reduction using a two-dimensional subalgebra, let us consider the algebra $\langle \mathbf{v}_t + a\mathbf{v}_x + c\mathbf{v}_\psi, \mathbf{v}_y + b\mathbf{v}_\psi \rangle$. In what follows v is a function of p. The invariants of this algebra are

$$p = x - at, \quad v = \psi - by - \frac{c}{a}x, \tag{2.8}$$

provided that $a \neq 0$. The corresponding reduced ODE of (2.1) then reads

$$(a+b)v_{ppp} - Fav_p = 0,$$

with the general solution

$$v = v_1 \exp\left(\sqrt{\frac{Fa}{a+b}}p\right) + v_2 \exp\left(-\sqrt{\frac{Fa}{a+b}}p\right) + v_3,$$

where $v_i = \text{const}$, $i = 1, 2, 3$. Transforming back to the original variables, renaming the constants v_i and applying the transformation (2.4), we obtain the invariant solution

$$\psi = \psi_3 + \left(b + \frac{\beta}{F}\right)y + \frac{c}{a}\left(x + \frac{\beta}{F}t\right) + \psi_1 \exp\left(\sqrt{\frac{Fa}{a+b}}\left(x + \frac{\beta}{F}t - at\right)\right) +$$

$$\psi_2 \exp\left(-\sqrt{\frac{Fa}{a+b}}\left(x+\frac{\beta}{F}t-at\right)\right)$$

which for $Fa/(a+b)<0$ gives rise to a travelling wave solution.

For the singular case $a=0$, we cannot use ansatz (2.8). Instead, we have the ansatz

$$p=x, \quad v=\psi-ct-by,$$

and the reduced vorticity equation reads

$$Fc+bv_{ppp}=0,$$

which gives rise to a polynomial solution in the case $b\neq 0$. If $b=0$, we get the condition that $c=0$ and the ansatz $\psi=v(x)$ itself is the solution of (2.1).

2.5 Summary and further comments

In the sections above, we discussed distinct cases of reduction by using inequivalent Lie subalgebras. The main advantages of this systematic approach are the following:

- Simplification of equation (2.1) since we only have to consider the case $F\neq 0$, $\beta=0$.
- Considering a minimal number of essential Lie subalgebras for reduction.
- Simplifying the ansatz for the reduced equations.
- Optimal preparation of the reduced equations for further investigations.

Upon using this approach, we have discussed all possible reductions by means of one-dimensional Lie subalgebras. We have to note that it is not possible to use the subalgebra $\langle \mathbf{v}_\psi \rangle$ for obtaining group-invariant solutions, since in this case there is no way to make an ansatz for ψ. In principle, the remaining two-dimensional Lie subalgebras can be used for reduction as well.

Moreover, the differential equations obtained by reduction could again be investigated by means of symmetry techniques. In general, some of the symmetries of these reduced equations will be induced by the symmetries of the original equation. However, sometimes there may be additional symmetries that are not induced in this way [67, 115] and which are called hidden symmetries [1]. They usually play an important role in the study of differential equations, as they may allow to reduce equations further than initially expected.

We may also note that it is still possible to generalise some of the results of this paper. By considering eqn. (2.1) as a system of two PDEs in the two dependent variables ψ and ζ, it is possible to construct partially invariant solutions [118]. For this class of exact solutions, one needs at least two dependent variables. For the first set of these dependent variables, it is possible to introduce new invariant variables, for the second set we keep the old noninvariant variables. The resulting reduced equations then also split in two sets of equations which have to be solved one after another. For this purpose, we could e.g. use subalgebras containing the operator $\mathbf{v}_\psi$. In this case, it is still not possible to make an ansatz for ψ but it is possible to do so for ζ. The resulting reduced system of differential equations may give rise to a much wider class of exact solutions than pure group-invariant solutions. An investigation of this class of solutions for the case $F=0$, $\beta\neq 0$ will be given elsewhere.

Chapter 3

Lie symmetries and exact solutions of the barotropic vorticity equation

Abstract Lie group methods are used for the study of various issues related to symmetries and exact solutions of the barotropic vorticity equation. The Lie symmetries of the barotropic vorticity equations on the f- and β-planes, as well as on the sphere in rotating and rest reference frames, are determined. A symmetry background for reducing the rotating reference frame to the rest frame is presented. The one- and two-dimensional inequivalent subalgebras of the Lie invariance algebras of both equations are exhaustively classified and then used to compute invariant solutions of the vorticity equations. This provides large classes of exact solutions, which include both Rossby and Rossby–Haurwitz waves as special cases. We also discuss the possibility of partial invariance for the β-plane equation, thereby further extending the family of its exact solutions. This is done in a more systematic and complete way than previously available in literature.

3.1 Introduction

The governing equations of geophysical fluid dynamics are mainly nonlinear partial differential equations (PDE). Since there is no general theory available for solving such equations, it is known to be very difficult to systematically construct their exact solutions. In meteorology, this problem is usually overcome by solving the governing equations numerically. However, as models become more sophisticated, it may be difficult to directly evaluate the quality of these numerical results. Moreover, it is dissatisfactory to rely solely on numerical modeling when studying the physics of the atmosphere. It is thus to be expected that exact solutions can both enhance our understanding of atmospheric processes and provide consistency tests for numerical models.

The classical method of reduction of PDEs by using its Lie symmetries [115, 118] and the extension to partially invariant solutions [118] provides a manageable way to systematically construct exact solutions. It is the goal of this paper to carry out a comprehensive symmetry investigation of the barotropic vorticity equation both on the β-plane and on the sphere. Although there are already a number of works on the β-plane equation [15, 21, 60, 63, 68, 69], none of them gives a systematic and complete symmetry analysis. In Ref. [15] the classification of inequivalent subalgebras is done only for the one-dimensional case. The symmetry properties were used in Ref. [21] in order to obtain new solutions from the known ones. In a recent paper [60], the procedure of group-invariant reduction is done without reference to the algebraic aspects of the classification problem. Consequently, some of the reductions presented in Ref. [60] are overly complicated, and hence in some cases, these authors were only able to obtain some particular solutions (most notably of the well-known Rossby wave class). This reveals that the vorticity equation has classes of completely integrable reduced PDEs, as shown in the present paper. Finally, Refs. [68, 69] (see also Ref. [63, pp. 221–225]) also contain only a nonsystematic list of some group-invariant solutions. To the best of our knowledge, the spherical equation has not been investigated in light of its symmetries at all so far.

We divide this paper into two main parts: the first dealing with the symmetry analysis of the equation on the β-plane and the second considering the spherical version. For both equations, we determine the maximal Lie invariance algebras and classify their one- and two-dimensional subalgebras. Based on this classification, we give a complete list of group-invariant reduced equations and then demonstrate that Rossby (Rossby–Haurwitz) waves can be realized as group-invariant solutions of the barotropic vorticity equation on the plane (the sphere). Also, by means of algebraic inspection of the Lie symmetry algebras, it is shown that for the spherical equation there is no need to consider rotation of the Earth. Finally, some examples for partially invariant solutions will be given for the β-plane equation.

3.2 The β-plane equation

This section contains the classical symmetry analysis of the barotropic vorticity equation on the β-plane (βBVE).

3.2.1 The model

Assuming the two-dimensional velocity field $\mathbf{v}$ to be nondivergent, it is possible to cast the Euler equations of an ideal fluid in a rotating reference frame as the conservation law of absolute vorticity $\eta = \zeta + f$, where $\zeta = \mathbf{k} \cdot (\nabla \times \mathbf{v})$ is the vertical component of the vorticity vector (relative vorticity) and f denotes the vertical Coriolis parameter, which depends only on y. In what follows, we approximate f by its truncated Taylor series, $f = f_0 + \mathrm{d}f/\mathrm{d}y|_0\, y =: f_0 + \beta y$, which leads to the β-plane approximation [57]. The Euler equations can then be equivalently written as the βBVE

$$\zeta_t + \psi_x \zeta_y - \psi_y \zeta_x + \beta \psi_x = 0, \tag{3.1}$$

where we have used the shorthand notation $\zeta_t = \partial\zeta/\partial t$, etc. The stream function $\psi = \psi(t, x, y)$ generates a nondivergent flow. It is related to the vorticity by means of the Laplacian,

$$\zeta := \psi_{xx} + \psi_{yy}.$$

Rescaling allows us to set $\beta = 1$, but for physical reasons this is not desired here.

3.2.2 The symmetries

The barotropic vorticity equation can be considered as a submodel of the ideal Euler equations, which have been thoroughly investigated in light of their symmetries (see, e.g., Refs. [6, 126]).

Nevertheless, it is instructive to consider the symmetries of the barotropic vorticity equation separately to work out the peculiarities of large scale, two-dimensional fluid dynamics. It is quite common for different models of incompressible fluids that they admit infinite dimensional maximal Lie invariance algebras of a special structure. Techniques for handling such infinite-dimensional Lie algebras in order to solve hydrodynamic equations are given, e.g., in Ref. [41]. To the best of our knowledge, the symmetry algebra of the barotropic vorticity equation in the regular case $\beta \neq 0$, as well as some exact solutions, was first computed in Refs. [68, 69] (see also Refs. [21, 63]). The fact that the singular case $\beta = 0$ admits nontrivial symmetries has been known for a long time [13]. The corresponding maximal Lie symmetry algebra was rigorously calculated in Ref. [7] (see also Ref. [6]). It is significantly larger than for the regular case of β.

We recomputed the symmetry algebras for our purposes and checked them with the computer algebra programs MuLie [51] and DESOLV [28]. In the singular case $\beta = 0$ corresponding to dynamics on the f-plane, the vorticity equation admits the infinite dimensional Lie symmetry algebra $\mathcal{B}_0^\infty$ with the basis generators

$$\begin{aligned}
&\mathcal{D}_1 = t\partial_t - \psi\partial_\psi && \mathcal{D}_2 = 2\psi\partial_\psi + x\partial_x + y\partial_y \\
&\mathrm{J} = -y\partial_x + x\partial_y && \mathrm{J}^t = -ty\partial_x + tx\partial_y + \tfrac{1}{2}(x^2+y^2)\partial_\psi \\
&\partial_t && \mathcal{Z}(g) = g(t)\partial_\psi \\
&\mathcal{X}(f) = f(t)\partial_x - f'(t)y\partial_\psi && \mathcal{Y}(h) = h(t)\partial_y + h'(t)x\partial_\psi
\end{aligned}$$

where h, f and g run through the set of real-valued time-dependent functions. The shorthand notation for partial derivatives, e.g., $\partial_t = \partial/\partial t$ is used. The physical significance of these generators is as follows: $\mathcal{D}_1$ and $\mathcal{D}_2$ are scaling operators, ∂_t generates time translations, and J and J^t correspond to rotations and time-dependent rotations in the horizontal plane. The operators $\mathcal{Y}(h)$ and $\mathcal{X}(f)$ are the infinitesimals of generalized transformations on a time-dependent moving coordinate system in the y- and x-directions, respectively. The generator $\mathcal{Z}(g)$ represents gauging the stream function.

Likewise, in the case $\beta \neq 0$, eqn. (3.1) admits an infinite-dimensional Lie symmetry algebra $\mathcal{B}_1^\infty$, which is a subalgebra of $\mathcal{B}_0^\infty$. The basis generators of $\mathcal{B}_1^\infty$ are

$$\mathcal{D} = \mathcal{D}_1 - \mathcal{D}_2 = t\partial_t - x\partial_x - y\partial_y - 3\psi\partial_\psi, \qquad \partial_t, \qquad \mathcal{Y}(1) = \partial_y, \qquad \mathcal{X}(f), \qquad \mathcal{Z}(g).$$

The physical importance of this algebra is obvious from the $\beta = 0$ case. As shown in the next section, this remarkable difference between the cases of vanishing and nonvanishing β has no counterpart in spherical coordinates.

For the sake of completeness, we mention that the vorticity equation admits the discrete symmetries $(t, x, y, \psi) \mapsto (-t, -x, y, \psi)$ and $(t, x, y, \psi) \mapsto (t, x, -y, -\psi)$ as well as their composition and their compositions with continuous symmetries.

3.2.3 Classification of subalgebras

For an efficient and systematic computation of invariant solutions of PDEs, it is crucial to classify their Lie symmetry subalgebras. This is done upon using the adjoint action of a Lie group on its Lie algebra, which allows to determine the simplest representatives of equivalent subalgebras. The adjoint action of $\exp(\varepsilon\mathbf{v})$ on $\mathbf{w}_0$ is defined as the Lie series,

$$\mathbf{w}(\varepsilon) = \mathrm{Ad}(\exp(\varepsilon\mathbf{v}))\mathbf{w}_0 := \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!}\{\mathbf{v}^n, \mathbf{w}_0\},$$

where we introduced a shorthand notation for nested commutators: $\{\mathbf{v}^0, \mathbf{w}_0\} := \mathbf{w}_0$, $\{\mathbf{v}^n, \mathbf{w}_0\} := (-1)^n[\mathbf{v}, \{\mathbf{v}^{n-1}, \mathbf{w}_0\}]$. Alternatively, the adjoint representation can also be calculated by integrating the initial value problem

$$\frac{\mathrm{d}\mathbf{w}(\varepsilon)}{\mathrm{d}\varepsilon} = [\mathbf{w}(\varepsilon), \mathbf{v}], \quad \mathbf{w}(0) = \mathbf{w}_0.$$

The nonidentical adjoint actions with basis elements of the algebra $\mathcal{B}_1^\infty$ are exhausted by the following list:

$$\begin{aligned}
&\mathrm{Ad}(e^{\varepsilon\partial_t})\mathcal{D} = \mathcal{D} - \varepsilon\partial_t && \mathrm{Ad}(e^{\varepsilon\mathcal{X}(f)})\mathcal{D} = \mathcal{D} + \varepsilon\mathcal{X}(tf' + f)\\
&\mathrm{Ad}(e^{\varepsilon\partial_y})\mathcal{D} = \mathcal{D} + \varepsilon\partial_y && \mathrm{Ad}(e^{\varepsilon\mathcal{Z}(g)})\mathcal{D} = \mathcal{D} + \varepsilon\mathcal{Z}(tg' + 3g)\\
&\mathrm{Ad}(e^{\varepsilon\mathcal{D}})\partial_t = e^{\varepsilon}\partial_t && \mathrm{Ad}(e^{\varepsilon\mathcal{Z}(g)})\partial_t = \partial_t + \varepsilon\mathcal{Z}(g')\\
&\mathrm{Ad}(e^{\varepsilon\mathcal{D}})\partial_y = e^{-\varepsilon}\partial_y && \mathrm{Ad}(e^{\varepsilon\mathcal{X}(f)})\partial_t = \partial_t + \varepsilon\mathcal{X}(f')\\
&\mathrm{Ad}(e^{\varepsilon\mathcal{X}(f)})\partial_y = \partial_y - \varepsilon\mathcal{Z}(f') && \mathrm{Ad}(e^{\varepsilon\partial_y})\mathcal{X}(f) = \mathcal{X}(f) + \varepsilon\mathcal{Z}(f')\\
&\mathrm{Ad}(e^{\varepsilon\mathcal{D}})\mathcal{X}(f) = \mathcal{X}(\tilde{f}), \quad \tilde{f} = e^{-\varepsilon}f(e^{-\varepsilon}t) && \mathrm{Ad}(e^{\varepsilon\partial_t})\mathcal{X}(f) = \mathcal{X}(\tilde{f}), \quad \tilde{f} = f(t-\varepsilon)\\
&\mathrm{Ad}(e^{\varepsilon\mathcal{D}})\mathcal{Z}(g) = \mathcal{Z}(\tilde{g}), \quad \tilde{g} = e^{-3\varepsilon}g(e^{-\varepsilon}t) && \mathrm{Ad}(e^{\varepsilon\partial_t})\mathcal{Z}(g) = \mathcal{Z}(\tilde{g}), \quad \tilde{g} = g(t-\varepsilon).
\end{aligned}$$

They are used subsequently to classify the one- and two-dimensional subalgebras of $\mathcal{B}_1^\infty$.

The approach to the classification of *one-dimensional subalgebras* is fairly inductive: one takes the most general form of an infinitesimal generator of $\mathcal{B}_1^\infty$, $\mathbf{v} = a_D\mathcal{D} + a_t\partial_t + a_y\partial_y + \mathcal{X}(f) + \mathcal{Z}(g)$, and subsequently tries to simplify it using adjoint actions and scaling by a nonvanishing constant multiplier [115]. This is done under additional assumptions on the constants a_D, a_t, and a_y and the functions $f(t)$ and $g(t)$. Finally, the optimal set of conjugacy inequivalent one-dimensional subalgebras of $\mathcal{B}_1^\infty$ reads

$$\langle\mathcal{D}\rangle, \qquad \langle\partial_t + b\partial_y\rangle, \qquad \langle\partial_y + \mathcal{X}(f)\rangle, \qquad \langle\mathcal{X}(f) + \mathcal{Z}(g)\rangle, \tag{3.2}$$

where $b \in \{-1, 0, 1\}$. By means of using the discrete symmetry $(t, x, y, \psi) \mapsto (t, x, -y, -\psi)$ we can further assume $b \in \{0, 1\}$. Moreover, due to adjoint actions, there are additional equivalences inside the third and fourth cases. In the third case, we can apply the adjoint actions $\mathrm{Ad}(e^{\varepsilon\mathcal{D}})$ to rescale the argument t and the function f and $\mathrm{Ad}(e^{\varepsilon\partial_t})$ to shift the argument t of the function f, respectively. In the fourth class, the additional equivalences are generated by $\mathrm{Ad}(e^{\varepsilon\partial_t})$, $\mathrm{Ad}(e^{\varepsilon\mathcal{D}})$, $\mathrm{Ad}(e^{\varepsilon\partial_y})$ and scaling the basis elements. So, the subalgebras $\langle\mathcal{X}(f) + \mathcal{Z}(g)\rangle$ and $\langle\mathcal{X}(\tilde{f}) + \mathcal{Z}(\tilde{g})\rangle$ are equivalent if and only if $\tilde{f}(t) = af(e^{\varepsilon_2}t + \varepsilon_1)$, $\tilde{g}(t) = ag(e^{\varepsilon_2}t + \varepsilon_1) + \varepsilon_3 f'(e^{\varepsilon_2}t + \varepsilon_1)$ for some constants ε_1, ε_2, ε_3, and a, where $a \neq 0$.

The procedure for the classification of *two-dimensional subalgebras* is quite the same as in the one-dimensional case: one takes the most general form of two (linearly independent) infinitesimal generators $\mathbf{v}^i = a_D^i\mathcal{D} + a_t^i\partial_t + a_y^i\partial_y + \mathcal{X}(f^i) + \mathcal{Z}(g^i)$ with $i = 1, 2$ and tries to simultaneously cast them in a simpler form. This can be done by taking their nondegenerate linear combinations and acting on them by adjoint actions under different assumptions on the constants a_D^i, a_t^i, and a_y^i and/or functions f^i and g^i. Since the generators $\mathbf{v}^i$ form a subalgebra, one additionally has to ensure that the commutator of $\mathbf{v}^1$ and $\mathbf{v}^2$ lies in their span. This places further restrictions on both the constants a_D^i, a_t^i, and a_y^i and on the functions $f^i(t)$, $g^i(t)$.

Since the classification of two-dimensional subalgebras is somewhat lengthy, we only give the final result here. The list of inequivalent algebras reads

$$\begin{aligned}
&\langle\mathcal{D},\ \partial_t\rangle, \quad \langle\mathcal{D},\ \partial_y + \mathcal{X}(a)\rangle, \quad \big\langle\mathcal{D},\ \mathcal{X}(|t|^a) + c\mathcal{Z}(|t|^{a-2})\big\rangle, \quad \big\langle\mathcal{D},\ \mathcal{Z}(|t|^{a-2})\big\rangle,\\
&\langle\partial_t,\ \partial_y + \mathcal{X}(a) + \mathcal{Z}(b)\rangle, \quad \big\langle\partial_t + b\partial_y,\ \mathcal{X}(e^{at}) + \mathcal{Z}((abt + c)e^{at})\big\rangle, \quad \big\langle\partial_t + b\partial_y,\ \mathcal{Z}(e^{at})\big\rangle,
\end{aligned}$$

$$\langle \partial_y + \mathcal{X}(f^1),\ \mathcal{X}(1) + \mathcal{Z}(g^2) \rangle, \quad \langle \partial_y + \mathcal{X}(f^1),\ \mathcal{Z}(g^2) \rangle,$$
$$\langle \mathcal{X}(f^1) + \mathcal{Z}(g^1),\ \mathcal{X}(f^2) + \mathcal{Z}(g^2) \rangle,$$

where $f^i = f^i(t)$ and $g^i = g^i(t)$, $i = 1, 2$, are arbitrary smooth functions and a, b and c are constants. In the last subalgebra, the pairs of functions (f^1, g^1) and (f^2, g^2) have to be linearly independent. Again, within of the above classes there are additional equivalences due to adjoint actions and changes of the basis. In particular, we can set additional restrictions for constants only in the fifth, sixth and seventh subalgebras: $b \in \{-1, 0, 1\}$ (resp. $b \in \{0, 1\}$ if discrete symmetries also are taken into account) in all these subalgebras; in the sixth and seventh subalgebras also $a \in \{-1, 0, 1\}$ (resp. $a \in \{0, 1\}$) if $b = 0$; $c = 0$ if $a \neq 0$ and $c \in \{-1, 0, 1\}$ (resp. $c \in \{0, 1\}$) if $a = b = 0$.

Computing a complete set of inequivalent group-invariant solutions is based on the above classification of subalgebras.

3.2.4 Group-invariant reduction with one-dimensional subalgebras

In what follows, we give the list of reduced equations obtained by imposing invariance under the one-parameter groups associated with the Lie algebras (3.2). In all four cases, p and q denote the new independent variables, while $v = v(p, q)$ is the new dependent variable. In the last case, an ansatz for ψ exists only for the values of t, where $f \neq 0$.

1	$\langle \mathcal{D} \rangle$	$\psi = t^{-3} v$	$p = tx, \quad q = ty$
	$-w + pw_p + qw_q + v_p w_q - v_q w_p + \beta v_p = 0, \quad w := v_{pp} + v_{qq}$		
2	$\langle \partial_t + b\partial_y \rangle$	$\psi = v$	$p = x, \quad q = y - bt$
	$-bw_q + v_p w_q - v_q w_p + \beta v_p = 0, \quad w := v_{pp} + v_{qq}$		
3	$\langle \partial_y + \mathcal{X}(f) \rangle$	$\psi = v - \frac{1}{2} f' y^2$	$p = x - fy, \quad q = t$
	$(1 + f^2) w_q + 2ff'w + \beta v_p - f'' = 0, \quad w := v_{pp}$		
4	$\langle \mathcal{X}(f) + \mathcal{Z}(g) \rangle$	$\psi = v - \frac{f'}{f} xy + \frac{g}{f} x$	$p = y, \quad q = t$
	$w_q + \left(\frac{g}{f} - \frac{f'}{f} p \right) w_p + \beta \left(\frac{g}{f} - \frac{f'}{f} p \right) = 0, \quad w := v_{pp}$		

We now discuss some properties and/or explicit solutions derived from the different cases considered above.

Case **1**. The reduced equation admits the maximal Lie invariance algebra $\langle \partial_p + q\partial_v,\, \partial_v \rangle$. The basis operators $\partial_p + q\partial_v$ and ∂_v are induced by the operators $\mathcal{X}(t^{-1})$ and $\mathcal{Z}(t^{-3})$ from $\mathcal{B}_1^\infty$, respectively. Hence, there are no hidden symmetries related to this reduction.

Case **2**. The reduced equation admits the maximal Lie invariance algebra $\langle \partial_p,\, \partial_q,\, \partial_v \rangle$ or $\langle p\partial_p + q\partial_q + 3v\partial_v,\, \partial_p,\, \partial_q,\, \partial_v \rangle$ if $b = 1$ or $b = 0$, respectively. Any operator from this algebra is obviously induced by an operator from $\mathcal{B}_1^\infty$. For the basis operators, we have the following correspondence: $\partial_p \leftarrow \mathcal{X}(1)$, $\partial_q \leftarrow \partial_y$, $\partial_v \leftarrow \mathcal{Z}(1)$, $p\partial_p + q\partial_q + 3v\partial_v \leftarrow -\mathcal{D}$. Again, we have no related hidden symmetries.

As a result, further Lie reductions in both of the above cases give no new solutions in comparison to Lie reduction with respect to two-dimensional subalgebras.

Case **3** admits an exhaustive description for its general solution. Integrating once with respect to p and using a change of coordinates yields

$$\tilde{v}_{\tilde{p}\tilde{q}} + \beta\tilde{v} = 0, \tag{3.3}$$

where

$$\tilde{v} = \frac{v}{1+f^2} - \frac{f''}{\beta}p + \frac{h}{\beta} + \frac{((1+f^2)f'')'}{\beta^2}, \qquad \tilde{q} = \int \frac{\mathrm{d}q}{1+f^2}, \qquad \tilde{p} = p,$$

and $h = h(q)$ is an arbitrary smooth function of $q = t$. Eqn. (3.3) is the one-dimensional Klein–Gordon equation (presented in the light-cone variables). For a list of exact solutions of this equation see, e.g. [125]. It is straightforward to recover the famous Rossby wave solution upon using a harmonic ansatz for $\tilde{v}$ and choosing either $f = 0$ (one-dimensional Rossby waves) or $f = \mathrm{const}$ (two-dimensional Rossby waves) [15].

Case **4** is completely integrable by quadratures. First, we determine w by solving the characteristic system. Afterwards, we integrate twice with respect to p to determine v. Finally, substituting the expression so obtained for v into the ansatz for ψ, we arrive at the corresponding group-invariant solution

$$\psi = \frac{1}{f^2}F(\theta) - \frac{1}{6}\beta y^3 + h^1 y + h^0 - \frac{f'}{f}xy + \frac{g}{f}x,$$

where f, g, h^1, and h^0 are arbitrary smooth functions of t, F is an arbitrary smooth function of $\theta = fy - \int g\,\mathrm{d}t$. The functions h^1 and h^0 can be set equal to 0 by symmetry transformations generated by an operator of the form $\mathcal{X}(f) + \mathcal{Z}(g)$.

3.2.5 Group-invariant reduction with two-dimensional subalgebras

Having considered reduction with one-dimensional subalgebras, it is not overly difficult to investigate reduction with two-dimensional subalgebras as well. Namely, the general solutions of cases **3** and **4** from section 3.2.4 are completely described. That is, it is not necessary to consider reduction with two-dimensional subalgebras containing the generators $\partial_y + \mathcal{X}(f)$ and $\mathcal{X}(f) + \mathcal{Z}(g)$. Moreover, since all algebras containing $\mathcal{Z}(g)$ cannot be used for a classical Lie reduction, the number of cases that need to be examined reduces to:

1	$\langle \mathcal{D}, \partial_t \rangle$	$\psi = \sqrt{(x^2+y^2)^3}\,v(\varphi)$	$\varphi = \arctan \frac{y}{x}$
		$v(w+\beta\sin\varphi)_\varphi - \frac{1}{3}v_\varphi(w+\beta\sin\varphi) = 0, \quad w := v_{\varphi\varphi} + 9v$	

The first of the above equations implies the following functional relation between w and v: $w + \beta\sin\varphi = c_0 v^{\frac{1}{3}}$. If $c_0 = 0$, the second equation can be easily integrated with respect to v. This leads to the invariant solution

$$\psi = c_1(x^2 - 3y^2)x + c_2(3x^2 - y^2)y - \frac{\beta}{8}(x^2+y^2)y$$

of (3.1). In the case $c_0 \neq 0$, we find particular solutions of the second equation which give rise to the invariant solutions

$$\psi = \frac{\beta}{2}(x^2+y^2)^{\frac{3}{2}}\sin^3\left(\frac{1}{3}\arctan\frac{y}{x}\right), \qquad \psi = -\frac{\beta}{2}(x^2+y^2)^{\frac{3}{2}}\sin^3\left(\frac{1}{3}\arctan\frac{y}{x} \pm \frac{\pi}{3}\right).$$

3.2.6 Partially invariant solutions

For a system with at least two dependent variables, it is possible to determine partially invariant solutions [118]. The construction of partially invariant solutions has already been extensively considered in hydrodynamics [6, 47, 52, 96, 126]. In this part, we compute some partially invariant solutions for the βBVE. First of all, it is noted that any single equation can be split into a system of multiple equations in various ways introducing a new dependent variable for each additional equation desired. We consider the βBVE as the system of two PDEs

$$\zeta_t + \psi_x\zeta_y - \psi_y\zeta_x + \beta\psi_x = 0, \qquad \zeta = \psi_{xx} + \psi_{yy}, \tag{3.4}$$

where both ψ and ζ are treated as dependent variables. The splitting of eqn. (3.1) into system (3.4) is quite natural since both ψ and ζ have an obvious physical importance. Of course, it is not unique. Another natural splitting is given by the system in terms of the usual velocity variables together with the condition of vanishing divergence. However, here we will not pursue any other splittings further.

It is an important property of the chosen splitting that the maximal Lie invariance algebra $\mathcal{B}_{1\mathrm{s}}^{\infty}$ of (3.4) is isomorphic to the algebra $\mathcal{B}_1^{\infty}$. More precisely, every operator from $\mathcal{B}_{1\mathrm{s}}^{\infty}$ is a prolongation of an operator from $\mathcal{B}_1^{\infty}$. This is why for the construction of partially invariant solutions we can use the lists of subalgebras obtained above.

As an example for a partially invariant solution, we use the subalgebra $\langle \mathcal{X}(1), \mathcal{Z}(g)\rangle$. Due to the generator $\mathcal{Z}(g)$, we cannot make an ansatz for ψ. However, we can make an ansatz for ζ and because of the generator ∂_x, we have $\psi = \psi(t,x,y)$ and $\zeta = \zeta(t,y)$. Therefore, (3.4) is reduced to $\zeta_t + \psi_x(\zeta_y + \beta) = 0$, $\zeta = \psi_{xx} + \psi_{yy}$. Introducing the absolute vorticity $\eta = \zeta + \beta y$ and setting $\psi = \Psi(t,x,y) + \tilde\zeta(t,y)$ with $\tilde\zeta_{yy} = \zeta$, we find $\eta_t + \Psi_x\eta_y = 0$, and $\Psi_{xx} + \Psi_{yy} = 0$. If $\eta_y = 0$, we have $\eta_t = 0$ and, consequently, $\eta = \mathrm{const}$. The stream function constructed in this way then reads as

$$\psi = \Psi - \frac{1}{6}\beta y^3 + \frac{1}{2}\eta y^2$$

where $\Psi(t,x,y)$ is an arbitrary solution of the Laplace equation $\Psi_{xx} + \Psi_{yy} = 0$.

In case $\eta_y \neq 0$, we find that the stream function has the form

$$\psi = \frac{1}{(g^1)^2}F(\omega) - \frac{1}{6}\beta y^3 - \frac{g^1_t y + g^0_t}{g^1}x + f^1 y + f^0,$$

where $\omega = g^1 y + g^0$ and g^1, g^0, f^1, and f^0 are functions of t.

To present one more example of a partially invariant solution, we take the subalgebra $\langle \partial_y, \mathcal{Z}(g)\rangle$. Similar to the previous case, we now have $\zeta = \zeta(t,x)$ and $\psi = \psi(t,x,y)$. Then, (3.4) is reduced to $\zeta_t - \psi_y\zeta_x + \beta\psi_x = 0$ and $\zeta = \psi_{xx} + \psi_{yy}$. Introducing $\psi = \phi(t,x,y) + \sigma(t,x)$, $\sigma_{xx} = \zeta$, this set of equations yields

$$\phi_{xx} + \phi_{yy} = 0, \qquad \sigma_{xxt} - \sigma_{xxx}\phi_y + \beta(\phi_x + \sigma_x) = 0. \tag{3.5}$$

We now have to distinguish different cases for the integration of this system.

1. $\sigma_{xxx} = 0$. In this instance, the solution for the stream function reads as

$$\psi = -\frac{1}{\beta}(2\chi^1_t x + \chi^2) + \chi^1 y^2 + \chi^3 y,$$

where χ^1, χ^2 and χ^3 are smooth functions of t. The functions χ^2 and χ^3 can be set equal to 0 by symmetry transformations generated by an operator of the form $\mathcal{X}(f) + \mathcal{Z}(g)$.

2. $\sigma_{xxx} \neq 0$. We set $\phi = H - \beta^{-1}\sigma_{xt} - \sigma$ and substitute into the second equation of (3.5), which yields a characteristic system for H. Solving this system, we find that $H = H(t, \eta)$, where $\eta = \sigma_{xx} + \beta y$ is again the absolute vorticity. From the first equation of (3.5), we then derive

$$H_{\eta\eta}((\sigma_{xxx})^2 + \beta^2) + H_\eta \sigma_{xxxy} - \frac{\sigma_{xxxt}}{\beta} - \sigma_{xx} = 0. \tag{3.6}$$

If we fix x in the above equation, we can write $h^2(t)H_{\eta\eta} + h^1(t)H_\eta + h^0(t) = 0$. We now have to distinguish whether there are two independent equations of this type or only one.

In case of two equations we have $H_{\eta\eta} = 0$ and consequently $H = \alpha(t)\eta + \gamma(t)$. Substituting this into (3.6), solving the resulting PDE and transforming back to the original variables, we find $\psi = \Sigma(t, x) + \alpha(t)\beta y$, $\alpha\Sigma_{xx} - \beta^{-1}\Sigma_{xt} + \beta^{-1}\delta(t) - \Sigma = 0$. By means of symmetry transformations generated by $\mathcal{X}(f)$ and $\mathcal{Z}(g)$ we can set $\alpha = \delta = 0$ and again arrive at the Klein-Gordon equation. This illustrates the fact that in some cases the ansatz for a partially invariant solution effectively reduces to a usual group-invariant reduction.

If we only have one independent equation in H, $h^2 \neq 0$, and the equation

$$H_{\eta\eta}((\sigma_{xxx})^2 + \beta^2) + H_\eta \sigma_{xxxx} - \frac{\sigma_{xxxt}}{\beta} - \sigma_{xx} = \lambda(h^2(t)H_{\eta\eta} + h^1(t)H_\eta + h^0(t)),$$

where $\lambda = \lambda(t, x)$, holds identically in H. Splitting this equation with respect to H leads to the three equations

$$(\sigma_{xxx})^2 + \beta^2 = \lambda h^2, \qquad \sigma_{xxxx} = \lambda h^1, \qquad -\beta^{-1}\sigma_{xxxt} - \sigma_{xx} = \lambda h^0.$$

Since $h^2 \neq 0$, we can express λ from the first equation. Provided that $h^1 = 0$, we integrate the second equation to find $\sigma = \sigma^3(t)x^3 + \sigma^2(t)x^2 + \sigma^1(t)x + \sigma^0(t)$. Inserting this expression in the third equation and splitting with respect to x then yields $\sigma^3 = 0$, i.e., $\sigma_{xxx} = 0$, contradicting the initial assumption for this case. For $h^1 \neq 0$, we integrate the second equation once with respect to x and then substitute the resulting expression for σ_{xx} into the third equation. This leads to a contradiction in the system constructed by splitting with respect to x, and hence no solution is obtained also under the assumption $h^1 \neq 0$.

3.3 The spherical equation

3.3.1 The model

The barotropic vorticity equation on the sphere (sBVE) is given by (e.g. Ref. [124])

$$\zeta_t + \frac{1}{R^2}(\psi_\lambda \zeta_\mu - \psi_\mu \zeta_\lambda) + \frac{2\Omega}{R^2}\psi_\lambda = 0, \quad \zeta := \frac{1}{R^2}\left[\frac{1}{1-\mu^2}\psi_{\lambda\lambda} + \left((1-\mu^2)\psi_\mu\right)_\mu\right], \tag{3.7}$$

where ψ is the (spherical) stream function and ζ the (spherical) vorticity. They are related through the Laplacian on the sphere. Instead of using the latitude φ as an independent variable, in practice, it is convenient to rather use $\mu = \sin\varphi$. The value of μ ranges from -1 (South Pole) to 1 (North Pole). By λ we denote the longitude, R is the mean radius of the Earth and Ω the absolute value of the Earth's angular rotation vector.

3.3.2 The symmetries

We aim to start with (3.7) in a nonrotating reference frame ($\Omega = 0$). Note in passing that it is possible to scale the radius R of the Earth to 1 by including R in the stream function via setting $\tilde{\psi} = \psi/R^2$.

The corresponding Lie symmetry algebra $\mathcal{S}_0^\infty$ is infinite dimensional and a suitable basis is provided by

$$\begin{aligned} &\mathcal{D} = t\partial_t - \psi\partial_\psi, \qquad \partial_t, \qquad \mathcal{Z}(g) = g(t)\partial_\psi, \qquad \mathrm{J}_1 = \partial_\lambda, \\ &\mathrm{J}_2 = \mu\frac{\sin\lambda}{\sqrt{1-\mu^2}}\partial_\lambda + \sqrt{1-\mu^2}\cos\lambda\partial_\mu, \qquad \mathrm{J}_3 = \mu\frac{\cos\lambda}{\sqrt{1-\mu^2}}\partial_\lambda - \sqrt{1-\mu^2}\sin\lambda\partial_\mu. \end{aligned} \tag{3.8}$$

As for the physical meaning of these basis elements, we find that $\mathcal{D}$ is the generator of scaling in t and ψ and ∂_t corresponds to time translations. The generators $\mathcal{J}_i$, $i = 1, 2, 3$, correspond to rotations in angular coordinates. This follows since they satisfy the commutation relations of the Lie algebra $\mathfrak{so}(3)$, $[\mathrm{J}_i, \mathrm{J}_j] = \sum_{k=1}^3 \varepsilon_{ijk}\mathrm{J}_k$, where $i, j = 1, 2, 3$ and ε_{ijk} is the Levi–Civita symbol. $\mathcal{Z}(g)$ again represents gauging of the stream function.

The algebra $\mathcal{S}_0^\infty$ has the structure of $\mathfrak{so}(3) \oplus (\mathfrak{g}_2 \Subset \langle\mathcal{Z}(g)\rangle)$, where $\mathfrak{g}_2 = \langle\mathcal{D}, \partial_t\rangle$ is the two-dimensional non-Abelian algebra and $\langle\mathcal{Z}(g)\rangle$ is an infinite-dimensional Abelian ideal in $\mathcal{S}_0^\infty$.

Now turning to the rotating case ($\Omega \neq 0$). Eqn. (3.7) admits the infinite-dimensional Lie invariance algebra $\mathcal{S}_\Omega^\infty$

$$\begin{aligned} &\mathcal{D} = t\partial_t - (\psi - \Omega\mu)\partial_\psi - \Omega t\partial_\lambda, \qquad \partial_t, \qquad \mathcal{Z}(g) = g(t)\partial_\psi, \qquad \mathrm{J}_1 = \partial_\lambda, \\ &\mathrm{J}_2 = \mu\frac{\sin(\lambda+\Omega t)}{\sqrt{1-\mu^2}}\partial_\lambda + \frac{\cos(\lambda+\Omega t)}{\sqrt{1-\mu^2}}\left((1-\mu^2)\partial_\mu + \Omega\partial_\psi\right), \\ &\mathrm{J}_3 = \mu\frac{\cos(\lambda+\Omega t)}{\sqrt{1-\mu^2}}\partial_\lambda - \frac{\sin(\lambda+\Omega t)}{\sqrt{1-\mu^2}}\left((1-\mu^2)\partial_\mu + \Omega\partial_\psi\right). \end{aligned}$$

The physical interpretation of the basis elements is obvious from those of the case $\Omega = 0$. Moreover, straightforward calculation shows that both Lie symmetry algebras $\mathcal{S}_0^\infty$ and $\mathcal{S}_\Omega^\infty$ are isomorphic and can be mapped to each other by means of the change in the coordinates,

$$\tilde{t} = t, \quad \tilde{\mu} = \mu, \quad \tilde{\lambda} = \lambda + \Omega t, \quad \tilde{\psi} = \psi - \Omega\mu. \tag{3.9}$$

Furthermore, it is possible to transform (3.7) into the corresponding equation in the rest frame ($\Omega = 0$) upon using (3.9). This recovers, in a systematic way, the transformation used by Platzman [124] to reduce the spherical vorticity equation to a reference frame with zero angular momentum.

Note that this mapping is possible due to the special form of the Laplacian in spherical coordinates. In particular, it is impossible to obtain a similar result for the vorticity equation in Cartesian coordinates since in this case the respective Lie symmetry algebras are nonisomorphic. Consequently, no transformation can be found that maps the vorticity equation on the β-plane to the vorticity equation on the f-plane. This indicates that the traditional β-plane approximation significantly distorts the geometry of the more natural spherical vorticity dynamics.

Again there are two independent discrete symmetries, given by $(t, \lambda, \mu, \psi) \mapsto (-t, -\lambda, \mu, \psi)$ and $(t, \lambda, \mu, \psi) \mapsto (t, \lambda, -\mu, -\psi)$, respectively.

3.3.3 Classification of subalgebras

The classification of subalgebras of $\mathcal{S}_0^\infty$ is done in the same fashion as for the $\mathcal{B}_1^\infty$. The nonidentical adjoint actions involving basis elements of the algebra $\mathcal{S}_0^\infty$ are exhausted by the following list:

$$\begin{aligned}
&\mathrm{Ad}(e^{\varepsilon\partial_t})\mathcal{D} = \mathcal{D} - \varepsilon\partial_t && \mathrm{Ad}(e^{\varepsilon \mathrm{J}_1})\mathrm{J}_2 = \mathrm{J}_2\cos\varepsilon + \mathrm{J}_3\sin\varepsilon\\
&\mathrm{Ad}(e^{\varepsilon\mathcal{Z}(g)})\mathcal{D} = \mathcal{D} + \varepsilon\mathcal{Z}(tg' + g) && \mathrm{Ad}(e^{\varepsilon \mathrm{J}_1})\mathrm{J}_3 = -\mathrm{J}_2\sin\varepsilon + \mathrm{J}_3\cos\varepsilon\\
&\mathrm{Ad}(e^{\varepsilon\mathcal{D}})\partial_t = e^{\varepsilon}\partial_t && \mathrm{Ad}(e^{\varepsilon \mathrm{J}_2})\mathrm{J}_3 = \mathrm{J}_3\cos\varepsilon + \mathrm{J}_1\sin\varepsilon\\
&\mathrm{Ad}(e^{\varepsilon\mathcal{Z}(g)})\partial_t = \partial_t + \varepsilon\mathcal{Z}(g') && \mathrm{Ad}(e^{\varepsilon \mathrm{J}_2})\mathrm{J}_1 = -\mathrm{J}_3\sin\varepsilon + \mathrm{J}_1\cos\varepsilon\\
&\mathrm{Ad}(e^{\varepsilon\mathcal{D}})\mathcal{Z}(g) = \mathcal{Z}(\tilde g), \quad \tilde g = e^{-\varepsilon}g(e^{-\varepsilon}t) && \mathrm{Ad}(e^{\varepsilon \mathrm{J}_3})\mathrm{J}_1 = \mathrm{J}_1\cos\varepsilon + \mathrm{J}_2\sin\varepsilon\\
&\mathrm{Ad}(e^{\varepsilon\partial_t})\mathcal{Z}(g) = \mathcal{Z}(\tilde g), \quad \tilde g = g(t-\varepsilon) && \mathrm{Ad}(e^{\varepsilon \mathrm{J}_3})\mathrm{J}_2 = -\mathrm{J}_1\sin\varepsilon + \mathrm{J}_2\cos\varepsilon.
\end{aligned}$$

Similar to the case of the βBVE, we start with the most general form of an infinitesimal generator $\mathbf{v} = a_D\mathcal{D} + a_t\partial_t + a_1\mathrm{J}_1 + a_2\mathrm{J}_2 + a_3\mathrm{J}_3 + \mathcal{Z}(g)$. In the same manner, by acting with the adjoint actions given above, we can determine the following list of conjugacy inequivalent *one-dimensional subalgebras* of $\mathcal{S}_0^\infty$:

$$\langle\mathcal{D} + a\mathrm{J}_1\rangle, \qquad \langle\partial_t + a\mathrm{J}_1\rangle, \qquad \langle\mathrm{J}_1 + \mathcal{Z}(g)\rangle, \qquad \langle\mathcal{Z}(g)\rangle, \tag{3.10}$$

where $a \in \mathbb{R}$ and $a \in \{-1, 0, 1\}$ for the first and second cases, respectively. Unlike the case of the βBVE, there is no discrete symmetry allowing placement of additional restrictions on the values of a. There are equivalence relations within the two last families of subalgebras, generated by adjoint actions of the scaling transformations, time translations, and within the last family, changes of algebra bases.

Using the same procedure as described in the second part of section 3.2.3 we find the following list of conjugacy inequivalent *two-dimensional subalgebras* of (3.8):

$$\begin{aligned}
&\langle\mathcal{D} + a\mathrm{J}_1,\, \partial_t\rangle, \qquad \langle\mathcal{D},\, \mathrm{J}_1 + \mathcal{Z}(at^{-1})\rangle, \qquad \langle\mathcal{D} + a\mathrm{J}_1,\, \mathcal{Z}(|t|^b)\rangle,\\
&\langle\partial_t,\, \mathrm{J}_1 + \mathcal{Z}(c)\rangle, \qquad \langle\partial_t + c\mathrm{J}_1,\, \mathcal{Z}(e^{\tilde c t})\rangle, \qquad \langle\mathrm{J}_1 + \mathcal{Z}(g^1),\, \mathcal{Z}(g^2)\rangle, \qquad \langle\mathcal{Z}(g^1),\, \mathcal{Z}(g^2)\rangle,
\end{aligned}$$

where $a, b \in \mathbb{R}$, $c \in \{-1, 0, 1\}$; $\tilde c \in \{-1, 0, 1\}$ if $c = 0$. There are additional equivalence relations within the last two series of subalgebras, generated by adjoint actions of the scale transformations, time translations and changes in the algebra bases.

3.3.4 Group-invariant reduction with one-dimensional subalgebras

Based on the above classification of one-dimensional algebras, below we present the corresponding list of reduced differential equations obtained from the sBVE here. Again, p, q denote the new independent variables, while $v = v(p, q)$ is the new dependent variable.

1	$\langle \mathcal{D} + a\mathrm{J}_1 \rangle$	$\psi = t^{-1}v$	$p = \lambda - a\ln t, \quad q = \mu$
		$w + aw_p - v_p w_q + v_q w_p = 0, \quad w := \frac{1}{1-q^2}v_{pp} + ((1-q^2)v_q)_q$	
2	$\langle \partial_t + a\mathrm{J}_1 \rangle$	$\psi = v$	$p = \lambda - at, \quad q = \mu$
		$-(aq+v)_q w_p + (aq+v)_p w_q = 0, \quad w := \frac{1}{1-q^2}v_{pp} + ((1-q^2)v_q)_q$	
3	$\langle \mathrm{J}_1 + \mathcal{Z}(g) \rangle$	$\psi = v + g(t)\lambda$	$p = t, \quad q = \mu$
		$w_p + gw_q = 0, \quad w := ((1-q^2)v_q)_q$	
4	$\langle \mathcal{Z}(g) \rangle$	No group-invariant reduction is possible in this case	

All Lie symmetries of the reduced equations of Cases **1** and **2** are induced by Lie symmetries of the sBVE. This is why further Lie reductions of these cases give no new solutions in comparison to Lie reductions with respect to two-dimensional subalgebras.

We now give some examples for solutions obtained upon using the above ansätze:

Case 2 includes the well-known Rossby–Haurwitz wave solutions. To show this, we construct a class of exact solutions upon using invariance of the sBVE under the algebra $\langle \partial_t + a\mathrm{J}_1 \rangle$. In particular, the corresponding reduced vorticity equation implies that $w = F$, where F is a function of $v + aq$. Hence, we have

$$\frac{1}{1-q^2}v_{pp} + (1-q^2)v_{qq} - 2qv_q = F. \tag{3.11}$$

Eqn. (3.11) is, in general, a nonlinear Poisson equation in spherical coordinates. To obtain the Rossby–Haurwitz wave solution from this equation, we set $F = c(v + aq)$, $c = \text{const}$, that is, we make a homogeneous linear ansatz for F. Separation of the variables gives the ansatz

$$v(p,q) = Ae^{imp}P_n^m(q) + Be^{imp}Q_n^m(q) - \frac{acq}{c+2},$$

with $A, B = \text{const}$, where $P_n^m(q)$ and $Q_n^m(q)$ are the associated Legendre functions of the first and second kind, respectively, and the degree n is given by

$$n = \frac{1}{2}(\sqrt{1-4c} - 1). \tag{3.12}$$

For the sake of brevity, we now set $B = 0$. Reverting to the original variables and employing transformation (3.9) to map the solution of the sBVE with vanishing rotation to a solution of the sBVE with rotation, we find

$$\psi(t,\lambda,\mu) = AP_n^m(\mu)e^{im(\lambda-(a-\Omega)t)} - \frac{ac\mu}{c+2} + \Omega\mu. \tag{3.13}$$

To derive pure wave solutions, we require $a = \Omega(c+2)/c$, which, upon inserting in (3.13) and considering (3.12), allows us to arrive at the well-known phase relation for a single Rossby–Haurwitz wave (e.g. Refs. [50, 105, 124]):

$$c_{\text{phase}} := a - \Omega = -\frac{2\Omega}{n(n+1)}. \tag{3.14}$$

Since the integer constant m is arbitrary and only linear cases of eqn. (3.11) are considered, we may extend the solution (3.13) by superposition of single solutions with m ranging from

$-n$ to n. This recovers—upon using (3.14)—the classical ansatz for the stream function of Rossby–Haurwitz waves.

Moreover, note that the class of solutions that may be obtained upon employing symmetry methods is again much wider than that obtained upon using the usual ansatz for the stream function. In fact, it can be seen that Rossby–Haurwitz waves correspond to particular simple solutions of the reduced spherical vorticity equation (3.11), but there is an infinite class of other solutions invariant under the same generator $\partial_t + a\mathrm{J}_1$.

Case 3 is completely integrable by quadratures. The general solution for the stream function reads

$$\psi = g(t)\lambda + f(t) + h(t)\mathrm{arctanh}\mu + \int \frac{\int w(\theta)\mathrm{d}\mu}{1-\mu^2}\mathrm{d}\mu, \qquad \theta := \mu - \int g(t)\mathrm{d}t.$$

3.3.5 Group-invariant reduction with two-dimensional subalgebras

As was discussed in section 3.2.5 it is, in general, not necessary to investigate reductions with the complete set of two-dimensional inequivalent subalgebras. Namely, if some of the equations obtained from one-dimensional reduction are completely integrable, we can avoid the computation of reduction with two-dimensional subalgebras if these algebras contain the generators that enabled the complete integration in the first place. For the sBVE, case **3** is integrable and case **4** does not allow to compute classical group-invariant solutions. Hence, we again have only the reduction in one two-dimensional subalgebra, which is not trivial in view of the reductions based on one-dimensional subalgebras: $\langle \mathcal{D} + a\mathrm{J}_1, \partial_t \rangle$. Note, however, that it is not possible to use this subalgebra in the case $a = 0$ for a classical Lie reduction since no proper ansatz for ψ can be constructed. Rather, it can only be used for the construction of partially invariant solutions. If $a \neq 0$, this subalgebra leads to invariant solutions that are obtainable as particular cases of reduction with the algebra $\langle \partial_t \rangle$. The corresponding ansatz $\psi = e^{b\lambda}v(\mu)$ reduces the sBVE to the equation $vw_\mu - v_\mu w = 0$, where $b = -1/a$, $w := b^2(1-\mu^2)^{-1}v + ((1-\mu^2)v_\mu)_\mu$. The reduced equation implies the following linear constraint between v and w: $w = Cv$, where C is an arbitrary constant, i.e., we have the equation

$$((1-\mu^2)v_\mu)_\mu + \frac{b^2}{1-\mu^2}v = Cv$$

which is integrable in terms of Legendre functions.

Chapter 4

Point symmetry group of the barotropic vorticity equation

Abstract The complete point symmetry group of the barotropic vorticity equation on the β-plane is computed using the direct method supplemented with two different techniques. The first technique is based on the preservation of any megaideal of the maximal Lie invariance algebra of a differential equation by the push-forwards of point symmetries of the same equation. The second technique involves a priori knowledge on normalization properties of a class of differential equations containing the equation under consideration. Both of these techniques are briefly outlined.

4.1 Introduction

It is well known that it is much easier to determine the continuous part of the complete point symmetry group of a differential equation than the entire group including discrete symmetries. The computation of continuous (Lie) symmetries is possible using infinitesimal techniques, which amounts to solving an overdetermined system of linear partial differential equations (referred to as *determining equations*) for coefficients of vector fields generating one-parameter Lie symmetry groups. Owing to the algorithmic nature of this problem, the automatic computation of Lie symmetries is already implemented in a number of symbolic calculation packages, see, e.g., papers [28, 51, 138] for detail description of certain packages and reviews [26, 53].

The relative simplicity of finding Lie symmetries of differential equations is also a primary reason why the overwhelming part of research on symmetries is devoted to symmetries of this kind. See, e.g., the textbooks [22, 23, 97, 115, 118] for general theory and numerous examples and additionally the works [6, 17, 18, 41, 96] for several applications of Lie methods in hydrodynamics and meteorology.

As continuous symmetries, also discrete symmetries are of practical relevance in a number of fields such as dynamical system theory, quantum mechanics, crystallography and solid state physics. They can also be helpful in some issues related to Lie symmetries, e.g. allowing for a simplification of optimal lists of inequivalent subalgebras, and due to enabling the construction of new solutions of differential equations from known ones. It is not possible, in general, to determine the whole point symmetry group in terms of finite transformations by usage of infinitesimal techniques. On the other hand, the direct computation of point symmetries based on their definition boils down to solving a cumbersome nonlinear system of determining equations, which is difficult to be integrated. Similar determining equations also arise under calculations of equivalence groups and sets of admissible transformations of classes of differential equations by means of employing the direct method. In order to simplify the derivation of the determining equations, different special techniques have been developed involving, in particular, the implicit representation of unknown functions, the combined splitting with respect to old and new variables and the inverse expression of old derivative via new ones [127, 131, 134].

There exist two particular techniques that can be applied for *a priori* simplification of calculations concerning the point symmetry groups of differential equations.

The first technique was presented in [61] for equations whose maximal Lie invariance algebras are finite dimensional. It is based on the fact that the push-forwards of point symmetries of a given system of differential equations to vector fields on the space of dependent and independent variables are automorphisms of the maximal Lie invariance algebra of the same system. This condition yields restrictions for those point transformations that can qualify as symmetries of the system of differential equations under consideration. We will adopt this technique to the infinite dimensional case using the notion of megaideals of Lie algebras, which are the most invariant algebraic structures.

The second technique involves available information on the set of admissible transformations of a class of differential equations [131], which contains the investigated equation.

In the present paper, we will demonstrate both of these techniques by computing the complete point symmetry group of the barotropic vorticity equation on the β-plane. This is one of the most classical models which are used in geophysical fluid dynamics. The techniques to be employed are briefly described in Section 4.2. The actual computations using the method based on the corresponding Lie invariance algebra and that involving a priori knowledge on admissible transformations of a class of generalized vorticity equations are presented in Section 4.3 and 4.4, respectively. A short summary concludes the paper.

4.2 Techniques of calculation of complete point symmetry groups

Both the techniques described in this section should be considered merely as tools for deriving preliminary restrictions on point symmetries. In either case, calculations must finally be carried out within the framework of the direct approach.

4.2.1 Using megaideals of Lie invariance algebra

The most refined version of the technique involving Lie symmetries in the calculations of complete point symmetry groups was applied in [61]. It is outlined as follows: Given a system of differential equations $\mathcal{L}$ whose maximal Lie invariance algebra $\mathfrak{g}$ is n-dimensional with a basis $\{e_1,\dots,e_n\}$, $n<\infty$, one has to compute the entire automorphism group of $\mathfrak{g}$, $\mathrm{Aut}(\mathfrak{g})$. Supposing that $\mathcal{T}$ is a transformation from the complete point symmetry group G of $\mathcal{L}$, one has the condition $\mathcal{T}_*e_j=\sum_{i=1}^n e_ia_{ij}$ for $j=1,\dots,n$, where $\mathcal{T}_*$ denotes the push-forward of vector fields induced by $\mathcal{T}$ and (a_{ij}) is the matrix of an automorphism of $\mathfrak{g}$ in the chosen basis. This condition implies constraints on the transformation $\mathcal{T}$ which are then taken into account in further calculations with the direct method.

The method we propose here is different to those described in the previous paragraph. In fact, it uses only the minimal information on the automorphism group $\mathrm{Aut}(\mathfrak{g})$ in the form of a set of megaideals of $\mathfrak{g}$. Due to this, it is applicable also in the case when the maximal Lie invariance algebra is infinite dimensional. The notion of megaideals was introduced in [128].

Definition 4.1. A *megaideal* $\mathfrak{i}$ is a vector subspace of $\mathfrak{g}$ that is invariant under any transformation from the automorphism group $\mathrm{Aut}(\mathfrak{g})$ of $\mathfrak{g}$.

That is, we have $\mathfrak{T}\mathfrak{i} = \mathfrak{i}$ for a megaideal $\mathfrak{i}$ of $\mathfrak{g}$, whenever $\mathfrak{T}$ is a transformation from $\mathrm{Aut}(\mathfrak{g})$. Any megaideal of $\mathfrak{g}$ is an ideal and characteristic ideal of $\mathfrak{g}$. Both the improper subalgebras of $\mathfrak{g}$ (the zero subspace and $\mathfrak{g}$ itself) are megaideals of $\mathfrak{g}$. The following assertions are obvious.

Proposition 4.1. *If* $\mathfrak{i}_1$ *and* $\mathfrak{i}_2$ *are megaideals of* $\mathfrak{g}$ *then so are* $\mathfrak{i}_1 + \mathfrak{i}_2$, $\mathfrak{i}_1 \cap \mathfrak{i}_2$ *and* $[\mathfrak{i}_1, \mathfrak{i}_2]$, *i.e., sums, intersections and Lie products of megaideals are again megaideals.*

Proposition 4.2. *If* $\mathfrak{i}_2$ *is a megaideal of* $\mathfrak{i}_1$ *and* $\mathfrak{i}_1$ *is a megaideal of* $\mathfrak{g}$ *then* $\mathfrak{i}_2$ *is a megaideal of* $\mathfrak{g}$, *i.e., megaideals of megaideals are also megaideals.*

Corollary 4.1. *All elements of the derived, upper and lower central series of a Lie algebra are its megaideals. In particular, the center and the derivative of a Lie algebra are its megaideals.*

Corollary 4.2. *The radical* $\mathfrak{r}$ *and nil-radical* $\mathfrak{n}$ *(i.e., the maximal solvable and nilpotent ideals, respectively) of* $\mathfrak{g}$ *as well as different Lie products, sums and intersections involving* $\mathfrak{g}$, $\mathfrak{r}$ *and* $\mathfrak{n}$ *(*$[\mathfrak{g}, \mathfrak{r}]$, $[\mathfrak{r}, \mathfrak{r}]$, $[\mathfrak{g}, \mathfrak{n}]$, $[\mathfrak{r}, \mathfrak{n}]$, $[\mathfrak{n}, \mathfrak{n}]$, *etc.) are megaideals of* $\mathfrak{g}$.

Suppose that $\mathfrak{g}$ is finite dimensional and possesses a megaideal $\mathfrak{i}$ which, without loss of generality, can be assumed to be spanned by the first k basis elements, $\mathfrak{i} = \langle e_1, \dots, e_k \rangle$. Then the matrix (a_{ij}) of any automorphism of $\mathfrak{g}$ has block structure, namely, $a_{ij} = 0$ for $i > k$. In other words, in the finite dimensional case we take into account only the block structure of automorphism matrices. This is reasonable as the entire automorphism group $\mathrm{Aut}(\mathfrak{g})$ (which should be computed within the method from [61]) may be much wider than the group of automorphisms of $\mathfrak{g}$ induced by elements of the point symmetry group G of $\mathcal{L}$. Moreover, it seems difficult to find the entire group $\mathrm{Aut}(\mathfrak{g})$ if the algebra $\mathfrak{g}$ is infinite dimensional. At the same time, in view of the above assertions it is easy to determine a set of megaideals for any Lie algebra.

4.2.2 Direct method and admissible transformations

The initial point of the second technique is to consider a given pth order system $\mathcal{L}^0$ of l differential equations for m unknown functions $u = (u^1, \dots, u^m)$ of n independent variables $x = (x_1, \dots, x_n)$ as an element of a class $\mathcal{L}|_\mathcal{S}$ of similar systems $\mathcal{L}_\theta$: $L(x, u_{(p)}, \theta(x, u_{(p)})) = 0$ parameterized by a tuple of pth order differential functions (arbitrary elements) $\theta = (\theta^1(x, u_{(p)}), \dots, \theta^k(x, u_{(p)}))$. Here $u_{(p)}$ denotes the set of all the derivatives of u with respect to x of order not greater than p, including u as the derivatives of order zero. The class $\mathcal{L}|_\mathcal{S}$ is determined by two objects: the tuple $L = (L^1, \dots, L^l)$ of l fixed functions depending on x, $u_{(p)}$ and θ and θ running through the set $\mathcal{S}$. Within the framework of symmetry analysis of differential equations, the set $\mathcal{S}$ is defined as the set of solutions of an auxiliary system consisting of a subsystem $S(x, u_{(p)}, \theta_{(q)}(x, u_{(p)})) = 0$ of differential equations with respect to θ and a non-vanish condition $\Sigma(x, u_{(p)}, \theta_{(q)}(x, u_{(p)})) \neq 0$ with another differential function Σ of θ. In the auxiliary system, x and $u_{(p)}$ play the role of independent variables and $\theta_{(q)}$ stands for the set of all the partial derivatives of θ of order not greater than q with respect to the variables x and $u_{(p)}$. In view of the purpose of our consideration we should have that $\mathcal{L}^0 = \mathcal{L}_{\theta_0}$ for some $\theta_0 \in \mathcal{S}$.

Following [131], for $\theta, \tilde\theta \in \mathcal{S}$ we denote by $\mathrm{T}(\theta, \tilde\theta)$ the set of point transformations which map the system $\mathcal{L}_\theta$ to the system $\mathcal{L}_{\tilde\theta}$. The maximal point symmetry group G_θ of the system $\mathcal{L}_\theta$ coincides with $\mathrm{T}(\theta, \theta)$.

Definition 4.2. $\mathrm{T}(\mathcal{L}|_\mathcal{S}) = \{(\theta, \tilde\theta, \varphi) \mid \theta, \tilde\theta \in \mathcal{S}, \varphi \in \mathrm{T}(\theta, \tilde\theta)\}$ is called the *set of admissible transformations in* $\mathcal{L}|_\mathcal{S}$.

Sets of admissible transformations were first systematically described by Kingston and Sophocleous for a class of generalized Burgers equations [73] and Winternitz and Gazeau for a class of variable coefficient Korteweg–de Vries equations [161], in terms of *form-preserving* [73, 74, 75] and *allowed* [161] transformations, respectively. The notion of admissible transformations can be considered as a formalization of their approaches.

Any point symmetry transformation of an equation $\mathcal{L}_\theta$ from the class $\mathcal{L}|_\mathcal{S}$ generates an admissible transformation in this class. Therefore, it obviously satisfies all restrictions which hold for admissible transformations [74]. For example, it is known for a long time that for any point (and even contact) transformation connecting a pair of $(1+1)$-dimensional evolution equations its component corresponding to t depends only on t, cf. [89]. The equations in the pair can also coincide. As a result, the same restriction should be satisfied by any point or contact symmetry transformation of every $(1+1)$-dimensional evolution equation.

The simplest description of admissible transformations is obtained for normalized classes of differential equations. Roughly speaking, a class of (systems of) differential equations is called *normalized* if any admissible transformation in this class is induced by a transformation from its equivalence group. Different kinds of normalization can be defined depending on what kind of equivalence group (point, contact, usual, generalized, extended, etc.) is considered. Thus, the *usual equivalence group* $G^\sim$ of the class $\mathcal{L}|_\mathcal{S}$ consists of those point transformations in the space of variables and arbitrary elements, which are projectable on the variable space and preserve the whole class $\mathcal{L}|_\mathcal{S}$. The class $\mathcal{L}|_\mathcal{S}$ is called normalized in the usual sense if the set $\mathrm{T}(\mathcal{L}|_\mathcal{S})$ is generated by the usual equivalence group $G^\sim$. As a consequence, all generalizations of the equivalence group within the framework of point transformations are trivial for this class. See [131] for precise definitions and further explanations. If the class $\mathcal{L}|_\mathcal{S}$ is normalized in certain sense with respect to point transformations, the point symmetry group G_{θ_0} of any equation $\mathcal{L}_{\theta_0}$ from this class is contained in the projection of the corresponding equivalence group of $\mathcal{L}|_\mathcal{S}$ to the space of independent and dependent variables (taken for the value $\theta = \theta_0$ in the case when the generalized equivalence group is considered).

As a rule, calculations of certain common restrictions on admissible transformations of the entire normalized class or its normalized subclasses or point symmetry transformations of a single equation from this class have the same level of complexity. For example, in order to derive the restriction that the transformation component corresponding to t depends only on t, we should carry out approximately the same operations, independently of considering the whole class of $(1+1)$-dimensional evolution equations, any well-defined subclass from this class or any single evolution equation. This is why it is worthwhile to first construct nested series of normalized classes of differential equations by starting from a quite general, obviously normalized class, imposing on each step additional auxiliary conditions on the arbitrary elements and then studying the complete point symmetries of a single equation from the narrowest class of the constructed series.

In the way outlined above we have already investigated hierarchies of normalized classes of generalized nonlinear Schrödinger equations [131], $(1+1)$-dimensional linear evolution equations [132], (1 + 1)-dimensional third-order evolution equations including variable-coefficient Korteweg–de Vries and modified Korteweg–de Vries equations [133] and generalized vorticity equations arising in the study of local parameterization schemes for the barotropic vorticity equation [127].

If an equation does not belong to a class whose admissible transformations have been studied earlier, one can try to map this equation using a point transformation to an equation from a class for which constraints on its admissible transformations are known a priori. Then one can either map the known constraints on admissible transformations back and then complete the calculations of point symmetries of the initial equation using the direct method or calculate the point symmetry group of the mapped equation using the direct method and then map this group back. The example on the application of this trick to the barotropic vorticity equation in presented in Section 4.4.

4.3 Calculations based on Lie invariance algebra of the barotropic vorticity equation

The barotropic vorticity equation on the β-plane reads

$$\zeta_t + \psi_x\zeta_y - \psi_y\zeta_x + \beta\psi_x = 0, \tag{4.1}$$

where $\psi = \psi(t, x, y)$ is the stream function and $\zeta := \psi_{xx} + \psi_{yy}$ is the relative vorticity, which is the vertical component of the vorticity vector. The barotropic vorticity equation in the formulation (4.1) is valid in situations where the two-dimensional wind field can be regarded as almost non-divergent and the motion in North–South direction is confined to a relatively small region. It is then convenient to use a local Cartesian coordinate system. In such a coordinate system, the effect of the sphericity of the Earth is conveniently taken into account by approximating the normal component of the vorticity due to the rotation of the Earth, $2\Omega\sin\varphi$, by its linear Taylor series expansion, where Ω is the angular rotation of the Earth and φ is the geographic latitude. This linear approximation at some reference latitude φ_0 is given by $2\Omega\sin\varphi_0 + \beta y$, where $\beta = 2\Omega\cos\varphi/a$ and a is the radius of the Earth. This is the traditional β-plane approximation, see [122] for further details. Then, taking the vertical component of the curl of the two-dimensional ideal Euler equations and using the β-plane approximation leads to Eq. (4.1).

It is straightforward to determine the maximal Lie invariance algebra $\mathfrak g$ of Eq. (4.1) using infinitesimal techniques:

$$\mathfrak g = \langle \mathcal D, \partial_t, \partial_y, \mathcal X(f), \mathcal Z(g)\rangle,$$

where $\mathcal D = t\partial_t - x\partial_x - y\partial_y - 3\psi\partial_\psi$, $\mathcal X(f) = f(t)\partial_x - f_t(t)y\partial_\psi$ and $\mathcal Z(g) = g(t)\partial_\psi$, and f and g run through the space of smooth functions of t. (In fact, the precise interpretation of $\mathfrak g$ as a Lie algebra strongly depends on what space of smooth functions is chosen for f and g, cf. Note A.1 in [41, p. 178].) This result was first obtained in [68] and is now easily accessible in the handbook [63, p. 223]. See also [17] for related discussions and the exhaustive study of the classical Lie reductions of Eq. (4.1).

The nonzero commutation relations of the algebra $\mathfrak g$ in the above basis are exhausted by the following ones:

$$[\partial_t, \mathcal D] = \partial_t, \quad [\partial_y, \mathcal D] = -\partial_y,$$

$$[\mathcal{D},\mathcal{X}(f)]=\mathcal{X}(tf_t+f),\quad [\mathcal{D},\mathcal{Z}(g)]=\mathcal{Z}(tg_t+3g),$$
$$[\partial_t,\mathcal{X}(f)]=\mathcal{X}(f_t),\quad [\partial_t,\mathcal{Z}(g)]=\mathcal{Z}(g_t),\quad [\partial_y,\mathcal{X}(f)]=-\mathcal{Z}(f_t).$$

It is easy to see from the commutation relations that the Lie algebra $\mathfrak{g}$ is solvable since

$$\mathfrak{g}'=[\mathfrak{g},\mathfrak{g}]=\langle\partial_t,\partial_y,\mathcal{X}(f),\mathcal{Z}(g)\rangle,$$
$$\mathfrak{g}''=[\mathfrak{g}',\mathfrak{g}']=\langle\mathcal{X}(f),\mathcal{Z}(g)\rangle,$$
$$\mathfrak{g}'''=[\mathfrak{g}'',\mathfrak{g}'']=0.$$

Therefore, the radical $\mathfrak{r}$ of $\mathfrak{g}$ coincides with the entire algebra $\mathfrak{g}$. The nil-radical of $\mathfrak{g}$ is the ideal

$$\mathfrak{n}=\langle\partial_y,\mathcal{X}(f),\mathcal{Z}(g)\rangle.$$

Indeed, this ideal is a nilpotent subalgebra of $\mathfrak{g}$ since

$$\mathfrak{n}^{(2)}=\mathfrak{n}'=[\mathfrak{n},\mathfrak{n}]=\langle\mathcal{Z}(g)\rangle,\quad \mathfrak{n}^{(3)}=[\mathfrak{n},\mathfrak{n}']=0.$$

It can be extended to a larger ideal of $\mathfrak{g}$ only with two sets of elements, $\{\partial_t\}$ and $\{\mathcal{D},\partial_t\}$. Both resulting ideals are not nilpotent. In other words, $\mathfrak{n}$ is the maximal nilpotent ideal.

Continuous point symmetries of Eq. (4.1) are determined from the elements of $\mathfrak{g}$ by integration of the associated Cauchy problems. It is obvious that Eq. (4.1) also possesses two discrete symmetries, $(t,x,y,\psi)\mapsto(-t,-x,y,\psi)$ and $(t,x,y,\psi)\mapsto(t,x,-y,-\psi)$, which are independent up to their composition and their compositions with continuous symmetries. The proof that the above symmetries generate the entire point symmetry group was, however, outstanding.

Theorem 4.1. *The complete point symmetry group of the barotropic vorticity equation on the β-plane* (4.1) *is formed by the transformations*

$$\mathcal{T}\colon\quad \tilde t=T_1t+T_0,\quad \tilde x=\frac{1}{T_1}x+f(t),\quad \tilde y=\frac{\varepsilon}{T_1}y+Y_0,$$
$$\tilde\psi=\frac{\varepsilon}{(T_1)^3}\psi-\frac{\varepsilon}{(T_1)^2}f_t(t)y+g(t),$$

where $T_1\neq0$, $\varepsilon=\pm1$ and f and g are arbitrary functions of t.

Proof. The discrete symmetries of the barotropic vorticity equation on the β-plane are computed as described in section 4.2.1. The general form of a point transformation of the vorticity equation is:

$$\mathcal{T}\colon\quad (\tilde t,\tilde x,\tilde y,\tilde\psi)=(T,X,Y,\Psi),$$

where T, X, Y and Ψ are regarded as functions of t, x, y and ψ, whose joint Jacobian does not vanish. To obtain the constrained form of $\mathcal{T}$, we use the above four proper nested megaideals of $\mathfrak{g}$, namely $\mathfrak{n}'$, $\mathfrak{g}''$, $\mathfrak{n}$ and $\mathfrak{g}'$, and $\mathfrak{g}$ itself. Recall once more that the transformation $\mathcal{T}$ must satisfy the conditions $\mathcal{T}_*\mathfrak{n}'=\mathfrak{n}'$, $\mathcal{T}_*\mathfrak{g}''=\mathfrak{g}''$, $\mathcal{T}_*\mathfrak{n}=\mathfrak{n}$, $\mathcal{T}_*\mathfrak{g}'=\mathfrak{g}'$ and $\mathcal{T}_*\mathfrak{g}=\mathfrak{g}$ in order to qualify as a point symmetry of the vorticity equation, where $\mathcal{T}_*$ denotes the push-forward of $\mathcal{T}$ to vector fields. In other words, we have

$$\mathcal{T}_*\mathcal{Z}(g)=g(T_\psi\partial_{\tilde t}+X_\psi\partial_{\tilde x}+Y_\psi\partial_{\tilde y}+\Psi_\psi\partial_{\tilde\psi})=\tilde{\mathcal{Z}}(\tilde g^g),\tag{4.2}$$
$$\mathcal{T}_*\mathcal{X}(f)=\tilde{\mathcal{X}}(\tilde f^f)+\tilde{\mathcal{Z}}(\tilde g^f),\tag{4.3}$$

$$\mathcal{T}_*\partial_t = T_t\partial_{\tilde t} + X_t\partial_{\tilde x} + Y_t\partial_{\tilde y} + \Psi_t\partial_{\tilde\psi} = a_1\partial_{\tilde t} + a_2\partial_{\tilde y} + \tilde{\mathcal{X}}(\tilde f) + \tilde{\mathcal{Z}}(\tilde g), \tag{4.4}$$

$$\mathcal{T}_*\partial_y = T_y\partial_{\tilde t} + X_y\partial_{\tilde x} + Y_y\partial_{\tilde y} + \Psi_y\partial_{\tilde\psi} = b_1\partial_{\tilde y} + \tilde{\mathcal{X}}(\tilde f^y) + \tilde{\mathcal{Z}}(\tilde g^y), \tag{4.5}$$

$$\mathcal{T}_*\mathcal{D} = c_1\tilde{\mathcal{D}} + c_2\partial_{\tilde t} + c_3\partial_{\tilde y} + \tilde{\mathcal{X}}(\tilde f^D) + \tilde{\mathcal{Z}}(\tilde g^D), \tag{4.6}$$

where all $\tilde f$'s and $\tilde g$'s are smooth functions of $\tilde t$ which are determined, as the constant parameters a_1, a_2, b_1, c_1, c_2 and c_3, by $\mathcal{T}_*$ and the operator from the corresponding left-hand side.

We will derive constraints on $\mathcal{T}_*$, consequently equating coefficients of vector fields in conditions (4.2)–(4.6) and taking into account constraints obtained on previous steps. Thus Eq. (4.2) immediately implies $T_\psi = X_\psi = Y_\psi = 0$ (hence $\Psi_\psi \neq 0$) and $g\Psi_\psi = \tilde g^g$. Evaluating the last equation for $g = 1$ and $g = t$ and combining the results gives $t = \tilde g^t(T)/\tilde g^1(T)$, where $\tilde g^t = \tilde g^g|_{g=t}$ and $\tilde g^1 = \tilde g^g|_{g=1}$. As the derivative with respect to T in the right hand side of this equality does not vanish, the condition $T = T(t)$ must hold. This implies that Ψ_ψ depends only on t.

As then $\mathcal{T}_*\mathcal{X}(f) = fX_x\partial_{\tilde x} + fY_x\partial_{\tilde y} + (f\Psi_x - f_ty\Psi_\psi)\partial_{\tilde\psi}$, it follows from Eq. (4.3) that $Y_x = 0$ and

$$fX_x = \tilde f^f, \quad f\Psi_x - f_ty\Psi_\psi = -\tilde f^f_{\tilde t}Y + \tilde g^f.$$

Evaluating the first of the displayed equalities for $f = 1$, we derive that $X_x = \tilde f^1(T) =: X_1(t)$. Therefore, $\tilde f^f(T) = f(t)X_1(t)$. The second equality then reads

$$f\Psi_x - f_ty\Psi_\psi = -\frac{(fX_1)_t}{T_t}Y + \tilde g^f.$$

Setting $f = 1$ and $f = t$ in the last equality and combining the resulting equalities yields $y\Psi_\psi = (T_t)^{-1}X_1Y + t\tilde g^1 - \tilde g^t$, where $\tilde g^t = \tilde g^f|_{f=t}$ and $\tilde g^1 = \tilde g^f|_{f=1}$. As $X_1 \neq 0$ this equation implies that $Y = Y_1(t)y + Y_0(t)$.

After analyzing Eq. (4.4), we find $T_t = \text{const}$, $Y_t = \text{const}$, which leads to $Y_1 = \text{const}$, $X_t = \tilde f(T)$ and thus $X_{tx} = 0$, i.e., $X_1 = \text{const}$. Finally, Eq. (4.4) also implies $\Psi_t = -\tilde f_{\tilde t}Y + \tilde g$. In a similar manner, upon taking into account the restrictions already derived so far, collecting coefficients in Eq. (4.5) gives the constraint $X_y = \tilde f^y =: X_2 = \text{const}$ since $X_{yt} = 0$. Moreover, $\Psi_y = \tilde g^y$, as $\tilde f^y_{\tilde t} = 0$.

The final restrictions on $\mathcal{T}$ based on the preservation of $\mathfrak{g}$ are derivable from Eq. (4.6), where

$$\begin{aligned}\mathcal{T}_*\mathcal{D} = {} & tT_t\partial_{\tilde t} + (tX_t - xX_x - yX_y)\partial_{\tilde x} + (tY_t - yY_y)\partial_{\tilde y} \\ & + (t\Psi_t - x\Psi_x - y\Psi_y - 3\psi\Psi_\psi)\partial_{\tilde\psi}.\end{aligned}$$

Collecting the coefficients of $\partial_{\tilde t}$ and $\partial_{\tilde y}$, we obtain that $c_1 = 1$ and $Y_t = 0$. Similarly, equating the coefficients of $\partial_{\tilde\psi}$ and further splitting with respect to x implies that $\Psi_x = 0$.

The results obtained so far lead to the following constrained form of the general point symmetry transformation of the vorticity equation (4.1)

$$\begin{aligned}& T = T_1t + T_0, \quad X = X_1x + X_2y + f(t), \quad Y = Y_1y + Y_0, \\ & \Psi = \Psi_1\psi + \Psi_2(t)y + \Psi_4(t),\end{aligned} \tag{4.7}$$

where T_0, T_1, X_1, X_2, Y_0, Y_1 and Ψ_1 are arbitrary constants, $T_1X_1Y_1\Psi_1 \neq 0$, and $f(t)$, $\Psi_2(t)$ and $\Psi_4(t)$ are arbitrary time-dependent functions. The form (4.7) takes into account all constraints on point symmetries of (4.1), which follow from the preservation of the maximal Lie invariance algebra $\mathfrak{g}$ by the associated push-forward of vector fields.

Now the direct method should be applied. We carry out a transformation of the form (4.7) in the vorticity equation. For this aim, we calculate the transformation rules for the partial derivative operators:

$$\partial_{\tilde t}=\frac{1}{T_1}\left(\partial_t-\frac{f_t}{X_1}\partial_x\right),\quad \partial_{\tilde x}=\frac{1}{X_1}\partial_x,\quad \partial_{\tilde y}=\frac{1}{Y_1}\left(\partial_y-\frac{X_2}{X_1}\partial_x\right).$$

Further restrictions on $\mathcal{T}$ can be imposed upon noting that the term ψ_{txy} can only arise in the expression for $\tilde\psi_{\tilde t\tilde y\tilde y}$, which is

$$\tilde\psi_{\tilde t\tilde y\tilde y}=-\frac{2\Psi_1}{T_1Y_1}\frac{X_2}{X_1}\psi_{txy}+\cdots.$$

This obviously implies that $X_2=0$. In a similar fashion, the expression for $\tilde\zeta_{\tilde t}$ is

$$\tilde\zeta_{\tilde t}=\frac{\Psi_1}{T_1}\left(\frac{1}{(X_1)^2}\zeta_t+\left(\frac{1}{(Y_1)^2}-\frac{1}{(X_1)^2}\right)\psi_{yyt}\right)+\cdots,$$

upon using $\psi_{xxt}=\zeta_t-\psi_{yyt}$. Hence $(X_1)^2=(Y_1)^2$ as there are no other terms with ψ_{yyt} in the invariance condition. After taking into account these two more restrictions on $\mathcal{T}$, it is straightforward to expand the transformed version of the vorticity equation. This yields

$$\begin{aligned}&\frac{\Psi_1}{T_1(X_1)^2}\zeta_t-\frac{f_t\Psi_1}{T_1(X_1)^3}\zeta_x+\frac{(\Psi_1)^2}{(X_1)^3Y_1}\psi_x\zeta_y-\left(\frac{\Psi_1}{Y_1}\psi_y+\frac{\Psi_2}{Y_1}\right)\frac{\Psi_1}{(X_1)^3}\zeta_x\\&+\beta\frac{\Psi_1}{X_1}\psi_x=\frac{\Psi_1}{T_1(X_1)^2}\left(\zeta_t+\psi_x\zeta_y-\psi_y\zeta_x+\beta\psi_x\right).\end{aligned}$$

The invariance condition is fulfilled provided that the constraints

$$\Psi_2=-\frac{Y_1}{T_1}f_t,\quad X_1=T_1(X_1)^2,\quad \frac{(\Psi_1)^2}{(X_1)^3Y_1}=\frac{\Psi_1}{T_1(X_1)^2}.$$

hold. This completes the proof of the theorem. □

Corollary 4.3. *The barotropic vorticity equation on the β-plane possesses only two independent discrete point symmetries, which are given by*

$$\Gamma_1\colon (t,x,y,\psi)\mapsto(-t,-x,y,\psi),\quad \Gamma_2\colon (t,x,y,\psi)\mapsto(t,x,-y,-\psi).$$

They generate the group of discrete symmetry transformations of the barotropic vorticity equation on the β-plane, which is isomorphic to $\mathbb{Z}^2\times\mathbb{Z}^2$, where $\mathbb{Z}^2$ denotes the cyclic group of two elements.

4.4 Direct method and admissible transformations of classes of generalized vorticity equations

The construction of the complete point symmetry group G of the barotropic vorticity equation (4.1) by means of using only the direct method involves cumbersome and sophisticated calculations. As Eq. (4.1) is a third-order PDE in three independent variables, the system of determining equations for transformations from G is an overdetermined nonlinear system of PDEs in four independent variables, which should be solved by taking into account the nonsingularity condition of the point transformations. This is an extremely challenging task. Fortunately, a hierarchy of normalized classes of generalized vorticity equations was recently constructed [127] that allows us to strongly simplify the whole investigation. Eq. (4.1) belongs only to the narrowest class of this hierarchy, which is quite wide and consists of equations of the general form

$$\zeta_t = F(t,x,y,\psi,\psi_x,\psi_y,\zeta,\zeta_x,\zeta_y,\zeta_{xx},\zeta_{xy},\zeta_{yy}), \quad \zeta := \psi_{xx}+\psi_{yy}, \tag{4.8}$$

where $(F_{\zeta_x},F_{\zeta_y},F_{\zeta_{xx}},F_{\zeta_{xy}},F_{\zeta_{yy}}) \neq (0,0,0,0,0)$. The equivalence group $G_1^\sim$ of this class is formed by the transformations

$$\tilde t = T(t), \quad \tilde x = Z^1(t,x,y), \quad \tilde y = Z^2(t,x,y), \quad \tilde\psi = \Upsilon(t)\psi + \Phi(t,x,y),$$
$$\tilde F = \frac{1}{T_t}\left(\frac{\Upsilon}{L}F + \left(\frac{\Upsilon}{L}\right)_0\zeta + \left(\frac{\Phi_{ii}}{L}\right)_0 - \frac{Z^i_tZ^i_j}{L}\left(\frac{\Upsilon}{L}\zeta_j + \left(\frac{\Upsilon}{L}\right)_j\zeta + \left(\frac{\Phi_{ii}}{L}\right)_j\right)\right),$$

where T, Z^i, Υ and Φ are arbitrary smooth functions of their arguments, satisfying the conditions $Z^1_kZ^2_k=0$, $Z^1_kZ^1_k=Z^2_kZ^2_k:=L$ and $T_t\Upsilon L\neq 0$. The subscripts 1 and 2 denote differentiation with respect to x and y, respectively, the indices i and j run through the set $\{1,2\}$ and the summation over repeated indices is understood. As Eq. (4.1) is an element of the class (4.8) and this class is normalized, the point symmetry group G of Eq. (4.1) is contained in the projection $\hat G_1^\sim$ of the equivalence group $G_1^\sim$ of the class (4.8) to the variable space (t,x,y,ψ). At the same time, the group G is much narrower than the group $\hat G_1^\sim$, and in order to single out G from $\hat G_1^\sim$ we should still derive and solve a quite cumbersome system of additional constraints. Instead of this we use the trick described in the end of Section 4.2.2. Namely, by the transformation

$$\check\psi = \psi + \frac{\beta}{6}y^3, \tag{4.9}$$

which identically acts on the independent variables and which is prolonged to the vorticity according to the formula $\check\zeta = \zeta + \beta y$, we map Eq. (4.1) to the equation

$$\check\zeta_t + \check\psi_x\check\zeta_y - \check\psi_y\check\zeta_x = -\frac{\beta}{2}y^2\check\zeta_x. \tag{4.10}$$

Eq. (4.10) belongs to the subclass of class (4.8) that is singled out by the constraints $F_\psi=0$, $F_\zeta=0$, $F_{\psi_x}=-\zeta_y$ and $F_{\psi_y}=\zeta_x$, i.e., the class consisting of the equations of the form

$$\zeta_t + \psi_x\zeta_y - \psi_y\zeta_x = H(t,x,y,\zeta_x,\zeta_y,\zeta_{xx},\zeta_{xy},\zeta_{yy}), \quad \zeta := \psi_{xx}+\psi_{yy}, \tag{4.11}$$

where H is an arbitrary smooth function of its arguments, which is assumed as an arbitrary element instead of $F = H - \psi_x\zeta_y + \psi_y\zeta_x$. The class (4.11) also is a member of the above hierarchy of normalized classes. Its equivalence group $G_2^\sim$ is much narrower than $G_1^\sim$ and is formed by the transformations

$$\tilde t = \tau, \quad \tilde x = \lambda(x\mathfrak c - y\mathfrak s) + \gamma^1, \quad \varepsilon\tilde y = \lambda(x\mathfrak s + y\mathfrak c) + \gamma^2,$$
$$\tilde\psi = \varepsilon\frac{\lambda}{\tau_t}\left(\lambda\psi + \frac{\lambda}{2}\theta_t(x^2{+}y^2) - \gamma^1_t(x\mathfrak s{+}y\mathfrak c) + \gamma^2_t(x\mathfrak c{-}y\mathfrak s)\right) + \delta + \frac{\sigma}{2}(x^2{+}y^2),$$
$$\tilde H = \frac{\varepsilon}{{\tau_t}^2}\left(H - \frac{\lambda_t}{\lambda}(x\zeta_x + y\zeta_y) + 2\theta_{tt}\right) - \frac{\delta_y{+}\sigma y}{\tau_t\lambda^2}\zeta_x + \frac{\delta_x{+}\sigma x}{\tau_t\lambda^2}\zeta_y + \frac{2}{\tau_t}\left(\frac{\sigma}{\lambda^2}\right)_t,$$

where $\varepsilon=\pm1$, $\mathfrak c=\cos\theta$, $\mathfrak s=\sin\theta$; τ, λ, θ, γ^i and σ are arbitrary smooth functions of t satisfying the conditions $\lambda>0$, $\tau_{tt}=0$ and $\tau_t\neq0$ and $\delta=\delta(t,x,y)$ runs through the set of solutions of the Laplace equation $\delta_{xx}+\delta_{yy}=0$.

In order to derive the additional constraints that are satisfied by the group parameters of transformations from the point symmetry group G_2 of Eq. (4.10), we substitute the values $H = -\beta y^2\zeta_x/2$ and $\tilde H = -\beta\tilde y^2\tilde\zeta_{\tilde x}/2$ as well as expressions for the transformed variables and derivatives via the initial ones into the transformation component for H and then make all possible splitting in the obtained equality. As a result, we derive the additional constraints

$$\theta = \gamma^2_t = 0, \quad \lambda = \frac{1}{\tau_t}, \quad \sigma = \frac{\varepsilon\beta\gamma^2}{2{\tau_t}^2}, \quad \delta_x = -\sigma x, \quad \delta_y = \sigma y + \frac{\varepsilon\beta(\gamma^2)^2}{2\tau_t}.$$

After projecting transformations from $G_2^\sim$ on the variable space (t, x, y, ψ), constraining the group parameters using the above conditions and taking the adjoint action of the inverse of the transformation (4.9), we obtain, up to re-denoting, the transformations from Theorem 4.1.

4.5 Conclusion

In this paper, we have computed the complete point symmetry group of the barotropic vorticity equation on the β-plane. It is obvious that both of the techniques presented in this paper are applicable to general systems of differential equations.

Despite of the apparent simplicity of the techniques employed above, there are a number of features that should be discussed properly. In particular, the relation between discrete symmetries of a differential equation and discrete automorphisms of the corresponding maximal Lie invariance algebra is neither injective nor surjective. This is why it can be misleading to restrict the consideration to discrete automorphism when trying to finding discrete symmetries. This and related issues will be investigated and discussed more thoroughly in a forthcoming work.

Chapter 5

Lie symmetry analysis and exact solutions of the quasi-geostrophic two-layer problem

Abstract The quasi-geostrophic two-layer model is of superior interest in dynamic meteorology since it is one of the easiest ways to study baroclinic processes in geophysical fluid dynamics. The complete set of point symmetries of the two-layer equations is determined. An optimal set of one- and two-dimensional inequivalent subalgebras of the maximum Lie invariance algebra is constructed. On the basis of these subalgebras we exhaustively carry out group-invariant reduction and compute various classes of exact solutions. Where possible, reference to the physical meaning of the exact solutions is given.

5.1 Introduction

There is a long history in dynamic meteorology to use simplified models of the atmosphere rather than the complete set of hydro-thermodynamical equations to study only selected phenomenon instead of accounting for the whole variety of weather and climate at once. The greatest simplification which is still capable to qualitatively (and, under some conditions, also quantitatively) describe the behavior of large-scale geophysical dynamics is the barotropic vorticity equation. While this equation indeed allows one to explain the propagation of the mid-latitude Rossby waves, it cannot be used to elucidate the occurrence of developing weather regimes. The reason for this substantial lack is that the barotropic vorticity equation is a single equation valid only in one atmospheric (pressure) layer. However, development in the atmosphere is usually associated with the vertical structure of, e.g., the entropy field and hence a single-layer consideration is at once limited.

Of course, the atmosphere is continuously stratified and hence it is in fact three-dimensional. However, the main process of *baroclinic instability*, which is the dominant mechanism responsible for the formation of mid-latitude weather systems can already be qualitatively understood by considering only two coupled atmospheric layers. Moreover, the studies of layer models are the basis for the more complicated investigation of the three-dimensional governing equations. For this reason, many results for the two-layer model are available in dynamic meteorology, making this model particularly interesting for a systematic mathematical investigation.

The aim of this paper is to carry out a Lie symmetry analysis of the quasi-geostrophic two-layer model. There already exists a number of papers dealing with symmetries and exact solutions of the simpler barotropic vorticity equations [17, 18, 63, 68, 69] as well as the more complicated Euler or Navier–Stokes equations (see, e.g., [6, 41, 96, 97, 126] and references therein). At the same time, the intermediate two-layer model has not been considered in this light so far.

The organization of this paper is the following: The model equations are presented in Section 5.2, together with some of their known properties. Section 5.3 contains the results on Lie symmetries of these equations. The theorem on the complete point symmetry group of the two-layer model is proved in Section 5.4. A list of one- and two-dimensional inequivalent subalgebras of the corresponding maximal Lie invariance algebra is constructed in Sections 5.5. In Sections 5.6 and 5.7 systems obtained under reduction upon using the constructed one- and two-dimensional subalgebras are derived and investigated. Related boundary-value problems are discussed in Section 5.8. Section 5.9 summarizes the most important results of this paper. Finally, in the Appendix 5.10 we give a new symmetry interpretation of a method for finding exact solutions of linear systems of PDEs.

5.2 The two-layer model

The first impulse to the investigation of baroclinic instability in the two-layer model was given in [123]. Since two layers are considered, the model is capable of studying the interaction of the barotropic mode and the first baroclinic mode, which is sufficient for describing the basic mechanism of baroclinic instability. The model consists of two copies of the barotropic potential vorticity, evaluated at two different atmospheric levels [122]:

$$\begin{aligned}\frac{\partial Q^1}{\partial t} + \{\psi^1, Q^1\} = 0,\\ \frac{\partial Q^2}{\partial t} + \{\psi^2, Q^2\} = 0,\end{aligned} \tag{5.1a}$$

where ψ^1 and ψ^2 are the stream functions in the upper and lower layer,

$$\{f,g\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}$$

is the usual Poisson bracket of the functions f and g with respect to the variables x and y and

$$\begin{aligned}Q^1 = \nabla^2\psi^1 + \beta y - F(\psi^1 - \psi^2),\\ Q^2 = \nabla^2\psi^2 + \beta y + F(\psi^1 - \psi^2),\end{aligned} \tag{5.1b}$$

are the respective potential vorticities, with the constants β and F being the Rossby parameter and internal rotational Froude number, respectively. For the purpose of simplicity, we set the Froude numbers of the two layers to be equal, i.e. $F_1 = F_2 = F$, thereby assuming both layers to be of equal depth. Moreover, we have assumed flat topography. For the configuration to be stably stratified, the lower layer must be denser than the upper layer, i.e. $\rho_2 > \rho_1$.

Due to equivalence transformations of scaling and alternating signs in the class of equations of form (5.1), it would be possible to set $F \in \{-1, 0, 1\}$ and $\beta \in \{0, 1\}$. Since F and β are positive meteorological quantities, it would imply that $F = \beta = 1$ but we will not use this scaling below.

5.3 Lie symmetries

In the case $F\beta \neq 0$, which is the single subject of the present paper, system (5.1) admits the maximal Lie invariance algebra $\mathfrak{g}$ generated by the following basis elements:

$$\partial_t, \quad \partial_y, \quad \mathcal{X}(f) = f(t)\partial_x - f'(t)y(\partial_{\psi^1}+\partial_{\psi^2}), \quad \mathcal{F} = \partial_{\psi^1}-\partial_{\psi^2}, \quad \mathcal{Z}(g) = g(t)(\partial_{\psi^1}+\partial_{\psi^2}), \tag{5.2}$$

where f and g run through the set of real-valued functions of t. For $\mathfrak{g}$ to really be a Lie algebra, additional restrictions on the smoothness of the parameter functions f and g should be imposed [41].

Physically, these generators are exponentiated to give time and north–south translation symmetry, generalized Galilean transformations with respect to x, as well as shifts and gauging of the stream functions.

The structure of the Lie symmetry algebra suggests the introduction of the new dependent variables $\psi^+ = \psi^1 + \psi^2$ and $\psi^- = \psi^1 - \psi^2$. This change transforms the set of generators to

$$\partial_t, \quad \partial_y, \quad \mathcal{X}(f) = f(t)\partial_x - 2f'(t)y\partial_{\psi^+}, \quad \mathcal{F} = 2\partial_{\psi^-}, \quad \mathcal{Z}(g) = 2g(t)\partial_{\psi^+}.$$

From the meteorological point of view, the new variables have a sound physical meaning. Since the two-dimensional wind fields at both the levels can be represented as the derivatives of the respective stream functions, it follows that the component ψ^+ gives rise to the mean of these fields. This part is usually referred to as the barotropic part of the flow. In turn, derivatives of ψ^- give rise to the difference in the wind field between the two-layers. In meteorology, this difference is called the *thermal wind*, which is a measure of the baroclinity of the fluid. Therefore, these variables are commonly used in the investigation of baroclinic instability, e.g., in the study of the linearized two-layer model [57]. However, from the group-theoretical point of view, their usage is already suggested by the special form of Lie symmetry operators (5.2).

Using the variables ψ^+ and ψ^-, the model (5.1) is represented as

$$\begin{aligned} &\frac{\partial Q^+}{\partial t} + \frac{1}{2}\left(\{\psi^+, Q^+\} + \{\psi^-, Q^-\}\right) = 0, \\ &\frac{\partial Q^-}{\partial t} + \frac{1}{2}\left(\{\psi^+, Q^-\} + \{\psi^-, Q^+\}\right) = 0, \end{aligned} \tag{5.3a}$$

where

$$\begin{aligned} Q^+ &= \nabla^2\psi^+ + 2\beta y, \\ Q^- &= \nabla^2\psi^- - 2F\psi^-, \end{aligned} \tag{5.3b}$$

are the barotropic and baroclinic potential vorticities, respectively. In the present paper, we will use both the forms of the two-layer model, i.e. employing both the "layered variables" ψ^1 and ψ^2 and the "barotropic/baroclinic variables" ψ^+ and ψ^-. The precise usage depends on whether we study linear or nonlinear submodels of the two-layer model.

5.4 Complete point symmetry group

The complete point symmetry group of a system of differential equations, which includes both continuous and discrete symmetries, is conventionally calculated by the *direct method*. The outlines of this method are quite simple. Supposing the most general form of a point transformation in the associated space of independent and dependent variables, one expresses all involved derivatives of the new (transformed) dependent variables via the old variables, substitutes the obtained expressions into the system written in terms of the transformed variables, then excludes the derivatives which are assumed constrained due to the system (*principal derivatives*) and splits with respect to the unconstrained (*parametric*) derivatives. As a result, one obtains a system of determining equations for point symmetry transformations, which is nonlinear in contrast to a similar system arising by application of the infinitesimal Lie method and, therefore, is much more complicated for solving. This is why different special techniques (the implicit representation for unknown functions, the combined splitting with respect to old and new variables, inverse expression of old derivative via new ones, etc.) are applied to the derivation of determining equations and their a priori simplification.

Here we aim to use an approach similar as described in [61], which is based on the knowledge of the Lie symmetries of a given differential equation. This method rests on the fact that any point symmetry generates an automorphism of the maximal Lie invariance algebra. By factoring out the continuous symmetries from the whole point symmetry group, the discrete symmetries of the differential equation can be determined.

The computation of the complete point symmetry group can be considerably simplified by noting that any automorphism of a Lie algebra $\mathfrak{g}$ leaves invariant, by definition, all *megaideals* of $\mathfrak{g}$ [128]. Recall that a megaideal of $\mathfrak{g}$ is a vector subspace of $\mathfrak{g}$ which is invariant under any transformation from the group of automorphisms of $\mathfrak{g}$. Therefore, by determining megaideals of the maximal Lie invariance algebra of the given differential equation and imposing the invariance condition of these megaideals under the push-forwards of vector fields associated with the point symmetries allows to restrict the general form of possible point symmetries already *before* transforming the differential equation itself. After taking into account these initial restrictions, it is usually much simpler to proceed with the splitting of the variables in the transformed differential equation as described in the first paragraph.

In this section, the approach just outlined is demonstrated for the two-layer equation in barotropic/baroclinic variables.

Theorem 5.1. *The point symmetry group G of system* (5.3) *consists of transformations of the form*

$$\tilde t = \varepsilon_1 t + T_0, \quad \tilde x = \varepsilon_1 x + f(t), \quad \tilde y = \varepsilon_2 y + Y_0,$$
$$\tilde\psi^- = \varepsilon_3\psi^- + \Psi_0^-, \quad \tilde\psi^+ = \varepsilon_2\psi^+ - 2\varepsilon_1\varepsilon_2 f_t y + g(t),$$

where $\varepsilon_i = \pm1$, $i = 1, 2, 3$; $T_0, Y_0, \Psi_0^- \in \mathbb{R}$ and f and g are arbitrary smooth functions of t.

Proof. Recall that the maximal Lie invariance algebra of system (5.3) is the infinite dimensional algebra $\mathfrak{g} = \langle \partial_t, \partial_y, \mathcal{X}(f), \mathcal{F}, \mathcal{Z}(g)\rangle$, where f and g run through the set of smooth functions of t. It is a solvable algebra since $\mathfrak{g}' = \langle \mathcal{X}(f), \mathcal{Z}(g)\rangle$ and hence $\mathfrak{g}'' = \{0\}$. In other words, the radical $\mathfrak{r}$ of $\mathfrak{g}$ coincides with the entire $\mathfrak{g}$. The algebra $\mathfrak{g}$ has the nontrivial center $\mathfrak{z} = \langle \mathcal{X}(1), \mathcal{F}, \mathcal{Z}(1)\rangle$.

The nil-radical of $\mathfrak{g}$ is the ideal $\mathfrak{n} = \langle \partial_y, \mathcal{X}(f), \mathcal{F}, \mathcal{Z}(g)\rangle$. Indeed, this ideal is a nilpotent subalgebra of $\mathfrak{g}$ since $\mathfrak{n}^{(2)} = \mathfrak{n}' = [\mathfrak{n}, \mathfrak{n}] = \langle \mathcal{Z}(g)\rangle$ and $\mathfrak{n}^{(3)} = [\mathfrak{n}, \mathfrak{n}'] = 0$. A unique ideal of $\mathfrak{g}$ properly containing $\mathfrak{n}$ is the entire algebra $\mathfrak{g}$ itself, which is not nilpotent. This means that $\mathfrak{n}$ is the maximal nilpotent ideal.

In the calculations of G, we use the following megaideals of $\mathfrak{g}$: the entire algebra $\mathfrak{g}$, the derived algebra $\mathfrak{g}'$, the nil-radical $\mathfrak{n}$, its derivative $\mathfrak{n}'$, the center $\mathfrak{z}$ and their proper intersections, $\mathfrak{z} \cap \mathfrak{g}' = \langle \mathcal{X}(1), \mathcal{Z}(1)\rangle$ and $\mathfrak{z} \cap \mathfrak{n}' = \langle \mathcal{Z}(1)\rangle$.

For the general point transformation

$$\mathcal{T}\colon \quad (\tilde t, \tilde x, \tilde y, \tilde\psi^-, \tilde\psi^+) = (T, X, Y, \Psi^-, \Psi^+),$$

where T, X, Y, Ψ^-, Ψ^+ are functions of t, x, y, ψ^-, ψ^+ with the Jacobian not equal to zero, to be a point symmetry of system (5.3), the associated push-forward $\mathcal{T}_*$ of vector fields must be an automorphism of $\mathfrak{g}$. In particular, $\mathcal{T}_*\mathfrak{g} = \mathfrak{g}$, $\mathcal{T}_*\mathfrak{g}' = \mathfrak{g}'$, $\mathcal{T}_*\mathfrak{n} = \mathfrak{n}$, $\mathcal{T}_*\mathfrak{n}' = \mathfrak{n}'$, $\mathcal{T}_*\mathfrak{z} = \mathfrak{z}$, $\mathcal{T}_*(\mathfrak{z}\cap\mathfrak{g}') = \mathfrak{z}\cap\mathfrak{g}'$ and $\mathcal{T}_*(\mathfrak{z}\cap\mathfrak{n}') = \mathfrak{z}\cap\mathfrak{n}'$.

Investigating the restrictions on $\mathcal{T}$ imposed by the invariance of $\mathfrak{z}\cap\mathfrak{n}'$ under $\mathcal{T}_*$, we have

$$\mathcal{T}_*\mathcal{Z}(1) = 2(T_{\psi^+}\partial_{\tilde t} + X_{\psi^+}\partial_{\tilde x} + Y_{\psi^+}\partial_{\tilde y} + \Psi^-_{\psi^+}\partial_{\tilde\psi^-} + \Psi^+_{\psi^+}\partial_{\tilde\psi^+}) = \tilde{\mathcal{Z}}(a),$$

where $a = \text{const}$. This equation implies that $T_{\psi^+} = X_{\psi^+} = Y_{\psi^+} = \Psi^-_{\psi^+} = 0$, $\Psi^+_{\psi^+} = a = \text{const}$, and $a \neq 0$. Then, from the transformation of the elements of $\mathfrak{n}'$ we conclude

$$\mathcal{T}_*\mathcal{Z}(g) = 2ag\partial_{\tilde\psi^+} = \tilde{\mathcal{Z}}(\tilde g^g),$$

where $\tilde g^g$ is a smooth function of $\tilde t$ related to g. Comparing coefficients, we find that $ag(t) = \tilde g^g(T)$. As g is an arbitrary smooth function of t, we can set $g = t$ to obtain $2at = \tilde g^t(T)$. Because of $a \neq 0$ and hence $\tilde g^t_{\tilde t} \neq 0$, this implies that $T = T(t)$ and $T_t \neq 0$.

In a similar manner, the condition

$$\mathcal{T}_*\mathcal{X}(1) = \tilde{\mathcal{X}}(b_1) + \tilde{\mathcal{Z}}(b_2),$$

with $b_1, b_2 = \text{const}$ follows from the invariance of $\mathfrak{z}\cap\mathfrak{g}'$ with respect to $\mathcal{T}_*$. It is spit into the equations $Y_x = \Psi^-_x = 0$, $X_x = b_1 = \text{const}$ and $\Psi^+_x = 2b_2 = \text{const}$, where $b_1 \neq 0$. The invariance of $\mathfrak{g}'$ implies the condition

$$\mathcal{T}_*\mathcal{X}(f) = \tilde{\mathcal{X}}(\tilde f^f) + \mathcal{Z}(\tilde g^f),$$

where $\tilde f^f$ and $\tilde g^f$ are smooth functions of $\tilde t$ related to f. Comparing coefficients in the last condition immediately gives the additional equations $b_1 f = \tilde f^f(T)$ and $b_2 f - a f_t y = -\tilde f^f_{\tilde t}(T)Y + \tilde g^f(T)$. Taking into account the equality $\tilde f^f_{\tilde t} = b_1 f_t/T_t$, we obtain that $Y = Y^1(t)y + Y^0(t)$, where $Y^1 = aT_t/b_1 \neq 0$ and the precise expression of Y^0 is not essential for this time.

The push-forward of the remaining basis operator $\mathcal{F}$ of the center $\mathfrak{z}$ by $\mathcal{T}$ implies

$$\mathcal{T}_*\mathcal{F} = \tilde{\mathcal{X}}(c_1) + c_2\mathcal{F} + \tilde{\mathcal{Z}}(c_3),$$

where again $c_1, c_2, c_3 = \text{const}$. From this condition, we can conclude that $X_{\psi^-} = c_1 = \text{const}$, $\Psi^-_{\psi^-} = c_2 = \text{const}$ and $\Psi^+_{\psi^-} = 2c_3 = \text{const}$.

It remains to investigate the restrictions imposed by the push-forward of the basis operators ∂_t and ∂_y. The operator ∂_t does not lie in the above proper megaideals. Hence, its push-forward can be represented only as a general element of $\mathfrak{g}$:

$$\mathcal{T}_*\partial_t = d_1\partial_{\tilde t} + d_2\partial_{\tilde y} + d_3\tilde{\mathcal{F}} + \tilde{\mathcal{X}}(\tilde f) + \tilde{\mathcal{Z}}(\tilde g),$$

for real constants d_1, d_2, d_3 and smooth functions $\tilde f$ and $\tilde g$ of $\tilde t$. It is then straightforward to find $T_t = d_1 = \text{const}$, $Y_t = d_2 = \text{const}$ and thus $Y^1 = \text{const}$ and $Y^0 = d_2 t + d_4$, where $d_4 = \text{const}$. Moreover, $\Psi^-_t = 2d_3 = \text{const}$, $X_t = \tilde f(T)$ is a function of t and $\Psi^+_t = -2\tilde f_{\tilde t}(T)\tilde y + \tilde g(T)$.

Since the operator ∂_y belongs to the megaideal $\mathfrak{n}$, its push-forward by $\mathcal{T}$ should take the form

$$\mathcal{T}_*\partial_y = e_1\partial_{\tilde y} + e_2\mathcal{F} + \tilde{\mathcal{X}}(\tilde f^y) + \tilde{\mathcal{Z}}(\tilde g^y),$$

with $e_1, e_2 = \text{const}$ and $\tilde f^y$ and $\tilde g^y$ again being smooth functions of $\tilde t$. Comparing coefficients implies $Y^1 = e_1$, $\Psi^-_y = 2e_2$, $\Psi^+_y = -2\tilde f^y_{\tilde t}(T)\tilde y + 2\tilde g^y(T)$ and $X_y = \tilde f^y(T)$, which is a function of t. As $X_{ty} = 0$, $X_y = \text{const}$ holds. Therefore, $\tilde f^y_{\tilde t} = 0$ and Ψ^+_y depends only on t.

Collecting all constraints obtained so far and re-denoting the involved values, we obtain the representation of the point transformations inducing automorphisms of $\mathfrak{g}$:

$$\begin{aligned}
&T = T_1t + T_0, \quad X = X_1x + X_2y + X_3\psi^- + f(t), \quad Y = Y_1y + Y_2t + Y_0,\\
&\Psi^- = \Psi^-_1\psi^- + \Psi^-_2y + \Psi^-_3t + \Psi^-_0,\\
&\Psi^+ = \Psi^+_1\psi^+ + \Psi^+_2x + \Psi^+_3\psi^- + \varphi(t)y + g(t),
\end{aligned} \tag{5.4}$$

where T_0, T_1, X_1, X_2, X_3, Y_0, Y_1, Y_2, $\Psi^-_0, \dots, \Psi^-_3$, $\Psi^+_1, \dots, \Psi^+_3$ are constants, $T_1X_1Y_1\Psi^-_1\Psi^+_1 \neq 0$, $X_1Y_1 = \Psi^+_1T_1$ and $\varphi_t = -2Y_1f_{tt}$.

We now have to take into account that the transformation $\mathcal{T}$ is a point symmetry of system (5.3). Therefore, we should find explicit expressions for the derivatives of $\tilde\psi^-$ and $\tilde\psi^+$ with respect to the new variables $\tilde t$, $\tilde x$, $\tilde y$. In view of the representation (5.4), the transformation rules for the partial derivative operators read

$$\begin{aligned}
&\partial_{\tilde t} = \frac{1}{T_1}\left(\mathrm{D}_t - \frac{\mathrm{D}_tX}{\mathrm{D}_xX}\mathrm{D}_x - \frac{Y_2}{Y_1}\left(\mathrm{D}_y - \frac{\mathrm{D}_yX}{\mathrm{D}_xX}\mathrm{D}_x\right)\right),\\
&\partial_{\tilde x} = \frac{1}{\mathrm{D}_xX}\mathrm{D}_x, \quad \partial_{\tilde y} = \frac{1}{Y_1}\left(\mathrm{D}_y - \frac{\mathrm{D}_yX}{\mathrm{D}_xX}\mathrm{D}_x\right).
\end{aligned}$$

The derivative ψ^+_{txy} can only arise in the expression for $\tilde\psi^+_{\tilde t\tilde y\tilde y}$:

$$\tilde\psi^+_{\tilde t\tilde y\tilde y} = -\frac{2}{T_1Y_1^2}\frac{\mathrm{D}_yX}{\mathrm{D}_xX}\psi^+_{txy} + \cdots.$$

Since in the first equation of (5.3) there is no term with ψ^+_{txy}, we consequently have $\mathrm{D}_yX = 0$, which implies $X_2 = X_3 = 0$. Using this result, the transformation rules for the partial derivative operators are essentially simplified:

$$\partial_{\tilde t} = \frac{1}{T_1}\left(\partial_t - \frac{f_t}{X_1}\partial_x - \frac{Y_2}{Y_1}\partial_y\right), \quad \partial_{\tilde x} = \frac{1}{X_1}\partial_x, \quad \partial_{\tilde y} = \frac{1}{Y_1}\partial_y.$$

Upon the substitution $\psi^+_{xx} = \nabla^2\psi^+ - \psi^+_{yy}$, we obtain

$$\tilde\nabla^2\tilde\psi^+_{\tilde t} = \frac{1}{T_1}\left(\frac{1}{X_1^2}\nabla^2\psi^+_t + \left(\frac{1}{Y_1^2} - \frac{1}{X_1^2}\right)\psi^+_{tyy}\right) + \cdots.$$

Since there is no extra term ψ^+_{tyy} in the first equation of (5.3), we have $X_1^2 = Y_1^2 =: \sigma \neq 0$.

To plug the transformed variables into system (5.3), it is convenient to write down the expressions for the transformed potential vorticities $\tilde Q^-$ and $\tilde Q^+$:

$$\begin{aligned}
&\tilde Q^+ = \frac{\Psi^+_1}{\sigma}(Q^+ - 2\beta y) + \frac{\Psi^+_3}{\sigma}(Q^- + 2F\psi^-) + 2\beta(Y_1x + Y_2t + Y_0),\\
&\tilde Q^- = \frac{\Psi^-_1}{\sigma}(Q^- + 2F\psi^-) - 2F(\Psi^-_1\psi^- + \Psi^-_2y + \Psi^-_3t + \Psi^-_0).
\end{aligned}$$

The further restrictions on the transformation $\mathcal{T}$ can be found by splitting the resulting equations with respect to the variables t, x, y, $\psi^\pm_x$, $\psi^\pm_y$, $Q^\pm_x$ and $Q^\pm_y$.

We start with the restrictions imposed from the transformation of the second equation of system (5.3). The term with $\psi^-_x Q^-_y$ arises after the expansion of $\tilde\psi^+_{\tilde x}\tilde Q^-_{\tilde y}+\tilde\psi^-_{\tilde x}\tilde Q^+_{\tilde y}$. As $\Psi^-_1\neq 0$, the corresponding coefficient, which is equal to $\Psi^+_3\Psi^-_1/(\sigma X_1Y_1)$, vanishes if and only if $\Psi^+_3=0$. A term with ψ^-_t is contained only in the expression for $\tilde Q^-_{\tilde t}$. The corresponding coefficient $2FT_1^{-1}\Psi^-_1(\sigma^{-1}-1)$ also must be equal to zero, i.e., $\sigma=1$ and hence $X_1=\varepsilon_1=\pm1$ and $Y_1=\varepsilon_2=\pm1$. Splitting in a similar way with respect to ψ^+_x and ψ^-_y, we respectively find that $\Psi^-_2=0$ and $Y_1=\Psi^+_1$. As we already know that $Y^1=\Psi^+_1T_1/X_1$, this implies that $T_1=X_1=\varepsilon_1$.

From these restrictions, we can conclude the following form of the transformations:

$$
\begin{aligned}
&T=\varepsilon_1 t+T_0,\quad X=\varepsilon_1 x+f(t),\quad Y=\varepsilon_2 y+Y_2t+Y_0,\\
&\Psi^-=\Psi^-_1\psi^-+\Psi^-_3t+\Psi^-_0,\quad \Psi^+=\varepsilon_2\psi^++\Psi^+_2x+\varphi(t)y+g(t),\\
&\tilde Q^-=\Psi^-_1Q^--2F(\Psi^-_3t+\Psi^-_0),\quad \tilde Q^+=\varepsilon_2Q^++2\beta(Y_2t+Y_0).
\end{aligned}
$$

Substituting these transformations into the second equation of system (5.3), we find

$$
\begin{aligned}
&\frac{\Psi^-_1}{\varepsilon_1}\left(Q^-_t-\frac{f_t}{\varepsilon_1}Q^-_x-\frac{Y_2}{\varepsilon_2}Q^-_y\right)-\frac{2F}{\varepsilon_1}\Psi^-_3+\frac{1}{2\varepsilon_1\varepsilon_2}(\varepsilon_2\psi^+_x+\Psi^+_2)\Psi^-_1Q^-_y-(\varepsilon_2\psi^+_y+\varphi)\Psi^-_1Q^-_x)\\
&\quad+\frac{\Psi^-_1}{2\varepsilon_1}(\psi^-_xQ^+_y-\psi^-_yQ^+_x)=\frac{\Psi^-_1}{\varepsilon_1}\left(Q^-_t+\frac12(\psi^+_xQ^-_y-\psi^+_y\Psi^-_1Q^-_x)+\frac12(\psi^-_xQ^+_y-\psi^-_yQ^+_x)\right).
\end{aligned}
$$

Splitting of this equation immediately gives $\Psi^-_3=0$, $\varphi=-2\varepsilon_1^{-1}\varepsilon_2f_t$ and $Y_2=\frac12\Psi^+_2$.

In a similar fashion, plugging these transformations into the first equation of system (5.3) we obtain

$$
\begin{aligned}
&\frac{\varepsilon_2}{\varepsilon_1}\left(Q^+_t-\frac{f_t}{\varepsilon_1}Q^+_x-\frac{Y_2}{\varepsilon_2}Q^+_y\right)+\frac{2\beta}{\varepsilon_1}Y_2+\frac{1}{2\varepsilon_1\varepsilon_2}((\varepsilon_2\psi^+_x+\Psi^+_2)\varepsilon_2Q^+_y-(\varepsilon_2\psi^+_y+\varphi)\varepsilon_2Q^+_x)\\
&\quad+\frac{(\Psi^-_1)^2}{2\varepsilon_1\varepsilon_2}(\psi^-_xQ^-_y-\psi^-_yQ^-_x)=\frac{\varepsilon_2}{\varepsilon_1}\left(Q^+_t+\frac12(\psi^+_xQ^+_y-\psi^+_yQ^+_x)+\frac12(\psi^-_xQ^-_y-\psi^-_yQ^-_x)\right).
\end{aligned}
$$

The symmetry condition implies $\Psi^-_1=\varepsilon_3=\pm1$, $Y_2=0$ and consequently $\Psi^+_2=0$. This completes the proof of the theorem. □

Remark 5.1. The continuous transformations generated by elements of the center $\mathfrak{z}$ and only such transformations from the point symmetry group of system (5.3) induce the identical automorphism of the algebra $\mathfrak{g}$.

Remark 5.2. Comparing the results presented in Theorem 5.1 and Section 5.3 implies that besides Lie point symmetries system (5.3) admits discrete point symmetries. The group of discrete symmetries is generated by the mirror symmetries $(t,x,y,\psi_1,\psi_2)\mapsto(-t,-x,y,\psi_1,\psi_2)$, $(t,x,y,\psi_1,\psi_2)\mapsto(t,x,-y,-\psi_1,-\psi_2)$ and $(t,x,y,\psi_1,\psi_2)\mapsto(t,x,y,\psi_2,\psi_1)$. Under the change of the "barotropic/baroclinic variables" (ψ^+,ψ^-) by the "layered variables" (ψ^1,ψ^2), these discrete symmetries are changed to the transformations mapping (t,x,y,ψ^+,ψ^-) to $(-t,-x,y,\psi^+,\psi^-)$, $(t,x,-y,-\psi^+,-\psi^-)$ and $(t,x,y,\psi^+,-\psi^-)$, respectively. They exhaust the independent discrete symmetries of system (5.3) up to mutual composing and composing with continuous symmetries. In spite of the proof of this claim involves cumbersome calculations, the discrete symmetries are not essential for further consideration and hence will be neglected.

5.5 Inequivalent subalgebras

To classify the inequivalent subalgebras of the maximal Lie invariance algebra $\mathfrak{g}$ of the system (5.1) (resp. the system (5.3)), we need the adjoint representation of the corresponding Lie group on $\mathfrak{g}$. See e.g. [41, 115, 118] for details of classification techniques. We list only the nontrivial actions associated with basis elements:

$$
\begin{aligned}
&\mathrm{Ad}(e^{\varepsilon\mathcal{Z}(g)})\partial_t = \partial_t + \varepsilon\mathcal{Z}(g'), && \mathrm{Ad}(e^{\varepsilon\partial_t})\mathcal{Z}(g) = Z(g(t-\varepsilon)),\\
&\mathrm{Ad}(e^{\varepsilon\mathcal{X}(f)})\partial_t = \partial_t + \varepsilon\mathcal{X}(f'), && \mathrm{Ad}(e^{\varepsilon\partial_t})\mathcal{X}(f) = X(f(t-\varepsilon)),\\
&\mathrm{Ad}(e^{\varepsilon\mathcal{X}(f)})\partial_y = \partial_y - \varepsilon\mathcal{Z}(f'), && \mathrm{Ad}(e^{\varepsilon\partial_y})\mathcal{X}(f) = \mathcal{X}(f) + \varepsilon\mathcal{Z}(f').
\end{aligned}
$$

The classification of inequivalent one-dimensional subalgebras is straightforward. An optimal set of such subalgebras reads

$$
\mathcal{A}^1_1 = \langle \partial_t + a\partial_y + b\mathcal{F}\rangle, \qquad \mathcal{A}^1_2 = \langle \partial_y + \mathcal{X}(f) + b\mathcal{F}\rangle, \qquad \mathcal{A}^1_3 = \langle \mathcal{X}(f) + \mathcal{Z}(g) + b\mathcal{F}\rangle, \tag{5.5}
$$

where $a, b \in \mathbb{R}$ and f and g are arbitrary functions of t.

The classification of the two-dimensional subalgebras yields the following list:

$$
\begin{aligned}
&\mathcal{A}^2_1 = \langle \partial_t + \kappa\mathcal{F}, \partial_y + \mathcal{X}(\nu) + \mathcal{Z}(\mu) + \rho\mathcal{F}\rangle,\\
&\mathcal{A}^2_2 = \langle \partial_t + \nu\partial_y + \kappa\mathcal{F}, \mathcal{X}(e^{\sigma t}) + \mathcal{Z}(\nu\sigma t e^{\sigma t})\rangle, \sigma \neq 0\\
&\mathcal{A}^2_{-1} = \langle \partial_t + \nu\partial_y + \kappa\mathcal{F}, \mathcal{Z}(e^{\sigma t})\rangle,\\
&\mathcal{A}^2_3 = \langle \partial_t + \nu\partial_y + \kappa\mathcal{F}, \mathcal{X}(1) + \mathcal{Z}(\mu) + \rho\mathcal{F}\rangle,\\
&\mathcal{A}^2_{-2} = \langle \partial_t + \nu\partial_y + \kappa\mathcal{F}, \mathcal{Z}(1) + \rho\mathcal{F}\rangle, \quad \mathcal{A}^2_{-3} = \langle \partial_t + \nu\partial_y, \mathcal{F}\rangle,\\
&\mathcal{A}^2_4 = \langle \partial_y + \mathcal{X}(f) + \kappa\mathcal{F}, \mathcal{X}(1) + \mathcal{Z}(g) + \rho\mathcal{F}\rangle, \ \kappa\rho = 0,\\
&\mathcal{A}^2_{-4} = \langle \partial_y + \mathcal{X}(f), \mathcal{Z}(g) + \mathcal{F}\rangle, \quad \mathcal{A}^2_{-5} = \langle \partial_y + \mathcal{X}(f) + \kappa\mathcal{F}, \mathcal{Z}(g)\rangle,\\
&\mathcal{A}^2_{-6} = \langle \mathcal{X}(f^1) + \mathcal{Z}(g^1) + \kappa\mathcal{F}, \mathcal{X}(f^2) + \mathcal{Z}(g^2) + \rho\mathcal{F}\rangle,
\end{aligned}
\tag{5.6}
$$

where a, b, κ, μ, ν, ρ, σ = const and f, f^1, f^2, g, g^1 and g^2 are arbitrary functions of t. In the final subalgebra the tuples (f^1, g^1, κ) and (f^2, g^2, ρ) must be linearly independent for the subalgebra to really be two-dimensional. By the same reason, the parameter function g is not identically equal to zero in $\mathcal{A}^2_{-5}$.

In fact, the above subalgebras are not single subalgebras but rather represent parameterized classes of subalgebras. This is why it would be beneficial to indicate the list of parameters in the notation of the corresponding algebras, but for the sake of brevity we omit this whenever possible. For the classes $\mathcal{A}^2_4$, $\mathcal{A}^2_{-4}$, $\mathcal{A}^2_5$, $\mathcal{A}^2_{-5}$ and $\mathcal{A}^2_{-6}$ the adjoint actions and linear combinations of basis elements induce equivalence relations on the corresponding sets of parameters. For example, the subalgebras $\mathcal{A}^2_{-4}(f,g)$ and $\mathcal{A}^2_{-4}(\tilde{f},\tilde{g})$ are equivalent if and only if $\tilde{f}(t) = f(t-\varepsilon)$ and $\tilde{g}(t) = g(t-\varepsilon)$ for some $\varepsilon \in \mathbb{R}$.

5.6 Invariant reduction with one-dimensional subalgebras

We present the complete list of submodels obtained under reduction using the list (5.5). For each submodel, we again determine their Lie symmetries, thereby seeking for hidden symmetries of the initial model. For a general discussion of the problem of hidden symmetries, see e.g. [1]. Throughout this section v^1, v^2, v^+ and v^- will be assumed as functions of p and q.

5.6.1 Subalgebra $\mathcal{A}_1^1$

A suitable ansatz for reduction of (5.1) under the first subalgebra of (5.5) is given by $\psi^1 = v^1 + bt$, $\psi^2 = v^2 - bt$, where $p = x$, $q = y - at$. The corresponding reduced equations read

$$aw_q^1 - Fa\left(v_q^1 - v_q^2 - 2\frac{b}{a}\right) - v_p^1(w_q^1 + \beta - F(v_q^1 - v_q^2)) + v_q^1(w_p^1 - F(v_p^1 - v_p^2)) = 0,$$
$$aw_q^2 + Fa\left(v_q^1 - v_q^2 - 2\frac{b}{a}\right) - v_p^2(w_q^2 + \beta + F(v_q^1 - v_q^2)) + v_q^2(w_p^2 + F(v_p^1 - v_p^2)) = 0,$$

where $w^i = v_{pp}^i + v_{qq}^i$, $i = 1, 2$. Considering again the admitted Lie symmetries of this system, we find that the symmetry algebra is $\mathfrak{g}_1 = \langle \partial_p, \partial_q, \partial_{v^1}, \partial_{v^2} \rangle$. All operators from the algebra $\mathfrak{g}_1$ are induced by Lie symmetry operators of the original system (5.1) and hence there are no purely hidden symmetries. This is why we do not have to further reduce the above system by using the Lie method. The Lie reductions of the reduced system with respect to one-dimensional subalgebras of $\mathfrak{g}_1$ are equivalent to Lie reductions of system (5.1) with respect to one of the listed two-dimensional subalgebras of $\mathfrak{g}$. The two-dimensional reductions of system (5.1) are exhaustively discussed in section 5.7.

5.6.2 Subalgebra $\mathcal{A}_2^1$

Reduction using $\mathcal{A}_2^1$. An ansatz associated with this subalgebra reads $\psi^1 = v^1 - \frac{1}{2}f'y^2 + by$ and $\psi^2 = v^2 - \frac{1}{2}f'y^2 - by$, where $p = x - f(t)y$ and $q = t$. It reduces system (5.1) to the system

$$(Hv_{pp}^1)_q - f'' - F(v_q^1 - v_q^2) - bF(v_p^1 + v_p^2) - bHv_{ppp}^1 + \beta v_p^1 = 0,$$
$$(Hv_{pp}^2)_q - f'' + F(v_q^1 - v_q^2) + bF(v_p^1 + v_p^2) + bHv_{ppp}^2 + \beta v_p^2 = 0,$$

where it was convenient to introduce the new notation

$$H = 1 + f^2.$$

To simplify this system, we use the above mentioned barotropic/baroclinic variables, which is particularly obvious for this submodel, since it is a linear system of differential equations. By introducing $w = v^1 + v^2$ and $v = v^1 - v^2$ we are able to rewrite the resulting system via:

$$\begin{aligned} &(Hw_{pp})_q - 2f'' - bHv_{ppp} + \beta w_p = 0, \\ &(Hv_{pp})_q - 2Fv_q - 2bFw_p - 2bHw_{ppp} + \beta v_p = 0. \end{aligned} \tag{5.7}$$

Note that this system may be derived directly by means of reduction of (5.3) under the ansatz $\psi^+ = w(p, q) - f'y^2$ and $\psi^- = v(p, q) + 2by$, where p and q are defined as above.

The resulting system is now simplified in a way similar as presented in [17]. Namely, we integrate once the first equation with respect to p yielding

$$(Hw_p)_q - 2f''p - bHv_{pp} + \beta w + h(q) = 0,$$

where h is an arbitrary function of $q = t$. Then we apply the transformations of the unknown functions

$$w = \hat{w} - 2\frac{(Hf'')'}{\beta^2} + \frac{2f''p}{\beta} - \frac{h}{\beta}, \quad v = \hat{v} - \frac{2bf'}{\beta}$$

and obtain the following system:

$$\begin{aligned}&(H\hat{w}_p)_q+\beta\hat{w}-bH\hat{v}_{pp}=0,\\&(H\hat{v}_{pp})_q-2F\hat{v}_q+\beta\hat{v}_p-2b(H\hat{w}_{ppp}+F\hat{w}_p)=0.\end{aligned}\tag{5.8}$$

First of all, we determine Lie symmetries of this system. As system (5.8) in fact is a class of systems parameterized by the arbitrary function $f=f(q)$ and the arbitrary constant b, it is necessary to solve a group classification problem [118, 129, 131]. That is, for the complete description of Lie symmetries it is necessary to seek for possible extensions of the Lie invariance algebras for special values of the parameters f and b, respectively. Recall that β and F are constant parameters which can be set to 1, so it is not required to also take into account the classification problem with respect to β and F.

Group classification of the reduced systems. Conventionally, the first step in the procedure of group classification is the identification of the kernel $G^{\rm ker}$ of the maximal Lie invariance groups of equations from class (5.8), i.e. the group which is admitted for any value of f and b. The Lie algebra $\mathfrak{g}^{\rm ker}$ corresponding to $G^{\rm ker}$ can be obtained by solving the determining equations for Lie symmetries under the assumption of arbitrariness of f and b. The part of the determining equations not including f and b can be immediately integrated yielding

$$\xi^p=ap+c,\quad \xi^q=aq+d,\quad \eta^{\hat{v}}=k_1\hat{v}+g^1(p,q),\quad \eta^{\hat{w}}=k_2\hat{w}+g^2(p,q),$$

which are the coefficients of the most general infinitesimal generator of Lie symmetries $\xi^p\partial_p+\xi^q\partial_q+\eta^{\hat{v}}\partial_{\hat{v}}+\eta^{\hat{w}}\partial_{\hat{w}}$, where a, c, d, k_1, k_2 = const. The part of the determining equations explicitly including the parameters of (5.8) (the classifying part) in turn is:

$$\begin{aligned}&(aq+d)H'-2aH=0,\quad (aq+d)H''-aH'=0,\\&aHH'+(aq+d)HH''-(aq+d)H'^2=0,\quad b(k_1-k_2)=0,\\&(Hg^2_p)_q+\beta g^2-bHg^1_{pp}=0,\\&(Hg^1_{pp})_q-2Fg^1_q+\beta g^1_p-2b(Fg^2_p+Hg^2_{ppp})=0.\end{aligned}\tag{5.9}$$

It is straightforward to recover system (5.8) in the two last equations of system (5.9). For the general values of f and b splitting of the above system yields the conditions $a=0$, $d=0$ and $k_1=k_2$ and hence gives rise to the Lie invariance algebra $\mathfrak{g}^{\rm gen}_{f,b}$ generated by the operators

$$\partial_p,\quad \mathcal{I}=\hat{v}\partial_{\hat{v}}+\hat{w}\partial_{\hat{w}},\quad \mathcal{L}(g^1,g^2)=g^1(p,q)\partial_{\hat{v}}+g^2(p,q)\partial_{\hat{w}},$$

where functions g^1 and g^2 run through the set of solutions of the system (5.8) for the fixed values f and b. The Lie symmetry operators $\mathcal{I}$ and $\mathcal{L}(g^1,g^2)$ arise due to linearity of (5.8).

To investigate the problem of induced symmetries of system (5.8), we need to consider the same problem for system (5.7) at first. Up to linear combining, for general values of f and b the Lie symmetry operators of (5.7) induced by operators from $\mathfrak{g}$ are exhausted by the operators ∂_p, $2\partial_{\hat{v}}$ and $g(q)\partial_{\hat{w}}$, which are induced by $\mathcal{X}(1)$, $\mathcal{F}$ and $\mathcal{Z}(g)$, respectively. Here g runs through the set of smooth functions of q. It is obvious that the operator $\partial_y+\mathcal{X}(f)+b\mathcal{F}$ induces the zero operator. Additionally, if f = const the operator ∂_t induces ∂_q. Under integration of the first equation of system (5.7), Lie symmetry transformations generated by $g(q)\partial_{\hat{w}}$ for any fixed g become equivalence transformations of the resulting system. By the above transformation to the unknown functions $\hat{v}$ and $\hat{w}$, we gauge the function h arising under integration to zero and hence break the invariance of system (5.8) with respect to operators of the form $g(q)\partial_{\hat{w}}$. This is why any operator from $\mathfrak{g}^{\rm gen}_{f,b}$ lying in the complement of the linear span of the operators ∂_p, $2\partial_{\hat{v}}$ and $g(q)\partial_{\hat{w}}$ and, additionally, ∂_q in the case of $f=\mathrm{const}$ is a hidden symmetry of the initial system.

The kernel algebra $\mathfrak{g}^{\rm ker}$ of class (5.8) is generated only by the operators ∂_p, $\mathcal{I}$ and $\mathcal{L}(1,0)$. This is because the set of generators $\mathcal{L}(g^1,g^2)$ is different for every representative of the class (5.8) as the form of systems from the class depends on values of f and b. However, it is not feasible to linearly combine solutions of different systems from the class (5.8) that contradicts belonging of $\mathcal{L}(g^1,g^2)$ to $\mathfrak{g}^{\rm ker}$ for arbitrary values of g^1 and g^2. Therefore, under solving the group classification problem for the class (5.8) it is natural to investigate extensions with respect to $\mathfrak{g}^{\rm gen}_{f,b}$ rather than with respect to $\mathfrak{g}^{\rm ker}$. In other words, we should find all inequivalent values of the arbitrary elements f and b for which $\mathfrak{g}^{\rm gen}_{f,b}$ is not the maximal Lie invariance algebra of the corresponding system of the form (5.8). Here the inequivalence is to be understood with respect to the equivalence group of the class (5.8).

For this purpose, it is necessary to solve the classifying part (5.9) of the determining equations by taking into account for which forms of f and b an extension of $\mathfrak{g}^{\rm gen}_{f,b}$ is admitted. It is obvious that for $b=0$, the generator $\mathcal{I}$ from $\mathfrak{g}^{\rm gen}_{f,b}$ splits into the two generators $\hat{v}\partial_{\hat{v}}$ and $\hat{w}\partial_{\hat{w}}$. This splitting of $\mathcal{I}$ corresponds to the decoupling of the two equations (5.8). As the remaining classifying part of the determining equations is independent of b, the extensions possible for different values of H are essentially not affected whether or not the two equations (5.8) are coupled. In case of $b=0$ we simply consider extensions with respect to $\mathfrak{g}^{\rm gen}_{f,0}=\langle\partial_p,\hat{v}\partial_{\hat{v}},\hat{w}\partial_{\hat{w}},\mathcal{L}(g^1,g^2)\rangle$ rather than to $\mathfrak{g}^{\rm gen}_{f,b}$. It is crucial to remark that from the first three equations of system (5.9) only the first equation is independent. The other two equations are its differential consequences. As the investigation of extensions must be done up to equivalence, it would be necessary to compute the equivalence group of the class (5.8). However, it is obvious that this class admits scalings and shifts of q as equivalence transformations. For this reason, we only have to distinguish between the cases $a\neq0$ and $a=0$. In the case $a\neq0$, we can set $a=1$ and $d=0$ by dividing the equation by a and shifting of q. As a result, we have $H=\varkappa q^2$, i.e., $f=\pm\sqrt{\varkappa q^2-1}$. The extension of $\mathfrak{g}^{\rm gen}_{f,b}$ is then given by the basis element $p\partial_p+q\partial_q$ which is a hidden symmetry of the initial system (5.3). If $a=0$, we have $f=\mathrm{const}$ and $\mathfrak{g}^{\rm max}_{f,b}=\mathfrak{g}^{\rm gen}_{f,b}+\langle\partial_q\rangle$. Recall that operator ∂_q in this case is induced by the operator ∂_t.

The uncoupled system ($b=0$). We now proceed by studying the case $b=0$, which leads to a decoupling of system (5.8):

$$(H\hat{w}_p)_q+\beta\hat{w}=0,\quad (H\hat{v}_{pp})_q-2F\hat{v}_q+\beta\hat{v}_p=0. \tag{5.10}$$

The change of variables

$$\bar{p}=p,\quad \bar{q}=\int\frac{\mathrm{d}q}{H(q)},\quad \bar{v}=H(q)\hat{v},\quad \bar{w}=H(q)\hat{w},$$

allows to transform this system to

$$\bar{w}_{\bar{p}\bar{q}}+\beta\bar{w}=0,\quad \bar{v}_{\bar{p}\bar{p}\bar{q}}-2F(H^{-1}\bar{v})_{\bar{q}}+\beta\bar{v}_{\bar{p}}=0. \tag{5.11}$$

Analog to [17], in the first equation we recover the Klein–Gordon equation in light-cone variables. For this system, we only have one possibility for Lie reduction. Namely, we can reduce (5.10) by using the subalgebra $\langle\partial_{\bar{p}}+\lambda_1\bar{v}\partial_{\bar{v}}+\lambda_2\bar{w}\partial_{\bar{w}}\rangle$, where we take into account that for the decoupled case $b=0$ the generator $\hat v\partial_{\hat v}+\hat w\partial_{\hat w}$ of (5.8) splits into the two single operators $\hat v\partial_{\hat v}$ and $\hat w\partial_{\hat w}$ (note that $\hat v\partial_{\hat v}=\bar v\partial_{\bar v}$ and $\hat w\partial_{\hat w}=\bar w\partial_{\bar w}$). The ansatz for reduction reads $\bar v=\tilde v(\bar q)e^{\lambda_1\bar p}$ and $\bar w=\tilde w(\bar q)e^{\lambda_2\bar p}$. It leads to the following system of ordinary differential equations

$$\lambda_2\tilde w_{\bar q}+\beta\tilde w=0,\quad \lambda_1^2\tilde v_{\bar q}-2F(H^{-1}\tilde v)_{\bar q}+\beta\lambda_1\tilde v=0.$$

The integration of the resulting system, the substitution of the obtained solution to the ansatz and the inverse change of variables yields

$$\hat w=\frac{c_1}{H}\exp\left(\lambda_2 p-\frac{\beta}{\lambda_2}\int\frac{\mathrm{d}q}{H}\right),\quad \hat v=\frac{c_2}{H}\exp\left(\lambda_1 p+\int\frac{2H_qF+\beta\lambda_1H}{2F-\lambda_1^2H}\frac{\mathrm{d}q}{H}\right).$$

Recall that $H=1+f^2$. For the values $f=-l/k$, where the constants k and l are wave numbers and $\lambda_1=\lambda_2=ik$, $i^2=-1$, this solution reduces to the well-known Rossby waves in the two-layer model. The solution for the barotropic mode $\hat w$ describes a single barotropic Rossby wave, which is independent of the vertical structure of the two-layer setting. In turn, the solution for $\hat v$ describes the evolution of the first baroclinic mode and explicitly depends on the vertical layer structure via the parameter dependency on F, which accounts for the density difference between the layers. Note that for a more general choice of the function f, this solution allows to derive series of wave solutions in a similar way as it was possible for the barotropic vorticity equation [17]. However, for $f\neq\text{const}$ there arises an additional term proportional to y^2 in the solution of ψ^+, which may violate the boundary conditions. Although such a violation of the boundaries does not exist for the solution of ψ^-, the global (i.e. large-scale) realization of generalized Rossby waves is at once limited due to this restriction.

The case $b=0$ shows that the classical Rossby wave solution can be recovered in two steps: Firstly reducing with respect to the operator $\partial_y+\mathcal{X}(f)+b\mathcal{F}$ and secondly performing reduction to a system of ordinary differential equations using a hidden symmetry of the submodel received in the first step.

Reduction with respect to the additional symmetries for special values $f(q)=\text{const}$ and $f(q)=\pm\sqrt{Cq^2-1}$ will not be considered here, because for a decoupled system it is be better to construct exact solutions of each equation separately and then compose them to a solution of the entire system. In our particular case, the single Klein–Gordon equation has a wider maximal Lie invariance algebra than system (5.11), given by $\langle\partial_{\bar p},\ \partial_{\bar q},\ \bar p\partial_{\bar p}-\bar q\partial_{\bar q},\ \bar w\partial_{\bar w},\ g(\bar p,\bar q)\partial_{\bar w}\rangle$, where g runs through the set of solutions of the Klein–Gordon equation. That is, we have more possibilities for finding exact solutions by Lie methods. Fortunately, it is not necessary to do this in view of large classes of exact solutions already known for the Klein–Gordon equation [125].

For the split system (5.11), it remains to determine the Lie symmetries and perform Lie reductions of the second equation,

$$\bar v_{\bar p\bar p\bar q}-2F(A\bar v)_{\bar q}+\beta\bar v_{\bar p}=0, \tag{5.12}$$

where $A=H^{-1}$. The determining equations for the coefficients of the Lie symmetry operator $Q=\xi^{\bar p}\partial_{\bar p}+\xi^{\bar p}\partial_{\bar p}+\eta\partial_{\bar v}$ of Eq. (5.12) not involving A can be integrated to give

$$\xi^{\bar p}=-a\bar p+c,\quad \xi^{\bar q}=a\bar q+d,\quad \eta=k\bar v+g(\bar p,\bar q).$$

The remaining classifying part of the determining equations reads

$$(a\bar q+d)A_{\bar q}-2aA=0,\quad aAA_{\bar q}-(a\bar q+d)A_{\bar q}^2+(a\bar q+d)AA_{\bar q\bar q}=0. \tag{5.13}$$

Again, the second equation is a differential consequence of the first equation. For general A, splitting of (5.13) leads to the two essential Lie symmetry generators $\partial_{\bar{p}}$ and $\bar{v}\partial_{\bar{v}}$ together with the linearity operator $\mathcal{L}(g)$, where g runs through the set of solutions of (5.12).

Then, upon again solving the group classification problem, we find that the general solution of (5.13) is $A = C(a\bar{q}+d)^2$, where $C = \text{const}$. Note that at least one of the constants a and d must not be equal to zero to guarantee that (5.13) is really a system in A. We then distinguish two cases. (i) $a \neq 0$. We can scale $a = 1$ and shift q to set $d = 0$ and obtain the additional generator $\bar{q}\partial_{\bar{q}} - \bar{p}\partial_{\bar{p}}$. (ii) $a = 0$, $d \neq 0$. In this case we find the additional generator $\partial_{\bar{q}}$. The first operator again is a hidden symmetry, while the second generator is induced by ∂_t. Note that the case of $A = 0$ would yield wider symmetry extensions but cannot be realized in the present case since by definition $A \neq 0$.

We now consider the Lie reductions of Eq. (5.12). For general A, the only nontrivial possibility for reduction is given by $\partial_{\bar{p}} + \lambda\bar{v}\partial_{\bar{v}}$, which was already considered above. The solution is

$$\bar{v} = c\exp\left(\lambda\bar{p} - \int \frac{\beta\lambda - 2FA_{\bar{q}}}{\lambda^2 - 2FA}\mathrm{d}\bar{q}\right),$$

which can be combined with arbitrary solutions of the Klein–Gordon equation to yield a solution of the decoupled system (5.11). It remains to investigate the Lie reductions due to the two extensions $\bar{q}\partial_{\bar{q}} - \bar{p}\partial_{\bar{p}}$ and $\partial_{\bar{q}}$.

In the first case we set $A = Cq^2$. The maximal Lie invariance algebra of this case is given by $\langle \bar{q}\partial_{\bar{q}} - \bar{p}\partial_{\bar{p}}, \partial_{\bar{p}}, \bar{v}\partial_{\bar{v}}, \mathcal{L}(g)\rangle$. There is one nontrivial one-dimensional subalgebra of this algebra, which reads $\langle \bar{q}\partial_{\bar{q}} - \bar{p}\partial_{\bar{p}} + \lambda\bar{v}\partial_{\bar{v}}\rangle$. The ansatz for reduction is $\bar{v} = \tilde{v}(r)q^{\lambda}$, where $r = \bar{p}\bar{q}$. Correspondingly, the reduced form of Eq. (5.12) is given by

$$r\tilde{v}_{rrr} + (\lambda+2)\tilde{v}_{rr} + (\beta - 2CFr)\tilde{v}_r - 2CF(\lambda+2)\tilde{v} = 0.$$

For $\lambda = -2$, this equation has a solution in terms of Whittaker functions $M_{m,n}(r)$, $W_{m,n}(r)$, which reads

$$\tilde{v} = C_1\int M_{m,n}(\sqrt{8CF}r)\,\mathrm{d}r + C_2\int W_{m,n}(\sqrt{8CF}r)\,\mathrm{d}r + C_3,$$

where $m = \beta(8CF)^{-1/2}$ and $n = 1/2$. Moreover, for $\lambda = -k$, $k \in \mathbb{N}$, this equation admits polynomial solutions.

In the second case of extension, we have $A = \text{const}$. Then, the maximal Lie invariance algebra is generated by $\langle \partial_{\bar{q}}, \partial_{\bar{p}}, \bar{v}\partial_{\bar{v}}, \mathcal{L}(g)\rangle$. Again, one nontrivial one-dimensional subalgebra can be used to carry out Lie reduction, which is $\langle \partial_{\bar{q}} + \kappa\partial_{\bar{p}} + \lambda\bar{v}\partial_{\bar{v}}\rangle$. An appropriate ansatz for reduction is $\bar{v} = \tilde{v}(r)e^{\lambda\bar{q}}$, where $r = \bar{p} - \kappa\bar{q}$. Plugging this ansatz into (5.12), we find

$$\kappa\tilde{v}_{rrr} - \lambda\tilde{v}_{rr} - (2AF\kappa + \beta)\tilde{v}_r + 2AF\lambda\tilde{v} = 0.$$

This is a linear, third-order ordinary differential equation with constant coefficients and thus can be solved by standard methods. In particular, as κ, λ and A are arbitrary constants, we can determine them upon prescribing a solution of the associated characteristic equation. This allows to generate wide classes of solutions with rather different type, such as e.g. periodic wave solutions.

The coupled system ($b \neq 0$). For the coupled case $b \neq 0$ the Lie reduction of system (5.8) is quite similar to the case $b = 0$. For the sake of completeness, we list here all the reduced models that are possible for different values of H.

For general H, the only possibility for reduction is due to the generator $Q = \partial_p + \lambda\mathcal{I}$. The corresponding reduction ansatz is $\hat{v} = \bar{v}(q)e^{\lambda p}$, $\hat{w} = \bar{w}(q)e^{\lambda p}$. Plugging the ansatz into system (5.8), one obtains

$$\lambda H\bar{w}_q + (\lambda H_q + \beta)\bar{w} - b\lambda^2 H\bar{v} = 0,$$
$$(\lambda^2 H - 2F)\bar{v}_q + (\lambda^2 H_q + \beta\lambda)\bar{v} - 2b\lambda(\lambda^2 + F)\bar{w} = 0.$$

For $H = \varkappa q^2$, the additional reduction using $Q = p\partial_p + q\partial_q + \lambda\mathcal{I}$ is possible. Utilizing the ansatz $\hat{v} = \bar{v}(r)q^\lambda$, $\hat{w} = \bar{w}(r)q^\lambda$, where $r = pq^{-1}$ leads to the system

$$(\lambda + 2)\bar{w}_r - r\bar{w}_{rr} + \beta\bar{w} - b\bar{v}_{rr} = 0,$$
$$r\hat{v}_{rrr} - \lambda\hat{v}_{rr} + 2F(\lambda - r)\hat{v}_r - \beta\hat{v}_r + 2b(F\hat{w}_r + \hat{w}_{rrr}) = 0.$$

For $H = \text{const}$ we can also reduce (5.8) using $Q = \partial_q + \kappa\partial_p + \lambda\mathcal{I}$. The ansatz for reduction is $\hat{v} = \bar{v}(r)e^{\lambda q}$, $\hat{w} = \bar{w}(r)e^{\lambda q}$ where $r = p - \kappa q$. The resulting model is

$$H(\kappa\bar{w}_{rr} - \lambda\bar{w}_r) - \beta\bar{w} + bH\bar{v}_{rr} = 0,$$
$$H(\kappa\bar{v}_{rrr} - \lambda\bar{v}_{rr}) - (2F\kappa + \beta)\bar{v}_r + 2F\lambda\bar{v} + 2b(H\bar{w}_{rrr} + F\bar{w}_r) = 0.$$

5.6.3 Subalgebra $\mathcal{A}_3^1$

Reduction using $\mathcal{A}_3^1$. For this subalgebra, it is convenient to start with barotropic/baroclinic variables from the beginning. Since in the case $f = 0$ no Lie reduction is possible, we assume that $f \neq 0$. An appropriate ansatz for reduction then reads

$$\psi^+ = v^+ - 2\frac{f'y - g}{f}x, \quad \psi^- = v^- + 2\frac{b}{f}x,$$

where $p = y$, $q = t$. Plugging this ansatz into system (5.3) gives reduction to the system:

$$\begin{aligned} &v^+_{ppq} - \frac{f'p - g}{f}(v^+_{ppp} + 2\beta) + \frac{b}{f}v^-_{ppp} = 0, \\ &v^-_{ppq} - 2Fv^-_q - \frac{f'p - g}{f}(v^-_{ppp} - 2Fv^-_p) + \frac{b}{f}(v^+_{ppp} + 2Fv^+_p + 2\beta) = 0, \end{aligned} \tag{5.14a}$$

where it can be seen that the coupling between the barotropic and baroclinic parts is again provided only due to the existence of generator $\mathcal{F}$. To solve this system, we integrate the first equation twice with respect to p to yield

$$v^+_q - \frac{f'p - g}{f}v^+_p + 2\frac{f'}{f}v^+ + \frac{b}{f}v^-_p - \frac{1}{3}\frac{f'}{f}\beta p^3 + \frac{\beta g}{f}p^2 + h^1(q)p + h^0(q) = 0, \tag{5.14b}$$

where h^1 and h^0 are arbitrary smooth functions of q. By means of the change of unknown functions

$$v^+ = \hat{v}^+ + \gamma^2 p^3 + \gamma^1(q)p + \gamma^0(q), \quad v^- = \hat{v}^- + \delta^2(q)p^2 + \delta^1(q)p + \delta^0(q),$$

where

$$\gamma^2 = -\frac{\beta}{3}, \qquad \delta^2 = -b\beta f^2\int\frac{1}{f^3}\mathrm{d}q,$$

$$\gamma^1 = -\frac{1}{f}\int (2b\delta^2 + fh^1)\mathrm{d}q, \qquad \delta^1 = -2f\int \frac{g\delta^2}{f^2}\mathrm{d}q,$$
$$\gamma^0 = -\frac{1}{f^2}\int f(g\gamma^1 + b\delta^1 + fh^0)\mathrm{d}q, \qquad \delta^0 = \frac{1}{F}\int \frac{bF\gamma^1 - gF\delta^1 + f\delta^2_q}{f}\mathrm{d}q,$$

we are able to reduce system (5.14) to the corresponding homogeneous form:

$$\begin{aligned} &\hat{v}^+_q - \frac{f'p-g}{f}\hat{v}^+_p + 2\frac{f'}{f}\hat{v}^+ + \frac{b}{f}\hat{v}^-_p = 0, \\ &\hat{v}^-_{ppq} - 2F\hat{v}^-_q - \frac{f'p-g}{f}(\hat{v}^-_{ppp} - 2F\hat{v}^-_p) + \frac{b}{f}(\hat{v}^+_{ppp} + 2F\hat{v}^+_p) = 0. \end{aligned} \tag{5.15}$$

This set of equations can be simplified further using the transformation

$$\tilde{p} = f(q)p - \int g(q)\,\mathrm{d}q, \quad \tilde{q} = q, \quad \tilde{v}^+ = \hat{v}^+, \quad \tilde{v}^- = \hat{v}^-.$$

In the new variables, system (5.15) becomes

$$\begin{aligned} &(f^2\tilde{v}^+)_{\tilde{q}} + b(f^2\tilde{v}^-)_{\tilde{p}} = 0, \\ &(f^2\tilde{v}^-_{\tilde{p}\tilde{p}} - 2F\tilde{v}^-)_{\tilde{q}} + b(f^2\tilde{v}^+_{\tilde{p}\tilde{p}} + 2F\tilde{v}^+)_{\tilde{p}} = 0. \end{aligned} \tag{5.16}$$

The first equation of (5.16) can be used to introduce a potential variable, via $V_{\tilde{p}} = f^2\tilde{v}^+$ and $V_{\tilde{q}} = -bf^2\tilde{v}^-$. Upon introducing $\bar{q} = \int f^2\mathrm{d}\tilde{q}$, the second equation then becomes

$$f^2(f^2V_{\tilde{p}\tilde{p}\bar{q}})_{\bar{q}} - 2Ff^2V_{\bar{q}\bar{q}} - b^2\left(V_{\tilde{p}\tilde{p}\tilde{p}\tilde{p}} + \frac{2F}{f^2}V_{\tilde{p}\tilde{p}}\right) = 0. \tag{5.17}$$

The decoupled system ($b = 0$). The general solution of the decoupled system (5.16) is

$$\hat{v}^+ = \frac{\zeta^1(\tilde{p})}{f^2}, \quad \hat{v}^- = \zeta^2(\tilde{p}) + \vartheta^1(q)e^{\sqrt{2F}p} + \vartheta^2(q)e^{-\sqrt{2F}p},$$

where ζ^1 and ζ^2 are arbitrary functions of $\tilde{p} = f(q)p - \int g(q)\,\mathrm{d}q$ and ϑ^1 and ϑ^2 are arbitrary functions of q.

The coupled system ($b \neq 0$). Since it is possible to obtain the general solution for the decoupled case of system (5.16) it is not necessary to investigate further Lie reductions for this case. We hence perform this reduction solely for the case of $b \neq 0$. For this reason, it is necessary to solve the group classification problem for system (5.16) under the assumption that $b \neq 0$. As system (5.16) is linear, the procedure of group classification is done as described in the previous section. The general form of a Lie symmetry operator of system (5.16) is $\xi^{\tilde{p}}\partial_{\tilde{p}} + \xi^{\tilde{q}}\partial_{\tilde{q}} + \eta^+\partial_{\tilde{v}^+} + \eta^-\partial_{\tilde{v}^-}$, where the coefficients are functions of $\tilde{p}$, $\tilde{q}$, $\tilde{v}^+$ and $\tilde{v}^-$. The solution of the determining equation gives

$$\xi^{\tilde{p}} = a\tilde{p} + c, \quad \xi^{\tilde{q}} = a\tilde{q} + d, \quad \eta^+ = k\tilde{v}^+ + g^1(\tilde{q}, \tilde{p}), \qquad \eta^- = k\tilde{v}^- + g^2(\tilde{q}, \tilde{p}).$$

Additionally there is the single classifying equation $(aq + d)f_q - af = 0$ (up to its differential consequences). The case of arbitrary f leads to $a = d = 0$. The maximal Lie invariance algebra in this case reads

$$\langle \partial_{\tilde{p}},\ \mathcal{I} = \tilde{v}^+\partial_{\tilde{v}^+} + \tilde{v}^-\partial_{\tilde{v}^-},\ \mathcal{L}(\tilde{g}^1, \tilde{g}^2) = g^1(\tilde{p}, \tilde{q})\partial_{\tilde{v}^+} + g^2(\tilde{p}, \tilde{q})\partial_{\tilde{v}^-} \rangle,$$

where (g^1, g^2) run through the set of solutions of system (5.16). The possible inequivalent extensions of this algebra are given by $f = \text{const}$ and $f = Cq$. The first case leads to the extension of the above invariance algebra by $\partial_{\tilde q}$, the second case gives an extension by $\tilde p\partial_{\tilde p} + \tilde q\partial_{\tilde q}$.

It was shown before that by introducing a potential variable, system (5.16) can be converted into the single equation (5.17) in conserved form. Since such an equation can have additional symmetries compared to the original system, we also perform group classification of (5.17) (again under the assumption that $b \neq 0$). Solving the determining equations for the coefficients of a Lie symmetry operator $\xi^{\tilde p}\partial_{\tilde p} + \xi^{\bar q}\partial_{\bar q} + \eta^V\partial_V$ of (5.17) gives

$$\xi^{\tilde p} = a\tilde p + c, \quad \xi^{\bar q} = 3a\bar q + d, \quad \eta^V = \alpha V + g(\tilde p, \bar q),$$

together with the classifying equation $(3a\bar q + d)f' - af = 0$. For arbitrary f, this immediately implies that $a = d = 0$ and hence gives rise to the maximal Lie invariance algebra $\langle \partial_{\tilde p},\, V\partial_V,\, g(\tilde p, \bar q)\partial_V \rangle$, where g is an arbitrary solution of (5.17). There are two inequivalent extensions of this algebra, depending on either $a \neq 0$, $d = 0$ or $a = 0$, $d \neq 0$. Since scalings and shifts in q are equivalence transformations, we may first set $a = 1/3$, $d = 0$ with $f = C\sqrt[3]{\bar q}$. The additional generator then reads $\tilde p\partial_{\tilde p} + 3\bar q\partial_{\bar q}$. The second case of extension is given by $a = 0$, $d = 1$ leading to $f = \text{const}$ with the corresponding additional generator $\partial_{\tilde q}$.

Comparing this result with the classification of system (5.16), we see that all Lie symmetries of the potential equation (5.17) are induced by Lie symmetries of (5.16) (note the change of the variable $\tilde q$ in the potential case). That is, for the reduced system (5.16) no purely potential symmetries associated with the potential equation (5.17) exist.

Now that we have investigated all symmetry extensions for particular values of $f(q)$, it remains to present the corresponding Lie reductions of (5.16). The only feasible way for reduction for general f is due to the operator $\partial_{\tilde p} + \lambda\mathcal{I}$. The ansatz for reduction then is $\tilde v^+ = \bar v^+(\tilde q)e^{\lambda\tilde p}$ and $\tilde v^- = \bar v^-(\tilde q)e^{\lambda\tilde p}$. The system of reduced equation reads

$$u_{\tilde q} + bf^2\lambda\bar v^- = 0, \quad (f^2\lambda^2\bar v^- - 2F\bar v^-)_{\tilde q} + b\lambda\left(\lambda^2 + \frac{2F}{f^2}\right)u = 0,$$

where $u = f^2\bar v^+$. The case of $\lambda = 0$ gives only a trivial solution and will not be considered here. For $\lambda \neq 0$ it is possible to solve the first equation for $\bar v^-$. Plugging the corresponding expression into the second equation, the following homogeneous second order ordinary differential equations with variable coefficients for u is obtained

$$u_{\tilde q\tilde q} + \vartheta^1(\tilde q)u_{\tilde q} + \vartheta^0(\tilde q)u = 0, \qquad \vartheta^1 = -\frac{2f'}{f(1-f^2)}, \quad \vartheta^0 = \frac{b^2\lambda^2}{1-f^2}\frac{f^2\lambda^2 + 2F}{\lambda^2 - 2F}.$$

Two more Lie reductions are possible for the particular values $f = C = \text{const}$ and $f = Cq$. In the first case, the maximal Lie invariance algebra reads $\langle \partial_{\tilde q}, \partial_{\tilde p}, \mathcal{I}, \mathcal{L}(\tilde g^1, \tilde g^2)\rangle$. We aim to reduce (5.16) using the generator $Q = \partial_{\tilde q} + \kappa\partial_{\tilde p} + \lambda\mathcal{I}$. The ansatz for reduction is $\tilde v^+ = \bar v^+(r)e^{\lambda\tilde q}$, where $r = \tilde p - \kappa\tilde q$ and $\tilde v^- = \bar v^-(r)e^{\lambda\tilde q}$. Plugging this ansatz into (5.16), we obtain

$$\begin{aligned}
&\kappa\bar v^+_r - \lambda\bar v^+ - b\bar v^-_r = 0,\\
&C^2(\kappa\bar v^-_{rrr} - \lambda\bar v^-_{rr}) - 2F(\kappa\bar v^-_r - \lambda\bar v^-) - b(C^2\bar v^+_{rrr} + 2Fv^+_r) = 0.
\end{aligned}$$

This is a coupled system of third order ordinary differential equations with constant coefficients and thus can be solved explicitly using standard techniques.

In the second case, the maximal Lie invariance algebra is given by $\langle \tilde{q}\partial_{\tilde{q}}+\tilde{p}\partial_{\tilde{p}}, \partial_{\tilde{p}}, \mathcal{I}, \mathcal{L}(\tilde{g}^1, \tilde{g}^2)\rangle$. One-dimensional reduction is feasible using the generator $Q = \tilde{q}\partial_{\tilde{q}}+\tilde{p}\partial_{\tilde{p}}+\lambda\mathcal{I}$. The corresponding ansatz for reduction is $\tilde{v}^+ = \bar{v}^+(r)q^\lambda$ and $\tilde{v}^- = \bar{v}^-(r)q^\lambda$, where $r = pq^{-1}$. This ansatz leads to the following reduction of system (5.16):

$$r\bar{v}^+_r - (\lambda+2)\bar{v}^+ - b\bar{v}^-_r = 0,$$
$$r\bar{v}^-_{rrr} - (2C^2 + C(\lambda-2))\bar{v}^-_{rr} - 2F(r\bar{v}^-_r - \lambda v^-) - b(C^2\bar{v}^+_{rrr} + 2F\bar{v}^+_r) = 0.$$

Remark 5.3. Substituting back the solutions of the reduced equations into the ansatz for the original unknown functions ψ^+ and ψ^-, we find that all solutions have time-dependent polynomial parts in x and y. Since this part is not compatible with typical boundaries of the two-layer equations (as discussed in section 5.8), only solutions for the restricted case of $b = g = 0$, $f = \text{const}$ might give candidate solutions that could be realized in the framework of geophysical fluid dynamics. However, since the solution of the decoupled case $b = 0$ for $g = 0$, $f = \text{const}$ is rather trivial, it is not too interesting from the physical point of view.

5.7 Invariant reduction with two-dimensional subalgebras

In this section we present the reduction of (5.1) or (5.3) using the optimal set of two-dimensional inequivalent subalgebras. We note in the beginning of this part, that not all subalgebras give rise to classical group-invariant reduction. Namely, all subalgebras of list (5.6) with negative subscript cannot be used in this respect. This is due to the impossibility of constructing ansatzes for the dependent variables in these cases. However, these algebras would be well-suited for the construction of partially-invariant solutions, but we do not pursue this idea further in this paper.

5.7.1 Subalgebra $\mathcal{A}^2_1$

An ansatz for group-invariant reduction using this subalgebra is $\psi^1 = v^1(p) + \kappa t + (\mu+\rho)y$ and $\psi^2 = v^2(p) - \kappa t + (\mu-\rho)y$, where $p = x - \nu y$. System (5.1) is reduced under this ansatz to

$$\begin{aligned} -(\rho+\mu)(1+\nu^2)v^1_{ppp} + F\mu(v^1_p - v^2_p) - F\rho(v^1_p + v^2_p) - 2F\kappa + \beta v^1_p = 0,\\ (\rho-\mu)(1+\nu^2)v^2_{ppp} - F\mu(v^1_p - v^2_p) + F\rho(v^1_p + v^2_p) + 2F\kappa + \beta v^2_p = 0. \end{aligned} \tag{5.18}$$

Integrating once with respect to p, the above system becomes an inhomogeneous system of two second order linear ordinary differential equations with constants coefficients, provided we screen out the singular cases of $\rho = \mu = 0$, $\rho = \mu \geq 0$, $\rho = \mu < 0$, $\rho = -\mu \geq 0$ and $\rho = -\mu < 0$. The solution of this system in the nonsingular case is straightforward but a bit lengthy and is therefore omitted here. Hence, we will focus on the listed singular cases only.

Case $\rho = \mu = 0$. The solution of this case is

$$v^1 = \frac{2F\kappa}{\beta}p + c_1, \qquad v^2 = -\frac{2F\kappa}{\beta}p + c_2,$$

which in the original variables ψ^1, ψ^2 gives a solution linear in x, y and thus represents a constant wind field in both layers.

Case $\rho = \mu \geq 0$. This case leads to the semi-coupled system

$$2\mu(1+\nu^2)v^1_{ppp} + 2F\mu v^2_p + 2F\kappa - \beta v^1_p = 0, \quad 2F\mu v^2_p + 2F\kappa + \beta v^2_p = 0,$$

which is integrated to yield

$$v^1 = c_1 \exp\left(\sqrt{\frac{\beta}{2\mu(1+\nu^2)}}p\right) + c_2 \exp\left(-\sqrt{\frac{\beta}{2\mu(1+\nu^2)}}p\right) + \frac{2F\kappa p}{2F\mu+\beta} + c_3,$$
$$v^2 = -\frac{2F\kappa p}{2F\mu+\beta} + c_4.$$

In the original variables, this represents a simple exponential solution and is thus unphysical.

Case $\rho = \mu < 0$. The general solution in this case is

$$v^1 = c_1 \cos\left(\sqrt{\frac{\beta}{2|\mu|(1+\nu^2)}}p\right) + c_2 \sin\left(\sqrt{\frac{\beta}{2|\mu|(1+\nu^2)}}p\right) - \frac{2F\kappa p}{2F|\mu|-\beta} + c_3,$$
$$v^2 = \frac{2F\kappa p}{2F|\mu|-\beta} + c_4,$$

which in the original variables represents a single (stationary) Rossby-wave in the above layer and a constant velocity field in the lower layer. This is a typical situation, in the study of baroclinic instability: An initial disturbance in the middle of the troposphere may start to grow while the lower part of the troposphere does not exhibit any peculiarities. It is not before the upper Rossby-wave starts unstable growth, that the lower layer also begins to show some wave-like disturbances, which subsequently may lead to the onset of cyclogenesis. Hence, the above exact solution may characterize the situation at the onset of baroclinic instability, where due to external forcing a Rossby wave in the upper layer is generated, while the wind field in the lower layer is still unaffected.

The cases $\rho = -\mu \geq 0$ and $\rho = -\mu < 0$ give the same solutions as in the two previous cases, except for interchanging the two layers, i.e. $v^1 \leftrightarrow v^2$ and permuting the sign of the linear in p term.

5.7.2 Subalgebra $\mathcal{A}_2^2$

It is convenient to use at once the quasi-geostrophic equations in terms of barotropic/baroclinic variables to perform the reduction. The ansatz we choose is: $\psi^+ = v^1(p) - 2\sigma p x$ and $\psi^- = v^2(p) + 2\kappa t$, where $p = y - \nu t$. Using these variables, the resulting submodel of (5.1) is:

$$(\nu+\sigma p)v^1_{ppp} + 2b\beta p = 0, \quad (\nu+\sigma p)v^2_{ppp} - 2F(\nu+\sigma p)v^2_p + 4F\kappa = 0,$$

which is a decoupled system of third order linear ODEs. The case of $b = 0$ is trivial and will not be considered here. For $b \neq 0$ the general solution is

$$v^1 = \frac{\nu}{\sigma}\beta\ln(\nu+\sigma p)\left(p^2 + 2\frac{\nu}{\sigma}p + \frac{\nu^2}{\sigma^2}\right) - \frac{1}{3}\beta\left(p^3 + \frac{9\nu}{2\sigma}p^2 + \frac{6\nu^2}{\sigma^2}p + \frac{3\nu^3}{2\sigma^3}\right)$$
$$+ c_1p^2 + c_2p + c_3,$$
$$v^2 = \frac{\kappa}{\sigma}\Big(2\ln(\nu+\sigma p) + \mathrm{Ei}\left(\tfrac{\sqrt{2F}}{\sigma}(\nu+\sigma p)\right)e^{\frac{\sqrt{2F}}{\sigma}(\nu+\sigma p)} + \mathrm{Ei}\left(-\tfrac{\sqrt{2F}}{\sigma}(\nu+\sigma p)\right)e^{-\frac{\sqrt{2F}}{\sigma}(\nu+\sigma p)}\Big)$$
$$+ c_4e^{\sqrt{2F}p} + c_5e^{-\sqrt{2F}p} + c_6,$$

where $\mathrm{Ei}(z) = \int_z^\infty t^{-1}e^{-t}\mathrm{d}t$ denotes the exponential integral. In terms of the ψ^+/ψ^- variables this solution is the superposition of some polynomial with an exponential function in y-direction. From the meteorological point of view, this solution does not seem to be relevant.

5.7.3 Subalgebra $\mathcal{A}_3^2$

Using the barotropic/baroclinic variables, the ansatz for reduction under this subalgebra is $\psi^+ = v^1(p) + 2\mu x$,$\psi^- = v^2(p) + 2\kappa t + 2\rho x$, where $p = y - \nu t$. The reduced system then is

$$
\begin{aligned}
&(\nu-\mu)v^1_{ppp} - \rho v^2_{ppp} - 2\beta\mu = 0,\\
&(\nu-\mu)v^2_{ppp} - \rho v^1_{ppp} - 2F(\nu-\mu)v^2_p - 2F\rho v^1_p + 4F\kappa - 2\beta\rho = 0.
\end{aligned}
$$

This system is now completely integrable. The case $\rho = 0$, $\mu \neq \nu$ leads to a decoupled system of equations which can be integrated easily. The same also holds in the case $\rho \neq 0$, $\nu = \mu$. Hence, we focus on the case where $\rho \neq 0$ and ν, μ are arbitrary. The first equation can be integrated at once three times, yielding the relation between v^1 and v^2, given by

$$
v^2 = \frac{1}{\rho}\left((\nu-\mu)v^1 - \frac{1}{3}\beta\mu p^3 + c_1p^2 + c_2p + c_3\right),
$$

where c_1, c_2 and c_3 are arbitrary constants. Then, integrating once the second equation and substituting the expression for v^2 produces a second order ordinary differential equation with constant coefficients, from which we determine v^1:

$$
v^1 = A^1\sin\sqrt{\frac{\gamma^2}{\gamma^1}}p + A^2\cos\sqrt{\frac{\gamma^2}{\gamma^1}}p - \frac{1}{\gamma^2}\left(\delta^3p^3 + \delta^2p^2 + \delta^1p + \delta^0\right) + \frac{\gamma^1}{(\gamma^2)^2}(6\delta^3p + 2\delta^2)
$$

where

$$
\begin{aligned}
\gamma^2 &= -2F\left(\frac{1}{\rho}(\nu-\mu)^2 + \rho\right), & \gamma^1 &= \frac{1}{\rho}(\nu-\mu)^2 - \rho,\\
\delta^3 &= \frac{2F\beta\mu}{3\rho}(\nu-\mu), & \delta^2 &= -\frac{2c_1F}{\rho}(\nu-\mu),\\
\delta^1 &= -2\left(\frac{\beta\mu + c_2F}{\rho}(\nu-\mu) - (2F\kappa - \beta\rho)\right), & \delta^0 &= \frac{2(c_1 - c_3F)}{\rho}(\nu-\mu) + c_4.
\end{aligned}
$$

provided that $\gamma^2/\gamma^1 > 0$. In the particular case of $\nu = \mu$, which leads to a considerable simplification of the above solution, this condition is verified. In the case of $\gamma^2/\gamma^1 < 0$ we can find a solution in terms of exponential functions, which is not presented here. Plugging this solution into the ansatz for the original unknown functions, the above solution gives the combination of a traveling wave in y-direction with a third order time-dependent polynomial in x and y. For usual fixed boundaries in north–south direction, this is an unphysical solution.

5.7.4 Subalgebra $\mathcal{A}_4^2$

An appropriate ansatz for this subalgebra is $\psi^+ = v^1(p) - f'y^2 - 2g(fy - x)$, $\psi^- = v^2(p) + 2\kappa y - 2\rho(fy - x)$, $p = t$, where we have again employed the barotropic/baroclinic variables. This ansatz enables reduction of (5.3) to the system

$$
f'' - \beta g = 0, \quad v^2_p + 2\kappa g - \frac{\beta\rho}{F} = 0.
$$

The general solution of this system is

$$
v^1 = \theta(p), \quad v^2 = -\frac{2\kappa}{\beta}f' + \frac{\beta\rho}{F}p + c,
$$

where θ is an arbitrary function of p and c is an arbitrary constant. The first equation of the reduced system is the compatibility condition of the initial system (5.3) and the invariant surface equation corresponding to subalgebra $\mathcal{A}_4^2$. It implies the constraint $g = f''/\beta$ for the parameter functions f and g, which is the necessary and sufficient condition for system (5.3) to have solutions invariant with respect to the algebra $\mathcal{A}_4^2$. This solution has no obvious physical importance in dynamic meteorology. The reason is that this is a simple polynomial solution with time-dependent coefficients. We note that the function θ can be set to zero due to gauging of the stream functions generated by $\mathcal{Z}(g)$.

5.8 Invariant reduction of boundary value problems

In this part, we aim to discuss admitted symmetries in the presence of boundaries. It is commonly assumed that group-invariant solutions may describe the behavior of a system that is far away from boundaries and hence a consideration of restrictions imposed by boundaries is usually omitted. However, as was shown e.g. in [55, 80], there may be situations where the system without boundaries is not simply the limit of a system with very distant boundaries. Consequently, consideration of boundaries may be necessary even for a conceptual understanding of the model evolution. Moreover, as was noted in the two previous sections, some of the group-invariant solutions corresponding to the optimal sets of inequivalent subalgebras give rise to unphysical solutions due to a violation of boundaries. We now compute those symmetries that are admitted by the boundaries and hence discuss which solutions could be compatible with the boundary value problem.

In the atmospheric sciences, for equations on the β-plane commonly a channel flow is assumed, which implies rigid boundaries in north–south direction. In east–west direction, one usually assumes periodic boundaries or an infinitely extended domain. In this setting, imposed conditions for the two-layer model are

$$\frac{\partial \psi_i}{\partial x} = 0, \quad \frac{\partial}{\partial t}\frac{1}{2L}\int_{-L}^{L} \frac{\partial \psi_i}{\partial y}\mathrm{d}x = 0 \qquad \text{for} \qquad y \in \{0, Y\}. \tag{5.19}$$

The second condition implies conservation of circulation at the boundaries. According to [23] for a boundary value problem to be invariant, three conditions must be satisfied: (i) invariance of the equation, (ii) invariance of the domain, (iii) invariance of the values on the boundaries. The first condition was already established in Section 5.3, so it remains to verify (ii) and (iii). Although it is possible to solve this problem on the stage of the Lie algebra using the infinitesimal method [23], we find it more comfortable to work with the finite group transformations. The most general form of a continuous symmetry transformation of (5.1) is given by

$$(t, x, y, \psi_1, \psi_2) \mapsto (t + \varepsilon_1, x + f, y + \varepsilon_2, \psi_1 - f'y + g + \varepsilon_3, \psi_2 - f'y + g - \varepsilon_3).$$

This transformation has to preserve both the domain and the boundary values. We now discuss the channel flow with three possibilities for boundaries in east–west direction.

Infinite-domain. If there are no sidewalls in east–west direction, that is $L \to \infty$, the most general symmetry preserving the boundary value problem is given by

$$(t, x, y, \psi_1, \psi_2) \mapsto (t + \varepsilon_1, x + h(t), y, \psi_1 - h'(t)y + g + \varepsilon_3, \psi_2 - h'(t) + g - \varepsilon_3).$$

where h is an arbitrary function of t with $h'' = 0$. Hence, we have $h = \varepsilon_4 t + \varepsilon_5$, leading to the important result that Galilean boosts preserve the boundary value problem.

Periodic boundaries. For periodic boundaries, we have $\psi_i(t, -L, y) = \psi_i(t, L, y)$. Similar calculations as above imply that the boundary-preserving symmetry group is the same as for an infinite-domain.

This shows that the Rossby wave solution is admitted by the boundary value problem. This may serve as a "symmetry explanation" for the prominent occurrence of this solution in geophysical fluid dynamics.

Limited domain. The model of the limited domain in east–west direction is very natural in oceanography but can be also realized in the atmospheric sciences as flow in a mountainous region. For the purpose of simplicity, we assume a rectangular domain. Besides (5.19), this setting requires the additional conditions

$$\frac{\partial \psi_i}{\partial y} = 0, \quad \frac{\partial}{\partial t}\frac{1}{Y}\int_0^Y \frac{\partial \psi_i}{\partial x}\mathrm{d}y = 0 \qquad \text{for} \qquad x \in \{-L, L\},$$

where L and Y denote the length and the width of the rectangle, respectively. Symmetries that are compatible with this boundary value problem are

$$(t, x, y, \psi_1, \psi_2) \mapsto (t + \varepsilon_1, x, y, \psi_1 + g + \varepsilon_3, \psi_2 + g - \varepsilon_3).$$

Hence, the only group-invariant solution that can be realized on this domain is a stationary solution.

5.9 Conclusion

In this paper, we have considered the baroclinic two-layer model from the viewpoint of symmetries. Lie point symmetries and discrete mirror symmetries were given and the former were classified with respect to the adjoint action. Using the optimal sets of inequivalent one- and two-dimensional subalgebras we performed reduction in one and two variables. This completely solved the classical Lie problem for the two-layer equations. The procedure lead to various classes of exact solutions, some of which are well-known in the atmospheric sciences, including barotropic and baroclinic Rossby waves. Finally, also the two-layer boundary value problem was investigated in the light of admitted Lie symmetries. We have obtained an analog result as was found in [135] (see also [118, pp. 379]) for the Navier–Stokes equations, namely that periodic boundary conditions admit Galilean boosts as symmetry transformations.

Although there is still a large number of obviously unphysical solutions, the study of the classical Lie problem is a necessary first step for the consideration of partially invariant solutions and nonclassical symmetries, which we save for future investigations. In addition, these solutions are of undeniable value when it comes to a numerical implementation of the two-layer equations, which can employ several kinds of boundary conditions. In this case, the obtained solutions can be used as benchmark tests to assess the quality of the numerical scheme involved by addressing issues such as convergence rates and the reproduction of correct phase space velocities of wave-like solutions. Moreover, the Lie symmetries determined in this paper can be used to compute differential invariants, which can again be used to extend the set of exact solutions, e.g. by construction of differentially invariant solutions [46].

From the mathematical point of view it is interesting to note that in all cases of reduction the coupling of the two reduced equations is due to the "baroclinic" operator $\mathcal{F} = \partial_{\psi^1} - \partial_{\psi^2}$. Moreover, in all cases of reduction, the equation for the baroclinic part of the system is structurally more complicated than those for the barotropic part. It should also be stressed that in the second and in the third case of reduction in one variable, the resulting $(1+1)$-dimensional systems of partial differential equations became linear. That is, the linear superposition principle is available for these equations. This allows to generate wide sets of exact solutions by linearly combining different solutions and substituting them back into the ansatz for the original unknown functions. Furthermore, there exists a number of special techniques for such linear equations. One more method for obtaining exact solutions of linear partial differential equations can be found in appendix 5.10.

Since the two-layer model is only capable of resolving the barotropic mode and the first baroclinic mode, it would be interesting to study symmetry properties of multi-layer models. This would be a preliminary step on the way to the investigation of a continuously stratified atmosphere. On the other hand, from the standpoint of application, a deeper investigation of layer models may even be more important than the three-dimensional system of governing equations. This is true since numerical utilization of these equations calls for some discretization, hence naturally leading back to the model of a layered atmosphere. Therefore, the present investigation of the two-layer model may not only be interesting for historical reasons.

5.10 Appendix: Extended Lie reduction of linear PDEs

Consider a linear partial differential equation $\mathcal{L}$: $Lu = 0$ in the unknown function u of n independent variables $x = (x_1, \ldots, x_n)$, where L is the associated linear differential operator. In what follows we use the summation convention for repeated indices. The indices i, j and k run from 1 to n, the indices a and b run from 1 to m.

Suppose that the equation $\mathcal{L}$ possesses a nontrivial Lie symmetry operator Q_0 of the form $Q_0 = \xi^i(x)\partial_{x_i} + \eta(x)u\partial_u$, where $\xi^i\xi^i \neq 0$. Then for an arbitrary constant λ the equation $\mathcal{L}$ obviously possesses also the vector field $Q_\lambda = Q_0 + \lambda u\partial_u$ as a nontrivial Lie symmetry operator. By $\widehat{Q}_\lambda$ we denote the differential operator acting on functions of x and associated with the operator Q_λ, i.e., $\widehat{Q}_\lambda = -\xi^i(x)\partial_{x_i} + \eta(x) + \lambda$. For any $m \in \mathbb{N}$ the differential function $(\widehat{Q}_\lambda)^m u$ is well known to be a characteristic of a generalized symmetry of $\mathcal{L}$, and hence any associated generalized ansatz reduces the equation $\mathcal{L}$ to a system of m linear differential equations in m new unknown functions of $n-1$ new independent variables invariant with respect to the operator Q_λ. In order to construct an ansatz, we should integrate the partial differential equation $(\widehat{Q}_\lambda)^m u = 0$. The general solution of this equation gives the ansatz

$$u = h(x)e^{\lambda\theta} \sum_{a=1}^{m} \varphi^a(\omega) \frac{\theta^{m-a}}{(m-a)!}, \tag{5.20}$$

where $\omega = (\omega^1(x), \ldots, \omega^{n-1}(x))$ is a tuple of functionally independent solutions of the equation $\xi^i u_{x_i} = 0$, which are assumed to be invariant independent variables, $\theta = \theta(x)$ is a particular solution of the equation $\xi^i u_{x_i} = 1$, $h = h(x)$ is a particular nonvanishing solution of the equation $\xi^i u_{x_i} = \eta u$ and $\varphi^a = \varphi^a(\omega)$ play the role of new unknown functions.

In view of the Lie invariance with respect to the operator Q_0, the equation $\mathcal{L}$ is mapped by the point transformation

$$\tilde{x}_1 = \omega^1(x), \quad \dots, \quad \tilde{x}_{n-1} = \omega^{n-1}(x), \quad \tilde{x}_n = \theta(x), \quad \tilde{u} = \frac{u}{h(x)}$$

to the equation $\tilde{L}\tilde{u} = 0$, where the coefficients of the operator $\tilde{L}$ do not depend on $\tilde{x}_n$. In the new variables $(\tilde{x}, \tilde{u})$ the ansatz (5.20) takes the form

$$\tilde{u} = e^{\lambda \tilde{x}_n} \sum_{a=1}^{m} \varphi^a(\tilde{x}_1, \dots, \tilde{x}_{n-1}) \frac{\tilde{x}_n^{m-a}}{(m-a)!}. \tag{5.21}$$

After substituting the ansatz (5.21) into the transformed equation $\tilde{L}\tilde{u} = 0$, dividing the resulting equation by $e^{\lambda \tilde{x}_n}$ and subsequently splitting with respect to different powers of the variable $\tilde{x}_n$, we obtain, at least for the general value of λ, the system $\mathcal{R}$ of m differential equations with respect to the functions φ^a in $n-1$ independent variables $(\tilde{x}_1, \dots, \tilde{x}_{n-1})$. In singular cases, for certain values of λ some of the equations are identities. The same reduction is obtained by the substitution of the ansatz (5.20) into the initial equation $\mathcal{L}$.

If the basic field is real, we can consider complex values of λ, construct the corresponding complex exact solution and then take its real and imagine parts in order to obtain real solutions.

The above consideration has a nice interpretation within the framework of Lie symmetries. Introducing the new dependent $v^a = (\widehat{Q}_\lambda)^{m-a}u$, instead of the single mth order linear partial differential equation $(\widehat{Q}_\lambda)^m u = 0$ for finding a generalized ansatz, we obtain the system of m first order linear partial differential equations

$$\widehat{Q}_\lambda v^1 = 0, \quad \widehat{Q}_\lambda v^a = v^{a-1}, \quad a = 2, \dots, m.$$

As Q_λ is a Lie symmetry operator of the equation $\mathcal{L}$, each function v^a satisfies this equation. To give the interpretation, we consider the system $\mathcal{S}$ of m copies of the initial equation $\mathcal{L}$

$$Lv^1 = 0, \quad \dots, \quad Lv^m = 0.$$

This system obviously possesses the operators $\bar{Q}_0 = \xi^i(x)\partial_{x_i} + \eta(x)v^a\partial_{v^a}$ and $v^b\partial_{v^a}$ as its Lie symmetry operators. Consider a linear combination of these operators, $\bar{Q}_\Lambda = \bar{Q}_0 + \Lambda_{ab}v^b\partial_{v^a}$, which is also a Lie symmetry operator of the system $\mathcal{S}$. Here and in what follows Λ_{ab} are constants. Up to the equivalence generated by adjoint action of the Lie symmetry group of $\mathcal{S}$ on the corresponding Lie invariance algebra and due to the linear superposition principle, we can assume without loss of generality that the matrix $\Lambda = (\Lambda_{ab})$ is the single $m \times m$ Jordan block with an eigenvalue λ,

$$\Lambda = J^m_\lambda = \begin{pmatrix} \lambda & 1 & 0 & 0 \cdots & 0 \\ 0 & \lambda & 1 & 0 \cdots & 0 \\ 0 & 0 & \lambda & 1 \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 \cdots & 1 \\ 0 & 0 & 0 & 0 \cdots & \lambda \end{pmatrix}.$$

The invariant surface condition for the operator $\bar{Q}_\Lambda$ with $\Lambda = J^m_\lambda$ consists of the equations

$$\xi^i v^1 = (\eta + \lambda)v^1, \quad \xi^i v^a = (\eta + \lambda)v^a + v^{a-1}, \quad a = 2, \dots, m,$$

and an ansatz constructed with this operator has the form

$$v^a = h(x)e^{\lambda\theta}\sum_{b=1}^{a}\varphi^b(\omega)\frac{\theta^{a-b}}{(a-b)!}, \tag{5.22}$$

where the notation from the ansatz (5.21) is used. According to the general theory of Lie reduction [115], the ansatz (5.22) necessarily reduces the system $\mathcal{S}$ to a system in the functions φ^a, which obviously coincides with the system $\mathcal{R}$ obtained by reducing the single equation $\mathcal{L}$ using the generalized ansatz (5.20).

If the equation $\mathcal{L}$ is considered over the real field and the eigenvalue λ is complex, it is not necessary to pass to the associated real Jordan block. As above, we can find the corresponding complex exact solution and then take its real and imagine parts in order to construct real solutions. This additionally justifies the usage of complex values of λ in the real case.

Example. For the general value of A, the second equation of system (5.11) admits only one independent nontrivial Lie symmetry operator, ∂_p. Consider a system of m copies of this equation:

$$v^a_{ppq} - 2(Av^a)_q + v^a_p = 0, \tag{5.23}$$

where for simplicity we have omitted bars over the variables and scaled $F = 1$ and $\beta = 1$. This system admits the Lie symmetry operator $\bar{Q}_\Lambda = \partial_p + \Lambda_{ab}v^b\partial_{v^a}$, where $\Lambda = J^m_\lambda$. The invariant surface condition associated with $\bar{Q}_\Lambda$ reads

$$v^1_p = \lambda v^1, \quad v^a_p = \lambda v^a + v^{a-1}, \quad a = 2, \dots, m.$$

Its general solution provides us with an appropriate ansatz for Lie reduction:

$$v^a = \exp(\lambda p)\sum_{b=1}^{a}\varphi^b(\omega)\frac{p^{a-b}}{(a-b)!},$$

where $\omega = q = t$ is the invariant independent variable. Substituting this ansatz into system (5.23) yields the system of ordinary differential equations for φ^a

$$\begin{aligned}
&\mathrm{L}\varphi^1 = 0,\\
&\mathrm{L}\varphi^2 + 2\lambda\varphi^1_q + \varphi^1 = 0,\\
&\mathrm{L}\varphi^k + 2\lambda\varphi^{k-1}_q + \varphi^{k-1} + \varphi^{k-2}_q = 0, \qquad k = 3, \dots, m,
\end{aligned}$$

where the operator L is given by $\mathrm{L} := (\lambda^2 - 2A)\partial_q - 2A_q + \lambda$.

The solution of the above system is:

$$\begin{aligned}
&\varphi^1 = c_1e^{-\zeta},\\
&\varphi^2 = c_2e^{-\zeta} + e^{-\zeta}\int\frac{\varphi^1 + 2\lambda\varphi^1_q}{2A - \lambda^2}e^{\zeta}\,\mathrm{d}q,\\
&\varphi^k = c_ke^{-\zeta} + e^{-\zeta}\int\frac{\varphi^{k-1} + 2\lambda\varphi^{k-1}_q + \varphi^{k-2}_q}{2A - \lambda^2}e^{\zeta}\,\mathrm{d}q, \qquad k = 3, \dots, m,
\end{aligned}$$

where

$$\zeta = \int\frac{2A_q - \lambda}{2A - \lambda^2}\mathrm{d}q.$$

In the special cases $A = \mathrm{const}$ and $A = C_1(q+C_0)^2$, where $C_0, C_1 = \mathrm{const}$, we can make more generalized reductions of the equation under consideration, which involves operators extending the Lie invariance algebra of the general case, cf. Section 5.6.2.

Chapter 6

Summary and conclusions

In this part of the thesis we have used Lie symmetries to obtain exact solutions of some models utilized in the atmospheric sciences. This is maybe the most classical way to employ symmetries of partial differential equations, namely by carrying out group-invariant reduction. The entire procedure is in principle fairly algorithmic. The Lie symmetries of a differential equation are computable by solving an overdetermined linear system of partial differential equations. Automatic solving algorithms for such systems have been implemented in all major computer algebra systems since the last twenty years [26, 28, 51, 138]. The determination of the invariants of subalgebras of the maximal Lie invariance algebra and the subsequent reduction of the given differential equation are also straightforward tasks. Provided the differential equation is simplified enough to be explicitly integrated, it is again quite artless to produce *some* exact solutions of the initial differential equation.

However, owing to the principally rather algorithmic nature of Lie reduction, there exist a seemingly endless list of papers with unsystematic, incomplete, superfluous or even incorrect results, see, e.g., the list of commented papers in the review [133] to get an idea of the extend of this problem only in the most recent years. Needless to say that systematically performing group-invariant reduction is not that simple. The computation of the optimal lists of inequivalent subalgebras can be an error-prone task for complicated Lie invariance algebras. In some instances, this system might even be amenable to further simplifications taking into account discrete symmetries, which in principle also should be computed rigorously. Moreover, it has to be kept in mind that a list of subalgebras that might be optimal from the algebraic point of view, might not be optimal from the point of view of the reduction procedure. The construction of an appropriate ansatz for reduction is also in some sense an elaborate exercise. It can happen that a seemingly complicated ansatz simplifies the resulting reduced differential equation considerably. The construction of the "optimal" ansatz for reduction therefore sometimes involves several steps of substitution into the equation followed by modifications of the ansatz. It can as well be the case that the reduction using several (classes of) Lie subalgebras, which are essentially different from the algebraic point of view, can be treated jointly by introducing parameters in the ansatz. Different values of these parameters then correspond to the single (classes of) subalgebras that are considered together. This can shorten the list of reduced differential equations and thus allows for a more efficient presentation of essential submodels. Furthermore, for each of the reduced submodels, the Lie symmetries should again be computed since at this stage one may find symmetries that are not induced by the symmetries of the original differential equations (hidden symmetries). This was the case in a number of the reductions carried out in this part. These additional symmetries can be used for further Lie reductions and often help to extend the list of exact solutions of a given model. This is especially true for the reduction of a nonlinear differential equation to a linear one as there always arise the hidden symmetries corresponding to the linear superposition principle. While these symmetries for themselves cannot be used for further reduction (but they can in combination with nontrivial symmetries as shown in Section 5.10), they allow one to linearly combine arbitrary solutions of the reduced linear differential equation, which might lead to rather different solutions of the original nonlinear differential equation.

Although group-invariant solutions are often the only ones that can be obtained systematically for a given differential equation, it can be frequently observed that a great majority of these solutions has no obvious physical importance. It should also be noted that a number of solutions computed in this part is unphysical or only of secondary interest. In our case, the main problem arises with the specific boundaries in the atmospheric sciences, such as those associated to a channel flow. They at once render it impossible to realize the various polynomial solutions obtained before on the entire domain of the fluid. The problem with specific boundaries can be avoided by investigating invariant boundary value problems as discussed, e.g., in the textbook [23] and applied to the baroclinic two-layer problem in Section 5.8. However, we do not necessarily favor this approach. Firstly, it usually greatly restricts the number of admitted symmetries and hence the possibility to carry out group-invariant reduction. Even if solutions of differential equations might not be physically meaningful they are nevertheless of pure mathematical value. In addition, they still can be used as benchmark tests for numerical discretization schemes where all kinds of boundary conditions can be implemented. This can be an essential monitoring in the development of such numerical codes. Secondly, it is not impossible that although some solution might violate the boundary conditions they could still be locally realizable, i.e. far away from the boundaries. Atmospheric flow is extremely complex and not all of its features are yet completely understood. With Rossby waves and Rossby–Haurwitz waves we have re-derived two very important exact solutions of two-dimensional incompressible geophysical fluid mechanics using symmetry methods. There is no argument, why these solutions should be the only ones obtainable using the Lie symmetry approach. Indeed, there are numerous examples of invariant and partially invariant solutions of physical interest in hydrodynamics, see e.g. the solutions discussed in the textbook [6]. It thus could be a potentially interesting task to evaluate real atmospheric data, searching for specific pattern that resemble some of the solutions derived in this part. Of course this is not a simple problem. It is left open for now but might be investigated in some later study.

It should be stressed that the previous paragraph might appear overly negative. Indeed, symmetry methods have already proved their importance in geophysical fluid dynamics for a long time. In fact, various scaling laws can be obtained by symmetry reductions using scale symmetries.[1] In dynamic meteorology, it is notable to mention the logarithmic wind law in the planetary boundary layer in this context. Other scaling laws have been recently derived in [114] for specific problems of hydrodynamics using Lie symmetries. In contrast to the rather intuitive finding of scaling laws and exact solutions, which prevailed in meteorology over almost the past hundred years, the classical Lie symmetry method has the great benefit to provide a unifying and exhaustive way for obtaining all the invariant solutions of a differential equation. This systematic and universal nature of the Lie symmetry approach, together with its reported

[1] For a review on scaling laws in connection with Lie symmetries, see the sections on the Pi-Theorem in the textbooks [23, 115].

success in classical hydrodynamics is why we consider it indispensable to further apply symmetry methods and extend their range of applicability in the atmospheric sciences. Two further fields in this direction are presented in the second and in the third part of the thesis. Hopefully other fields will follow in the future.

Chapter 7

Supplementary material

In the thesis, we have presented several optimal lists of one- and two-dimensional inequivalent subalgebras. While the computation of one-dimensional subalgebras is relatively straightforward for the Lie algebras we considered in the thesis, the classification of two-dimensional subalgebras takes some greater effort. In order to easily allow a verification of the calculations in the previous chapters, we subsequently present two examples of classifications of two-dimensional subalgebras. We detail the computations of the respective optimal lists of the barotropic potential vorticity equation and the barotropic vorticity equation on the β-plane. The first equation admits an finite-dimensional Lie invariance algebra while the second equation possesses an infinite-dimensional Lie invariance algebra. Since the classification of subalgebras of the other equations in this part works in a very similar manner, we omit the corresponding details here.

7.1 Optimal list of subalgebras of the barotropic potential vorticity equation

The classification of inequivalent one-dimensional subalgebras of the barotropic potential vorticity equation was already given in Section 2.3.1. Since in the thesis we aimed to at most reduce the given partial differential equations in three independent variables to ordinary differential equations, for the potential vorticity equation we in addition only have to construct the optimal list of two-dimensional subalgebras.

The general strategy for the classification of higher-dimensional subalgebras is similar to the classification of one-dimensional subalgebras as described in [115]. The additional restriction is that the linear span of each resulting set of operators must be closed under commutation. Thus, besides simplification by means of the adjoint actions, also closure of the Lie algebra structure has to be assured.

The maximal Lie invariance algebra of the barotropic potential vorticity equation (2.1) reads

$$\mathcal{D} = t\partial_t - \psi\partial_\psi, \quad \mathbf{v}_r = -y\partial_x + x\partial_y, \quad \mathbf{v}_t = \partial_t, \quad \mathbf{v}_x = \partial_x, \quad \mathbf{v}_y = \partial_y, \quad \mathbf{v}_\psi = \partial_\psi.$$

The nonzero commutation relations are

$$[\mathbf{v}_t, \mathcal{D}] = \mathbf{v}_t, \quad [\mathbf{v}_\psi, \mathcal{D}] = -\mathbf{v}_\psi, \quad [\mathbf{v}_x, \mathbf{v}_r] = \mathbf{v}_y, \quad [\mathbf{v}_y, \mathbf{v}_r] = -\mathbf{v}_x.$$

The nonidentical adjoint actions involving basis elements are exhausted by the followings:

$$\mathrm{Ad}(e^{\varepsilon\mathbf{v}_t})\mathcal{D} = \mathcal{D} - \varepsilon\mathbf{v}_t, \qquad \mathrm{Ad}(e^{\varepsilon\mathbf{v}_\psi})\mathcal{D} = \mathcal{D} + \varepsilon\mathbf{v}_\psi,$$

$$\begin{aligned}
&\mathrm{Ad}(e^{\varepsilon\mathcal{D}})\mathbf{v}_t = e^{\varepsilon}\mathbf{v}_t, && \mathrm{Ad}(e^{\varepsilon\mathcal{D}})\mathbf{v}_\psi = e^{-\varepsilon}\mathbf{v}_\psi,\\
&\mathrm{Ad}(e^{\varepsilon\mathbf{v}_x})\mathbf{v}_r = \mathbf{v}_r - \varepsilon\mathbf{v}_y, && \mathrm{Ad}(e^{\varepsilon\mathbf{v}_y})\mathbf{v}_r = \mathbf{v}_r + \varepsilon\mathbf{v}_x,\\
&\mathrm{Ad}(e^{\varepsilon\mathbf{v}_r})\mathbf{v}_x = \mathbf{v}_x\cos\varepsilon + \mathbf{v}_y\sin\varepsilon, && \mathrm{Ad}(e^{\varepsilon\mathbf{v}_r})\mathbf{v}_y = -\mathbf{v}_x\sin\varepsilon + \mathbf{v}_y\cos\varepsilon.
\end{aligned}$$

The starting point of our investigations are two linearly independent copies of the most general infinitesimal generator,

$$\begin{aligned}
\mathbf{v}^1 &= a^1_D\mathcal{D} + a^1_r\mathbf{v}_r + a^1_t\mathbf{v}_t + a^1_x\mathbf{v}_x + a^1_y\mathbf{v}_y + a^1_\psi\mathbf{v}_\psi,\\
\mathbf{v}^2 &= a^2_D\mathcal{D} + a^2_r\mathbf{v}_r + a^2_t\mathbf{v}_t + a^2_x\mathbf{v}_x + a^2_y\mathbf{v}_y + a^2_\psi\mathbf{v}_\psi,
\end{aligned} \tag{7.1}$$

where $a^i_\mu \in \mathbb{R}$, $i = 1, 2$ and $\mu \in \{D, r, t, x, y, \psi\}$. The final purpose of the classification is a list of inequivalent subalgebras that are maximally simplified due to an application of the above adjoint actions.

Remark 7.1. In the classification of subalgebras, it is necessary to determine an appropriate ordering of the single basis elements according to their importance for the algebra. The ordering of operators was already suggested in Section 2.3.1, but without giving a precise explanations for the choice made. In the present case, the ordering is suggested by the solvable structure of the maximal Lie invariance algebra (2.3). The presence of the operator $\mathcal{D}$ in a subalgebra allows to apply a number of adjoint actions which lead to simplifications in the respective algebra (compare with the list of one-dimensional inequivalent subalgebras in Section 2.3.1 and with the two-dimensional inequivalent subalgebras in Section 2.3.2 and below). Similar arguments also hold for the operator $\mathbf{v}_r$. This justifies their respective ranks as the most principal operators. Moreover, for the present purpose we aim to use the inequivalent subalgebras solely for the sake of classical Lie reduction. Since reductions using algebras involving either $\mathbf{v}_D$ or $\mathbf{v}_t$ have a similar degree of complexity, we rank $\mathbf{v}_t$ as the third operator and before $\mathbf{v}_x$ and $\mathbf{v}_y$. Another argument is the following: In the classification of subalgebras it is necessary to make vanishing and non-vanishing assumptions on the coefficients of basis elements of the maximal Lie invariance algebra. These assumptions should be preserved under any adjoint action used in the course of classification. However, due to the adjoint action $\mathrm{Ad}(e^{\varepsilon\mathbf{v}_r})$, vanishing or non-vanishing conditions on a^i_x and a^i_y are not preserved. This is why it is beneficial to rank $\mathbf{v}_x$ and $\mathbf{v}_y$ rather in the end of $\mathbf{v}$. Finally, although from the algebraic point of view the operators $\mathbf{v}_t$ and $\mathbf{v}_\psi$ have equal importance, from the point of view of reduction, the operator $\mathbf{v}_t$ is more important. This is true, since the single operator $\mathbf{v}_\psi$ cannot be used for the purpose of Lie reduction. This is also the reason why we list $\mathbf{v}_x$ and $\mathbf{v}_y$ before $\mathbf{v}_\psi$.

In the subsequent part, we aim to use the following notation to shorten the resulting expressions in the cases arising under classification of two-dimensional subalgebras,

$$A_{\mu_1\cdots\mu_n} := \begin{pmatrix} a^1_{\mu_1} & \cdots & a^1_{\mu_n} \\ a^2_{\mu_1} & \cdots & a^2_{\mu_n} \end{pmatrix}, \tag{7.2}$$

where $\mu_i \in \{D, r, t, x, y, \psi\}$ and $n < 6$, is the $2 \times n$ matrix consisting of the coefficients of the respective basis elements in the generators $\mathbf{v}^1$ and $\mathbf{v}^2$. Correspondingly, the right-hand side in vanishing conditions of the form $A_{\mu_1\cdots\mu_n} = 0$ should be understand as a zero matrix of appropriate dimension.

1. $A_D \neq 0$. This is the first main assumption with which we start. Depending on the ranks of the matrices $A_{D\cdots\mu_n}$ and $A_{D\mu_{n+1}}$, we obtain several inequivalent cases of two-dimensional subalgebras that are exhaustively presented below.

   (a) $\det A_{Dr} \neq 0$. In this case, by a change of the basis we can set $A_{Dr} = \mathbb{I}$, where $\mathbb{I}$ is the identity matrix. By applying the adjoint actions $\mathrm{Ad}(e^{\varepsilon \mathbf{v}_t})$, $\mathrm{Ad}(e^{\varepsilon \mathbf{v}_\psi})$, $\mathrm{Ad}(e^{\varepsilon \mathbf{v}_x})$ and $\mathrm{Ad}(e^{\varepsilon \mathbf{v}_y})$, the two generators in (7.1) are simplified to[1]

$$\mathbf{v}^1 = \mathcal{D} + a_x^1 \mathbf{v}_x + a_y^1 \mathbf{v}_y,$$
$$\mathbf{v}^2 = \mathbf{v}_r + a_t^2 \mathbf{v}_t + a_\psi^2 \mathbf{v}_\psi.$$

   It is now necessary to assure closure of the algebra generated by $\mathbf{v}^1$ and $\mathbf{v}^2$. That is, we must have $[\mathbf{v}^1, \mathbf{v}^2] = a\mathbf{v}^1 + b\mathbf{v}^2$, where $a, b \in \mathbb{R}$. Commuting the two vector fields we obtain

$$[\mathbf{v}^1, \mathbf{v}^2] = -a_t^2 \mathbf{v}_t + a_\psi^2 \mathbf{v}_\psi + a_x^1 \mathbf{v}_y - a_y^1 \mathbf{v}_x = a\mathbf{v}^1 + b\mathbf{v}^2.$$

   This relation holds true only in case of $a = b = 0$, from which we conclude $a_t^2 = a_\psi^2 = a_x^1 = a_y^1 = 0$. Hence, the representative of the class of equivalent subalgebras with the assumption that $\det A_{Dr} \neq 0$ is $\langle \mathcal{D}, \mathbf{v}_r \rangle$.

   (b) $\det A_{Dr} = 0$, $\det A_{Dt} \neq 0$. These assumptions allow us to set $A_{Dt} = \mathbb{I}$ upon changing the basis. Applying the adjoint action $\mathrm{Ad}(e^{\varepsilon \mathbf{v}_\psi})$ to the resulting generators $\mathbf{v}^1$ and $\mathbf{v}^2$ in (7.1), we obtain

$$\begin{aligned} \mathbf{v}^1 &= \mathcal{D} + a_r^1 \mathbf{v}_r + a_x^1 \mathbf{v}_x + a_y^1 \mathbf{v}_y, \\ \mathbf{v}^2 &= \mathbf{v}_t + a_x^2 \mathbf{v}_x + a_y^2 \mathbf{v}_y + a_\psi^2 \mathbf{v}_\psi. \end{aligned} \tag{7.3}$$

   It follows from (7.3) and the condition that $[\mathbf{v}^1, \mathbf{v}^2] \in \langle \mathbf{v}^1, \mathbf{v}^2 \rangle$ that the commutation relation must be $[\mathbf{v}^1, \mathbf{v}^2] = -\mathbf{v}^2$. Once more, two cases must be distinguished.

      i. $a_r^1 \neq 0$. By using the adjoint actions $\mathrm{Ad}(e^{\varepsilon \mathbf{v}_x})$ and $\mathrm{Ad}(e^{\varepsilon \mathbf{v}_y})$, the operators (7.3) are simplified to

$$\mathbf{v}^1 = \mathcal{D} + a_r^1 \mathbf{v}_r,$$
$$\mathbf{v}^2 = \mathbf{v}_t + a_x^2 \mathbf{v}_x + a_y^2 \mathbf{v}_y + a_\psi^2 \mathbf{v}_\psi.$$

      Commuting these two basis operators, we obtain

$$[\mathbf{v}^1, \mathbf{v}^2] = -\mathbf{v}_t + a_\psi^2 \mathbf{v}_\psi - a_r^1 a_x^2 \mathbf{v}_y + a_r^1 a_y^2 \mathbf{v}_x = -\mathbf{v}^2,$$

      which holds provided that $a_\psi^2 = a_x^2 = a_y^2 = 0$. The resulting subalgebra is $\langle \mathcal{D} + a_r^1 \mathbf{v}_r, \mathbf{v}_t \rangle$.

      ii. $a_r^1 = 0$. We can use the adjoint action $\mathrm{Ad}(e^{\varepsilon \mathbf{v}_r})$ to set $a_y^1 = 0$. Commutation of the resulting operators (7.3) yields

$$[\mathbf{v}^1, \mathbf{v}^2] = -\mathbf{v}_t + a_\psi^2 \mathbf{v}_\psi = -\mathbf{v}^2,$$

      from which we conclude $a_x^2 = a_y^2 = a_\psi^2 = 0$. The corresponding subalgebra then reads $\langle \mathcal{D} + a_x^1 \mathbf{v}_x, \mathbf{v}_t \rangle$.

[1] In principle, the coefficients of the basis elements in these two generators are not the same as before applying the adjoint actions. To not overly confuse notation we have nevertheless decided to always use the same symbols for these coefficients in the course of the calculations.

(c) $\text{rank}(A_{Drt}) = 1$, $\text{rank}(A_{Dxy}) = 2$. Using changes of basis and adjoint action $\text{Ad}(e^{\varepsilon \mathbf{v}_r})$ we can obtain $A_{Dx} = \mathbb{I}$ and $a_y^2 = 0$. By acting on the resulting operators with the adjoint actions $\text{Ad}(e^{\varepsilon \mathbf{v}_t})$ and $\text{Ad}(e^{\varepsilon \mathbf{v}_\psi})$ this simplifies (7.1) to

$$\mathbf{v}^1 = \mathcal{D} + a_r^1 \mathbf{v}_r + a_y^1 \mathbf{v}_y,$$
$$\mathbf{v}^2 = \mathbf{v}_x + a_\psi^2 \mathbf{v}_\psi.$$

Commutation of these basis elements gives

$$[\mathbf{v}^1, \mathbf{v}^2] = -a_\psi^2 \mathbf{v}_\psi - a_r^1 \mathbf{v}_y = a\mathbf{v}^1 + b\mathbf{v}^2,$$

and thus $a_r^1 = a_\psi^2 = 0$. The associated two-dimensional Lie subalgebra is $\langle \mathcal{D} + a_y^1 \mathbf{v}_y, \mathbf{v}_x \rangle$.

(d) $\text{rank}(A_{Drtxy}) = 1$, $\det A_{D\psi} \neq 0$. In this case we can assume that $A_{D\psi} = \mathbb{I}$ by linearly combining $\mathbf{v}^1$ and $\mathbf{v}^2$. Further simplification is possible by applying the adjoint operator $\text{Ad}(e^{\varepsilon \mathbf{v}_t})$, which allows to set $a_t^1 = 0$. The simplified generators then are

$$\begin{aligned} \mathbf{v}^1 &= \mathcal{D} + a_r^1 \mathbf{v}_r + a_x^1 \mathbf{v}_x + a_y^1 \mathbf{v}_y, \\ \mathbf{v}^2 &= \mathbf{v}_\psi. \end{aligned} \tag{7.4}$$

Again, splitting into two cases is necessary.

i. $a_r^1 \neq 0$. In this case, we can act on (7.4) by $\text{Ad}(e^{\varepsilon \mathbf{v}_x})$ and $\text{Ad}(e^{\varepsilon \mathbf{v}_y})$, which allows us to set $a_x^1 = a_y^1 = 0$. Commutation places no further restriction on a_r^1 and hence the resulting subalgebra is $\langle \mathcal{D} + a_r^1 \mathbf{v}_r, \mathbf{v}_\psi \rangle$.

ii. $a_r^1 = 0$. We can set $a_y^1 = 0$ upon acting on by the adjoint action $\text{Ad}(e^{\varepsilon \mathbf{v}_r})$. Commuting the resulting generators yields zero and hence does not restrict a_x^1. The associated subalgebra reads $\langle \mathcal{D} + a_x^1 \mathbf{v}_x, \mathbf{v}_\psi \rangle$.

2. $A_D = 0$, $A_r \neq 0$. There follow three different inequivalent subalgebras from these assumptions.

(a) $\det A_{rt} \neq 0$. By changing the basis and applying the adjoint actions $\text{Ad}(e^{\varepsilon \mathbf{v}_x})$ and $\text{Ad}(e^{\varepsilon \mathbf{v}_y})$ we arrive at

$$\mathbf{v}^1 = \mathbf{v}_r + a_\psi^1 \mathbf{v}_\psi,$$
$$\mathbf{v}^2 = \mathbf{v}_t + a_x^2 \mathbf{v}_x + a_y^2 \mathbf{v}_y + a_\psi^2 \mathbf{v}_\psi.$$

Taking the commutator of $\mathbf{v}^1$ and $\mathbf{v}^2$ gives

$$[\mathbf{v}^1, \mathbf{v}^2] = -a_x^2 \mathbf{v}_y + a_y^2 \mathbf{v}_x \in \langle \mathbf{v}^1, \mathbf{v}^2 \rangle,$$

and thus $a_x^2 = a_y^2 = 0$. By means of the adjoint operator $\text{Ad}(e^{\varepsilon \mathcal{D}})$ we can scale $a_\psi^1 \in \{-1, 0, 1\}$, or in the case of $a_\psi^1 = 0$ we can scale $a_\psi^2 \in \{-1, 0, 1\}$. The corresponding subalgebra hence is $\langle \mathbf{v}_r + a_\psi^1 \mathbf{v}_\psi, \mathbf{v}_t + a_\psi^2 \mathbf{v}_\psi \rangle$, with scalings of a_ψ^1 and a_ψ^2, respectively, as described before.

(b) $\det A_{rt} = 0$, $\text{rank}(A_{rxy}) = 2$. The restriction that $[\mathbf{v}^1, \mathbf{v}^2] \in \langle \mathbf{v}^1, \mathbf{v}^2 \rangle$ is not satisfied for this case. The initial assumptions therefore lead to a contradiction.

(c) $\operatorname{rank}(A_{rtxy}) = 1$, $\det A_{r\psi} \neq 0$. Changing the basis and using the adjoint actions $\operatorname{Ad}(e^{\varepsilon \mathbf{v}_x})$ and $\operatorname{Ad}(e^{\varepsilon \mathbf{v}_y})$ we obtain

$$\mathbf{v}^1 = \mathbf{v}_r + a_t^1 \mathbf{v}_t,$$
$$\mathbf{v}^2 = \mathbf{v}_\psi,$$

which form an Abelian subalgebra. Acting on these generators by $\operatorname{Ad}(e^{\varepsilon \mathcal{D}})$ we get the subalgebra $\langle \mathbf{v}_r + a_t^1 \mathbf{v}_t, \mathbf{v}_\psi \rangle$, where $a_t^1 \in \{-1, 0, 1\}$.

3. $A_D = A_r = 0$, $A_t \neq 0$. The splitting into two subcases is once more necessary.

(a) $\operatorname{rank}(A_{txy}) = 2$. Simultaneously changing the basis and using the adjoint action $\operatorname{Ad}(e^{\varepsilon \mathbf{v}_r})$, we obtain

$$\mathbf{v}^1 = \mathbf{v}_t + a_y^1 \mathbf{v}_y + a_\psi^1 \mathbf{v}_\psi,$$
$$\mathbf{v}^2 = \mathbf{v}_x + a_\psi^2 \mathbf{v}_\psi.$$

Since this subalgebra is again Abelian, commuting $\mathbf{v}^1$ and $\mathbf{v}^2$ places no restrictions on the coefficients a_y^1, a_ψ^1 and a_ψ^2. However, using $\operatorname{Ad}(e^{\varepsilon \mathcal{D}})$ we can scale $a_\psi^1 \in \{-1, 0, 1\}$, or, in case of $a_\psi^1 = 0$, we can scale $a_\psi^2 \in \{-1, 0, 1\}$ or $a_y^1 \in \{-1, 0, 1\}$. The resulting subalgebra is $\langle \mathbf{v}_t + a_y^1 \mathbf{v}_y + a_\psi^1 \mathbf{v}_\psi, \mathbf{v}_x + a_\psi^2 \mathbf{v}_\psi \rangle$.

(b) $\operatorname{rank}(A_{txy}) = 1$, $\det A_{t\psi} \neq 0$. Changing the basis and acting on $\mathbf{v}^1$ and $\mathbf{v}^2$ by $\operatorname{Ad}(e^{\varepsilon \mathbf{v}_r})$ results in

$$\mathbf{v}^1 = \mathbf{v}_t + a_x^1 \mathbf{v}_x,$$
$$\mathbf{v}^2 = \mathbf{v}_\psi.$$

Due to the vanishing commutator of $\mathbf{v}^1$ and $\mathbf{v}^2$, no further restrictions are placed on a_x^1. The resulting algebra therefore is $\langle \mathbf{v}_t + a_x^1 \mathbf{v}_x, \mathbf{v}_\psi \rangle$. The action of $\operatorname{Ad}(e^{\varepsilon \mathcal{D}})$ again allows scalings $a_x^1 \in \{-1, 0, 1\}$.

4. $A_D = A_r = A_t = 0$, $A_{xy} \neq 0$. For a complete consideration of this case, splitting into two subcases is needed.

(a) $\det A_{xy} \neq 0$. Changing the basis, commuting $\mathbf{v}^1$ and $\mathbf{v}^2$ and finally employing $\operatorname{Ad}(e^{\varepsilon \mathcal{D}})$, we obtain the subalgebra $\langle \mathbf{v}_x + a_\psi^1 \mathbf{v}_\psi, \mathbf{v}_y + a_\psi^2 \mathbf{v}_\psi \rangle$, where $a_\psi^1 \in \{-1, 0, 1\}$, or, $a_\psi^2 \in \{-1, 0, 1\}$ in the case of $a_\psi^1 = 0$.

(b) $\det A_{xy} = 0$, $\det A_{x\psi} \neq 0$. Again changing the basis and using $\operatorname{Ad}(e^{\varepsilon \mathbf{v}_r})$, we recover the subalgebra $\langle \mathbf{v}_x, \mathbf{v}_\psi \rangle$.

This yields all the inequivalent two-dimensional subalgebras of the maximal Lie invariance algebra of the barotropic potential vorticity equation listed in Section 2.3.2.

7.2 Optimal list of subalgebras of the barotropic vorticity equation on the β-plane

The classification of an optimal list of one-dimensional subalgebras was already presented in our master thesis [15]. It hence again only remains to detail the calculations of the two-dimensional inequivalent subalgebras. We once more present the maximal Lie invariance algebra of the vorticity equation on the β-plane:

$$\mathcal{D} = t\partial_t - x\partial_x - y\partial_y - 3\psi\partial_\psi, \quad \partial_t, \quad \partial_y, \quad \mathcal{X}(f) = f(t)\partial_x - f'(t)y\partial_\psi, \quad \mathcal{Z}(g) = g(t)\partial_\psi.$$

To perform classification up to equivalence, we again need the nonidentical adjoint actions:

$$\begin{aligned}
&\mathrm{Ad}(e^{\varepsilon\partial_t})\mathcal{D} = \mathcal{D} - \varepsilon\partial_t, && \mathrm{Ad}(e^{\varepsilon\mathcal{X}(f)})\mathcal{D} = \mathcal{D} + \varepsilon\mathcal{X}(tf' + f),\\
&\mathrm{Ad}(e^{\varepsilon\partial_y})\mathcal{D} = \mathcal{D} + \varepsilon\partial_y, && \mathrm{Ad}(e^{\varepsilon\mathcal{Z}(g)})\mathcal{D} = \mathcal{D} + \varepsilon\mathcal{Z}(tg' + 3g),\\
&\mathrm{Ad}(e^{\varepsilon\mathcal{D}})\partial_t = e^{\varepsilon}\partial_t, && \mathrm{Ad}(e^{\varepsilon\mathcal{Z}(g)})\partial_t = \partial_t + \varepsilon\mathcal{Z}(g'),\\
&\mathrm{Ad}(e^{\varepsilon\mathcal{D}})\partial_y = e^{-\varepsilon}\partial_y, && \mathrm{Ad}(e^{\varepsilon\mathcal{X}(f)})\partial_t = \partial_t + \varepsilon\mathcal{X}(f'),\\
&\mathrm{Ad}(e^{\varepsilon\mathcal{X}(f)})\partial_y = \partial_y - \varepsilon\mathcal{Z}(f'), && \mathrm{Ad}(e^{\varepsilon\partial_y})\mathcal{X}(f) = \mathcal{X}(f) + \varepsilon\mathcal{Z}(f'),\\
&\mathrm{Ad}(e^{\varepsilon\mathcal{D}})\mathcal{X}(f) = \mathcal{X}(\tilde f), \quad \tilde f = e^{-\varepsilon}f(e^{-\varepsilon}t), && \mathrm{Ad}(e^{\varepsilon\partial_t})\mathcal{X}(f) = \mathcal{X}(\tilde f), \quad \tilde f = f(t-\varepsilon),\\
&\mathrm{Ad}(e^{\varepsilon\mathcal{D}})\mathcal{Z}(g) = \mathcal{Z}(\tilde g), \quad \tilde g = e^{-3\varepsilon}g(e^{-\varepsilon}t), && \mathrm{Ad}(e^{\varepsilon\partial_t})\mathcal{Z}(g) = \mathcal{Z}(\tilde g), \quad \tilde g = g(t-\varepsilon).
\end{aligned}$$

As before, we start with two linearly independent infinitesimal generators of the most general form

$$\begin{aligned}
\mathbf{v}^1 &= a^1_D\mathcal{D} + a^1_t\partial_t + a^1_y\partial_y + \mathcal{X}(f^1) + \mathcal{Z}(g^1),\\
\mathbf{v}^2 &= a^2_D\mathcal{D} + a^2_t\partial_t + a^2_y\partial_y + \mathcal{X}(f^2) + \mathcal{Z}(g^2),
\end{aligned} \tag{7.5}$$

where a^i_μ, $i = 1, 2$ and $\mu \in \{D, t, y\}$ are arbitrary real-valued constants and f^i, g^i, $i = 1, 2$ are arbitrary time-dependent functions.

Remark 7.2. The ordering of vector fields in $\mathbf{v}^1$ and $\mathbf{v}^2$ chosen above can again be explained with the solvable structure of the maximal Lie invariance algebra of the barotropic vorticity equation. The operator $\mathcal{D}$ acts on all other operators and is thus the most principal one (for using adjoint actions). The ordering of the remaining operators can be justified with a similar argument. This is well-agreed with the elements of the derived series, $\mathfrak{g}'$, $\mathfrak{g}''$ and $\mathfrak{g}'''$, given in Section 4.3. For the sake of Lie reduction, $\mathcal{Z}(g)$ is the least important operator, as it can only be used to derive partially invariant solutions using the splitting of the vorticity equation as discussed in Section 3.2.6.

The classification procedure is similar to those in the previous section. In particular, we aim to use the matrix notation (7.2). The main difference is that we now have to deal with infinite-dimensional Lie algebras. However, as is shown below, the main strategy of classification does not change substantially.

1. $A_D \neq 0$. We now investigate the different inequivalent subalgebras corresponding to this initial assumption. As in the section before, it is necessary to consider a couple of subcases.

   (a) $\det A_{Dt} \neq 0$. The change of basis allows us to put $A_{Dt} = \mathbb{I}$. Applying the adjoint actions $\mathrm{Ad}(e^{\varepsilon\partial_y})$, $\mathrm{Ad}(e^{\varepsilon\mathcal{X}(f)})$ and $\mathrm{Ad}(e^{\varepsilon\mathcal{Z}(g)})$, the set of operators (7.5) is simplified to

$$\mathbf{v}^1 = \mathcal{D} + \mathcal{X}(f^1) + \mathcal{Z}(g^1),$$

$$\mathbf{v}^2 = \partial_t + a_y^2\partial_y.$$

Commuting $\mathbf{v}^1$ and $\mathbf{v}^2$ leads to

$$[\mathbf{v}^1, \mathbf{v}^2] = -\partial_t + a_y^2\partial_y - \mathcal{X}(f_t^1) - \mathcal{Z}(g_t^1) + a_y^2\mathcal{Z}(f_t^1) \in \langle \mathbf{v}^1, \mathbf{v}^2\rangle.$$

This forms a genuine Lie subalgebra if and only if $a_y^2 = 0$, $f^1 = \text{const}$ and $g^1 = \text{const}$. Since f^1 and g^1 are constants, we can act on the resulting algebra by $\mathrm{Ad}(e^{\varepsilon\mathcal{X}(f)})$ and $\mathrm{Ad}(e^{\varepsilon\mathcal{Z}(g)})$ where f and g are suitable chosen constants allowing to set $f^1 = g^1 = 0$. Note that acting on the algebra by such operators has no effect on $\mathbf{v}^2$ and correspondingly does not complicate its form. Accordingly, the resulting subalgebra is $\langle \mathcal{D}, \partial_t\rangle$.

(b) $\det A_{Dt} = 0$, $\det A_{Dy} \neq 0$. Up to a change of the basis, it is possible to cast the associated operators $\mathbf{v}^1$ and $\mathbf{v}^2$ into the form with $A_{Dy} = \mathbb{I}$. The adjoint actions $\mathrm{Ad}(e^{\varepsilon\partial_t})$, $\mathrm{Ad}(e^{\varepsilon\mathcal{X}(f)})$ and $\mathrm{Ad}(e^{\varepsilon\mathcal{Z}(g)})$ simplify the corresponding version of (7.5) to

$$\mathbf{v}^1 = \mathcal{D},$$
$$\mathbf{v}^2 = \partial_y + \mathcal{X}(f^2) + \mathcal{Z}(g^2).$$

The commutation of $\mathbf{v}^1$ and $\mathbf{v}^2$ yields

$$[\mathbf{v}^1, \mathbf{v}^2] = \partial_y + \mathcal{X}(tf_t^2 + f^2) + \mathcal{Z}(tg_t^2 + 3g^2) = a\mathbf{v}^1 + b\mathbf{v}^2.$$

From these two equalities, we can conclude that $a = 0$, $b = 1$ and therefore $f_t^2 = 0$ or $f^2 = c = \text{const}$. In addition, g^2 must be a solution of the differential equation $tg_t^2 + 2g^2 = 0$, which is solved by $g^2 \propto t^{-2}$. By using the adjoint action $\mathrm{Ad}(e^{\varepsilon\mathcal{X}(f)})$ with $f \propto t^{-1}$ we can obtain $g^2 = 0$. Moreover, this adjoint action acts identically on $\mathbf{v}^1$. The subalgebra following from these assumptions thus is $\langle \mathcal{D}, \partial_y + \mathcal{X}(c)\rangle$.

(c) $\mathrm{rank}(A_{Dty}) = 1$, the tuples (a_D^1, f^1) and (a_D^2, f^2) are linearly independent. Since the last tuples are linearly independent it is possible to cast them into the form $(1, f^1)$ and $(0, f^2)$, respectively, where $f^2 \neq 0$. The adjoint actions $\mathrm{Ad}(e^{\varepsilon\partial_t})$, $\mathrm{Ad}(e^{\varepsilon\partial_y})$, $\mathrm{Ad}(e^{\varepsilon\mathcal{X}(f)})$ and $\mathrm{Ad}(e^{\varepsilon\mathcal{Z}(g)})$ then lead to the following simplifications of (7.5):

$$\mathbf{v}^1 = \mathcal{D},$$
$$\mathbf{v}^2 = \mathcal{X}(f^2) + \mathcal{Z}(g^2).$$

Under commutation we find

$$[\mathbf{v}^1, \mathbf{v}^2] = \mathcal{X}(tf_t^2 + f^2) + \mathcal{Z}(tg^2 + 3g^2) \in \langle \mathbf{v}^1, \mathbf{v}^2\rangle.$$

It first of all follows that $[\mathbf{v}^1, \mathbf{v}^2] \in \langle \mathbf{v}^2\rangle$. Furthermore, this commutation relation places restrictions on f^2 and g^2 as they have to satisfy the ordinary differential equations $tf_t^2 + f^2 = (a+1)f^2$ and $tg_t^2 + 3g^2 = (a+1)g^2$, where $a = \text{const}$. Solving these differential equations leads to the third algebra from the list of two-dimensional subalgebras in Section 3.2.3, $\langle \mathcal{D}, \mathcal{X}(|t|^a) + c\mathcal{Z}(|t|^{a-2})\rangle$.

(d) $\mathrm{rank}\,(A_{Dty}) = 1$, the tuples (a_D^1, f^1) and (a_D^2, f^2) are linearly dependent, the tuples (a_D^1, g^1) and (a_D^2, g^2) are linearly independent. This is the final case to be treated under the initial assumption $A_D \neq 0$. The triples (a_D^i, f^i, g^i), i=1,2, can then be brought into the form $a_D^1 = 1$, $a_D^2 = 0$, $f^2 = 0$ and $g^2 \neq 0$. Using the adjoint actions $\mathrm{Ad}(e^{\varepsilon\partial_t})$, $\mathrm{Ad}(e^{\varepsilon\partial_y})$, $\mathrm{Ad}(e^{\varepsilon\mathcal{X}(f)})$ and $\mathrm{Ad}(e^{\varepsilon\mathcal{Z}(g)})$ we obtain

$$\mathbf{v}^1 = \mathcal{D},$$
$$\mathbf{v}^2 = \mathcal{Z}(g^2).$$

The commutation relation places the same restriction on g^2 as before, so that we get the subalgebra $\langle\mathcal{D}, \mathcal{Z}(|t|^{a-2})\rangle$.

2. $A_D = 0$, $A_t \neq 0$. The subcases listed subsequently serve as an exhaustive investigation of the subalgebras following from these assumptions.

   (a) $\det A_{ty} \neq 0$. This case implies that it is possible to set $A_{ty} = \mathbb{I}$. Moreover, the adjoint actions $\mathrm{Ad}(e^{\varepsilon\mathcal{X}(f)})$ and $\mathrm{Ad}(e^{\varepsilon\mathcal{Z}(g)})$ allow to simplify the generators in the corresponding form of (7.5) to

   $$\mathbf{v}^1 = \partial_t,$$
   $$\mathbf{v}^2 = \partial_y + \mathcal{X}(f^2) + \mathcal{Z}(g^2).$$

   Taking the commutator gives

   $$[\mathbf{v}^1, \mathbf{v}^2] = \mathcal{X}(f_t^2) + \mathcal{Z}(g_t^2),$$

   which belongs to the linear span of $\mathbf{v}^1$ and $\mathbf{v}^2$ provided that $f^2 = a = \text{const}$ and $g^2 = b = \text{const}$. The associated two-dimensional subalgebra then is $\langle\partial_t, \partial_y + \mathcal{X}(a) + \mathcal{Z}(b)\rangle$.

   (b) $\det A_{ty} = 0$, the tuples (a_t^1, f^1) and (a_t^2, f^2) are linearly independent. By linearly combining $\mathbf{v}^1$ and $\mathbf{v}^2$ we can set $a_t^1 = 1$, $a_t^2 = a_y^2 = 0$. Again, the adjoint actions $\mathrm{Ad}(e^{\varepsilon\mathcal{X}(f)})$ and $\mathrm{Ad}(e^{\varepsilon\mathcal{Z}(g)})$ permit some simplification of the generator $\mathbf{v}^1$ in (7.5). We find

   $$\mathbf{v}^1 = \partial_t + a_y^1\partial_y,$$
   $$\mathbf{v}^2 = \mathcal{X}(f^2) + \mathcal{Z}(g^2).$$

   As the commutation of $\mathbf{v}^1$ and $\mathbf{v}^2$ should belong to their linear span we have

   $$[\mathbf{v}^1, \mathbf{v}^2] = \mathcal{X}(f_t^2) + \mathcal{Z}(g_t^2) - a_y^1\mathcal{Z}(f_t^2) = a\mathcal{X}(f^2) + a\mathcal{Z}(g^2).$$

   That is, the functions f^2 and g^2 satisfy the system of differential equations $f_t^2 = af^2$ and $g_t^2 = ag^2 + a_y^1 f_t^2$. Solving this system leads to the subalgebra $\langle\partial_t + b\partial_y, \mathcal{X}(e^{at}) + \mathcal{Z}((abt + c)e^{at})\rangle$, up to re-denoting $a_y^1 = b$.

   (c) The triples (a_t^1, a_y^1, f^1) and (a_t^2, a_y^2, f^2) are linearly dependent, the tuples (a_t^1, g^1) and (a_t^2, g^2) are linearly independent. This case implies that we can $a_t^1 = 1$, $a_t^2 = a_y^2 = 0$ and $f^2 = 0$. The simplification using $\mathrm{Ad}(e^{\varepsilon\mathcal{X}(f)})$ and $\mathrm{Ad}(e^{\varepsilon\mathcal{Z}(g)})$ is analog as before and therefore leads to

   $$\mathbf{v}^1 = \partial_t + a_y^1\partial_y,$$
   $$\mathbf{v}^2 = \mathcal{Z}(g^2).$$

   The restriction imposed by $[\mathbf{v}^1, \mathbf{v}^2]$ implies that $g^2 \propto e^{at}$ and by once more re-denoting $a_y^1 = b$ we recover the subalgebra $\langle\partial_t + b\partial_y, \mathcal{Z}(e^{at})\rangle$.

3. $A_D = A_t = 0$, $A_y \neq 0$. There follow two inequivalent classes of two-dimensional subalgebras from these assumptions.

   (a) The tuples (a_y^1, f^1) and (a_y^2, f^2) are linearly independent. By recombining, we can set them to be $(1, f^1)$ and $(0, f^2)$, respectively, where $f^2 \neq 0$. The adjoint action $\mathrm{Ad}(e^{\varepsilon \mathcal{X}(f)})$ allows to cancel the operator $\mathcal{Z}(g^1)$ in the first generator. The remaining nonzero parts of $\mathbf{v}^1$ and $\mathbf{v}^2$ in (7.5) read

   $$\mathbf{v}^1 = \partial_y + \mathcal{X}(f^1),$$
   $$\mathbf{v}^2 = \mathcal{X}(f^2) + \mathcal{Z}(g^2).$$

   Under commutation we obtain

   $$[\mathbf{v}^1, \mathbf{v}^2] = -\mathcal{Z}(f_t^2),$$

   from which we conclude that $f^2 = \mathrm{const}$. By rescaling $\mathbf{v}^2$ we can assume that $f^2 = 1$. The corresponding subalgebra is $\langle \partial_y + \mathcal{X}(f^1), \mathcal{X}(1) + \mathcal{Z}(g^2) \rangle$.

   (b) The tuples (a_y^1, f^1) and (a_y^2, f^2) are linearly dependent, the tuples (a_y^1, g^1) and (a_y^2, g^2) are linearly independent. This case leads to the subalgebra $\langle \partial_y + \mathcal{X}(f^1), \mathcal{Z}(g^2) \rangle$ using a change of the basis and the adjoint action $\mathrm{Ad}(e^{\varepsilon \mathcal{X}(f)})$.

4. $A_D = A_t = A_y = 0$. For each tuple of linearly independent functions (f^1, g^1) and (f^2, g^2) the final subalgebra reads $\langle \mathcal{X}(f^1) + \mathcal{Z}(g^1), \mathcal{X}(f^2) + \mathcal{Z}(g^2) \rangle$.

This is the complete classification of two-dimensional inequivalent subalgebras of the barotropic vorticity equation on the β-plane presented in Section 3.2.3.

Remark 7.3. All but the subalgebra $\langle \mathcal{D}, \partial_t \rangle$ in the above classification represent classes of subalgebras rather than single subalgebras. It was already indicated in Section 3.2.3 that changes of the basis and adjoint actions can generate additional equivalences inside the single elements of optimal lists of inequivalent subalgebras. This is also the case for the two-dimensional inequivalent subalgebras of the maximal Lie invariance algebra of the barotropic vorticity equation.

Part II

Symmetries and parameterization schemes

Chapter 8

Symmetry preserving parameterization schemes

Abstract General methods for the design of physical parameterization schemes that preserve prescribed invariance properties of averaged differential equations are proposed. These methods are based on techniques of group classification and provide means to determine expressions for unclosed terms arising in the course of averaging of nonlinear differential equations. For equations where no prescribed form of the unknown terms is given, symmetry subgroups of the original differential equations are used to determine suitable differential invariants that can be used to construct closure schemes for the averaged differential equations. This can be seen as an application of inverse group classification. For equations where a general functional relation between the known and unknown terms can be established, the direct group classification problem for classes of differential equations is solved. This includes the computation of the equivalence algebra and the set of admissible transformations. Ansatzes based on inequivalent subalgebras of the equivalence algebra are used to specify restricted forms of the given functional relation leading to closed equations with different symmetry properties. For classes that are normalized, this approach yields the entire set of possible invariant parameterization schemes. The different methods are exemplified with the barotropic vorticity equation. Parameterizations of the eddy vorticity flux in the averaged vorticity equation are constructed using differential invariants. A general class of parameterizations is proposed for which the equivalence algebra and the set of admissible transformations is determined. This class is shown to be normalized. The kernels of symmetry groups of two restricted classes of equations are computed. For these classes, parameterizations leading to equations admitting up to five-dimensional symmetry group extensions are constructed. The physical importance of these parameterizations is discussed.

8.1 Introduction

The problem of parameterization is one of the most important issues in modern dynamic meteorology and climate research [66, 148]. As even the most accurate present days numerical models are not capable to resolve all small scale features of the atmosphere, there is a necessity for finding ways to incorporate these unresolved processes in terms of the resolved ones. This technique is referred to as parameterization. The physical processes being parameterized in numerical weather and climate prediction models can be quite different, including e.g. cumulus convection, momentum, heat and moisture fluxes, gravity wave drag and vegetation effects. The general problem of parameterization is intimately linked to the design of closure schemes for averaged (or filtered) nonlinear equations. By averaging, a nonlinear differential equation becomes unclosed, that is, there arise additional terms for which no prognostic or diagnostic equation exist. These terms must hence be re-expressed in a physically reasonable way to be included in the averaged equations.

It has been noted in [150] that every parameterization scheme ought to retain some basic properties of the unresolved terms, which must be expressed by the resolved quantities. These properties include, just to mention a few, correct dimensionality, tensorial properties, invariance under changes of the coordinate system and invariance with respect to Galilean transformations. While the formulation of a parameterization scheme with correct dimensions is in general a straightforward task, not all parameterization schemes that have been used in practice are indeed Galilean invariant. An example for this finding is given by the classical Kuo convection scheme [78, 79]. In this scheme, it is assumed that the vertically integrated time-change of the water vapor at a point locally balances a fraction of the observed precipitation rate [38, pp. 528]. This also implies that the moisture convergence is proportional to the precipitation rate. However, while the precipitation rate is clearly a Galilean invariant quantity, the moisture convergence depends on the motion of the observer [70]. That is, the Kuo scheme does not properly account for pure symmetry constraints, which is a potential source of unphysical effects in the results of a numerical model integration.

The latter finding is the main motivation for the present investigations. Galilean invariance is an important example for a Lie symmetry, but it is by no means the only invariance characteristic that might be of importance in the course of the parameterization process. This is why it is reasonable to focus on parameterization schemes that also preserve other symmetries. This is not an academic task. Almost all real-world processes exhibit miscellaneous symmetry characteristics. These characteristics are reflected in the symmetry properties of differential equations and correspondingly should also be reflected in case where these processes cannot be explicitly modeled by differential equations, i.e. in the course of parameterizations. What is hence desirable is a constructive method for the design of symmetry-preserving parameterization schemes. It is the aim of this paper to demonstrate that techniques from group classification do provide such constructive methods. In particular, we state the following proposition:

The problem of finding invariant parameterizations is a group classification problem.

The implications following from the above proposition form the core of the present study. It appears that this issue was first opened in [112], dealing with the problem of turbulence closure of the averaged Navier–Stokes equations. We aim to build on this approach and extend it in several directions. As the equations of hydrodynamics and geophysical fluid dynamics usually possess wide symmetry groups [6, 17, 20, 41, 63], the design of symmetry-preserving parameterizations will in general lead to a great variety of different classes of invariant schemes.

Needless to say that the parameterization problem is too comprehensive both in theory and applications to be treated exhaustively in a single paper. Therefore, it is crucial to restrict to a setting that allows to demonstrate the basic ideas of invariant parameterizations without overly complicating the presentation by physical or technical details. This is the reason for illustrating the invariant parameterization procedure with the rather elementary barotropic vorticity equation. For the sake of simplicity, we moreover solely focus on local closure schemes in the present study. That is, the quantities to be parameterized at each point are substituted with known quantities defined at the same respective point [150]. This renders it possible to thoroughly use differential equations and hence it will not be necessary to pass to integro-differential equations, as would be the case for nonlocal closure schemes. On the other hand, this restriction at once excludes a number of processes with essential nonlocal nature, such as e.g. atmospheric convection. Nevertheless, there are several processes that can be adequately described within the framework of the present paper, most notably different kinds of turbulent transport phenomena.

The organization of this paper is the following: Section 8.2 discusses different possibilities for the usage of symmetry transformations in the parameterization procedure, most noteworthy the application of techniques of direct and inverse group classification. We restate some basic results from the theory of group classification and relate them to the parameterization problem. Section 8.3 is devoted to the construction of several parameterization schemes for the eddy vorticity flux of the two-dimensional vorticity equation. Parameterization schemes admitting subalgebras of the maximal Lie invariance algebra of the vorticity equation are constructed using the techniques introduced in Section 8.2. A short discussion of the results of this paper is presented in Section 8.4. In Appendix 8.5 details on the classification of the equivalence algebra presented in Section 8.3 can be found.

8.2 The general idea

Throughout this paper, the notation we adopt follows closely that presented in the textbook [115]. Let there be given a system of differential equations

$$\Delta^l(x, u_{(n)}) = 0, \qquad l = 1, \dots, m, \tag{8.1}$$

where $x = (x^1, \dots, x^p)$ denote the independent variables and the tuple $u_{(n)}$ includes all dependent variables $u = (u^1, \dots, u^q)$ as well as all derivatives of u with respect to x up to order n. Hereafter, subscripts of functions denote differentiation with respect to the corresponding variables.

Both numerical representations of (8.1) as well as real-time measurements are not able to capture the instantaneous value of u, but rather only provide some mean values. That is, to employ (8.1) in practice usually requires an averaging or filter procedure. For this purpose, u is separated according to

$$u = \bar{u} + u',$$

where $\bar{u}$ and u' refer to the averaged and the deviation quantities, respectively. The precise form of the averaging or filter method used determines additional calculation rules, e.g., $\overline{ab} = \bar{a}\bar{b} + \overline{a'b'}$ for the classical Reynolds averaging. At the present stage it is not essential to already commit oneself to a definite averaging method. For nonlinear system (8.1) averaging usually gives expressions

$$\tilde{\Delta}^l(x, \bar{u}_{(n)}, w) = 0, \qquad l = 1, \dots, m, \tag{8.2}$$

where $\tilde{\Delta}^l$ are smooth functions of their arguments whose explicit form is precisely determined by the form of Δ^l and the chosen averaging rule. The tuple $w = (w^1, \dots, w^k)$ includes all averaged nonlinear combinations of terms, which cannot be obtained by means of the quantities $\bar{u}_{(n)}$. These combinations typically include such expressions as $\overline{u'u'}$, $\overline{u'\bar{u}}$, $\overline{u'u'_x}$, etc., referred to as subgrid scale terms. Stated in another way, system (8.2) contains more unknown quantities than equations. To solve system (8.2), suitable assumptions on w have to be made. An adequate choice for these assumptions is the problem of parameterization.

The most straightforward way to tackle this issue is to directly express the unclosed terms w as functions of the variables x and $\bar{u}_{(r)}$ for some r which can be greater than n. In other words, system (8.2) is closed via

$$\tilde{\Delta}^l(x, \bar{u}_{(n)}, f(x, \bar{u}_{(r)})) = 0, \qquad l = 1, \dots, m, \tag{8.3}$$

using the relation $w^s = f^s(x, \bar{u}_{(r)})$, $s = 1, \dots, k$. The purpose of this paper is to discuss different paradigms for the choice of the functions $f = (f^1, \dots, f^k)$ within the symmetry approach, where k is the number of unclosed terms which are necessary to be parameterized. In other words, we should carry out, in different ways, group analysis of the class (8.3) with the arbitrary elements running through a set of differential functions. To simplify notation, we will omit bars over the dependent variables in systems where parameterization of w is already applied.

Remark. In the theory of group classification, any class of differential equations is considered in a jet space of a fixed order. That is, both the explicit part of the expression of the general equation from the class and the arbitrary elements can be assumed to depend on derivatives up to the same order. In contrast to this, for the construction of parameterization schemes it is beneficial to allow for varying the orders of arbitrary elements while the order of the explicitly resolved terms is fixed. This is why we preserve different notations for the orders of derivatives in the explicit part of the expression of the general equation and in the arbitrary elements of the class (8.3).

8.2.1 Parameterization via inverse group classification

Parameterizations based on Lie symmetries appear to have been first investigated for the Navier–Stokes equations. It was gradually realized that the consideration of symmetries plays a key role in the construction of subgrid scale models for the Navier–Stokes equations to allow for realistic simulations of flow evolution. See [112, 113] for a further discussions on this subject. The approach involving symmetries for the design of local closure schemes, was later extended in [136, 137] in order to also incorporate the second law of thermodynamics into the consideration.

For an arbitrary system of differential equations, this approach can be sketched as follows: First, determine the group of Lie point symmetries of the model to be investigated. For common models of hydro-thermodynamics these computations were already carried out and results can be found in collections like [63]. Subsequently, determine the differential invariants of this group. If the left hand side of system (8.2) is formulated in terms of these invariants by an adequate choice of the function f, it is guaranteed that the parameterized system will admit the same group of point symmetries as the unfiltered system. Usually the above procedure leads to classes of differential equations rather than to a single model. That is, among all models constructed this way it is be possible to select those which also satisfy other desired physical and mathematical properties.

The procedure outlined above can be viewed as a special application of techniques of *inverse group classification*. Inverse group classification starts with a prescribed symmetry group and aims to determine the entire class of differential equations admitting the given group as a symmetry group [118]. Thus, in [112, 113, 136, 137] it is assumed that the closure scheme for the subgrid scale terms leads to classes of differential equations admitting the complete Lie symmetry group of the Navier–Stokes equations. From the mathematical point of view, this assumption is justified as filtering (or averaging) of the Navier–Stokes equations introduces a turbulent friction term among the viscous friction term that already appears in the unfiltered equations. That is, filtering does not principally perturbs the structure of the Navier–Stokes equations. However, this assumption may not be as well justified if a model is chosen, where filtering leads to terms of forms not already included in the unfiltered model. In such cases, it may be more straightforward to solve the parameterization problem by inverse group classification only with respect to particular subgroups of the symmetry group of the initial model.

The approach of inverse group classification usually relies on the notion of differential invariants [115, 118]. Differential invariants are defined as the invariants of the prolonged action of a given symmetry group. They can be determined either with the infinitesimal method [46, 118] or with the technique of moving frames [31, 40, 116]. In the present paper we will use the former method which is briefly described here for this reason.

Let X be the p-dimensional space of independent and U be the q-dimensional space of dependent variables. The Lie group G acts locally on the space $X \times U$, with $\mathfrak{g}$ denoting the associated Lie algebra of infinitesimal generators. (The whole consideration is assumed local). Each element of $\mathfrak{g}$ is of the form $Q = \xi^i(x,u)\partial_{x^i} + \varphi^a(x,u)\partial_{u^a}$. In this section the indices i and j run from 1 to p while the indices a and b run from 1 to q, and the summation convention over repeated indices is used. The space $X \times U_{(r)}$ is the rth prolongation of the space $X \times U$, which is the space endowed with coordinates x_i and u^a_α, $|\alpha| := \alpha_1 + \cdots + \alpha_p < r$, where u^a_α stands for the variable corresponding to the derivative $\partial^{|\alpha|}u^a/\partial x_1^{\alpha_1}\dots\partial x_p^{\alpha_p}$, and $\alpha = (\alpha_1, \dots, \alpha_p)$ is an arbitrary multiindex, $\alpha_i \in \mathbb{N} \cup \{0\}$. The action of G can be extended to an action on $X \times U_{(r)}$ and so the elements of $\mathfrak{g}$ can be prolonged via

$$Q_{(r)} = Q + \sum_{\alpha>0} \varphi^{a\alpha}\partial_{u^a_\alpha}, \quad \varphi^{a\alpha} := \mathrm{D}_1^{\alpha_1}\cdots\mathrm{D}_p^{\alpha_p}(\varphi^a - \xi^i u^\alpha_i) + \xi^i u^a_{\alpha+\delta_i}. \tag{8.4}$$

Here $\mathrm{D}_i = \mathrm{D}_{x_i}$ denotes the operator of total differentiation with respect to the variable x_i, i.e. $\mathrm{D}_i = \partial_{x_i} + u^a_{\alpha+\delta_i}\partial_{u^a_\alpha}$, where δ_i is the multiindex whose ith entry equals 1 and whose other entries are zero. More details can be found in the textbooks [115, 117, 118].

A differential function f (i.e., a smooth function from $X \times U_{(r)}$ to $\mathbb{R}$ for some r) is called an (rth order) *differential invariant* of the point transformation group G if for any transformation $g\colon (x,u) \mapsto (\tilde{x},\tilde{u})$ from G we have that $f(\tilde{x},\tilde{u}_{(r)}) = f(x,u_{(r)})$. The function f is a differential invariant of the Lie group G if and only if the equality $Q_{(r)}f = 0$ holds for any $Q \in \mathfrak{g}$.

An operator $\delta^i\mathrm{D}_i$, where δ^i are differential functions, is called an *operator of invariant differentiation* for the group G if the result $\delta^i\mathrm{D}_i f$ of its action to any differential invariant f of G also is a differential invariant of G.

The *Fundamental Basis Theorem* states that any finite-dimensional Lie group (or, more generally, any Lie pseudo-group satisfying certain condition) acting on $X \times U$ possesses exactly p operators of invariant differentiation, which are independent up to linear combining with coefficients depending on differential invariants, and a finite basis of differential invariants, i.e., a finite set of differential invariants such that any differential invariant of the group can be obtained from basis invariants by a finite number of functional operations and actions by the chosen independent operators of invariant differentiation.

For a finite-dimensional Lie group G, the characteristic property of operators of invariant differentiation is that each of them commutes with every infinitely prolonged operator from the corresponding Lie algebra $\mathfrak{g}$, i.e., $[Q_{(\infty)}, h^i \mathrm{D}_i] = 0$ for any $Q \in \mathfrak{g}$. In this case the basis set of p operators of invariant differentiation can be found by solving, with respect to the differential functions $h^i = h^i(x, u_{(r)})$, the system of first-order quasi-linear partial differential equations

$$Q_{(r)} h^i = h^j \mathrm{D}_j \xi^i,$$

where Q runs through a basis of the corresponding Lie algebra $\mathfrak{g}$ and r equals the minimum order for which the rank of the prolonged basis operators of $\mathfrak{g}$ coincides with its dimension. Eventually, it may be convenient to determine h^i in the implicit form $\Omega^j(h^1, \dots, h^p, x, u_{(r)}) = 0$, where $\det(\Omega^j_{h_i}) \neq 0$ and Ω^j satisfy the associated system of homogeneous equations

$$\left(Q_{(r)} + (h^{i'} \mathrm{D}_{i'} \xi^i) \partial_{\lambda^i} \right) \Omega^j = 0.$$

A systematic approach to parameterization via inverse group classification hence consists of determining the basis differential invariants of a group together with the list of operators of invariant differentiation. Subsequently, there are infinitely many parameterizations that can be constructed, which admit the given group as a symmetry group.

8.2.2 Parameterization via direct group classification

The main assumption in the approach presented in [112, 137] is that a realistic subgrid scale model for the Navier–Stokes equations should admit the symmetry group of the original equations. However, this assumption is rather restrictive in more general situations. While it is true that a filtered model should be a realistic approximation of the unfiltered equations, parameterization schemes also have to take into account physical processes for which we may not have a precise understanding yet. That is, one eventually has to face the problem to deal with processes for which we may not even have a differential equation. This particularly means that a fixed set of symmetries (as for the Navier–Stokes equations) may not be obtainable.

On the other hand, symmetries do provide a useful guiding principle for the selection of physical models. As nature tends to prefer states with a high degree of symmetry, a general procedure for the derivation of symmetry-preserving parameterization schemes seems reasonable. The only crucial remark is, that we may not know in advance, which symmetries are most essential for capturing the characteristics of the underlying physical processes. For such problems, application of inverse group classification techniques is at once limited. Rather, it may be beneficial to derive parameterization schemes admitting different symmetry groups and subsequently test these various schemes to select among them those which best describe the processes under consideration. That is, instead of expressing the tuple w in system (8.2) using differential invariants of a symmetry group of the unfiltered equations (or another convenient symmetry group) from the beginning, we investigate symmetries of system (8.3) for different realizations of the functions f which are eventually required to satisfy some prescribed conditions. This way, we could be interested in special classes of parameterizations, such as, e.g., time- or spatially independent ones. This naturally leads back to the usual problem of *direct group classification*: Let there be given a class of differential equations, parameterized by arbitrary functions. First determine the symmetries admitted for all choices of these functions, leading to the kernel of symmetry groups of the class under consideration. Subsequently, investigate for which special values of these parameter-functions there are extensions of the kernel group [118, 131].

To systematically carry out direct group classification, it is necessary to determine the equivalence group of the class, i.e. the group of transformations mapping an equation from the class (8.3) to an equation from the same class. Classification of extensions of the kernel group is then done up to equivalence imposed by the equivalence group of the class (8.3).

The continuous part of the equivalence group can be found using infinitesimal methods in much the same way as Lie symmetries can be found using the infinitesimal invariance criterion. This firstly yields the equivalence algebra, the elements of which can then be integrated to give the continuous equivalence group. See [118, 131] for more details on this subject.

We now formalize the method reviewed in the previous paragraphs. Let there be given a class of differential equations of the form (8.3), $\tilde\Delta^l(x,\bar u_{(n)},f(x,\bar u_{(r)}))=0$, $l=1,\dots,m$. The arbitrary elements f usually satisfy an auxiliary system of equations $S(x,u_{(r)},f_{(\rho)}(x,u_{(r)}))=0$, $S=(S^1,\dots,S^s)$, and an inequality $\Sigma(x,u_{(r)},f_{(\rho)}(x,u_{(r)}))=0$, where $f_{(\rho)}$ denotes the collection of f and all derivatives of f with respect to the variables x and $u_{(r)}$ up to order ρ. The conditions $S=0$ and $\Sigma\neq 0$ restrict the generality of f and hence allow the design of specialized parameterizations. We denote the solution set of the auxiliary system by $\mathcal S$, the system of form (8.3) corresponding to an $f\in\mathcal S$ by $\mathcal L_f$ and the entire class of such system by $\mathcal L|_{\mathcal S}$.

The set of all point transformations that map a system $\mathcal L_f$ to a system $\mathcal L_{\tilde f}$, where both $f,\tilde f\in\mathcal S$ is denoted by $\mathrm T(f,\tilde f)$ and is referred to as the set of admissible transformations from the system $\mathcal L_f$ to the system $\mathcal L_{\tilde f}$. The collection of all point transformations relating at least two systems from the class $\mathcal L|_S$ gives rise to the set of admissible transformations of $\mathcal L|_S$.

Definition 8.1. The set of admissible transformations of the class $\mathcal L|_S$ is the set

$$\mathrm T(\mathcal L|_S)=\{(f,\tilde f,\varphi)\,|\,f,\tilde f\in\mathcal S,\varphi\in\mathrm T(f,\tilde f)\}.$$

That is, an admissible transformation is a triple, consisting of the initial system (with arbitrary elements f), the target system (with arbitrary elements $\tilde f$) and a mapping φ between these two systems.

The *usual equivalence group* $G^\sim=G^\sim(\mathcal L|_{\mathcal S})$ of the class $\mathcal L|_{\mathcal S}$ is defined in a rigorous way in terms of admissible transformations. Namely, any element Φ from $G^\sim$ is a point transformation in the space of $(x,u_{(r)},f)$, which is projectable on the space of $(x,u_{(r')})$ for any $0\le r'\le r$, so that the projection is the r'th order prolongation of $\Phi|_{(x,u)}$, the projection of Φ on the variables (x,u), and for any arbitrary elements $f\in\mathcal S$ we have that $\Phi f\in\mathcal S$ and $\Phi|_{(x,u)}\in\mathrm T(f,\Phi f)$. The admissible transformations of the form $(f,\Phi f,\Phi|_{(x,u)})$, where $f\in\mathcal S$ and $\Phi\in G^\sim$, are called induced by transformations from the equivalence group $G^\sim$. Needless to say, that in general not all admissible transformations are induced by elements from the equivalence group. Different generalizations of the notion of usual equivalence groups exist in the literature [95, 131]. By $\mathfrak g^\sim$ we denote the algebra associated with the equivalence group $G^\sim$ and call it the equivalence algebra of the class $\mathcal L|_{\mathcal S}$.

After clarifying the notion of admissible transformations and equivalence groups, we move on with the description of a common technique in the course of group analysis of differential equations, namely the algebraic method. Within this method one at first should classify inequivalent subalgebras of the corresponding equivalence algebra and then solve the inverse group classification problem for each of the subalgebras obtained. This procedure usually yields most of the cases of extensions and therefore leads to preliminary group classification (see, e.g., [64, 65, 156] for applications of this technique to various classes of differential equations).

The algebraic method rests on the following two propositions [36]:

Proposition 8.1. *Let $\mathfrak a$ be a subalgebra of the equivalence algebra $\mathfrak g^\sim$ of the class $\mathcal L|_{\mathcal S}$, $\mathfrak a\subset\mathfrak g^\sim$, and let $f^0(x,u_{(r)})\in\mathcal S$ be a value of the tuple of arbitrary elements f for which the algebraic* *equation $f = f^0(x, u_{(r)})$ is invariant with respect to $\mathfrak{a}$. Then the differential equation $\mathcal{L}|_{f^0}$ is invariant with respect to the projection of $\mathfrak{a}$ to the space of variables (x, u).*

Proposition 8.2. *Let $\mathfrak{a}_i$ be a subalgebra of the equivalence algebra $\mathfrak{g}^\sim$ and let $\mathcal{S}_i$ be the subset of $\mathcal{S}$ that consists of all arbitrary elements for which the corresponding equations admit projections of $\mathfrak{a}_i$ to the space (x, u) as Lie invariance algebras, $i = 1, 2$. Then the subalgebras $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are equivalent with respect to the adjoint action of $G^\sim$ if and only if the subsets $\mathcal{S}_1$ and $\mathcal{S}_2$ are mapped to each other by transformations from $G^\sim$.*

The result of preliminary group classification is a list of inequivalent (with respect to the equivalence group) members $\mathcal{L}_f$ of the class $\mathcal{L}|_\mathcal{S}$, admitting symmetry extension of the kernel of symmetry algebras using subalgebras of the equivalence algebra.

Although the algebraic method is a straightforward tool to derive cases of symmetry extensions for classes of differential equations with arbitrary elements, there remains the important question when it gives complete group classification, i.e., preliminary and complete group classification coincide. This question is of importance also for the problem of parameterization, as only complete group classification will lead to an exhaustive description of all possible parameterization schemes feasible for some class of differential equations. The answer is that the class under consideration should satisfy the following property: The span of maximal Lie invariance algebras of all equations from the class is contained in the projection of the corresponding equivalence algebra to the space of independent and dependent variables, $\langle \mathfrak{g}_f \mid f \in \mathcal{S} \rangle \subset \mathrm{P}\mathfrak{g}^\sim$.

The above property is satisfied by all normalized classes of differential equations [131].

Definition 8.2. The class of equations $\mathcal{L}|_\mathcal{S}$ is normalized if $\forall\,(f, \tilde{f}, \varphi) \in \mathrm{T}(\mathcal{L}_\mathcal{S}), \exists \Phi \in G^\sim : \tilde{f} = \Phi f$ and $\varphi = \Phi|_{(x,u)}$.

Additionally, if the class of equations $\mathcal{L}|_\mathcal{S}$ is normalized then the group classification of equations from this class up to $G^\sim$-equivalence coincides with the group classification using the general point transformation equivalence. Due to this fact we have no additional equivalences between cases obtained under the classification up to $G^\sim$-equivalence. As a result, solving the group classification problem for normalized classes of differential equations is especially convenient and effective.

In turn, depending on normalization properties of the given class (or their lacking), different strategies of group classification should be applied [131]. For a normalized class, the group classification problem is reduced, within the infinitesimal approach, to classification of subalgebras of its equivalence algebra. A class that is not normalized can eventually be embedded into a normalized class which is not necessarily minimal among the normalized superclasses. One more way to treat a non-normalized class is to partition it into a family of normalized subclasses and to subsequently classify each subclass separately. If a partition into normalized subclasses is difficult to construct due to the complicated structure of the set of admissible transformations, conditional equivalence groups and additional equivalence transformations may be involved in the group classification. In the case when the class is parameterized by constant arbitrary elements or arbitrary elements depending only on one or two arguments, one can apply the direct method of group classification based on compatibility analysis and integration of the determining equations for Lie symmetries up to $G^\sim$-equivalence [118].

8.3 Symmetry preserving parameterizations for vorticity equation

The inviscid barotropic vorticity equations in Cartesian coordinates reads

$$\zeta_t + \{\psi, \zeta\} = 0 \tag{8.5}$$

where $\{a, b\} = a_x b_y - a_y b_x$ denotes the usual Poisson bracket with respect the variables x and y. The vorticity ζ and the stream function ψ are related through the Laplacian, i.e. $\zeta = \nabla^2\psi$. The two-dimensional wind field $\mathbf{v} = (u, v, 0)^{\mathrm{T}}$ is reconstructed from the stream function via the relation $\mathbf{v} = \mathbf{k} \times \nabla\psi$, where $\mathbf{k}$ is the vertical unit vector.

The maximal Lie invariance algebra $\mathfrak{g}_0$ of the equation (8.5) is generated by the operators

$$\begin{aligned}
&\mathcal{D}_1 = t\partial_t - \psi\partial_\psi, \qquad \partial_t, \qquad \mathcal{D}_2 = x\partial_x + y\partial_y + 2\psi\partial_\psi,\\
&\mathrm{J} = -y\partial_x + x\partial_y, \qquad \mathrm{J}^t = -ty\partial_x + tx\partial_y + \tfrac{1}{2}(x^2+y^2)\partial_\psi,\\
&\mathcal{X}(\gamma^1) = \gamma^1(t)\partial_x - \gamma^1_t(t)y\partial_\psi, \qquad \mathcal{Y}(\gamma^2) = \gamma^2(t)\partial_y + \gamma^2_t(t)x\partial_\psi, \qquad \mathcal{Z}(\chi) = \chi(t)\partial_\psi,
\end{aligned} \tag{8.6}$$

where γ^1, γ^2 and χ run through the set of smooth functions of t. See, e.g. [6, 17] for further discussions.

Reynolds averaging the above equation leads to

$$\bar{\zeta}_t + \{\bar{\psi}, \bar{\zeta}\} = \nabla\cdot(\overline{\mathbf{v}'\zeta'}). \tag{8.7}$$

The term $\overline{\mathbf{v}'\zeta'} = (\overline{u'\zeta'}, \overline{u'\zeta'}, 0)^{\mathrm{T}}$ is the horizontal eddy vorticity flux. Its divergence provides a source term for the averaged vorticity equation. The presence of this source term destroys several of the properties of (8.5), such as, e.g., possessing conservation laws. In this paper we aim to find parameterizations of this flux term, which admit certain symmetries.

A simple choice for a parameterization of the eddy vorticity flux is given by the *down-gradient ansatz*

$$\overline{\mathbf{v}'\zeta'} = -K\nabla\bar{\zeta},$$

where the eddy viscosity coefficient K still needs to be specified. Physically, this ansatz accounts for the necessity of the vorticity flux to be directed down-scale, as enstrophy (integrated squared vorticity) is continuously dissipated at small scales. Moreover, this ansatz will lead to a uniform distribution of the mean vorticity field, provided there is no external forcing that counteracts this tendency [91]. The simplest form of the parameter K is apparently $K = K(x, y)$, i.e. the eddy viscosity coefficient is only a function of space. More advanced ansatzes for K assume dependence on $\overline{\zeta'^2}$, which is the eddy enstrophy [91] (see also the discussion in the recent paper [92]). This way, the strength of the eddy vorticity flux depends on the intensity of two-dimensional turbulence, which gives a more realistic model for the behavior of the fluid. There also exist a number of other parameterization schemes that can be applied to the vorticity equation, such as methods based on statistical mechanics [76] or the anticipated potential vorticity method [140, 158].

In the present framework, we exclusively focus on first order closure schemes. This is why we are only able to parameterize the eddy vorticity flux using the independent and dependent variables, as well as all derivatives of the dependent variables. This obviously excludes the more sophisticated and recent parameterization ansatzes of geophysical fluid dynamics from the present study. On the other hand, the basic method of invariant parameterization can already be demonstrated for this rather simple model. Indeed, symmetries of the vorticity equation employing the down-gradient ansatz or related parameterizations are investigated below using both inverse and direct group classification. Physically more advanced examples for parameterizations can be constructed following the methods outlined in Section 8.2 and exemplified subsequently.

8.3.1 Parameterization via inverse group classification

This is the technique by [112, 137] applied to the inviscid vorticity equation. In view of the description of Section 8.2.1 this approach consists of singling out subgroups (subalgebras) of the maximal Lie invariance group (algebra) of the vorticity equation and computation of the associated differential invariants (via a basis of differential invariants and operators of invariant differentiation). These differential invariants can then be used to construct different parameterizations of the eddy vorticity flux.

It is important to note that singling out subgroups of the maximal Lie invariance group of the vorticity equation is a meteorological way of group classification. This is why it is necessary to have a basic understanding of the processes to be parameterized before the selection of a particular group is done (otherwise, we would have to face the problem of how to combine these invariants to physically meaningful parameterizations). For the vorticity equation, we demonstrate the basic mechanisms of parameterizations via inverse group classification by singling out subgroups that allow to include the down-gradient ansatz of the previous section. This choice is of course not unique as there exist various other possibilities for parameterizations of the eddy vorticity flux. However, this choice allows us to demonstrate several of the issues of parameterization via inverse group classification.

Invariance under the whole symmetry group. The most general case where invariance of the parameterization under the whole symmetry group of the vorticity equation is desired can be neglected for physical reasons. This is since it is impossible to realized, e.g. the down-gradient ansatz described in the previous section within this framework. It can easily be checked that the corresponding vorticity equation with parameterized eddy vorticity flux only admits one scaling operator for any physically meaningful ansatz for K. In contrast to the example of the Navier–Stokes equations discussed in [112], the vorticity equation hence does not allow physical parameterizations leading to a closed model invariant under the same symmetry group as the original vorticity equation. This is why it is beneficial to single out several subgroups of the maximal Lie invariance group and consider the invariant parameterization problem only with respect to these subgroups.

Explicit spatial dependency. If the two-dimensional fluid is anisotropic and inhomogeneous the only subalgebra of (8.6) that can be admitted is

$$\partial_t, \qquad \mathcal{Z}(\chi) = \chi(t)\partial_\psi.$$

Operators of invariant differentiation are $\mathrm{D}_t, \mathrm{D}_x$ and D_y. A basis of invariants is formed by x, y, ψ_x and ψ_y. If we express the right hand side of (8.7) in terms of differential invariants of the above subalgebra, a possible representation reads

$$\zeta_t + \{\psi, \zeta\} = K(x, y)\nabla^2\zeta.$$

Hence we assembled our parameterization using the (differential) invariants x, y, $\mathrm{D}_x^3\psi_x = \psi_{xxxx}$, $\mathrm{D}_y^2\mathrm{D}_x\psi_x = \psi_{xxyy}$ and $\mathrm{D}_y^3\psi_y = \psi_{yyyy}$. This boils down to the usual gradient ansatz for the eddy flux term, where the eddy viscosity K explicitly depends on the position in the space. Note, however, that this ansatz is only one possibility which is feasible within this class of models.

Rotationally invariant fluid. In case the two-dimensional fluid is isotropic, the resulting parameterized system should also admit rotations. Hence, we seek for differential invariants of the subalgebra

$$\mathrm{J} = -y\partial_x + x\partial_y, \qquad \mathrm{J}^t = -ty\partial_x + tx\partial_y + \tfrac{1}{2}(x^2+y^2)\partial_\psi, \qquad \partial_t, \qquad \mathcal{Z}(\chi) = \chi(t)\partial_\psi.$$

The invariant differentiation operators are $x\mathrm{D}_x + y\mathrm{D}_y$, $-y\mathrm{D}_x + x\mathrm{D}_y$ and $\mathrm{D}_t - \psi_y\mathrm{D}_x + \psi_x\mathrm{D}_y$ and a basis of invariants consists of x^2+y^2 and $\psi_x^2+\psi_y^2$. Within this class

$$\zeta_t + \{\psi,\zeta\} = K(r)\nabla^2\zeta,$$

where $r = \sqrt{x^2+y^2}$ is realizable as a symmetry preserving parameterization.

In the same fashion it would be possible to derive classes of parameterizations that preserve, e.g. (generalized) Galilean symmetry or a scaling symmetry but we do not derive them in this paper.

8.3.2 Equivalence algebras of classes of generalized vorticity equations

In order to demonstrate different possible techniques, we present the details of the calculation of the usual equivalence algebra $\mathfrak{g}_1^\sim$ for the class of equations

$$\zeta_t + \{\psi,\zeta\} = \mathrm{D}_i f^i(t,x,y,\zeta_x,\zeta_y) = f^i_i + f^i_{\zeta_j}\zeta_{ij}, \quad \zeta := \psi_{ii}, \tag{8.8}$$

where for convenience we introduce another notation for the independent variables, $t = z_0$, $x = z_1$ and $y = z_2$, and omit bars over the dependent variables. Throughout the section the indices i, j and k range from 1 to 2, while the indices κ, λ, μ and ν run from 0 to 2. The summation over repeated indices is understood. A numerical subscript of a function denotes the differentiation with respect to the corresponding variable z_μ.

In fact, the equivalence algebra of class (8.8) can be easily obtained from the much more general results on admissible transformations, presented in Section 8.3.3. At the same time, calculations using the direct method applied for finding admissible transformations are too complicated and lead to solving nonlinear overdetermined systems of partial differential equations. This is why the infinitesimal approach is wider applied and realized within symbolic calculation systems. The usage of the infinitesimal approach for the construction of the equivalence algebra of (8.8) has specific features richly deserving to be demonstrated here.

Theorem 8.1. *The equivalence algebra $\mathfrak{g}_1^\sim$ of class* (8.8) *is generated by the operators*

$$
\begin{aligned}
&\tilde{\mathcal{D}}_1 = t\partial_t - \psi\partial_\psi - \zeta_x\partial_{\zeta_x} - \zeta_y\partial_{\zeta_y} - 2f^1\partial_{f^1} - 2f^2\partial_{f^2}, \qquad \partial_t,\\
&\tilde{\mathcal{D}}_2 = x\partial_x + y\partial_y + 2\psi\partial_\psi - \zeta_x\partial_{\zeta_x} - \zeta_y\partial_{\zeta_y} + f^1\partial_{f^1} + f^2\partial_{f^2},\\
&\tilde{\mathrm{J}}(\beta) = \beta x\partial_y - \beta y\partial_x + \frac{\beta_t}{2}(x^2+y^2)\partial_\psi + \beta(\zeta_x\partial_{\zeta_y} - \zeta_y\partial_{\zeta_x})\\
&\qquad\quad + (\beta_{tt}x - \beta f^2)\partial_{f^1} + (\beta_{tt}y + \beta f^1)\partial_{f^2},\\
&\tilde{\mathcal{X}}(\gamma^1) = \gamma^1\partial_x - \gamma^1_t y\partial_\psi, \qquad \tilde{\mathcal{Y}}(\gamma^2) = \gamma^2\partial_y + \gamma^2_t x\partial_\psi,\\
&\tilde{\mathcal{R}}(\sigma) = \frac{\sigma}{2}(x^2+y^2)(\partial_\psi + \zeta_y\partial_{f^1} - \zeta_x\partial_{f^2}) + \sigma_t x\partial_{f^1} + \sigma_t y\partial_{f^2},\\
&\tilde{\mathcal{H}}(\delta) = \delta(\partial_\psi + \zeta_y\partial_{f^1} - \zeta_x\partial_{f^2}), \qquad \tilde{\mathcal{G}}(\rho) = \rho_x\partial_{f^2} - \rho_y\partial_{f^1}, \qquad \tilde{\mathcal{Z}}(\chi) = \chi\partial_\psi,
\end{aligned}
\tag{8.9}
$$

where β, γ^i, σ and χ are arbitrary smooth functions of t solely, $\delta = \delta(t,x,y)$ is an arbitrary solution of the Laplace equation $\delta_{xx} + \delta_{yy} = 0$ and $\rho = \rho(t,x,y)$ is an arbitrary smooth function of its arguments.

Remark. Although the coefficients of ∂_{ζ_x} and ∂_{ζ_y} can be obtained by standard prolongation from the coefficients associated with the equation variables, it is necessary to include the corresponding terms in the representation of the basis elements (8.9) in order to guarantee that they commute in a proper way.

Remark. The operators $\tilde{\mathcal{G}}(\rho)$ and $\tilde{\mathcal{H}}(\chi) - \tilde{\mathcal{Z}}(\chi)$ arise due to the total divergence expression of the right hand side of the first equation in (8.8), leading to the gauge freedom in rewriting the right hand side of the class (8.8). They do not generate transformations of the independent and dependent variables and hence form the gauge equivalence subalgebra of the equivalence algebra (8.9) [131]. The parameter-function ρ is defined up to summand depending on t.

Proof. As coordinates in the underlying fourth-order jet space $\mathfrak{J}^{(4)}$, we choose the variables

$$
\begin{aligned}
&z_\mu, \quad \psi, \quad \psi_\mu, \quad \psi_{\mu\nu},\ \mu \leqslant \nu, \quad \psi_{\lambda\mu\nu},\ \lambda \leqslant \mu \leqslant \nu,\ (\mu,\nu) \neq (2,2), \quad \zeta_\mu,\\
&\psi_{\kappa\lambda\mu\nu},\ \kappa \leqslant \lambda \leqslant \mu \leqslant \nu,\ (\mu,\nu) \neq (2,2), \quad \zeta_{\mu\nu},\ \mu \leqslant \nu.
\end{aligned}
$$

(Variables of the jet space and related values are defined by their notation up to permutation of indices.) The variable ζ_0 of the jet space is assumed principal, i.e., it is expressed via the other coordinate variables (called the parametric ones) in view of equation (8.8). Under calculation we also carry out the substitutions $\psi_{22\mu} = \zeta_\mu - \psi_{11\mu}$. To avoid repetition of the above conditions for indices, in what follows we assume that the index tuples (μ,ν), (λ,μ,ν) and (κ,λ,μ,ν) satisfy these conditions by default.

Due to the special form of the arbitrary elements f^i, we have to augment equation (8.8) with the following auxiliary system for f^i:

$$
f^i_\psi = f^i_{\psi_\mu} = f^i_{\psi_{\mu\nu}} = f^i_{\psi_{\lambda\mu\nu}} = f^i_{\zeta_0} = f^i_{\psi_{\kappa\lambda\mu\nu}} = f^i_{\zeta_{\mu\nu}} = 0. \tag{8.10}
$$

As we compute the usual equivalence algebra rather than the generalized one [95] and the arbitrary elements f^i do not depend on fourth order derivatives of ψ, the elements of the algebra are assumed to be vector fields in the joint space of the variables of $\mathfrak{J}^{(3)}$ and the arbitrary elements f^i, which are projectable to both the spaces (t,x,y,ψ) and $\mathfrak{J}^{(3)}$. In other words, the algebra consists of vector fields of the general form

$$
Q = \xi^\mu\partial_\mu + \eta\partial_\psi + \eta^\mu\partial_{\psi_\mu} + \eta^{\mu\nu}\partial_{\psi_{\mu\nu}} + \eta^{\lambda\mu\nu}\partial_{\psi_{\lambda\mu\nu}} + \theta^\mu\partial_{\zeta_\mu} + \varphi^i\partial_{f^i},
$$

where $\xi^\mu=\xi^\mu(t,x,y,\psi)$, $\eta=\eta(t,x,y,\psi)$, the coefficients corresponding to derivatives of ψ are obtained by the standard prolongation (8.4) from ξ^μ and η, the coefficients θ^ν are obtained by the standard prolongation from ξ^μ and $\theta=\eta^{ii}$, and the coefficients φ^i depends on all the variables of $\mathfrak{J}^{(3)}$ and the arbitrary elements f^j. As a result, each element from the equivalence algebra is determined by its coefficients ξ^μ, η and φ^i. To act on the equations (8.8) and (8.10) by the operator Q, we should additionally prolong it to the variables $\psi_{\kappa\lambda\mu\nu}$ and $\zeta_{\mu\nu}$ in the conventional way and to the derivatives of f, assuming all the variables of $\mathfrak{J}^{(3)}$ as usual ones:

$$\begin{aligned}\bar{Q}=Q&+\eta^{\kappa\lambda\mu\nu}\partial_{\psi_{\kappa\lambda\mu\nu}}+\theta^{\mu\nu}\partial_{\zeta_{\mu\nu}}\\&+\varphi^{i\mu}\partial_{f^i_\mu}+\varphi^{i\psi}\partial_{f^i_\psi}+\varphi^{i\psi_\mu}\partial_{f^i_{\psi_\mu}}+\varphi^{i\psi_{\mu\nu}}\partial_{f^i_{\psi_{\mu\nu}}}+\varphi^{i\psi_{\lambda\mu\nu}}\partial_{f^i_{\psi_{\lambda\mu\nu}}}+\varphi^{i\zeta_\mu}\partial_{f^i_{\zeta_\mu}}.\end{aligned}$$

First we consider the infinitesimal invariance conditions associated with equations (8.10). The invariance condition for the equation $f^i_\psi=0$ is

$$\varphi^{i\psi}\big|_{\text{Eq. (8.10)}}=\varphi^i_\psi-\xi^\mu_\psi f^i_\mu-\theta^k_\psi f^i_{\zeta_k}=0.$$

Splitting with respect to derivatives of f^i in the latter equation implies that $\varphi^i_\psi=0$, $\xi^\mu_\psi=0$, $\theta^i_\psi=0$. As $\theta^i=\mathrm{D}_j\mathrm{D}_j\mathrm{D}_i(\eta-\xi^\mu\psi_\mu)+\xi^\mu\psi_{\mu jji}$, we additionally derive the simple determining equation $\eta_{\psi\psi}=0$.

In a similar way, the invariance conditions for the equations $f^i_{\psi_\mu}=0$, $f^i_{\psi_{\mu\nu}}=0$, $f^i_{\psi_{\lambda\mu\nu}}=0$ and $f^i_{\zeta_0}=0$ can be presented in the form

$$\begin{aligned}&\varphi^{i\psi_\mu}\big|_{\text{Eq. (8.10)}}=\varphi^i_{\psi_\mu}-\theta^k_{\psi_\mu}f^i_{\zeta_k}=0,\\&\varphi^{i\psi_{\mu\nu}}\big|_{\text{Eq. (8.10)}}=\varphi^i_{\psi_{\mu\nu}}-\theta^k_{\psi_{\mu\nu}}f^i_{\zeta_k}=0,\\&\varphi^{i\psi_{\lambda\mu\nu}}\big|_{\text{Eq. (8.10)}}=\varphi^i_{\psi_{\lambda\mu\nu}}-\theta^k_{\psi_{\lambda\mu\nu}}f^i_{\zeta_k}=0,\\&\varphi^{i\zeta_0}\big|_{\text{Eq. (8.10)}}=\varphi^i_{\zeta_0}-\theta^k_{\zeta_0}f^i_{\zeta_k}=0,\end{aligned}$$

which is split into $\varphi^i_{\psi_\mu}=0$, $\theta^k_{\psi_\mu}=0$; $\varphi^i_{\psi_{\mu\nu}}=0$, $\theta^k_{\psi_{\mu\nu}}=0$; $\varphi^i_{\psi_{\lambda\mu\nu}}=0$, $\theta^k_{\psi_{\lambda\mu\nu}}=0$; and $\varphi^i_{\zeta_0}=0$, $\theta^k_{\zeta_0}=0$, respectively. The equations $\theta^k_{\psi_\mu}=0$, $\theta^k_{\psi_{\lambda\mu\nu}}=0$ and $\theta^k_{\zeta_0}=0$ provide no essential restrictions on the coefficients ξ^μ, η and φ^i. From the equation $\theta^k_{\psi_{\lambda\mu\nu}}=0$ we derive that $\xi^0_j=0$, $\xi^1_2+\xi^2_1=0$ and $\xi^1_1-\xi^2_2=0$. Hence

$$\theta=\eta^{jj}=\eta_{jj}+2\eta_{j\psi}\psi_j+\eta_\psi\psi_{jj}-2\xi^i_j\psi_{ij}=\eta_{jj}+2\eta_{j\psi}\psi_j+(\eta_\psi-2\xi^1_1)\zeta.$$

It remains to solve the determining equations following from the invariance condition for equation (8.8). The invariance condition reads

$$\theta^0+\eta^1\zeta_2+\psi_1\theta^2-\eta^2\zeta_1+\psi_2\theta^1=\varphi^{ii}+\varphi^{i\zeta_j}\zeta_{ji}+f^i_{\zeta_j}\theta^{ji},$$

or explicitly

$$\begin{aligned}&\eta_{jjt}+\eta_{jj\psi}\psi_t+2\eta_{tj\psi}\psi_j+2\eta_{j\psi}\psi_{tj}+(\eta_{t\psi}-2\xi^1_{t1})\zeta\\&+(\eta_\psi-2\xi^1_1-\xi^0_t)(f^i_i+f^i_{\zeta_j}\zeta_{ij}-\psi_1\zeta_2+\psi_2\zeta_1)-\xi^i_t\zeta_i\\&+(\eta_1+\eta_\psi\psi_1-\xi^i_1\psi_i)\zeta_2+\psi_1(\eta_{jj2}+\eta_{jj\psi}\psi_2+2\eta_{2j\psi}\psi_j+2\eta_{j\psi}\psi_{2i}+(\eta_\psi-2\xi^1_1)\zeta_2-\xi^i_2\zeta_i)\\&-(\eta_2+\eta_\psi\psi_2-\xi^i_2\psi_i)\zeta_1-\psi_2(\eta_{jj1}+\eta_{jj\psi}\psi_1+2\eta_{1j\psi}\psi_j+2\eta_{j\psi}\psi_{1j}+(\eta_\psi-2\xi^1_1)\zeta_1-\xi^i_1\zeta_i)\\&=\varphi^i_i+\varphi^i_{f^j}f^j_i-\xi^j_if^i_j-\theta^k_if^i_{\zeta_k}+\zeta_{ij}(\varphi^i_{\zeta_j}+\varphi^i_{f^k}f^k_{\zeta_j}-\theta^k_{\zeta_j}f^i_{\zeta_k})+f^i_{\zeta_j}\theta^{ij}.\end{aligned}$$

Collecting the coefficients of ψ_{tj} gives $\eta_{j\psi} = 0$. This implies that $\theta_\psi = 0$. Similarly, the coefficients of $\psi_i\zeta_j$ lead to the equation $\eta_{ijj} = 0$ and $\eta_\psi - 2\xi^1_1 + \xi^0_t = 0$. As $\xi^0_i = 0$ and $\eta_{i\psi} = 0$, the second equation together with the relations $\xi^1_1 = \xi^2_2$ and $\xi^1_2 + \xi^2_1 = 0$ implies that $\xi^i_{jk} = 0$. Then, the coefficient of ζ gives $\xi^0_{tt} = 0$ and the coefficients of f^i_j lead to $\varphi^1_{f^2} = \xi^1_2$, $\varphi^2_{f^1} = \xi^2_1$ and $\varphi^1_{f^1} = \varphi^2_{f^2} = \xi^1_1 - 2\xi^0_t$. In view of the determining equations that we have already derived, the terms involving $f^i_{\zeta_j}$ are identically canceled. Note that the coefficients of $f^i_{\zeta_j}\zeta_{kl}$ simultaneously lead to the same set of equations as the coefficients of f^i_j.

The remaining part of the invariance condition is $\eta_{jjt} - \xi^i_t\zeta_i + \eta_1\zeta_2 - \eta_2\zeta_1 = \varphi^i_i + \zeta_{ij}\varphi^i_{\zeta_j}$. Splitting with respect to ζ_{ij} in this relation gives $\varphi^1_{\zeta_1} = \varphi^2_{\zeta_2} = 0$, $\varphi^1_{\zeta_2} + \varphi^2_{\zeta_1} = 0$ and

$$\varphi^i_i = \eta_{jjt} - \xi^i_t\zeta_i + \eta_1\zeta_2 - \eta_2\zeta_1.$$

Acting on the last equation by the operator $\partial_j\partial_{\zeta_j}$, we obtain $\xi^i_{it} = 0$. Further splitting with respect to ζ_1 and ζ_2 is not possible since φ^j may depend on them.

Finally, the reduced system of determining equations reads

$$\begin{aligned}
&\xi^0_\psi = \xi^0_i = \xi^0_{tt} = 0, \quad \xi^i_\psi = \xi^i_{jk} = 0, \quad \xi^i_{it} = 0, \quad \xi^1_1 = \xi^2_2, \quad \xi^1_2 + \xi^2_1 = 0,\\
&\eta_{\psi\psi} = 0, \quad \eta_\psi = 2\xi^1_1 - \xi^0_t, \quad \eta_{ijj} = 0,\\
&\varphi^i_\psi = 0, \quad \varphi^i_{\psi_\mu} = 0, \quad \varphi^i_{\psi_{\mu\nu}} = 0, \quad \varphi^i_{\psi_{\lambda\mu\nu}} = 0, \quad \varphi^i_{\zeta_0} = 0,\\
&\varphi^1_{f^2} = \xi^1_2, \quad \varphi^2_{f^1} = \xi^2_1, \quad \varphi^1_{f^1} = \varphi^2_{f^2} = \xi^1_1 - 2\xi^0_t,\\
&\varphi^1_{\zeta_1} = \varphi^2_{\zeta_2} = 0, \quad \varphi^1_{\zeta_2} + \varphi^2_{\zeta_1} = 0, \quad \varphi^i_i = \eta_{jjt} - \xi^i_t\zeta_i + \eta_1\zeta_2 - \eta_2\zeta_1.
\end{aligned}$$

The solution of this system provides the principal coefficients of the operators from the equivalence algebra of the class (8.8):

$$\begin{aligned}
&\xi^0 = c_1t + c_0, \qquad \xi^1 = c_2x - \beta y + \gamma^1, \qquad \xi^2 = \beta x + c_2y + \gamma^2,\\
&\eta = (2c_2 - c_1)\psi + \delta - \gamma^1_t y + \gamma^2_t x + \frac{\beta_t}{2}(x^2 + y^2) + \frac{\sigma}{2}(x^2 + y^2) + \chi,\\
&\varphi^1 = (c_2 - 2c_1)f^1 - \beta f^2 + \delta\zeta_y + \frac{\sigma}{2}(x^2 + y^2)\zeta_y + \beta_{tt}x + \sigma_t x - \rho_y,\\
&\varphi^2 = \beta f^1 + (c_2 - 2c_1)f^2 - \delta\zeta_x - \frac{\sigma}{2}(x^2 + y^2)\zeta_x + \beta_{tt}y + \sigma_t y + \rho_x,
\end{aligned} \tag{8.11}$$

where β, γ^i, σ and χ are real-valued smooth functions of t only, c_0, c_1 and c_2 are arbitrary constants, ρ is an arbitrary function of t, x and y and $\delta = \delta(t, x, y)$ is an arbitrary solution of the Laplace equation $\delta_{jj} = 0$.

Splitting with respect to parametric values in (8.11), we obtain the coefficients of the basis operators (8.9) of the algebra $\mathfrak{g}^\sim_1$. Recall that the coefficients η^μ, $\eta^{\mu\nu}$, $\eta^{\lambda\mu\nu}$ and θ^ν are calculated from ξ^μ and η via the standard procedure of prolongation and the coefficients φ^i do not depend on ψ_μ, $\psi_{\mu\nu}$, $\psi_{\lambda\mu\nu}$, and ζ_0. Therefore, both the operators from $\mathfrak{g}^\sim_1$ and their commutators are completely determined by the coefficients of ∂_μ, ∂_ψ, ∂_{ζ_i} and ∂_{f^j}. This is why in (8.9) and similar formulas we omit the other terms for sake of brevity. □

Remark. The auxiliary system for the arbitrary elements is an important component of the definition of a class of differential equations. Its choice is usually guided by some prior knowledge about the processes to be parameterized. We have decided to assume that the arbitrary elements f^1 and f^2 depend also on t, keeping in mind two more, purely mathematical, reasons. The first reason is that the projection of the corresponding equivalence algebra on the space

(t, x, y, ψ) contains the maximal Lie invariance algebra $\mathfrak{g}_0$ of the vorticity equation (8.5) which is the initial point of the entire consideration. The basis operators (8.6) of $\mathfrak{g}_0$ are obtained from (8.9) by

$$\mathcal{D}_1 = \mathrm{P}\tilde{\mathcal{D}}_1, \qquad \partial_t = \mathrm{P}\partial_t, \qquad \mathcal{D}_2 = \mathrm{P}\tilde{\mathcal{D}}_2, \qquad \mathrm{J} = \mathrm{P}\tilde{\mathrm{J}}(1), \qquad \mathrm{J}^t = \mathrm{P}\tilde{\mathrm{J}}(t),$$
$$\mathcal{X}(\gamma^1) = \mathrm{P}\tilde{\mathcal{X}}(\gamma^1), \qquad \mathcal{Y}(\gamma^2) = \mathrm{P}\tilde{\mathcal{Y}}(\gamma^2), \qquad \mathcal{Z}(\chi) = \mathrm{P}\tilde{\mathcal{Z}}(\chi),$$

where P denotes the projection operator on the space (t, x, y, ψ). (Though the expressions for the operator ∂_t (resp. $\tilde{\mathcal{X}}(\gamma^1)$, $\tilde{\mathcal{Y}}(\gamma^2)$ or $\tilde{\mathcal{Z}}(\chi)$) and its projection formally coincide, they in fact determine vector fields on different spaces.) The second reason is that the class (8.8) is normalized, cf. Section 8.3.3. This in particular implies that the maximal Lie invariance algebra of any equation from the class (8.8) is contained in the projection of the equivalence algebra $\mathfrak{g}_1^{\sim}$ of this class.

We also calculate the equivalence algebras of two subclasses of the class (8.8).

The first subclass corresponds to parameterizations not depending on time explicitly and, therefore, is singled out from the class (8.8) by the further auxiliary equation

$$f^i_t = 0,$$

which has no influence on splitting of the invariance conditions for the equations (8.8) and (8.10) and gives the additional determining equations $\varphi^i_t = \xi^i_t = \theta^i_t = 0$. These determining equations imply that β, γ^i and σ are constant, δ is a function only of x and y and ρ can be assumed as a function only of x and y. Therefore, the equivalence algebra of this subclass is

$$\langle \tilde{\mathcal{D}}_1,\, \partial_t,\, \tilde{\mathcal{D}}_2,\, \tilde{\mathrm{J}}(1),\, \tilde{\mathcal{X}}(1),\, \tilde{\mathcal{Y}}(1),\, \tilde{\mathcal{R}}(1),\, \tilde{\mathcal{H}}(\delta),\, \tilde{\mathcal{G}}(\rho),\, \tilde{\mathcal{Z}}(\chi)\rangle,$$

where the parameter-function $\delta = \delta(x, y)$ runs through the set of solutions of the Laplace equation $\delta_{xx} + \delta_{yy} = 0$ and $\rho = \rho(x, y)$ is an arbitrary function of its arguments.

The second subclass is associated with spatially independent parameterizations. Hence we additionally set

$$f^i_j = 0.$$

It has to be noted that after attaching this condition we cannot split with respect to f^i_j as we did in the course of solving the determining equations. However, precisely the same conditions obtained from splitting with respect to f^i_j can also be obtained from splitting with respect to $f^i_{\zeta_j}$. Hence the condition $f^i_j = 0$ only leads to the additional restriction $\varphi^i_j = 0$ and, therefore, we find that $\delta_i = 0$, $\sigma = 0$, $\beta_{tt} = 0$ and $\rho_{ij} = 0$. Without loss of generality we can set $\rho = \rho^i(t) z_i$, where ρ^i are arbitrary smooth functions of t. As a result, the equivalence algebra $\mathfrak{g}_2^{\sim}$ of the second subclass is generated by the operators

$$\tilde{\mathcal{D}}_1,\, \partial_t,\, \tilde{\mathcal{D}}_2,\, \tilde{\mathrm{J}}(1),\, \tilde{\mathrm{J}}(t),\, \tilde{\mathcal{X}}(\gamma^1),\, \tilde{\mathcal{Y}}(\gamma^2),\, \tilde{\mathcal{H}}(\delta),\, \tilde{\mathcal{G}}(\rho^1 x + \rho^2 y),\, \tilde{\mathcal{Z}}(\chi),$$

where γ^i, ρ^i, δ and χ are arbitrary smooth functions of t.

The intersection of the above subclasses corresponds to the set of parameterizations independent of both t and (x, y) and is singled out from the class (8.8) by the joint auxiliary system

$$f^i_t = f^i_j = 0.$$

Its equivalence algebra is the intersection of the equivalence algebras of the above subclasses and, therefore, equals

$$\langle \tilde{\mathcal{D}}_1,\, \partial_t,\, \tilde{\mathcal{D}}_2,\, \tilde{\mathrm{J}}(1),\, \tilde{\mathcal{X}}(1),\, \tilde{\mathcal{Y}}(1),\, \tilde{\mathcal{H}}(1),\, \tilde{\mathcal{G}}(\rho^1 x+\rho^2 y),\, \tilde{\mathcal{Z}}(\chi)\rangle,$$

where ρ^1, ρ^2 and χ are arbitrary smooth functions of t.

8.3.3 Normalized classes of generalized vorticity equations

In the course of computing the set of admissible transformations of a class of differential equations, it is often convenient to construct a hierarchy of normalized superclasses for this class [131, 133]. This is why here we also start with the quite general class of differential equations

$$\zeta_t - F(t,x,y,\psi,\psi_x,\psi_y,\zeta,\zeta_x,\zeta_y,\zeta_{xx},\zeta_{xy},\zeta_{yy}) = 0, \quad \zeta := \psi_{ii}, \tag{8.12}$$

where $(F_{\zeta_x}, F_{\zeta_y}, F_{\zeta_{xx}}, F_{\zeta_{xy}}, F_{\zeta_{yy}}) \neq (0,0,0,0,0)$, to assure that the generalized vorticity equations of the form (8.8) belong to this class. We use notations and agreements from the previous section. In particular, $z = (z_0, z_1, z_2) = (t, x, y)$, the indices i, j and k again run through $\{1, 2\}$, while the indices κ, λ, μ and ν range from 0 to 2.

Admissible transformations are determined using the direct method in terms of finite transformations. Namely, we aim to exhaustively describe point transformations of the form

$$\mathcal{T}\colon \quad \tilde{z}_\mu = Z^\mu(z,\psi), \quad \tilde{\psi} = \Psi(z,\psi), \quad \text{where} \quad J = \frac{\partial(Z^0, Z^1, Z^2, \Psi)}{\partial(z_0, z_1, z_2, \psi)} \neq 0,$$

which map an equation from class (8.12) to an equation from the same class. We express derivatives of the "old" dependent variable ψ with respect to the "old" independent variables z via derivatives of the "new" dependent variable $\tilde{\psi}$ with respect to the "new" independent variables $\tilde{z}$. The latter derivatives will be marked by tilde over ψ. Thus, the derivative of $\tilde{\psi}$ with respect to $\tilde{z}_\mu$ is briefly denoted by $\tilde{\psi}_\mu$, etc. Then we substitute the expressions for derivatives into the equation $\zeta_t - F = 0$, exclude the new principal derivative $\tilde{\psi}_{022}$ using the transformed equation $\tilde{\psi}_{022} = -\tilde{\psi}_{011} + \tilde{F}$, split with respect to parametric variables whenever this is possible and solve the obtained determining equations for Z^μ and Ψ supplemented with the inequality $J \neq 0$, considering all arising cases for values of the arbitrary element F and simultaneously finding the expression for $\tilde{F}$ via F, Z^μ and Ψ.

The first order derivatives ψ_μ are expressed in the following manner:

$$\psi_\mu = -\frac{\Psi_\mu - \tilde{\psi}_\nu Z^\nu_\mu}{\Psi_\psi - \tilde{\psi}_\nu Z^\nu_\psi} = -\frac{V_\mu}{V_\psi},$$

where we have introduced the notation $V = V(z,\psi,\tilde{z}) := \Psi(z,\psi) - \tilde{\psi}_\nu(\tilde{z}) Z^\nu(z,\psi)$ which is assumed as a function of the old dependent and independent variables and the new independent variables, so that $V_\mu = \Psi_\mu - \tilde{\psi}_\nu Z^\nu_\mu$ and $V_\psi = \Psi_\psi - \tilde{\psi}_\nu Z^\nu_\psi$. We will not try to express the old variables via the new variables by inverting the transformation. This is a conventional trick within the direct method, which essentially simplifies the whole consideration. In what follows we will also use three more abbreviations similar to V_μ:

$$U^{\mu\nu} := Z^\mu_\nu V_\psi - Z^\mu_\psi V_\nu, \quad W^{\mu\nu} := U^{\mu i} U^{\nu j} F_{\zeta_{ij}}, \quad P^\mu := U^{\mu 0} - U^{\mu i} F_{\zeta_i}.$$

Higher order derivatives are expressible in an analogous way. The Laplacian of ψ, e.g., reads

$$\psi_{ii} = V_\psi^{-3}(U^{\mu i}U^{\nu i}\tilde\psi_{\mu\nu} - V_\psi^2 V_{ii} + 2V_iV_\psi V_{i\psi} - V_i^2 V_{\psi\psi}).$$

For the class (8.12) considered here, we need the derivatives of the Laplacian up to second order. The highest derivatives required are of the form

$$\psi_{iijk} = V_\psi^{-5}U^{\mu i}U^{\nu i}U^{\kappa j}U^{\lambda k}\tilde\psi_{\mu\nu\kappa\lambda} + \dots,$$

where the tail contains only derivatives of $\tilde\psi$ up to order three.

Denote by G the left hand side of the equation obtained by substituting all the expressions for derivatives into (8.12). For the transformation $\mathcal{T}$ to be admissible, the condition $G_{\tilde\psi_{\mu\nu\kappa\lambda}} = 0$ has to be satisfied for any tuple of the subscripts (μ,ν,κ,λ) in which at least one of the subscripts equals 0. Under varying the subscripts, this condition leads to the following system:

$$\begin{aligned}
&G_{\tilde\psi_{0000}} = 0\colon \quad U^{0k}U^{0k}W^{00} = 0,\\
&G_{\tilde\psi_{000i}} = 0\colon \quad U^{0k}U^{0k}W^{0i} + U^{0k}U^{ik}W^{00} = 0,\\
&G_{\tilde\psi_{00ij}} = 0\colon \quad U^{0k}U^{0k}W^{ij} + 2U^{0k}U^{ik}W^{0j} + 2U^{0k}U^{jk}W^{0i} + U^{ik}U^{jk}W^{00} = 0.
\end{aligned}$$

Suppose that $U^{0k}U^{0k} \neq 0$. Then the above equations imply that $W^{\mu\nu} := U^{\mu i}U^{\nu j}F_{\zeta_{ij}} = 0$. If $\operatorname{rank}(U^{\mu i}) < 2$ then for any μ and ν

$$U^{\mu 1}U^{\nu 2} - U^{\mu 2}U^{\nu 1} = \left(\frac{\partial(Z^\mu, Z^\nu, \Psi)}{\partial(z_1, z_2, \psi)} - \tilde\psi_\kappa \frac{\partial(Z^\mu, Z^\nu, Z^\kappa)}{\partial(z_1, z_2, \psi)}\right) V_\psi = 0$$

and after splitting with respect to $\tilde\psi_\lambda$ we obtain that

$$\frac{\partial(Z^\mu, Z^\nu, \Psi)}{\partial(z_1, z_2, \psi)} = \frac{\partial(Z^\mu, Z^\nu, Z^\kappa)}{\partial(z_1, z_2, \psi)} = 0 \quad \text{or} \quad Z^\kappa_\psi = \Psi_\psi = 0,$$

but this contradicts the transformation nondegeneracy condition $J \neq 0$. Hence $\operatorname{rank}(U^{\mu i}) = 2$ and, therefore, the equation $U^{\mu i}U^{\nu j}F_{\zeta_{ij}} = 0$ sequentially implies that $U^{\nu j}F_{\zeta_{ij}} = 0$ and $F_{\zeta_{ij}} = 0$. Then, the necessary conditions $G_{\tilde\psi_{000}} = 0$ and $G_{\tilde\psi_{00i}} = 0$ for admissible transformations are respectively equivalent to the equations $U^{0k}U^{0k}P^0 = 0$ and $U^{0k}U^{0k}P^i + 2U^{0k}U^{ik}P^0 = 0$ which jointly gives in view of the condition $U^{0k}U^{0k} \neq 0$ that $P^\mu = 0$. Thus, we should have $\det(U^{\mu\nu}) = 0$. At the same time,

$$\begin{aligned}
\det(U^{\mu\nu}) &= {V_\psi}^2\big(|Z^0_\nu, Z^1_\nu, Z^2_\nu|V_\psi - |V_\nu, Z^1_\nu, Z^2_\nu|Z^0_\psi - |Z^0_\nu, V_\nu, Z^2_\nu|Z^1_\psi - |Z^0_\nu, Z^1_\nu, V_\nu|Z^2_\psi\big)\\
&= {V_\psi}^2 \frac{\partial(Z^0, Z^1, Z^2, V)}{\partial(z_0, z_1, z_2, \psi)} = {V_\psi}^2 J \neq 0
\end{aligned}$$

that leads to a contradiction. Therefore, the supposition $U^{0k}U^{0k} \neq 0$ is not true, i.e., $U^{0k}U^{0k} = 0$ and hence $U^{0k} = 0$. Substituting the expressions for U^{0k} and V into the last equation and splitting with respect to $\tilde\psi_\mu$, we derive the equations

$$Z^0_k Z^\mu_\psi = Z^0_\psi Z^\mu_k, \quad Z^0_k \Psi_\psi = Z^0_\psi \Psi_k.$$

The tuples (Z^μ_1, Ψ_1), (Z^μ_2, Ψ_2) and (Z^μ_ψ, Ψ_ψ) are not proportional since $J \neq 0$. This is why we finally obtain the first subset of determining equations $Z^0_k = Z^0_\psi = 0$. It follows from them that $Z^0_0 \neq 0$ (otherwise $J = 0$) and expressions for "old" derivatives with respect to only x and y

contain “new” derivatives only of the same type. In other words, derivatives of $\tilde\psi$ involving differentiation with respect to $\tilde t$ appear only in the expressions for ψ_{0aa} and we can simply split with respect to them via collecting their coefficients.

Equating the coefficients of $\tilde\psi_{012}$ leads, in view of the condition $Z^0_0\neq 0$, to the equation $U^{1k}U^{2k}=0$, i.e.,

$$\left(Z^1_k\Psi_\psi - Z^1_\psi\Psi_k + (Z^1_\psi Z^2_k - Z^1_k Z^2_\psi)\tilde\psi_2\right)\left(Z^2_k\Psi_\psi - Z^2_\psi\Psi_k - (Z^1_\psi Z^2_k - Z^1_k Z^2_\psi)\tilde\psi_1\right)=0. \quad (8.13)$$

We split equation (8.13) with respect to $\tilde\psi_1$ and $\tilde\psi_2$. Collecting the coefficients of $\tilde\psi_1\tilde\psi_2$ gives the equation $(Z^1_\psi Z^2_k - Z^1_k Z^2_\psi)(Z^1_\psi Z^2_k - Z^1_k Z^2_\psi)=0$, or equivalently $Z^1_\psi Z^2_k - Z^1_k Z^2_\psi=0$. As $\operatorname{rank}(Z^i_1, Z^i_2, Z^i_\psi)=2$, this implies that $Z^i_\psi=0$ and, therefore, $\Psi_\psi\neq 0$. Consequently, equation (8.13) is reduced to $Z^1_k Z^2_k=0$.

The derivative $\tilde\psi_{022}$ is assumed principal, $\tilde\psi_{022}=-\tilde\psi_{011}+\tilde F$. Hence another third order derivative of the above type appropriate for splitting is only $\tilde\psi_{011}$. The corresponding equation $Z^1_kZ^1_k=Z^2_kZ^2_k:=L$ joint with the equation $Z^1_kZ^2_k=0$ implies that the functions Z^1 and Z^2 satisfy the Cauchy–Riemann system $Z^1_1=\varepsilon Z^2_2$, $Z^1_2=-\varepsilon Z^2_1$, where $\varepsilon=\pm1$, and hence $Z^i_{kk}=0$. Note that $L\neq 0$ since $J\neq 0$.

Analogously, collecting the coefficients of $\tilde\psi_{0i}$ and further splitting with respect to $\tilde\psi_j$ lead to the equations $Z^i_kZ^j_k\Psi_{\psi\psi}=0$ and $Z^i_k\Psi_{k\psi}=0$. Therefore, $\Psi_{\psi\psi}=0$ and $\Psi_{k\psi}=0$. Here we take into account the inequalities $L\neq 0$ and $\det(Z^i_k)\neq 0$.

We do not have more possibilities for splitting. The derived system of determining equations consists of the equations

$$Z^0_k=Z^0_\psi=0,\quad Z^i_\psi=0,\quad Z^1_kZ^2_k=0,\quad Z^1_kZ^1_k=Z^2_kZ^2_k,\quad \Psi_{\psi\psi}=\Psi_{k\psi}=0.$$

The remaining terms determine the transformation rule for the arbitrary element F. This is why any point transformation satisfying the above determining equations maps every equation from class (8.12) to an equation from the same class and, therefore, belongs to the equivalence group $G^\sim_1$ of class (8.12). In other words, any admissible point transformation of class (8.12) is induced by a transformation from $G^\sim_1$, i.e., class (8.12) is normalized. As a result, we have the following theorem.

Theorem 8.2. *Class* (8.12) *is normalized. Its equivalence group $G^\sim_1$ consists of the transformations*

$$\tilde t=T(t),\quad \tilde x=Z^1(t,x,y),\quad \tilde y=Z^2(t,x,y),\quad \tilde\psi=\Upsilon(t)\psi+\Phi(t,x,y),$$
$$\tilde F=\frac{1}{T_t}\left(\frac{\Upsilon}{L}F+\left(\frac{\Upsilon}{L}\right)_0\zeta+\left(\frac{\Phi_{ii}}{L}\right)_0-\frac{Z^i_tZ^i_j}{L}\left(\frac{\Upsilon}{L}\zeta_j+\left(\frac{\Upsilon}{L}\right)_j\zeta+\left(\frac{\Phi_{ii}}{L}\right)_j\right)\right),$$

where T, Z^i, Υ and Φ are arbitrary smooth functions of their arguments, satisfying the conditions $Z^1_kZ^2_k=0$, $Z^1_kZ^1_k=Z^2_kZ^2_k:=L$, $T_t\Upsilon L\neq 0$, and the subscripts 1 and 2 denote differentiation with respect to x and y, respectively.

The expression for the transformed vorticity is also simple: $\tilde\zeta=L^{-1}(\Upsilon\zeta+\Phi_{ii})$.

Remark. The continuous component of unity of the group $G^\sim_1$ consists of the transformations from $G^\sim_1$ with $T_t>0$, $\varepsilon=1$ and $\Upsilon>0$. Therefore, a complete set of independent discrete transformations in $G^\sim_1$ is exhausted by the uncoupled changes of the signs of t, y and ψ. In particular, the value $\varepsilon=-1$ corresponds to alternating the sign of y.

Consider the subclass of class (8.12), singled out by the constraints $F_\psi = 0$, $F_{\psi_x} = -\zeta_y$ and $F_{\psi_y} = \zeta_x$, i.e., the class consisting of the equations of the form

$$\zeta_t + \psi_x\zeta_y - \psi_y\zeta_x = H(t, x, y, \zeta, \zeta_x, \zeta_y, \zeta_{xx}, \zeta_{xy}, \zeta_{yy}), \quad \zeta := \psi_{ii}, \tag{8.14}$$

where H is an arbitrary smooth function of its arguments, which is assumed as an arbitrary element instead of $F = H - \psi_x\zeta_y + \psi_y\zeta_x$. The class (8.14) is still a superclass of the class (8.8).

Theorem 8.3. *Class* (8.14) *is normalized. The equivalence group* $G_2^\sim$ *of this class is formed by the transformations*

$$\begin{aligned}
&\tilde t = \tau, \quad \tilde x = \lambda(x\mathfrak{c} - y\mathfrak{s}) + \gamma^1, \quad \varepsilon\tilde y = \lambda(x\mathfrak{s} + y\mathfrak{c}) + \gamma^2,\\
&\tilde\psi = \varepsilon\frac{\lambda}{\tau_t}\left(\lambda\psi + \frac{\lambda}{2}\beta_t(x^2+y^2) - \gamma^1_t(x\mathfrak{s} + y\mathfrak{c}) + \gamma^2_t(x\mathfrak{c} - y\mathfrak{s})\right) + \delta + \frac{\sigma}{2}(x^2+y^2),\\
&\tilde H = \frac{\varepsilon}{{\tau_t}^2}\left(H - \frac{\tau_{tt}}{\tau_t}\zeta - \frac{\lambda_t}{\lambda}(x\zeta_x + y\zeta_y) + 2\beta_{tt} - 2\frac{\tau_{tt}}{\tau_t}\beta_t\right) - \frac{\delta_y + \sigma y}{\tau_t\lambda^2}\zeta_x + \frac{\delta_x + \sigma x}{\tau_t\lambda^2}\zeta_y\\
&\qquad + \frac{2}{\tau_t}\left(\frac{\sigma}{\lambda^2}\right)_t,
\end{aligned} \tag{8.15}$$

where $\varepsilon = \pm1$, $\mathfrak{c} = \cos\beta$, $\mathfrak{s} = \sin\beta$; τ, λ, β, γ^i *and* σ *are arbitrary smooth functions of* t *satisfying the conditions* $\lambda > 0$ *and* $\tau_t \neq 0$; $\delta = \delta(t, x, y)$ *runs through the set of solutions of the Laplace equation* $\delta_{xx} + \delta_{yy} = 0$.

Proof. The class (8.14) is a subclass of the class (8.12) and the class (8.12) is normalized. Therefore, any admissible transformation of the class (8.14) is generated by a transformation from the equivalence group $G_1^\sim$ of the superclass. It is only necessary to derive the additional restrictions on transformation parameters caused by narrowing the class.

The group $G_1^\sim$ is a usual equivalence group [118], i.e., in contrast to different generalizations of equivalence groups [94, 131], it consists of point transformations of the joint space of the equation variables and arbitrary elements, and the components of transformations for the variables do not depend on the arbitrary elements. Any transformation from $G_1^\sim$ is additionally projectable to the space of the independent variables and the space of the single variable t. This is why it already becomes convenient, in contrast to the proof of Theorem 8.2, to express the new derivatives via old ones. Then we substitute the expressions for new derivatives into the transformed equation $\tilde\zeta_{\tilde t} + \tilde\psi_{\tilde x}\tilde\zeta_{\tilde y} - \tilde\psi_{\tilde y}\tilde\zeta_{\tilde x} = \tilde H$, exclude the principal derivative ψ_{tyy} using the equation

$$\psi_{tyy} = -\psi_{txx} - \psi_x\zeta_y + \psi_y\zeta_x + H,$$

split with respect to parametric variables whenever this is possible and solve the obtained determining equations. As equations from the class (8.14) involve derivatives ψ_x and ψ_y in an explicitly defined (linear) manner, we can split with respect to these derivatives, simply collecting their coefficients. Since these coefficients do not involve the arbitrary element H, we can further split them with respect to other derivatives. As a result, we obtain the equations

$$\Upsilon = \varepsilon\frac{L}{T_t}, \quad L_i = 0, \quad \Phi_{jji} = 0,$$

where $\varepsilon = \pm1$ and other notations are defined in the proof of Theorem 8.2. Therefore, L and Φ_{jj} are functions of t only. As $L > 0$, we can introduce the function $\lambda = \sqrt{L}$ of t. Acting by the Laplace operator ∂_{jj} on the conditions $Z^1_kZ^1_k = \lambda^2$ and $Z^2_kZ^2_k = \lambda^2$ and taking into account that Z^i are solutions of the Laplace equation, $Z^i_{kk}=0$, we derive the important differential consequences $Z^i_{jk}=0$, which imply that the functions Z^i are affine in (x,y). Hence there exists a function $\beta=\beta(t)$ such that $Z^1_1=\lambda\mathfrak{c}$ and $Z^1_2=-\lambda\mathfrak{s}$, where $\mathfrak{c}=\cos\beta$ and $\mathfrak{s}=\sin\beta$, and, therefore, $Z^1_1=\varepsilon\lambda\mathfrak{s}$ and $Z^1_2=\varepsilon\lambda\mathfrak{c}$. We re-denote T by τ for the sake of notation consistency and represent Φ in the following form[1]:

$$\Phi=\delta(t,x,y)+\frac{\sigma}{2}(x^2+y^2)+\varepsilon\frac{\lambda}{\tau_t}\left(\frac{\lambda}{2}\beta_t(x^2+y^2)-\gamma^1_t(x\mathfrak{s}+y\mathfrak{c})+\gamma^2_t(x\mathfrak{c}-y\mathfrak{s})\right),$$

where σ is a function of t and $\delta=\delta(t,x,y)$ is a solution of the Laplace equation $\delta_{xx}+\delta_{yy}=0$. Collecting the terms without ψ_x and ψ_y gives the transformation for the arbitrary element H.

Similarly to the proof of Theorem 8.2, any transformation from $G^\sim_1$ satisfying the above additional constraints maps every equation from the class (8.14) to an equation from the same class and, therefore, belongs to the equivalence group $G^\sim_2$ of the class (8.14). In other words, any admissible point transformation of the class (8.14) is induced by a transformation from $G^\sim_2$, i.e., the class (8.14) is normalized. □

Remark. The transformations from the equivalence group $G^\sim_2$, which are associated with the parameter-function δ depending only on t, and only such transformations identically act on the arbitrary element H and, therefore, their projections to the space of independent and dependent variables form the kernel (intersection) of point symmetry groups of the class (8.14).

Corollary 8.1. *The subclass of the class* (8.14) *singled out by the constraint $H_\zeta=0$ is normalized. Its equivalence group $G^\sim_3$ consists of the elements of $G^\sim_2$ with $\tau_{tt}=0$.*

Proof. As the vorticity and its derivatives are transformed by elements of $G^\sim_2$ according to the formulas

$$\tilde\zeta=\frac{\varepsilon}{\tau_t}(\zeta+\beta_t)+2\frac{\sigma}{\lambda^2},\quad \tilde\zeta_i=\frac{\varepsilon Z^i_j}{\tau_t\lambda^2}\zeta_j, \tag{8.16}$$

it follows from (8.15) under the constraints $H_\zeta=0$ and $\tilde H_{\tilde\zeta}=0$ that $\tau_{tt}=0$. The rest of the proof is similar to the end of the proof of Theorem 8.3. □

Corollary 8.2. *The subclass of the class* (8.14) *singled out by the constraints $H_i=0$ is normalized. Its equivalence group $G^\sim_4$ consists of the elements of $G^\sim_2$ with $\lambda_t=0$, $\sigma=0$ and $\delta_{ij}=0$.*

Proof. As any admissible transformation of the class (8.14) has the form (8.15) and, therefore, the vorticity and its derivatives are transformed according to (8.16), the system $\tilde H_{\tilde x}=0$, $\tilde H_{\tilde y}=0$ is equivalent to the system $\tilde H_x=0$, $\tilde H_y=0$. After differentiating the last equation in (8.15) with respect to x and y and splitting with respect to ζ_x and ζ_y, we derive all the above additional constraints on transformation parameters. The rest of the proof is similar to the end of the proof of Theorem 8.3. □

[1] There is an ambiguity in representations of Z^i and Φ. For example, the last summand in the representation of Φ can be omitted. The usage of the above complicated representations is motivated by a few reasons: the consistency with the notation of basis operators of the equivalence algebra $\mathfrak{g}^\sim_1$ from Theorem 8.1, the simplification of the expression for the transformed arbitrary element $\tilde H$ and the convenience of studying admissible transformations within subclasses of the class (8.14).

Corollary 8.3. *The subclass of the class* (8.14) *singled out by the constraints* $H_\zeta = 0$ *and* $H_i = 0$ *is normalized. Its equivalence group* $G_5^\sim$ *consists of the elements of* $G_2^\sim$ *with* $\tau_{tt} = 0$, $\lambda_t = 0$, $\sigma = 0$ *and* $\delta_{ij} = 0$.

Proof. The subclass under consideration is normalized as it is the intersection of the normalized subclasses from Corollaries 8.1 and 8.2. Therefore, we also have $G_5^\sim = G_3^\sim \cap G_4^\sim$. □

Remark. For the subclass from Corollary 8.3, the kernel of point symmetry groups is essentially extended in comparison with the whole class (8.14). It is formed by the projections of elements of the equivalence group $G_2^\sim$, associated with the parameter-functions γ^1 and γ^2 and the parameter-function δ depending only on t, to the space of independent and dependent variables, cf. Section 8.3.4.

A further narrowing is given by the condition that the arbitrary element H with $H_\zeta = 0$ is a total divergence with respect to the space variables, i.e., $H = \mathrm{D}_i f^i$ for some differential functions $f^i = f^i(t, x, y, \zeta_x, \zeta_y)$. The corresponding subclass rewritten in the terms of f^i coincides with the class (8.8) and is singled out from the class (8.14) by the constraints $H_\zeta = 0$ and $\mathsf{E}H = 0$, where $\mathsf{E} = \partial_\zeta - \mathrm{D}_i\partial_{\zeta_i} + \sum_{i\leqslant j} \mathrm{D}_i\mathrm{D}_j\partial_{\zeta_{ij}} + \dots$ is the associated Euler operator. In this Euler operator, the role of independent and dependent variables is played by (x, y) and ζ, respectively, and the variable t is assumed as a parameter. The vorticity ζ can be considered in E as the dependent variable instead of ψ since the arbitrary element H depends only on combinations of derivatives of ψ being derivatives of ζ.

Remark. It is obvious that the arbitrary element H satisfies the constraints $H_\zeta = 0$ and $\mathsf{E}H = 0$ if it is represented in the form $H = \mathrm{D}_i f^i$ for some differential functions $f^i = f^i(t, x, y, \zeta_x, \zeta_y)$. The converse claim should be proved. Thus, the constraint $\mathsf{E}H = 0$ implies the representation $H = \mathrm{D}_i f^i$ for some differential functions $f^i(t, x, y, \zeta, \zeta_x, \zeta_y)$, which may depend on ζ. Substituting this representation into the constraint $H_\zeta = 0$ and splitting the resulting equations with respect to the second derivatives of ζ, we obtain the following system of PDEs for the functions f^i: $f^i_{\zeta i} + f^i_{\zeta\zeta_i}\zeta_i = 0$, $f^1_{\zeta\zeta_1} = 0$, $f^2_{\zeta\zeta_2} = 0$, $f^1_{\zeta\zeta_2} + f^2_{\zeta\zeta_1} = 0$. Its general solution has the form $f^1 = D_2\Psi + \tilde f^1$ and $f^2 = -D_1\Psi + \tilde f^2$ for some smooth functions $\Psi = \Psi(t, x, y, \zeta)$ and $\tilde f^i = \tilde f^i(t, x, y, \zeta_x, \zeta_y)$. The first summands in the expressions for f^i can be neglected due to the gauge equivalence in the set of arbitrary elements (f^1, f^2). As a result, we construct the necessary representation for the arbitrary element H.

Corollary 8.4. *The class* (8.8) *is normalized. The equivalence group* $G_6^\sim$ *of this class represented in terms of the arbitrary element* H *consists of the elements of* $G_2^\sim$ *with* $\tau_{tt} = 0$ *and* $\lambda_t = 0$. *The arbitrary elements* f^i *are transformed in the following way:*

$$\begin{aligned}
\tilde f^1 &= \varepsilon\lambda\frac{f^1\mathfrak{c} - f^2\mathfrak{s}}{\tau_t^2} + \left(\frac{\delta}{\tau_t\lambda} + \frac{\sigma}{2\tau_t\lambda}(x^2+y^2) - \frac{\varepsilon\chi}{\lambda^2}\right)(\zeta_x\mathfrak{s} + \zeta_y\mathfrak{c})\\
&\quad + (\varepsilon\lambda^2\beta_{tt} + \tau_t\sigma_t)\frac{x\mathfrak{c} - y\mathfrak{s}}{{\tau_t}^2\lambda} - \varepsilon\frac{\rho_x\mathfrak{s} + \rho_y\mathfrak{c}}{\lambda^2},\\
\tilde f^2 &= \lambda\frac{f^1\mathfrak{s} + f^2\mathfrak{c}}{{\tau_t}^2} - \varepsilon\left(\frac{\delta}{\tau_t\lambda} + \frac{\sigma}{2\tau_t\lambda}(x^2+y^2) - \frac{\varepsilon\chi}{\lambda^2}\right)(\zeta_x\mathfrak{c} - \zeta_y\mathfrak{s})\\
&\quad + \varepsilon(\varepsilon\lambda^2\beta_{tt} + \tau_t\sigma_t)\frac{x\mathfrak{s} + y\mathfrak{c}}{{\tau_t}^2\lambda} + \frac{\rho_x\mathfrak{c} - \rho_y\mathfrak{s}}{\lambda^2},
\end{aligned} \tag{8.17}$$

where $\chi = \chi(t)$ *and* $\rho = \rho(t, x, y)$ *are arbitrary functions of their arguments.*

Proof. The class (8.8) is contained in the normalized subclass of the class (8.14) singled out by the constraint $H_\zeta = 0$. Therefore, any admissible transformation of the class (8.8) is generated by an element of $G_2^\sim$ with $\tau_{tt} = 0$, and the corresponding transformations of the space variables are affine with respect to these variables, $Z^i_{jk} = 0$. Then $\tilde{\mathrm{D}}_j \tilde f^j = \mathrm{D}_i(\lambda^{-2} Z^j_i \tilde f^j)$, i.e., the differential function $\tilde H$ is a total divergence with respect to the new space variables if and only if it is a total divergence with respect to the old space variables. Applying the Euler operator E to the last equality in (8.15) under the conditions $H_\zeta = 0$, $\tilde H_{\tilde\zeta} = 0$, $\mathsf{E}H = 0$, $\tilde{\mathsf{E}}\tilde H = 0$ and $\tau_{tt} = 0$, we derive the additional constraint $\lambda_t = 0$. The remaining part of the proof of normalization of the class (8.8) and its equivalence group is analogous to the end of the proof of Theorem 8.3.

In order to construct the transformations of the arbitrary elements f^i, we represent the right hand side of the last equality in (8.15) as a total divergence: $\tilde H = \mathrm{D}_i h^i$, where

$$h^1 = \frac{\varepsilon}{\tau_t^2}(f^1 + \beta_{tt}x) + \frac{\sigma_t}{\tau_t\lambda^2}x + \left(\delta + \frac{\sigma}{2}(x^2+y^2)\right)\frac{\zeta_y}{\tau_t\lambda^2},$$
$$h^2 = \frac{\varepsilon}{\tau_t^2}(f^2 + \beta_{tt}y) + \frac{\sigma_t}{\tau_t\lambda^2}y - \left(\delta + \frac{\sigma}{2}(x^2+y^2)\right)\frac{\zeta_x}{\tau_t\lambda^2},$$

As $\tilde H = \tilde{\mathrm{D}}_j \tilde f^j = \mathrm{D}_i h^i = \tilde{\mathrm{D}}_j Z^j_i h^i$, the pair of the differential functions $\tilde f^j - Z^j_i h^i$ is a null divergence, $\tilde{\mathrm{D}}_i(\tilde f^j - Z^j_i h^i) = 0$. In view of Theorem 4.24 from [115] there exists a differential function Q depending on t, x, y and derivatives of ζ such that $\tilde f^1 - Z^1_i h^i = -\tilde{\mathrm{D}}_2 Q$ and $\tilde f^2 - Z^2_i h^i = \tilde{\mathrm{D}}_2 Q$. As $\tilde{\mathrm{D}}_i Q$ and, therefore, $\mathrm{D}_i Q$ should be functions of t, x, y, ζ_x and ζ_y, the function Q is represented in the form $Q = \chi(t)\zeta + \rho(t,x,y)$ for some smooth functions $\chi = \chi(t)$ and $\rho = \rho(t,x,y)$. □

Remark. The equivalence transformations associated with the parameter-functions χ and ρ are identical with respect to both the independent and dependent variables, i.e., they transform only arbitrary elements with no effect on the corresponding equation and, therefore, are *trivial* [83, p. 53] or *gauge* [131, Section 2.5] equivalence transformations. These transformations arise due to the special representation of the arbitrary element H as a total divergence and form a normal subgroup of the entire equivalence group considered in terms of the arbitrary elements f^1 and f^2, called the *gauge equivalence group* of the class (8.8).

Remark. The continuous component of unity of the group $G_6^\sim$ is singled out from $G_6^\sim$ by the conditions $\tau_t > 0$ and $\varepsilon = 1$. Therefore, a complete set of independent discrete transformations in $G_6^\sim$ is exhausted by alternating signs either in the tuple (t,ψ) or in the tuple (y,ψ,f^1).

Consider the subclass of the class (8.8), singled out by the further auxiliary equation $f^i_j = 0$, i.e., the class of equations

$$\zeta_t + \{\psi,\zeta\} = \mathrm{D}_i f^i(t,\zeta_x,\zeta_y), \quad \zeta := \psi_{ii}, \tag{8.18}$$

with the arbitrary elements $f^i = f^i(t,\zeta_x,\zeta_y)$.

Remark. Rewritten in the terms of H, the class (8.18) is a well-defined subclass of (8.14). It is singled out from the class (8.14) by the constraints $\mathsf{E}H = 0$, $H_\zeta = 0$, $H_i = 0$ and $\zeta_{ij}H_{\zeta_{ij}} = H$. Indeed, the representation $H = \mathrm{D}_i f^i(t,\zeta_x,\zeta_y)$ obviously implies that the arbitrary element H does not depend on x, y and ζ, is annulated by the Euler operator E and is a (homogenous) linear function in the totality of the derivatives ζ_{ij}. Hence all the above constraints are necessary. Conversely, the constraint $\mathsf{E}H = 0$ implies that the arbitrary element H is affine in the totality of ζ_{ij} and, therefore, in view of the constraint $\zeta_{ij}H_{\zeta_{ij}} = H$ it is a (homogenous) linear function in these derivatives of ζ. As a result, we have the representation $H = h^{ij}\zeta_{ij}$, where the coefficients h^{ij}, $h^{12} = h^{21}$, depend solely on t, ζ_x and ζ_y since $H_\zeta = 0$ and $H_i = 0$. Then the constraint $\mathsf{E}H = 0$ is equivalent to the single equation

$$2h^{12}_{\zeta_1\zeta_2} = h^{11}_{\zeta_2\zeta_2} + h^{22}_{\zeta_1\zeta_1}$$

whose general solutions is represented in the form $h^{11} = f^1_{\zeta_1}$, $h^{12} = f^1_{\zeta_2} + f^2_{\zeta_1}$ and $h^{22} = f^2_{\zeta_2}$ for some differential functions $f^i = f^i(t, \zeta_x, \zeta_y)$. This finally gives the necessary representation for H.

Remark. In view of the previous remark, the subclass of the class (8.14), singled out by the constraints $\mathsf{E}H = 0$, $H_\zeta = 0$ and $H_i = 0$ is a proper superclass for the class (8.18) rewritten in the terms of H. This superclass of (8.18) is normalized since it is the intersection of the normalized class from Corollary 8.3 and the normalized class (8.8). Its equivalence group coincides with the group $G^\sim_5$ described in Corollary 8.3.

In a way analogous to the above proofs, the normalization of the superclass and formulas (8.15) and (8.17) imply the following assertion.

Corollary 8.5. *The class* (8.18) *is normalized. The equivalence group $G^\sim_7$ of this class represented in terms of the arbitrary element H consists of the elements of $G^\sim_2$ with $\tau_{tt} = 0$, $\lambda_t = 0$, $\beta_{tt} = 0$, $\sigma = 0$ and $\delta_i = 0$. The arbitrary elements f^i are transformed according to* (8.17)*, where additionally $\rho_{ij} = 0$.*

Remark. The above consideration of normalized classes is intended for the description of invariant parameterizations of the forms (8.8) and (8.18). The hierarchy of normalized classes constructed is, in some sense, minimal and optimal for this purpose. It can be easily extended with related normalized classes. For instance, the subclass singled out from the class (8.14) by the constraints $\mathsf{E}H = 0$ is normalized. Other hierarchies of normalized classes, which are related to the vorticity equation (8.5) and different from the hierarchy presented, can be constructed.

Remark. In fact, all subclasses of generalized vorticity equations studied in this section are strongly normalized, cf. [131].

8.3.4 Parameterization via direct group classification

As proved in Section 8.3.3, the class (8.18) is normalized. Its equivalence algebra $\mathfrak{g}^\sim_2$ (cf. Section 8.3.2) can be represented as a semidirect sum $\mathfrak{g}^\sim_2 = \tilde{\mathfrak{i}} \ni \tilde{\mathfrak{a}}$, where $\tilde{\mathfrak{i}} = \langle \tilde{\mathcal{X}}(\gamma^1),\ \tilde{\mathcal{Y}}(\gamma^2),\ \tilde{\mathcal{Z}}(\chi)\rangle$ and $\tilde{\mathfrak{a}} = \langle \tilde{\mathcal{D}}_1, \tilde{\mathcal{D}}_2, \partial_t, \tilde{\mathrm{J}}^1, \tilde{\mathrm{J}}^t,\ \tilde{\mathcal{K}}(\delta),\ \tilde{\mathcal{G}}(\rho^1 x + \rho^2 y)\rangle$ are an ideal and a subalgebra of $\mathfrak{g}^\sim_2$, respectively. Here γ^1, γ^2, ρ^1, ρ^2, δ and χ run through the set of smooth functions of the variable t and we use the notation $\tilde{\mathrm{J}}^1 = \tilde{\mathrm{J}}(1)$, $\tilde{\mathrm{J}}^t = \tilde{\mathrm{J}}(t)$ and $\tilde{\mathcal{K}}(\delta) = \tilde{\mathcal{H}}(\delta) - \tilde{\mathcal{Z}}(\delta)$. The intersection (*kernel*) of the maximal Lie invariance algebras of equations from class (8.18) is

$$\mathfrak{g}^\cap_2 = \langle \mathcal{X}(\gamma^1),\ \mathcal{Y}(\gamma^2),\ \mathcal{Z}(\chi)\rangle = \mathrm{P}\,\tilde{\mathfrak{i}}.$$

In other words, the complete infinite dimensional part $\mathrm{P}\,\tilde{\mathfrak{i}}$ of the projection of the equivalence algebra $\mathfrak{g}^\sim_2$ to the space of variables (t, x, y, ψ) is already a Lie invariance algebra for any equation from the class (8.18). Therefore, any Lie symmetry extension is only feasible via (finite-dimensional) subalgebras of the five-dimensional solvable algebra

$$\mathfrak{a} = \langle \mathcal{D}_1,\ \partial_t,\ \mathcal{D}_2,\ \mathrm{J},\ \mathrm{J}^t\rangle = \mathrm{P}\tilde{\mathfrak{a}}.$$

In other words, for any values of the arbitrary elements $f^i = f^i(t,\zeta_x,\zeta_y)$ the maximal Lie invariance algebra $\mathfrak{g}_f^{\max}$ of the corresponding equation $\mathcal{L}_f$ from the class (8.18) is represented in the form $\mathfrak{g}_f^{\max} = \mathfrak{g}_f^{\rm ext} \Subset \mathfrak{g}_2^{\cap}$, where $\mathfrak{g}_f^{\rm ext}$ is a subalgebra of $\mathfrak{a}$. A nonzero linear combination of the operators J and J^t is a Lie symmetry operator of the equation $\mathcal{L}_f$ if and only if this equation is invariant with respect to the algebra $\langle \mathrm{J}, \mathrm{J}^t\rangle$. Therefore, for any extension within the class (8.18) we have that either $\mathfrak{g}_f^{\rm ext} \cap \langle \mathrm{J}, \mathrm{J}^t\rangle = \{0\}$ or $\mathfrak{g}_f^{\rm ext} \supset \langle \mathrm{J}, \mathrm{J}^t\rangle$, i.e.,

$$\dim(\mathfrak{g}_f^{\rm ext} \cap \langle \mathrm{J}, \mathrm{J}^t\rangle) \in \{0, 2\}. \tag{8.19}$$

Moreover, as $\mathrm{P}\mathfrak{g}_2^{\sim} = \mathfrak{g}_0$, the maximal Lie invariance algebra of the inviscid barotropic vorticity equation (8.5), the normalization of class (8.18) means that only subalgebras of $\mathfrak{g}_0$ can be used to construct spatially independent parameterization schemes within the class (8.18). That is, for such parameterizations, the approach from [113] based on inverse group classification is quite natural and gives the same exhaustive result as direct group classification. Due to the normalization, the complete realization of preliminary group classification of equations from the class (8.18) is also equivalent to its direct group classification which can be carried out for this class with the algebraic method.

Note that the class (8.18) possesses the nontrivial gauge equivalence algebra

$$\mathfrak{g}^{\rm gauge} = \langle \tilde{\mathcal{K}}(\delta),\, \tilde{\mathcal{G}}(\rho^1 x + \rho^2 y)\rangle,$$

cf. the second remark after Theorem 8.1. As we have $\mathrm{P}\mathfrak{g}^{\rm gauge} = \{0\}$, the projections of operators from $\mathfrak{g}^{\rm gauge}$ obviously do not appear in $\mathfrak{g}_f^{\rm ext}$ for any value of f. At the same time, they are essential for finding all possible parameterizations that admit symmetry extensions.

Therefore two equivalent ways for the further use of the algebraic method in this problem depending on subalgebras of what algebra will be classified.

As a first impression, the optimal way is to construct a complete list of inequivalent subalgebras of the Lie algebra $\mathfrak{a}$ and then substitute basis operators of each obtained subalgebra to the infinitesimal invariance criterion in order to derive the associated system of equations for f^i that should be integrated. The algebra $\mathfrak{a}$ is finite dimensional and has the structure of a direct sum, $\mathfrak{a} = \langle \mathcal{D}_1,\, \partial_t,\, \mathrm{J},\, \mathrm{J}^t\rangle \oplus \langle \mathcal{D}_2\rangle$. The first summand is the four-dimensional Lie algebra $\mathfrak{g}_{4.8}^{-1}$ in accordance with Mubarakzyanov's classification of low-dimensional Lie algebras [103] whose nilradical is isomorphic to the Weyl (Bianchi II) algebra $\mathfrak{g}_{3.1}$. The classification of inequivalent subalgebra up to the equivalence relation generated by the adjoint action of the corresponding Lie group on $\mathfrak{a}$ is a quite simple problem. Moreover, the set of subalgebras to be used is reduced after taking into account the condition (8.19). At the same time, the derived systems for f^i consist of second order partial differential equations and have to be integrated up to $G_7^{\sim}$-equivalence.

This is why another way is optimal. It is based on the fact that $\mathfrak{g}_f^{\rm ext}$ coincides with a subalgebra $\mathfrak{g}$ of $\mathfrak{a}$ if and only if there exists a subalgebra $\tilde{\mathfrak{g}}$ of $\tilde{\mathfrak{a}}$ such that $\mathrm{P}\tilde{\mathfrak{g}} = \mathfrak{g}$ and the arbitrary elements f^i satisfy the equations

$$\xi^0 f^i_t + \theta^j f^i_{\zeta_j} = \varphi^i \tag{8.20}$$

for any operator $\tilde{\mathcal{Q}}$ from $\tilde{\mathfrak{g}}$, where ξ^0 and θ^j are coefficients of ∂_t and ∂_{ζ_j} in $\tilde{\mathcal{Q}}$, respectively. In fact, the system (8.20) is the invariant surface condition for the operator $\tilde{\mathcal{Q}}$ and the functions f^i depending only on t and ζ^j. This system is not compatible for any operator from $\tilde{\mathfrak{a}}$ of the form $\tilde{\mathcal{Q}} = \tilde{\mathcal{K}}(\delta) + \tilde{\mathcal{G}}(\rho^1 x + \rho^2 y)$, where at least one of the parameter-functions δ, ρ^1 or ρ^2 does not vanish. In other words, each operator from $\tilde{\mathfrak{g}}$ should have a nonzero part belonging to $\langle \tilde{\mathcal{D}}_1, \tilde{\mathcal{D}}_2, \partial_t, \tilde{\mathrm{J}}^1, \tilde{\mathrm{J}}^t\rangle$

and hence $\dim \mathrm{P}\tilde{\mathfrak{g}} = \dim \tilde{\mathfrak{g}} \leqslant 5$. Taking into account also the condition (8.19), we obtain the following algorithm for classification of possible Lie symmetry extensions within the class (8.18):

1. We classify $G_7^{\sim}$-inequivalent subalgebras of $\tilde{\mathfrak{a}}$ each of which satisfies the conditions $\dim \mathrm{P}\tilde{\mathfrak{g}} = \dim \tilde{\mathfrak{g}}$ and $\dim(\tilde{\mathfrak{g}} \cap \langle \mathrm{J}, \mathrm{J}^t \rangle) \in \{0, 2\}$. Adjoint actions corresponding to operators from $\tilde{\mathfrak{i}}$ can be neglected.
2. We fix a subalgebra $\tilde{\mathfrak{g}}$ from the list constructed in the first step. This algebra is necessarily finite dimensional, $\dim \tilde{\mathfrak{g}} \leqslant 5$. We solve the system consisting of equations of the form (8.20), where the operator $\tilde{\mathcal{Q}}$ runs through a basis of $\tilde{\mathfrak{g}}$. For every solution of this system we have $\mathfrak{g}_f^{\mathrm{ext}} = \mathrm{P}\tilde{\mathfrak{g}}$.
3. Varying $\tilde{\mathfrak{g}}$, we get the required list of values of the arbitrary elements (f^1, f^2) and the corresponding Lie symmetry extensions.

In order to realize the first step of the algorithm, we list the nonidentical adjoint actions related to basis elements of $\tilde{\mathfrak{a}}$:

$$
\begin{aligned}
&\mathrm{Ad}(e^{\varepsilon\partial_t})\mathcal{D}_1 = \mathcal{D}_1 - \varepsilon\partial_t, && \mathrm{Ad}(e^{\varepsilon\mathcal{D}_1})\partial_t = e^{\varepsilon}\partial_t,\\
&\mathrm{Ad}(e^{\varepsilon\mathrm{J}^t})\mathcal{D}_1 = \mathcal{D}_1 + \varepsilon\mathrm{J}^t, && \mathrm{Ad}(e^{\varepsilon\mathcal{D}_1})\mathrm{J}^t = e^{-\varepsilon}\mathrm{J}^t,\\
&\mathrm{Ad}(e^{\varepsilon\mathcal{K}(\delta)})\mathcal{D}_1 = \mathcal{D}_1 + \varepsilon\mathcal{K}(t\delta_t + \delta), && \mathrm{Ad}(e^{\varepsilon\mathcal{D}_1})\mathcal{K}(\delta) = \mathcal{K}(e^{-\varepsilon}\delta(e^{-\varepsilon}t)),\\
&\mathrm{Ad}(e^{\varepsilon\mathcal{G}(\rho)})\mathcal{D}_1 = \mathcal{D}_1 + \varepsilon\mathcal{G}(t\rho_t + 2\rho), && \mathrm{Ad}(e^{\varepsilon\mathcal{D}_1})\mathcal{G}(\rho) = \mathcal{G}(e^{-2\varepsilon}\rho(e^{-\varepsilon}t, x, y)),\\
&\mathrm{Ad}(e^{\varepsilon\mathcal{K}(\delta)})\partial_t = \partial_t + \varepsilon\mathcal{K}(\delta_t), && \mathrm{Ad}(e^{\varepsilon\partial_t})\mathrm{J}^t = \mathrm{J}^t - \varepsilon\mathrm{J}^1,\\
&\mathrm{Ad}(e^{\varepsilon\mathcal{G}(\rho)})\partial_t = \partial_t + \varepsilon\mathcal{G}(\rho_t), && \mathrm{Ad}(e^{\varepsilon\partial_t})\mathcal{K}(\delta) = \mathcal{K}(\delta(t-\varepsilon)),\\
&\mathrm{Ad}(e^{\varepsilon\mathrm{J}^t})\partial_t = \partial_t + \varepsilon\mathrm{J}, && \mathrm{Ad}(e^{\varepsilon\partial_t})\mathcal{G}(\rho) = \mathcal{G}(\rho(t-\varepsilon, x, y)),\\
&\mathrm{Ad}(e^{\varepsilon\mathcal{K}(\delta)})\mathcal{D}_2 = \mathcal{D}_2 + \varepsilon\mathcal{K}(2\delta), && \mathrm{Ad}(e^{\varepsilon\mathcal{D}_2})\mathcal{K}(\delta) = \mathcal{K}(e^{2\varepsilon}\delta(t)),\\
&\mathrm{Ad}(e^{\varepsilon\mathcal{G}(\rho)})\mathcal{D}_2 = \mathcal{D}_2 + \varepsilon\mathcal{G}(2\rho), && \mathrm{Ad}(e^{\varepsilon\mathcal{D}_2})\mathcal{G}(\rho) = \mathcal{G}(e^{-\varepsilon}\rho(t, e^{-\varepsilon}x, e^{-\varepsilon}y)),\\
&\mathrm{Ad}(e^{\varepsilon\mathcal{G}(\rho)})\mathrm{J}^1 = \mathrm{J}^1 + \varepsilon\mathcal{G}(\rho^2x - \rho^1y), && \mathrm{Ad}(e^{\varepsilon\mathrm{J}^1})\mathcal{G}(\rho) = \mathcal{G}(\hat{\rho}^{\varepsilon}),\\
&\mathrm{Ad}(e^{\varepsilon\mathcal{G}(\rho)})\mathrm{J}^t = \mathrm{J}^t + \varepsilon\mathcal{G}(t\rho^2x - t\rho^1y), && \mathrm{Ad}(e^{\varepsilon\mathrm{J}^t})\mathcal{G}(\rho) = \mathcal{G}(\check{\rho}^{\varepsilon}),
\end{aligned}
$$

where we omit tildes in the notation of operators and also omit arguments of parameter-functions if these arguments are not changed under the corresponding adjoint action, $\rho = \rho^1x + \rho^2y$, $\hat{\rho}^{\varepsilon} = (\rho^1x + \rho^2y)\cos\varepsilon + (\rho^1y - \rho^2x)\sin\varepsilon$, $\check{\rho}^{\varepsilon} = (\rho^1x + \rho^2y)\cos\varepsilon t + (\rho^1y - \rho^2x)\sin\varepsilon t$,

Based upon these adjoint actions, we derive the following list of $G_7^{\sim}$-inequivalent subalgebras of $\tilde{\mathfrak{a}}$ satisfying the above restrictions (we again omit tildes in the notation of operators):

one-dimensional subalgebras:

$$\langle \mathcal{D}_1 + b\mathcal{D}_2 + a\mathrm{J}^1\rangle, \quad \langle \partial_t + c\mathcal{D}_2 + \hat{c}\mathrm{J}^t\rangle, \quad \langle \mathcal{D}_2 + \mathrm{J}^t\rangle, \quad \langle \mathcal{D}_2 + a\mathrm{J}^1\rangle;$$

two-dimensional subalgebras:

$$
\begin{aligned}
&\langle \mathcal{D}_1 + b\mathcal{D}_2 + a\mathrm{J} + \mathcal{K}(c) + \mathcal{G}(\tilde{c}x), \partial_t\rangle, \quad \langle \mathcal{D}_1 + a\mathrm{J}^1, \mathcal{D}_2 + \hat{a}\mathrm{J}^1\rangle,\\
&\langle \partial_t + c\mathrm{J}^t, \mathcal{D}_2 + \hat{a}\mathrm{J}^1\rangle, \quad \langle \mathrm{J}^1 + \mathcal{K}(\delta^1(t)), \mathrm{J}^t + \mathcal{K}(\delta^2(t))\rangle;
\end{aligned}
$$

three-dimensional subalgebras:

$$\langle \mathcal{D}_1 + a\mathrm{J}^1, \partial_t, \mathcal{D}_2 + \hat{a}\mathrm{J}^1\rangle, \quad \langle \mathcal{D}_1 + b\mathcal{D}_2, \mathrm{J}^1 + \mathcal{K}(c|t|^{2b-1}), \mathrm{J}^t + \mathcal{K}(\hat{c}|t|^{2b})\rangle,$$

$$\langle \partial_t + \tilde{c}\mathcal{D}_2, \mathrm{J}^1 + \mathcal{K}(ce^{2\tilde{c}t}), \mathrm{J}^t + \mathcal{K}((ct+\hat{c})e^{2\tilde{c}t})\rangle, \quad \langle \mathcal{D}_2, \mathrm{J}^1, \mathrm{J}^t\rangle;$$

four-dimensional subalgebras:

$$\langle \mathcal{D}_1 + b\mathcal{D}_2 + \mathcal{K}(\nu_2), \partial_t, \mathrm{J}^1 + \mathcal{K}(\nu_1), \mathrm{J}^t + \mathcal{K}(\nu_1 t + \nu_0)\rangle,\ (2b-1)\nu_1 = 0,\ b\nu_0 = 0,$$
$$\langle \mathcal{D}_1, \mathcal{D}_2, \mathrm{J}^1, \mathrm{J}^t\rangle, \quad \langle \partial_t, \mathcal{D}_2, \mathrm{J}^1, \mathrm{J}^t\rangle;$$

five-dimensional subalgebra:

$$\langle \mathcal{D}_1, \partial_t, \mathcal{D}_2, \mathrm{J}^1, \mathrm{J}^t\rangle.$$

In the above subalgebras, due to adjoint actions we can put the following restrictions on the algebra parameters: $a \geq 0$, $c, \tilde{c} \in \{0, 1\}$, $\hat{a} \geq 0$ if $a = 0$ (resp. $c = 0$), $\hat{c} \in \{0, 1\}$ if $c = 0$; additionally, in the first two-dimensional subalgebra we can set $(1+2b)c = 0$ and $((1+b)^2 + a^2)\tilde{c} = 0$; in the first four-dimensional subalgebra one non-zero parameter among ν_0, ν_1, ν_2 can be set to 1. In the last two-dimensional subalgebra, the parameters δ^1 and δ^2 are arbitrary smooth functions of t. The subalgebras with parameter tuples (δ^1, δ^2) and $(\tilde{\delta}^1, \tilde{\delta}^2)$ are equivalent if and only if there exist constants ε_0, ε_1 and ε_2 such that $\tilde{\delta}^1 = e^{\varepsilon_2 - \varepsilon_1}\delta^1(e^{-\varepsilon_1}t + \varepsilon_0)$ and $\tilde{\delta}^2 = e^{\varepsilon_2}\delta^2(e^{-\varepsilon_1}t + \varepsilon_0)$.

Concerning the realization of the second step of the algorithm, we note that the system corresponding to the last two-dimensional subalgebra is compatible if and only if $\delta^2(t) = t\delta^1(t)$. We re-denote δ^1 by δ. As the general solution of the system is parameterized by functions of two arguments, we put the associated two-dimensional symmetry extension into Table 8.1, where the other extensions are one-dimensional. A similar remark is true for the three-dimensional subalgebra $\langle \mathcal{D}_2, \mathrm{J}^t, \mathrm{J}\rangle$, which is why we list it in Table 8.2 containing symmetry extensions parameterized by functions of a single argument.

The system associated with the first two-dimensional subalgebra is compatible if and only if $(a, b) \neq (0, -1)$. The solution of the system is split into three cases, (i) $b \neq -1, 1/2$, (ii) $b = 1/2$ and (iii) $b = -1$ and $a \neq 0$. We will use the notation $\mu = c/(2b-1)$ for $b \neq 1/2$ and $\mu = 2c/3$ in case of $b = 1/2$.

For the second and third three-dimensional subalgebras, the corresponding systems are compatible if and only if $c = \hat{c}$ and $\hat{c} = 0$, respectively.

For the reason of compatibility, in the first four-dimensional subalgebra we have $\nu_0 = 0$ and $b \neq -1$. Due to the condition $(2b-1)\nu_1 = 0$, the solution of the corresponding system should be split into the two cases $b \neq 1/2$ and $b = 1/2$. For simplicity of the representation of the results in Table 8.3 we introduce the notation $\mu = \nu_2/(2b-1)$ if $b \neq 1/2$ and $\tilde{\nu}_2 = -2\nu_2/3$ for $b = 1/2$.

In Tables 8.1–8.3,

$$\Phi = \arctan\frac{\zeta_y}{\zeta_x}, \quad R = \sqrt{\zeta_x^2 + \zeta_y^2}, \quad P_1 = \frac{\zeta_x I_1 - \zeta_y I_2}{\zeta_x^2 + \zeta_y^2}, \quad P_2 = \frac{\zeta_y I_1 + \zeta_x I_2}{\zeta_x^2 + \zeta_y^2}.$$

Moreover, $\alpha_1 = 3/(b+1)$ (for $b \neq -1$), $\alpha_2 = 3/a$ (for $b = -1$ and $a \neq 0$) and $\alpha_3 = 3/(\hat{a} - a)$ (for $\hat{a} \neq a$). In Table 8.1, δ is an arbitrary function of t. In Table 8.3, subalgebras I_1 and I_2 are two arbitrary constants.

Up to gauge equivalence, the single parameterization admitting five-dimensional symmetry extension within the class (8.18) is the trivial parameterization, $f^1 = f^2 = 0$, in which we neglect the eddy vorticity flux. This shows the limits of applicability of the method proposed in [112], cf. Section 8.3.1.

Table 8.1: Symmetry extensions parameterized by functions of two arguments

$\mathfrak{g}_f^{\text{ext}}$	Arguments of I_1, I_2		f^1, f^2
$\langle \mathcal{D}_1 + b\mathcal{D}_2 + a\text{J}\rangle$	$\lvert t\rvert^{b+1}(\zeta_x\cos\tau + \zeta_y\sin\tau)$, $\lvert t\rvert^{b+1}(\zeta_y\cos\tau - \zeta_x\sin\tau)$,	$\tau := a\ln\lvert t\rvert$	$\lvert t\rvert^{b-2}(I_1\cos\tau - I_2\sin\tau)$, $\lvert t\rvert^{b-2}(I_1\sin\tau + I_1\cos\tau)$
$\langle \partial_t + c\mathcal{D}_2 + \hat{c}\text{J}^t\rangle$	$e^{ct}(\zeta_x\cos\tau + \zeta_y\sin\tau)$, $e^{ct}(\zeta_y\cos\tau - \zeta_x\sin\tau)$,	$\tau := \dfrac{\hat{c}}{2}t^2$	$e^{ct}(I_1\cos\tau - I_2\sin\tau)$, $e^{ct}(I_1\sin\tau + I_1\cos\tau)$
$\langle \mathcal{D}_2 + \text{J}^t\rangle$	t, $Re^{\Phi/t}$		P_1, P_2
$\langle \mathcal{D}_2 + a\text{J}\rangle$	t, $R^a e^{\Phi}$		P_1, P_2
$\langle \text{J}, \text{J}^t\rangle$	t, R		$\zeta_x I^1 - \zeta_y I^2 + \delta(t)\zeta_y\Phi$, $\zeta_y I^1 + \zeta_x I^2 - \delta(t)\zeta_x\Phi$

Table 8.2: Symmetry extensions parameterized by functions of a single argument

$\mathfrak{g}_f^{\text{ext}}$	Argument of I_1, I_2	f^1, f^2
$\langle \mathcal{D}_1 + b\mathcal{D}_2 + a\text{J}, \partial_t\rangle$, $b \neq -1, \frac{1}{2}$	$R^a e^{(1+b)\Phi}$	$R^{\alpha_1}P_1 - \mu\zeta_y$, $R^{\alpha_1}P_2 + \mu\zeta_x$
$\langle \mathcal{D}_1 + \frac{1}{2}\mathcal{D}_2 + a\text{J}, \partial_t\rangle$	$R^a e^{3\Phi/2}$	$R^2P_1 - \mu\zeta_y\ln R$, $R^2P_2 + \mu\zeta_x\ln R$
$\langle \mathcal{D}_1 - \mathcal{D}_2 + a\text{J}, \partial_t\rangle$, $a \neq 0$	R	$e^{\alpha_2\Phi}P_1 - \mu\zeta_y$, $e^{\alpha_2\Phi}P_2 + \mu\zeta_x$
$\langle \mathcal{D}_1 + a\text{J}, \mathcal{D}_2 + \hat{a}\text{J}\rangle$	$\lvert t\rvert^{\hat{a}-a}R^{\hat{a}}e^{\Phi}$	$t^{-3}P_1$, $t^{-3}P_2$
$\langle \partial_t + c\text{J}^t, \mathcal{D}_2 + \hat{a}\text{J}\rangle$	$R^{\hat{a}}e^{\Phi - ct^2/2}$	P_1, P_2
$\langle \mathcal{D}_1 + b\mathcal{D}_2, \text{J}^1, \text{J}^t\rangle$	$\lvert t\rvert^{b+1}R$	$\lvert t\rvert^{2b-1}(\zeta_x I^1 - \zeta_y I^2 + c\zeta_y\Phi)$, $\lvert t\rvert^{2b-1}(\zeta_y I^1 + \zeta_x I^2 - c\zeta_x\Phi)$
$\langle \partial_t + \tilde{c}\mathcal{D}_2, \text{J}^1, \text{J}^t\rangle$	$e^{\tilde{c}t}R$	$e^{2\tilde{c}t}(\zeta_x I^1 - \zeta_y I^2 + c\zeta_y\Phi)$, $e^{2\tilde{c}t}(\zeta_y I^1 + \zeta_x I^2 - c\zeta_x\Phi)$
$\langle \mathcal{D}_2, \text{J}, \text{J}^t\rangle$	t	P_1, P_2

Table 8.3: Symmetry extensions parameterized by constants

$\mathfrak{g}_f^{\text{ext}}$	f^1, f^2
$\langle \mathcal{D}_1 + a\text{J}^1, \partial_t, \mathcal{D}_2 + \hat{a}\text{J}^1\rangle$, $\hat{a} \neq a$	$R^{\alpha_3\hat{a}}e^{\alpha_3\Phi}P_1$, $R^{\alpha_3\hat{a}}e^{\alpha_3\Phi}P_2$
$\langle \mathcal{D}_1 + b\mathcal{D}_2, \partial_t, \text{J}, \text{J}^t\rangle$, $b \neq -1, \frac{1}{2}$	$R^{\alpha_1}P_1 - \mu\zeta_y$, $R^{\alpha_1}P_2 + \mu\zeta_x$
$\langle \mathcal{D}_1 + \frac{1}{2}\mathcal{D}_2, \partial_t, \text{J}, \text{J}^t\rangle$	$R^2P_1 - (\tilde{\nu}_2\ln R + \nu_1\Phi)\zeta_y$, $R^2P_2 + (\tilde{\nu}_2\ln R + \nu_1\Phi)\zeta_x$
$\langle \mathcal{D}_1, \mathcal{D}_2, \text{J}, \text{J}^t\rangle$	$t^{-3}P_1$, $t^{-3}P_2$
$\langle \partial_t, \mathcal{D}_2, \text{J}, \text{J}^t\rangle$	P_1, P_2

8.3.5 Parameterization via preliminary group classification

The technique of preliminary group classification is based on classifications of extensions of the kernel Lie invariance algebra by operators obtained via projection of elements of the corresponding equivalence algebra to the space of independent and dependent variables [65]. It is illustrated here with the class (8.8) whose equivalence algebra $\mathfrak{g}_1^\sim$ is calculated in Section 8.3.2.

The kernel Lie invariance algebra of class (8.8) (i.e., the intersection of the maximal Lie invariance algebras of equations from the class) is $\langle \mathcal{Z}(\chi)\rangle$. In view of the classification of one-dimensional subalgebras of the equivalence algebra in Appendix 8.5 (list (8.21)) and since for preliminary group classification we are only concerned with extensions of the complement of $\langle \mathcal{Z}(\chi)\rangle$ in $\mathfrak{g}_1^\sim$, we essentially have to consider the inequivalent subalgebras

$$\langle \mathcal{D}_1 + a\mathcal{D}_2\rangle, \quad \langle \partial_t + b\mathcal{D}_2\rangle, \quad \langle \mathcal{D}_2 + \mathrm{J}(\beta) + \mathcal{R}(\sigma)\rangle, \quad \langle \mathrm{J}(\beta) + \mathcal{R}(\sigma)\rangle,$$
$$\langle \mathcal{X}(\gamma^1) + \mathcal{R}(\sigma)\rangle, \quad \langle \mathcal{R}(\sigma) + \mathcal{H}(\delta) + \mathcal{G}(\rho)\rangle.$$

It now remains to present the corresponding parameterization schemes, which can be found in the Table 8.4.

Table 8.4: One-dimensional symmetry algebra extensions for the case $f^i = f^i(t, x, y, \zeta_x, \zeta_y)$

$\mathfrak{g}_f^{\rm ext}$	Arguments of I_1, I_2	f^1, f^2
$\langle \mathcal{D}_1 + a\mathcal{D}_2\rangle$	$t^{-a}x,\ t^{-a}y,$ $t^{1+a}\zeta_x,\ t^{1+a}\zeta_y$	$t^{-3}P_1,\ t^{-3}P_2$
$\langle \partial_t + a\mathcal{D}_2\rangle$	$xe^{-at},\ ye^{-at},$ $\zeta_x e^{at},\ \zeta_y e^{at}$	$P_1,\ P_2$
$\langle \mathcal{D}_2 + \mathrm{J}(\beta) + \mathcal{R}(\sigma)\rangle$	$t,\ r^\beta e^{-\varphi},\ x\zeta_x + y\zeta_y,$ $y\zeta_x - x\zeta_y$	$xI^1 - yI^2 + \dfrac{\sigma}{2} r^2\zeta_y \ln r + h^1(t,x,y),$ $yI^1 + xI^2 - \dfrac{\sigma}{2} r^2\zeta_x \ln r + h^2(t,x,y)$
$\langle \mathrm{J}(\beta) + \mathcal{R}(\sigma)\rangle$	$t,\ r,\ x\zeta_x + y\zeta_y,$ $y\zeta_x - x\zeta_y$	$xI^1 - yI^2 + \dfrac{\sigma}{2\beta} r^2\zeta_y\varphi + \dfrac{\beta_{tt}+\sigma_t}{2\beta} x\varphi,$ $yI^1 + xI^2 - \dfrac{\sigma}{2\beta} r^2\zeta_x\varphi + \dfrac{\beta_{tt}+\sigma_t}{2\beta} y\varphi$
$\langle \mathcal{X}(\gamma^1) + \mathcal{R}(\sigma)\rangle$	$t,\ y,\ \zeta_x,\ \zeta_y$	$I_1 + \dfrac{1}{\gamma^1}\left(\sigma_t \dfrac{x^2}{2} + \dfrac{\sigma}{4}\left(\dfrac{x^3}{3} + xy^2\right)\zeta_y\right),$ $I_2 + \dfrac{1}{\gamma^1}\left(\sigma_t xy - \dfrac{\sigma}{4}\left(\dfrac{x^3}{3} + xy^2\right)\zeta_x\right)$
$\langle \mathcal{R}(\sigma) + \mathcal{H}(\delta) + \mathcal{G}(\rho)\rangle$	No ansatz possible	

In this table, P_1 and P_2 are the same as in Tables 8.1–8.3, $r = \sqrt{x^2+y^2}$ and $\varphi = \arctan y/x$. The functions h^1 and h^2 are $h^1 = x\varphi(\beta_{tt} + \sigma_t)/\beta$ and $h^2 = y\varphi(\beta_{tt}+\sigma_t)/\beta$ in the case of $\beta \neq 0$ and $h^1 = \sigma_t x \ln r$ and $h^2 = \sigma_t y \ln r$ for $\beta = 0$. In the last class of subalgebras no ansatz can be constructed due to the special form of functions f^i. Namely, as the variable ψ is not included in the list of arguments of f^i, any operator of the form $\mathcal{R}(\sigma) + \mathcal{H}(\delta) + \mathcal{G}(\rho) + \mathcal{Z}(\chi)$ belongs to the gauge equivalence algebra and hence its projection gives no extension of the kernel algebra.

8.4 Conclusion

In this paper we have addressed the question of symmetry-preserving parameterization schemes. It was demonstrated that the problem of finding invariant parameterization schemes can be treated as a group classification problem. In particular, the interpretation of parameterizations as particular elements of classes of differential equations renders it possible to use well-established methods of symmetry analysis for the design of general classes of closure schemes with prescribed symmetry properties. For parameterizations to admit selected subgroups of the maximal Lie invariance group of the related unaveraged differential equation, they should be expressed in terms of related differential invariants. These differential invariants can be computed either using infinitesimal methods or the method of moving frames, cf. Section 8.3.1. For parameterization ansatzes with prescribed functional dependence on the resolved quantities and no prescribed symmetry group, the direct group classification problem should be solved. In the case where the given class of differential equations is normalized (which can be checked by the computation of the set of admissible transformations), it is possible and convenient to carry out the classification using the algebraic method [131]. In the case where the class fails to be normalized (or in the case where it is impossible to compute the set of admissible transformations), an exhaustive investigation of parameterizations might be possible due to applying compatibility analysis of the corresponding determining equations or by combining the algebraic and compatibility methods. For more involved classes of differential equations at least symmetry extensions induced by subalgebras of the equivalence algebra can be found, i.e. preliminary group classification can be carried out.

Since the primary aim of this paper is a clear presentation of the variety of invariant parameterization methods, we focused on rather simple first order local closure schemes for the classical barotropic vorticity equation, cf. the introduction of Section 8.3. That is, we parameterized already the eddy vorticity flux $\overline{\mathbf{v}'\zeta'}$ using $\bar{\zeta}$ and its derivatives. Admittedly, this is a quite simple ansatz for one of the simplest physically relevant models in geophysical fluid dynamics. On the other hand, it can be seen that already for this particular simple example the computations involved were rather elaborate. This is in particular true for the computation of the set of admissible transformations for the various classes of vorticity equations considered in Section 8.3.3. Needless to say that irrespectively of practical computational problems the same technique would be applicable to higher order closure schemes as well. In designing such schemes it is necessary to explicitly include differential equations for the first or higher order correlation terms. In the case of the vorticity equations, a second order closure schemes is obtainable upon retaining the equations governing the evolution of $\overline{\mathbf{v}'\zeta'}$ and parameterize the higher order correlation terms arising in these equations. In practice, however, it becomes increasingly difficult to acquire real atmospheric data for such higher-order correlation quantities, which therefore makes it difficult to propose parameterization schemes based solely on physical considerations [150]. We argue that especially in such cases symmetries could provide a useful guiding principle to determine general classes of relevant parameterizations.

Up to now, we have restricted ourselves to the problem of invariant local closure schemes. Nonlocal schemes constructed using symmetry arguments should be investigated in a subsequent work. This extension is crucial in order to make general methods available that can be used in the development of parameterization schemes for all types of physical processes in atmosphere-ocean dynamics. A further perspective for generalization of the present work is the design of parameterization schemes that preserve conservation laws. This is another aspect that is of major importance in practical applications. For parameterizations of conservative processes, it is crucial that the corresponding closed differential equation preserves energy conservation. This is by no means self-evident. In fact, energy conservation is violated by various classes of down-gradient ansatzes [158], which is straightforward to check also for parameterizations constructed in this paper. The construction of parameterization schemes that retain conservation laws will call for the classification of conservation laws in the way similar as the usual group classification. A main complication is that there is no restriction on the order of conservation laws for general systems of partial differential equations (so far, such restrictions are only known for $(1+1)$-dimensional evolution equations of even order and some similar classes of equations).

8.5 Appendix: Inequivalent one-dimensional subalgebras of the equivalence algebra of class (8.8)

In this appendix, we classify one-dimensional subalgebras of the equivalence algebra $\mathfrak{g}_1^\sim$ with basis elements (8.9). For this means, we subsequently present the commutator table of $\mathfrak{g}_1^\sim$. In what follows we omit tildes in the notation of operators.

Table 8.5: Commutation relations for the algebra $\mathfrak{g}_1^\sim$

	$\mathcal{D}_1$	$\mathcal{D}_2$	∂_t	$\mathrm{J}(\beta)$	$\mathcal{X}(\gamma^1)$
$\mathcal{D}_1$	0	0	$-\partial_t$	$\mathrm{J}(t\beta_t)$	$\mathcal{X}(t\gamma^1_t)$
$\mathcal{D}_2$	0	0	0	0	$-\mathcal{X}(\gamma^1)$
∂_t	∂_t	0	0	$\mathrm{J}(\beta_t)$	$\mathcal{X}(\gamma^1_t)$
$\mathrm{J}(\tilde\beta)$	$-\mathrm{J}(t\tilde\beta_t)$	0	$-\mathrm{J}(\tilde\beta_t)$	0	$-\mathcal{Y}(\tilde\beta\gamma^1)+\mathcal{G}(\gamma^1\tilde\beta_{tt}y)$
$\mathcal{X}(\tilde\gamma^1)$	$-\mathcal{X}(t\tilde\gamma^1_t)$	$\mathcal{X}(\tilde\gamma^1)$	$-\mathcal{X}(\tilde\gamma^1_t)$	$\mathcal{Y}(\beta\tilde\gamma^1)-\mathcal{G}(\tilde\gamma^1\beta_{tt}y)$	0
$\mathcal{Y}(\tilde\gamma^2)$	$-\mathcal{Y}(t\tilde\gamma^2_t)$	$\mathcal{Y}(\tilde\gamma^2)$	$-\mathcal{Y}(\tilde\gamma^2_t)$	$-\mathcal{X}(\beta\tilde\gamma^2)+\mathcal{G}(\tilde\gamma^2\beta_{tt}x)$	$-\mathcal{Z}((\gamma^1\tilde\gamma^2)_t)$
$\mathcal{R}(\tilde\sigma)$	$-\mathcal{R}(t\tilde\sigma_t+\tilde\sigma)$	0	$-\mathcal{R}(\tilde\sigma_t)$	0	$-\mathcal{H}(\gamma^1\tilde\sigma x)+\mathcal{G}(\gamma^1\tilde\sigma_t y)$
$\mathcal{H}(\tilde\delta)$	$-\mathcal{H}(t\tilde\delta_t+\tilde\delta)$	$-\mathcal{H}(x\tilde\delta_x+y\tilde\delta_y-2\tilde\delta)$	$-\mathcal{H}(\tilde\delta_t)$	$-\mathcal{H}(\beta x\tilde\delta_y-\beta y\tilde\delta_x)$	$-\mathcal{H}(\gamma^1\tilde\delta_x)$
$\mathcal{G}(\tilde\rho)$	$-\mathcal{G}(t\tilde\rho_t+2\tilde\rho)$	$-\mathcal{G}(x\tilde\rho_x+y\tilde\rho_y+\tilde\rho)$	$-\mathcal{G}(\tilde\rho_t)$	$-\mathcal{G}(\beta x\tilde\rho_y-\beta y\tilde\rho_x)$	$\mathcal{G}(\gamma^1\tilde\rho_x)$
$\mathcal{Z}(\tilde\chi)$	$-\mathcal{Z}(t\tilde\chi_t+\tilde\chi)$	$2\mathcal{Z}(\tilde\chi)$	$-\mathcal{Z}(\tilde\chi_t)$	0	0

	$\mathcal{Y}(\gamma^2)$	$\mathcal{R}(\sigma)$	$\mathcal{H}(\delta)$	$\mathcal{G}(\rho)$	$\mathcal{Z}(\chi)$
$\mathcal{D}_1$	$\mathcal{Y}(t\gamma^2_t)$	$\mathcal{R}(t\sigma_t+\sigma)$	$\mathcal{H}(t\delta_t+\delta)$	$\mathcal{G}(t\rho_t+2\rho)$	$\mathcal{Z}(t\chi_t+\chi)$
$\mathcal{D}_2$	$-\mathcal{Y}(\gamma^2)$	0	$H(x\delta_x+y\delta_y-2\delta)$	$\mathcal{G}(x\rho_x+y\rho_y+\rho)$	$-2\mathcal{Z}(\chi)$
∂_t	$\mathcal{Y}(\gamma^2_t)$	$\mathcal{R}(\sigma_t)$	$\mathcal{H}(\delta_t)$	$\mathcal{G}(\rho_t)$	$\mathcal{Z}(\chi_t)$
$\mathrm{J}(\tilde\beta)$	$\mathcal{X}(\tilde\beta\gamma^2)-\mathcal{G}(\gamma^2\tilde\beta_{tt}x)$	0	$\mathcal{H}(\tilde\beta x\delta_y-\tilde\beta y\delta_x)$	$\mathcal{G}(\tilde\beta x\rho_y-\tilde\beta y\rho_x)$	0
$\mathcal{X}(\tilde\gamma^1)$	$\mathcal{Z}((\tilde\gamma^1\gamma^2)_t)$	$\mathcal{H}(\tilde\gamma^1\sigma x)-\mathcal{G}(\tilde\gamma^1\sigma_t y)$	$\mathcal{H}(\tilde\gamma^1\delta_x)$	$\mathcal{G}(\tilde\gamma^1\rho_x)$	0
$\mathcal{Y}(\tilde\gamma^2)$	0	$\mathcal{H}(\tilde\gamma^2\sigma y)+\mathcal{G}(\tilde\gamma^2\sigma_t x)$	$\mathcal{H}(\tilde\gamma^2\delta_y)$	$\mathcal{G}(\tilde\gamma^2\rho_y)$	0
$\mathcal{R}(\tilde\sigma)$	$-\mathcal{H}(\gamma^2\tilde\sigma y)-\mathcal{G}(\gamma^2\tilde\sigma_t x)$	0	0	0	0
$\mathcal{H}(\tilde\delta)$	$-\mathcal{H}(\gamma^2\tilde\delta_y)$	0	0	0	0
$\mathcal{G}(\tilde\rho)$	$-\mathcal{G}(\gamma^2\tilde\rho_y)$	0	0	0	0
$\mathcal{Z}(\tilde\chi)$	0	0	0	0	0

Based on Table 8.5, it is straightforward to recover the following nontrivial adjoint actions:

$$\mathrm{Ad}(e^{\varepsilon\partial_t})\mathcal{D}_1=\mathcal{D}_1-\varepsilon\partial_t, \qquad \mathrm{Ad}(e^{\varepsilon\mathcal{D}_1})\partial_t=e^{\varepsilon}\mathcal{D}_1,$$

$$\mathrm{Ad}(e^{\varepsilon\mathrm{J}(\beta)})\mathcal{D}_1=\mathcal{D}_1+\varepsilon\mathrm{J}(t\beta_t), \qquad \mathrm{Ad}(e^{\varepsilon\mathcal{D}_1})\mathrm{J}(\beta)=\mathrm{J}(\beta(e^{-\varepsilon}t)),$$

$$\mathrm{Ad}(e^{\varepsilon\mathcal{X}(\gamma^1)})\mathcal{D}_1=\mathcal{D}_1+\varepsilon\mathcal{X}(t\gamma^1_t), \qquad \mathrm{Ad}(e^{\varepsilon\mathcal{D}_1})\mathcal{X}(\gamma^1)=\mathcal{X}(\gamma^1(e^{-\varepsilon}t)),$$

$$\mathrm{Ad}(e^{\varepsilon\mathcal{Y}(\gamma^2)})\mathcal{D}_1=\mathcal{D}_1+\varepsilon\mathcal{Y}(t\gamma^2_t), \qquad \mathrm{Ad}(e^{\varepsilon\mathcal{D}_1})\mathcal{Y}(\gamma^2)=\mathcal{Y}(\gamma^2(e^{-\varepsilon}t)),$$

$$
\begin{aligned}
&\mathrm{Ad}(e^{\varepsilon\mathcal{R}(\sigma)})\mathcal{D}_1=\mathcal{D}_1+\varepsilon\mathcal{R}(t\sigma_t+\sigma), && \mathrm{Ad}(e^{\varepsilon\mathcal{D}_1})\mathcal{R}(\sigma)=\mathcal{R}(e^{-\varepsilon}\sigma(e^{-\varepsilon}t)),\\
&\mathrm{Ad}(e^{\varepsilon\mathcal{H}(\delta)})\mathcal{D}_1=\mathcal{D}_1+\varepsilon\mathcal{H}(t\delta_t+\delta), && \mathrm{Ad}(e^{\varepsilon\mathcal{D}_1})\mathcal{H}(\delta)=\mathcal{H}(e^{-\varepsilon}\delta(e^{-\varepsilon}t,x,y)),\\
&\mathrm{Ad}(e^{\varepsilon\mathcal{G}(\rho)})\mathcal{D}_1=\mathcal{D}_1+\varepsilon\mathcal{G}(t\rho_t+2\rho), && \mathrm{Ad}(e^{\varepsilon\mathcal{D}_1})\mathcal{G}(\rho)=\mathcal{G}(e^{-2\varepsilon}\rho(e^{-\varepsilon}t,x,y)),\\
&\mathrm{Ad}(e^{\varepsilon\mathcal{Z}(\chi)})\mathcal{D}_1=\mathcal{D}_1+\varepsilon\mathcal{Z}(t\chi_t+\chi), && \mathrm{Ad}(e^{\varepsilon\mathcal{D}_1})\mathcal{Z}(\chi)=\mathcal{Z}(e^{-\varepsilon}\chi(e^{-\varepsilon}t)),\\
&\mathrm{Ad}(e^{\varepsilon\mathrm{J}(\beta)})\partial_t=\partial_t+\varepsilon\mathrm{J}(\beta_t), && \mathrm{Ad}(e^{\varepsilon\partial_t})\mathrm{J}(\beta)=\mathrm{J}(\beta(t-\varepsilon)),\\
&\mathrm{Ad}(e^{\varepsilon\mathcal{X}(\gamma^1)})\partial_t=\partial_t+\varepsilon\mathcal{X}(\gamma^1_t), && \mathrm{Ad}(e^{\varepsilon\partial_t})\mathcal{X}(\gamma^1)=\mathcal{X}(\gamma^1(t-\varepsilon)),\\
&\mathrm{Ad}(e^{\varepsilon\mathcal{Y}(\gamma^2)})\partial_t=\partial_t+\varepsilon\mathcal{Y}(\gamma^2_t), && \mathrm{Ad}(e^{\varepsilon\partial_t})\mathcal{Y}(\gamma^2)=\mathcal{Y}(\gamma^2(t-\varepsilon)),\\
&\mathrm{Ad}(e^{\varepsilon\mathcal{R}(\sigma)})\partial_t=\partial_t+\varepsilon\mathcal{R}(\sigma_t), && \mathrm{Ad}(e^{\varepsilon\partial_t})\mathcal{R}(\sigma)=\mathcal{R}(\sigma(t-\varepsilon)),\\
&\mathrm{Ad}(e^{\varepsilon\mathcal{H}(\delta)})\partial_t=\partial_t+\varepsilon\mathcal{H}(\delta_t), && \mathrm{Ad}(e^{\varepsilon\partial_t})\mathcal{H}(\delta)=\mathcal{H}(\delta(t-\varepsilon,x,y)),\\
&\mathrm{Ad}(e^{\varepsilon\mathcal{G}(\rho)})\partial_t=\partial_t+\varepsilon\mathcal{G}(\rho_t), && \mathrm{Ad}(e^{\varepsilon\partial_t})\mathcal{G}(\rho)=\mathcal{G}(\rho(t-\varepsilon,x,y)),\\
&\mathrm{Ad}(e^{\varepsilon\mathcal{Z}(\chi)})\partial_t=\partial_t+\varepsilon\mathcal{Z}(\chi_t), && \mathrm{Ad}(e^{\varepsilon\partial_t})\mathcal{Z}(\chi)=\mathcal{Z}(\chi(t-\varepsilon)),\\
&\mathrm{Ad}(e^{\varepsilon\mathcal{X}(\gamma^1)})\mathcal{D}_2=\mathcal{D}_2-\varepsilon\mathcal{X}(\gamma^1), && \mathrm{Ad}(e^{\varepsilon\mathcal{D}_2})\mathcal{X}(\gamma^1)=\mathcal{X}(e^{\varepsilon}\gamma^1),\\
&\mathrm{Ad}(e^{\varepsilon\mathcal{Y}(\gamma^2)})\mathcal{D}_2=\mathcal{D}_2-\varepsilon\mathcal{Y}(\gamma^2), && \mathrm{Ad}(e^{\varepsilon\mathcal{D}_2})\mathcal{Y}(\gamma^2)=\mathcal{Y}(e^{\varepsilon}\gamma^2),\\
&\mathrm{Ad}(e^{\varepsilon\mathcal{H}(\delta)})\mathcal{D}_2=\mathcal{D}_2+\varepsilon\mathcal{H}(x\delta_x+y\delta_y-2\delta), && \mathrm{Ad}(e^{\varepsilon\mathcal{D}_2})\mathcal{H}(\delta)=\mathcal{H}(e^{2\varepsilon}\delta(t,e^{-\varepsilon}x,e^{-\varepsilon}y)),\\
&\mathrm{Ad}(e^{\varepsilon\mathcal{G}(\rho)})\mathcal{D}_2=\mathcal{D}_2+\varepsilon\mathcal{G}(x\rho_x+y\rho_y+\rho), && \mathrm{Ad}(e^{\varepsilon\mathcal{D}_2})\mathcal{G}(\rho)=\mathcal{G}(e^{-\varepsilon}\rho(t,e^{-\varepsilon}x,e^{-\varepsilon}y)),\\
&\mathrm{Ad}(e^{\varepsilon\mathcal{Z}(\chi)})\mathcal{D}_2=\mathcal{D}_2-2\varepsilon\mathcal{Z}(\chi), && \mathrm{Ad}(e^{\varepsilon\mathcal{D}_2})\mathcal{Z}(\chi)=\mathcal{Z}(e^{2\varepsilon}\chi),\\
&\mathrm{Ad}(e^{\varepsilon\mathcal{X}(\gamma^1)})\mathrm{J}(\beta)=A_1, && \mathrm{Ad}(e^{\varepsilon\mathrm{J}(\beta)})\mathcal{X}(\gamma^1)=A_3,\\
&\mathrm{Ad}(e^{\varepsilon\mathcal{Y}(\gamma^2)})\mathrm{J}(\beta)=A_2, && \mathrm{Ad}(e^{\varepsilon\mathrm{J}(\beta)})\mathcal{Y}(\gamma^2)=A_4,\\
&\mathrm{Ad}(e^{\varepsilon\mathcal{H}(\delta)})\mathrm{J}(\beta)=\mathrm{J}(\beta)+\varepsilon\mathcal{H}(\beta x\delta_y-\beta y\delta_x), && \mathrm{Ad}(e^{\varepsilon\mathrm{J}(\beta)})\mathcal{H}(\delta)=A_5,\\
&\mathrm{Ad}(e^{\varepsilon\mathcal{G}(\rho)})\mathrm{J}(\beta)=\mathrm{J}(\beta)+\varepsilon\mathcal{G}(\beta x\rho_y-\beta y\rho_x), && \mathrm{Ad}(e^{\varepsilon\mathrm{J}(\beta)})\mathcal{G}(\rho)=A_6,\\
&\mathrm{Ad}(e^{\varepsilon\mathcal{Y}(\gamma^2)})\mathcal{X}(\gamma^1)=\mathcal{X}(\gamma^1)+\varepsilon\mathcal{Z}((\gamma^1\gamma^2)_t), && \mathrm{Ad}(e^{\varepsilon\mathcal{X}(\gamma^1)})\mathcal{Y}(\gamma^2)=\mathcal{Y}(\gamma^2)-\varepsilon\mathcal{Z}((\gamma^1\gamma^2)_t),\\
&\mathrm{Ad}(e^{\varepsilon\mathcal{R}(\sigma)})\mathcal{X}(\gamma^1)=A_7, && \mathrm{Ad}(e^{\varepsilon\mathcal{X}(\gamma^1)})\mathcal{R}(\sigma)=A_8,\\
&\mathrm{Ad}(e^{\varepsilon\mathcal{H}(\delta)})\mathcal{X}(\gamma^1)=\mathcal{X}(\gamma^1)+\varepsilon\mathcal{H}(\gamma^1\delta_x), && \mathrm{Ad}(e^{\varepsilon\mathcal{X}(\gamma^1)})\mathcal{H}(\delta)=\mathcal{H}(\delta(t,x-\varepsilon\gamma^1,y)),\\
&\mathrm{Ad}(e^{\varepsilon\mathcal{G}(\rho)})\mathcal{X}(\gamma^1)=\mathcal{X}(\gamma^1)+\varepsilon\mathcal{G}(\gamma^1\rho_x), && \mathrm{Ad}(e^{\varepsilon\mathcal{X}(\gamma^1)})\mathcal{G}(\rho)=\mathcal{G}(\rho(t,x-\varepsilon\gamma^1,y)),\\
&\mathrm{Ad}(e^{\varepsilon\mathcal{R}(\sigma)})\mathcal{Y}(\gamma^2)=A_9, && \mathrm{Ad}(e^{\varepsilon\mathcal{Y}(\gamma^2)})\mathcal{R}(\sigma)=A_{10},\\
&\mathrm{Ad}(e^{\varepsilon\mathcal{H}(\delta)})\mathcal{Y}(\gamma^2)=\mathcal{Y}(\gamma^2)+\varepsilon\mathcal{H}(\gamma^2\delta_y), && \mathrm{Ad}(e^{\varepsilon\mathcal{Y}(\gamma^2)})\mathcal{H}(\delta)=\mathcal{H}(\delta(t,x,y-\varepsilon\gamma^2)),\\
&\mathrm{Ad}(e^{\varepsilon\mathcal{G}(\rho)})\mathcal{Y}(\gamma^2)=\mathcal{Y}(\gamma^2)+\varepsilon\mathcal{G}(\gamma^2\rho_y), && \mathrm{Ad}(e^{\varepsilon\mathcal{Y}(\gamma^2)})\mathcal{G}(\rho)=\mathcal{G}(\rho(t,x,y-\varepsilon\gamma^2)),
\end{aligned}
$$

where

$$
\begin{aligned}
&A_1:=\mathrm{J}(\beta)-\varepsilon\big(\mathcal{Y}(\beta\gamma^1)-\mathcal{G}(\beta_{tt}\gamma^1y)\big)+\tfrac12\varepsilon^2\mathcal{Z}\big((\beta(\gamma^1)^2)_t\big),\\
&A_2:=\mathrm{J}(\beta)+\varepsilon\big(\mathcal{X}(\beta\gamma^2)-\mathcal{G}(\beta_{tt}\gamma^2x)\big)+\tfrac12\varepsilon^2\mathcal{Z}\big((\beta(\gamma^2)^2)_t\big),\\
&A_3:=\mathcal{X}(\gamma^1\cos\beta\varepsilon)+\mathcal{Y}(\gamma^1\sin\beta\varepsilon)-\varepsilon\mathcal{G}\big(\gamma^1\beta_{tt}(-x\sin\beta\varepsilon+y\cos\beta\varepsilon)\big),\\
&A_4:=-\mathcal{X}(\gamma^2\sin\beta\varepsilon)+\mathcal{Y}(\gamma^2\cos\beta\varepsilon)+\varepsilon\mathcal{G}\big(\gamma^1\beta_{tt}(x\cos\beta\varepsilon+y\sin\beta\varepsilon)\big),\\
&A_5:=\mathcal{H}(\delta(t,x\cos\beta\varepsilon+y\sin\beta\varepsilon,-x\sin\beta\varepsilon+y\cos\beta\varepsilon)),\\
&A_6:=\mathcal{G}(\rho(t,x\cos\beta\varepsilon+y\sin\beta\varepsilon,-x\sin\beta\varepsilon+y\cos\beta\varepsilon)),\\
&A_7:=\mathcal{X}(\gamma^1)+\varepsilon\big(\mathcal{H}(\gamma^1\sigma x)-\mathcal{G}(\gamma^1\sigma_ty)\big),\\
&A_8:=\mathcal{R}(\sigma)-\varepsilon\left(\mathcal{H}(\gamma^1\sigma x)-\mathcal{G}(\gamma^1\sigma_ty)\right)+\tfrac12\varepsilon^2\mathcal{H}\left((\gamma^1)^2\sigma\right),
\end{aligned}
$$

$$A_9 := \mathcal{Y}(\gamma^2) + \varepsilon\left(\mathcal{H}\left(\gamma^2\sigma y\right) + \mathcal{G}(\gamma^2\sigma_t x)\right),$$
$$A_{10} := \mathcal{R}(\sigma) - \varepsilon\left(\mathcal{H}(\gamma^2\sigma y) + \mathcal{G}(\gamma^2\sigma_t x)\right) + \tfrac{1}{2}\varepsilon^2\mathcal{H}\left((\gamma^2)^2\sigma\right).$$

Using the above adjoint actions, we construct the following optimal list of inequivalent one-dimensional subalgebras of $\mathfrak{g}_1^{\sim}$:

$$\begin{gathered}\langle\mathcal{D}_1 + a\mathcal{D}_2\rangle, \quad \langle\partial_t + b\mathcal{D}_2\rangle, \quad \langle\mathcal{D}_2 + \mathrm{J}(\beta) + \mathcal{R}(\sigma)\rangle, \quad \langle\mathrm{J}(\beta) + \mathcal{R}(\sigma) + \mathcal{Z}(\chi)\rangle,\\ \langle\mathcal{X}(\gamma^1) + \mathcal{R}(\sigma)\rangle, \quad \langle\mathcal{R}(\sigma) + \mathcal{H}(\delta) + \mathcal{G}(\rho) + \mathcal{Z}(\chi)\rangle,\end{gathered} \tag{8.21}$$

where $a \in \mathbb{R}$, $b \in \{-1, 0, 1\}$. In fact, each element of the above list represents a class of subalgebras rather than a single subalgebra. Subalgebras within each of the four last classes can be equivalent. Thus, in the third class we can use adjoint action $\mathrm{Ad}(e^{\varepsilon\mathcal{D}_1})$ to rescale σ as well as the argument t of β and σ. Using $\mathrm{Ad}(e^{\varepsilon\partial_t})$ allows us to shift t in the functions β and σ. In the fourth class, equivalence is understood up to actions of $\mathrm{Ad}(e^{\varepsilon\mathcal{D}_1})$, $\mathrm{Ad}(e^{\varepsilon\mathcal{D}_2})$ and $\mathrm{Ad}(e^{\varepsilon\partial_t})$, which permit rescaling of σ, χ and their argument t, scaling of χ as well as shifts of t in β, σ and χ. Similar equivalence is also included in the fifth class. The last class comprises equivalence with respect to actions of $\mathrm{Ad}(e^{\varepsilon\mathrm{J}(\beta)})$, $\mathrm{Ad}(e^{\varepsilon\mathcal{X}(\gamma^1)})$ and $\mathrm{Ad}(e^{\varepsilon\mathcal{Y}(\gamma^2)})$. In the three last classes we can also rescale the entire basis elements.

Chapter 9

Summary and conclusions

In this part of the thesis we have presented several methods for the construction of physical parameterization schemes with prescribed symmetry properties. These methods have been demonstrated by the construction of invariant parameterization schemes for the eddy vorticity flux in the barotropic vorticity equation. It should be stressed that this model is indeed one of the simplest equations for which the parameterization problem can be illustrated. Additionally, we have only focussed on first order closure schemes. On the other hand, the computations involved in the construction of parameterizations for this equation were already rather elaborate. This is especially true in view of the calculations of the equivalence algebra and the set of admissible transformations for the considered setting. The situation gets even worse if one aims to investigate not only first order closure schemes as done in the present thesis. Currently, it is typical to employ third or higher order closure schemes in state of the art models, i.e. to construct parameterizations with explicit differential equations for the first and second order correlation terms [98, 150]. In such situations it is necessary to simultaneously treat a possibly large system of several differential equations. Although the methods proposed in this part can be applied to all kinds of local parameterization schemes, the practical computation of the equivalence algebra of these models by hand seems to be an extremely challenging task. This is why we regard the usage of suitable computer packages for an automatic computation of the equivalence algebra to be crucial in such situations. Similarly, the calculation of the set of admissible transformations and checking of the normalization property seem to be an almost hopeless task for systems of differential equations employing such higher order closure schemes. It should be added that there are so far no computer algebra packages available which allow for the computation of the set of admissible transformations.

At the same time, even if the set of admissible transformations of a class of the differential equations is incomputable (or if a given class fails to be normalized), there are several possibilities for the construction of invariant parameterization schemes. Using methods of inverse group classification only requires knowledge of the differential invariants of the maximal Lie invariance algebra of the system of differential equations not involving unknown terms. What is more, the computation of the maximal Lie invariance algebra is feasible even for large systems of differential equations by infinitesimal methods. For this reason we regard inverse group classification as suitable for state of the art models in which parameterizations are needed. But also direct group classification methods can prove valuable in cases where the set of admissible transformations cannot be obtained. For simple classes of parameterizations, the group classification problem can possibly be solved completely by employing compatibility methods to the system of determining equations of Lie symmetries. If the determining equations at hand are not suitable for a thorough compatibility analysis, invariant parameterizations can still be constructed using preliminary group classification, provided that the equivalence algebra is known. Since this technique is essentially algorithmic, it may also be well-suited for an application by researchers working in the field of parameterization without explicit background in symmetry analysis. Though such a treatment will not yield a complete description of all possible invariant parameterization schemes, it might nevertheless help finding a restricted form of the chosen parameterization functions. This might be especially fruitful for processes that are well-understood but have to be parameterized in numerical models. For such processes, the parameterization functions are already quite determined. In turn, the more restrictions on the parameterization functions are already imposed, the less possibilities exist for the construction of different invariant parameterizations. That is, in order to determine invariant parameterization schemes, it might not be necessary at all to carry out an exhaustive group classification of the given class of differential equations as several classes that could arise under classification can already be excluded for physical reasons. In this light, the exhaustive description of invariant parameterizations of the eddy vorticity flux in the vorticity equation should also be considered to be of pedagogical value going beyond the requirements, which might be finally needed in practice.

So far, the parameterization schemes proposed in this part have not been tested in practice. It should be clear that while the parameterizations in the Tables 8.1–8.4 are optimal from the mathematical viewpoint, they are far from being suitably prepared for an actual application in a numerical model. Firstly, as mentioned above, some of them are obviously unphysical (such as, e.g., the first and the second parameterization in Table 8.1). Secondly, parameterizations from these tables represent classes of invariant parameterizations. This means that there is still a freedom in choosing a particular realization from the single classes. At this level, other desirable physical properties should be incorporated into the parameterization. Finally, each of the parameterizations listed above is a representative of the respective class based on the classification of inequivalent subalgebras. It is hence possible to use transformations of the corresponding equivalence group to map the given representative to a new one, without leaving the current class of parameterization. This freedom in choosing an appropriate class representative can be crucial in order to find those parameterization that is best suited for practical applications.

All these remarks have to be taken into account before it becomes possible to attach such an invariant parameterization to a numerical integration model. Nevertheless, it will be interesting to test and assess the quality of parameterization schemes that are constructed using methods of group classification. This is one of our future perspective in the framework of invariant parameterization theory.

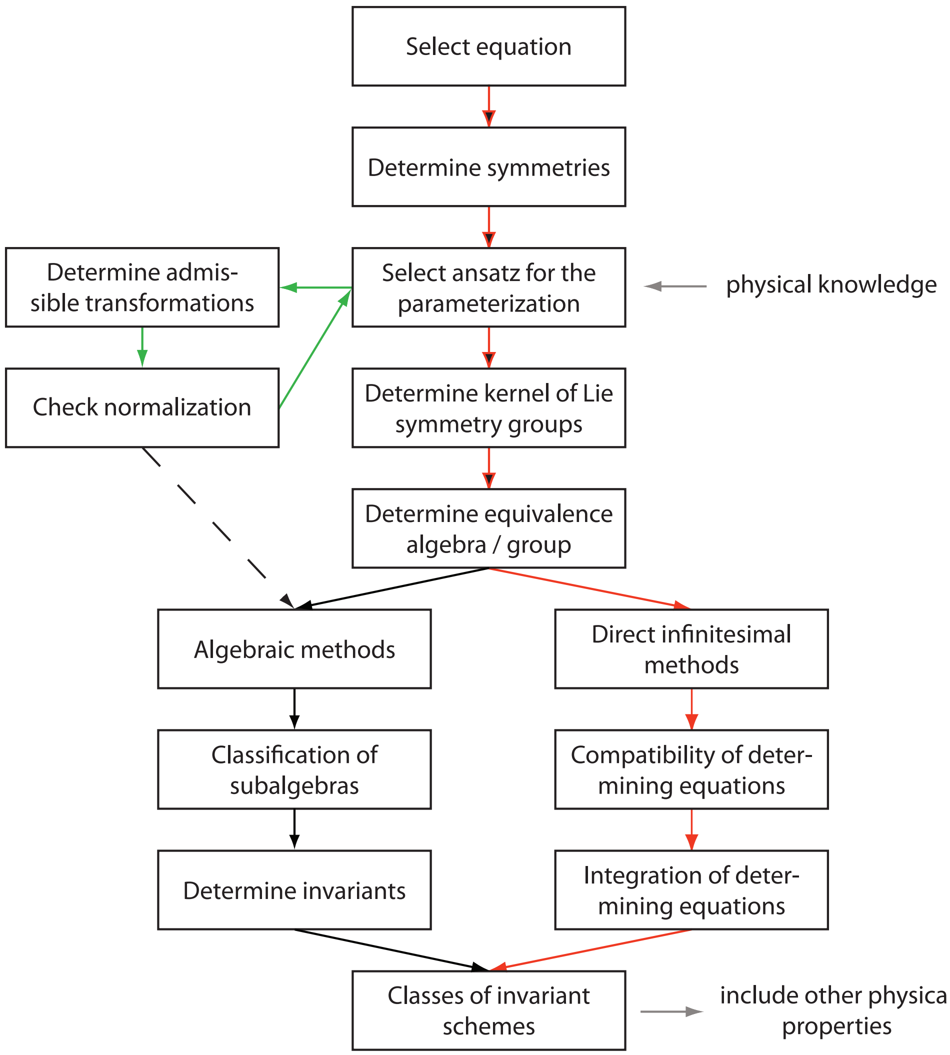


Figure 9.1: Schematic overview of the construction of invariant discretization schemes based on methods of direct group classification. Red path: Suitable for classes of parameterization schemes that can be investigated using compatibility analysis of the system of determining equations of Lie point symmetries. Black path: The algebraic method of group classification applied to the parameterization problem. This latter method is specially adapted for classes that possess the normalization property (which should be checked separately), eventually leading to a modification of the parameterization ansatz (green path). For classes that fail to be normalized, the result of this method gives a non-exhaustive list of invariant parameterizations, which still might be useful in practice.

Part III

Symmetries, Nambu mechanics and finite-mode models

Chapter 10

Rayleigh-Bénard Convection as a Nambu-metriplectic problem

Abstract The traditional Hamiltonian structure of the equations governing conservative Rayleigh-Bénard convection (RBC) is singular, i.e. it's Poisson bracket possesses nontrivial Casimir functionals. We show that a special form of one of these Casimirs can be used to extend the bilinear Poisson bracket to a trilinear generalised Nambu bracket. It is further shown that the equations governing dissipative RBC can be written as the superposition of the conservative Nambu bracket with a dissipative symmetric bracket. This leads to a Nambu-metriplectic system, which completes the geometrical picture of RBC.

10.1 Introduction

The noncanonical Hamiltonian form of the hydro-thermodynamical equations in Eulerian variables is typically singular. This gives rise to the existence of a special class of conserved quantity, the Casimir functionals. This singularity is a consequence of the reduction that takes place if one changes the coordinates from the (canonical) Lagrangian coordinates to the (noncanonical) Eulerian coordinates by means of the particle-relabeling symmetry, e.g. [155].

In the noncanonical Hamiltonian form the existence of additional conserved quantities is still hidden and hence it is natural to seek for a formulation which lets enter them the description in a similar way as the Hamiltonian does. For the great majority of hydrodynamical systems this is possible [106, 108] and leads to a form of description that formally resembles the structure of Nambu mechanics, which was first introduced by [104] for discrete systems.

We start with the classical definition of a discrete Nambu system [151]:

Definition: Let M be a smooth manifold and $C^\infty(M)$ the algebra of infinitely differentiable real valued functions defined on M. Then M is called a *Nambu-Poisson manifold* of order n if there exists a map $\{,\dots,\} : [C^\infty(M)]^{\otimes n} \to C^\infty(M)$ called the *Nambu bracket* that satisfies

1. $\{f_1, f_2, \dots, f_n\} = (-1)^{\epsilon(\sigma)}\{f_{\sigma(1)}, f_{\sigma(2)}, \dots, f_{\sigma(n)}\}$ where σ denotes an element of the symmetric group of n elements, with $\epsilon(\sigma)$ being it's parity.

2. $\{f_1 f_2, f_3, \dots, f_{n+1}\} = f_1\{f_2, f_3, \dots, f_{n+1}\} + \{f_1, f_3, \dots, f_{n+1}\}f_2$

3. $\{\{f_1, \dots, f_{n-1}, f_n\}, f_{n+1}, \dots, f_{2n-1}\} + \{f_n, \{f_1 \dots, f_{n-1}, f_{n+1}\}, f_{n+2}, \dots, f_{2n-1}\} + \dots + \{f_n, \dots, f_{2n-2}, \{f_1, \dots, f_{n-1}, f_{2n-1}\}\} = \{f_1, \dots, f_{n-1}, \{f_n, \dots, f_{2n-1}\}\}$

Note that the first property defines the skew-symmetry of the Nambu bracket, the second property is the Leibniz rule and the last one is a generalisation of the Jacobi identity, known as the Fundamental identity (FI) or Takhtajan identity.

The Nambu bracket defines the kinematic part of a Nambu system, which is supplemented by $n-1$ function $H_1, \ldots, H_{n-1}$, such that the evolution of a real-valued function on M is given by

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \{f, H_1, \ldots, H_{n-1}\}.$$

Due to the antisymmetry of the Nambu bracket, it follows that $H_1, \ldots, H_{n-1}$ are conserved by the Nambu flow.

The generalisation to field equations was done in [106, 108] by starting with the noncanonical Hamiltonian formulation of the respective model equations. They extended them by means of using *one* of their Casimirs as additional conserved quantity. Hence, their continuous Nambu formulation also only uses *trilinear* bracket structures. This kind of generalisation is the one we aim to use for the equations governing RBC.

Let us now turn to dissipative systems with a conservative Hamiltonian core. Although the notion of such so called metriplectic systems is not unique (see [49] for a review) the constituent parts of them are the antisymmetric Poisson and a symmetric (or gradient) structure, that accounts for dissipation. Adding both pieces together then describes the dynamics of the whole dissipative system. The metriplectic bracket hence reads

$$\langle\langle f, g\rangle\rangle := \{f, g\} + \langle f, g\rangle$$

where

$$\{f, g\} = -\{g, f\} \qquad \text{antisymmetric (Poisson) bracket}$$
$$\langle f, g\rangle = \langle g, f\rangle \qquad \text{symmetric (gradient) bracket}$$

holds.

In this paper we show that the equations of RBC can be cast in Nambu-metriplectic form, that is, the conservative part possesses a Nambu and the dissipative part a symmetric bracket structure.

10.2 The Nambu structure of conservative RBC

The equations of two-dimensional RBC in case of an incompressible fluid using the Boussinesq approximation in nondimensional form read [153]:

$$\frac{\partial \zeta}{\partial t} + [\psi, \zeta] = \frac{\partial T}{\partial x} + \nu \nabla^4 \psi, \qquad \frac{\partial T}{\partial t} + [\psi, T] = \frac{\partial \psi}{\partial x} + \kappa \nabla^2 T. \tag{10.1}$$

As usual, ψ is the stream function generating two-dimensional nondivergent flow in the x–z-plane, $\zeta = \nabla^2\psi$ is the vorticity, T is the temperature departure of a linear conduction profile, ν and κ are kinematic viscosity and thermal conductivity, respectively. $[a, b] := \partial a/\partial x\, \partial b/\partial z - \partial a/\partial z\, \partial b/\partial x$ denotes the Jacobian operator.

In the following, we will assume that the domain of interest is periodic in the x-direction. In the vertical, we either also assume periodicity or stress-free boundary conditions [145], i.e.

$$\psi = 0, \quad \zeta = 0, \quad \frac{\partial \psi}{\partial x} = 0 \qquad \text{at} \qquad z = 0, \quad z = z_{\text{top}}.$$

In either case, since in RBC a constant temperature difference is maintained externally between the top and bottom of the fluid, the appropriate boundary condition for the temperature deviation T is

$$T = 0 \qquad \text{at} \qquad z = 0, \quad z = z_{\text{top}}.$$

Determining the classical continuous Nambu form is possible only in the case of vanishing dissipation, i.e. in case $\nu = \kappa = 0$. Then, since the remaining terms on the right hand side can be written as $\partial T/\partial x = [T, z]$ and $\partial \psi/\partial x = [\psi, z]$, respectively, both equations may be arranged as

$$\frac{\partial \zeta}{\partial t} + [\psi, \zeta] + [z, T - z] = 0, \qquad \frac{\partial T}{\partial t} + [\psi, T - z] = 0. \tag{10.2}$$

This set of equations is Hamiltonian upon using

$$\mathcal{H} = \int_\Omega \left(\frac{1}{2} (\nabla \psi)^2 - Tz \right) \mathrm{d}f$$

as Hamiltonian functional and

$$\{\mathcal{F}, \mathcal{H}\} = \int_\Omega \left(\zeta \left[\frac{\delta \mathcal{F}}{\delta \zeta}, \frac{\delta \mathcal{H}}{\delta \zeta} \right] + (T - z) \left(\left[\frac{\delta \mathcal{F}}{\delta \zeta}, \frac{\delta \mathcal{H}}{\delta T} \right] + \left[\frac{\delta \mathcal{F}}{\delta T}, \frac{\delta \mathcal{H}}{\delta \zeta} \right] \right) \right) \mathrm{d}f \tag{10.3}$$

as a Poisson bracket. An analogue bracket arises also in magnetohydrodynamics (MHD) [102]. Note, however, that this Poisson bracket is singular, i.e. it possesses nonvanishing Casimir functionals. They are

$$\mathcal{C}_1 = \int_\Omega g(T - z)\, \mathrm{d}f, \qquad \mathcal{C}_2 = \int_\Omega \zeta h(T - z)\, \mathrm{d}f,$$

where g, h are arbitrary functions of $T - z$. For a physical interpretation of the analogue Casimirs in MHD, see [154]. In conservative RBC the first class of Casimirs $\mathcal{C}_1$ physically describes the preservation of $T - z$-contours. In fact, this class of Casimirs generally arises in nondivergent and inviscid fluid models. In turn, the second class $\mathcal{C}_2$ incorporates the Kelvin's circulation theorem, which in the case of RBC requires conservation of the vorticity ζ on closed contours of $T - z$.

To determine the conservative Nambu form, we are only interested in the special form of $\mathcal{C}_2$ with $h = T - z$:

$$\mathcal{C} = \int_\Omega \zeta (T - z)\, \mathrm{d}f.$$

It allows us to extend the Poisson bracket formulation of the governing equations to arrive at the their Nambu form:

$$\begin{aligned}
\frac{\partial \zeta}{\partial t} &= - \left[\frac{\delta \mathcal{C}}{\delta T}, \frac{\delta \mathcal{H}}{\delta \zeta} \right] - \left[\frac{\delta \mathcal{C}}{\delta \zeta}, \frac{\delta \mathcal{H}}{\delta T} \right] &&= \{\zeta, \mathcal{C}, \mathcal{H}\} \\
\frac{\partial T}{\partial t} &= - \left[\frac{\delta \mathcal{C}}{\delta \zeta}, \frac{\delta \mathcal{H}}{\delta \zeta} \right] &&= \{T, \mathcal{C}, \mathcal{H}\},
\end{aligned}$$

where the bracket $\{\cdot,\cdot,\cdot\}$ is defined for arbitrary functionals $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ by the equation

$$\{\mathcal{F}_1,\mathcal{F}_2,\mathcal{F}_3\} := -\int_\Omega \left(\frac{\delta\mathcal{F}_1}{\delta T}\left[\frac{\delta\mathcal{F}_2}{\delta\zeta},\frac{\delta\mathcal{F}_3}{\delta\zeta}\right] + \frac{\delta\mathcal{F}_1}{\delta\zeta}\left[\frac{\delta\mathcal{F}_2}{\delta T},\frac{\delta\mathcal{F}_3}{\delta\zeta}\right] + \frac{\delta\mathcal{F}_1}{\delta\zeta}\left[\frac{\delta\mathcal{F}_2}{\delta\zeta},\frac{\delta\mathcal{F}_3}{\delta T}\right]\right)\mathrm{d}f. \tag{10.4}$$

This bracket is easily seen to be totally antisymmetric in case we assume periodic boundary conditions in both directions. Another possibility that guarantees antisymmetry of (10.4) is provided by periodicity in x-direction and free boundaries in the vertical. For this choice, however, it is necessary to fix $\mathcal{F}_3 = \mathcal{H}$ in the Nambu bracket. This is explicitly shown in the appendix. Assuring antisymmetry is also possible by only considering those class of functionals that sufficiently rapidly go to zero towards the boundaries. All these choices guarantee that surface terms emerging from an integration by parts vanish.

The FI is proved by noting that the above Nambu bracket is simply the continuous analogue of the heavy top Nambu bracket. Hence, this bracket may indeed serve as a good Nambu bracket.

Note that $\mathcal{C}$ is indefinite with respect to sign. In this respect it is akin to the helicity, which is a Casimir for the three dimensional incompressible Euler equations [108]. To our knowledge, the latter model and RBC are the only known ones that need indefinite Casimirs to allow for a Nambu representation.

10.3 The Nambu-metriplectic structure of dissipative RBC

Now turning to the dissipative equations (i.e. $\nu \neq 0, \kappa \neq 0$), we aim to show that this model has Nambu-metriplectic form. Let us first note, that adding $0 = [T-z, T-z]$ to the first equation in (10.2) leads to the equivalent system

$$\frac{\partial\zeta}{\partial t} + [\psi,\zeta] + [T, T-z] = 0, \qquad \frac{\partial T}{\partial t} + [\psi, T-z] = 0.$$

which is Hamiltonian upon using (10.3) as Poisson bracket and

$$\mathcal{G} = \int_\Omega \left(\frac{1}{2}(\nabla\psi)^2 - Tz - \frac{1}{2}(T-z)^2\right)\mathrm{d}f = \mathcal{H} - \mathcal{S}$$

as Hamiltonian functional. Note that $\mathcal{S}$ is the particular realisation of the first class of Casimir functionals $\mathcal{C}_1$ with $g = 1/2(T-z)^2$. In [100], functionals like $\mathcal{G}$ are termed generalised free energy.

This formulation enables us to cast the equations governing dissipative RBC in Nambu-metriplectic form. Indeed, in this case the equations can be written as

$$\begin{aligned}
&\frac{\partial\zeta}{\partial t} + \left[\frac{\delta\mathcal{C}}{\delta T},\frac{\delta\mathcal{G}}{\delta\zeta}\right] + \left[\frac{\delta\mathcal{C}}{\delta\zeta},\frac{\delta\mathcal{G}}{\delta T}\right] = -\nu\nabla^4\frac{\delta\mathcal{G}}{\delta\zeta}\\
&\frac{\partial T}{\partial t} + \left[\frac{\delta\mathcal{C}}{\delta\zeta},\frac{\delta\mathcal{G}}{\delta\zeta}\right] = -\kappa\nabla^2\frac{\delta\mathcal{G}}{\delta T}
\end{aligned}$$

and it is possible to introduce an indefinite symmetric bracket that governs dissipation:

$$\langle\mathcal{F},\mathcal{G}\rangle := -\int_\Omega\left(\nu\frac{\delta\mathcal{F}}{\delta\zeta}\nabla^4\frac{\delta\mathcal{G}}{\delta\zeta} + \kappa\frac{\delta\mathcal{F}}{\delta T}\nabla^2\frac{\delta\mathcal{G}}{\delta T}\right)\mathrm{d}f. \tag{10.5}$$

The symmetry property of this bracket is assured if either periodic boundaries are assumed or only functionals that sufficiently rapidly decay to zero near the boundaries are considered. Adding together both brackets gives the entire dynamics of two-dimensional RBC:

$$\frac{\partial \zeta}{\partial t} = \{\zeta, \mathcal{C}, \mathcal{G}\} + \langle \zeta, \mathcal{G} \rangle \,, \qquad \frac{\partial T}{\partial t} = \{T, \mathcal{C}, \mathcal{G}\} + \langle T, \mathcal{G} \rangle \,.$$

Note that in some sense the Nambu-metriplectic formulation of (10.1) is geometrically the most complete, since representatives of *both* classes of Casimirs are needed to represent the whole dynamics.

The existence of dissipation generally spoils the conservation properties of conservative system. This is also the case in RBC, since we have

$$\frac{\partial \mathcal{G}}{\partial t} = \langle \mathcal{G}, \mathcal{G} \rangle \neq 0, \qquad \frac{\partial \mathcal{C}}{\partial t} = \langle \mathcal{C}, \mathcal{G} \rangle \neq 0.$$

That is, the evolution of $\mathcal{G}$ is determined solely by $\mathcal{G}$, whereas the evolution of $\mathcal{C}$ is determined both by $\mathcal{C}$ and $\mathcal{G}$.

10.4 Comments and outlook

In this work we have shown that the conservative part of (10.1) can be written in Nambu form, while the full dissipative system possesses a Nambu-metriplectic form. For both representations, the explicit usage of Casimir functionals of (10.3) is crucial. The Casimir $\mathcal{C}$ allows to extend the bilinear Poisson bracket to a trilinear Nambu bracket. In turn, the Casimir $\mathcal{S}$ can be subtracted from the Hamiltonian $\mathcal{H}$ to give the modified Hamiltonian $\mathcal{G}$. This doesn't alter the dynamics, since Casimirs are trivial conserved quantities and Hamiltonians are only determined up to Casimir functionals. But introducing $\mathcal{G}$ is essential to allow for the necessary symmetry property of the dissipative bracket (10.5).

The Poisson bracket (10.3) is an example of a Lie-Poisson system. Such systems are built from an underlying Lie algebra structure, typically owing to a reduction from a set of canonical to a set of noncanonical variables, e.g. [155]. In case of RBC it is the semi-direct extension of the algebra associated to the group of volume-preserving diffeomorphisms on some domain Ω with the vector space of real-valued functions on Ω [155]. As was shown in our work, the Nambu structure is compatible with this type of algebra extension. In particular, the Nambu bracket (10.4) allows to put (10.3) in a more symmetric form.

Studying algebras and extensions thereof in conjunction with Nambu structures offers a way to classify Nambu systems. Such work is currently in progress and will be published elsewhere.

Appendix

We aim to explicitly show here that assuming periodic boundaries in x-direction and free boundaries in the vertical guarantees the total antisymmetry of (10.4) provided we fix the Hamiltonian $\mathcal{H}$ in the bracket.

We only need to prove antisymmetry with respect to the first two functionals of the bracket (10.4). The antisymmetry in the other pair of arguments is obvious. An integration by parts gives

$$\int_\Omega \frac{\delta \mathcal{F}}{\delta T} \left[\frac{\delta \mathcal{C}}{\delta \zeta}, \frac{\delta \mathcal{H}}{\delta \zeta} \right] \mathrm{d}f = - \int_\Omega \frac{\delta \mathcal{C}}{\delta \zeta} \left[\frac{\delta \mathcal{F}}{\delta T}, \frac{\delta \mathcal{H}}{\delta \zeta} \right] \mathrm{d}f - \int_L \left(\frac{\delta \mathcal{F}}{\delta T} \frac{\delta \mathcal{C}}{\delta \zeta} \frac{\partial}{\partial x} \frac{\delta \mathcal{H}}{\delta \zeta} \right) \Big|_z \mathrm{d}x$$

$$\int_\Omega \frac{\delta\mathcal{F}}{\delta\zeta}\left[\frac{\delta\mathcal{C}}{\delta T},\frac{\delta\mathcal{H}}{\delta\zeta}\right]\mathrm{d}f = -\int_\Omega \frac{\delta\mathcal{C}}{\delta T}\left[\frac{\delta\mathcal{F}}{\delta\zeta},\frac{\delta\mathcal{H}}{\delta\zeta}\right]\mathrm{d}f - \int_L \left(\frac{\delta\mathcal{F}}{\delta\zeta}\frac{\delta\mathcal{C}}{\delta T}\frac{\partial}{\partial x}\frac{\delta\mathcal{H}}{\delta\zeta}\right)\Big|_z \mathrm{d}x$$
$$\int_\Omega \frac{\delta\mathcal{F}}{\delta\zeta}\left[\frac{\delta\mathcal{C}}{\delta\zeta},\frac{\delta\mathcal{H}}{\delta T}\right]\mathrm{d}f = -\int_\Omega \frac{\delta\mathcal{C}}{\delta\zeta}\left[\frac{\delta\mathcal{F}}{\delta\zeta},\frac{\delta\mathcal{H}}{\delta T}\right]\mathrm{d}f - \int_L \left(\frac{\delta\mathcal{F}}{\delta\zeta}\frac{\delta\mathcal{C}}{\delta\zeta}\frac{\partial}{\partial x}\frac{\delta\mathcal{H}}{\delta T}\right)\Big|_z \mathrm{d}x$$

were we have already taken into account periodicity in the x-direction. Due to the imposed boundary conditions

$$\frac{\partial}{\partial x}\left(\frac{\delta\mathcal{H}}{\delta\zeta}\right)\Big|_z = -\frac{\partial\psi}{\partial x}\Big|_z = 0, \qquad \frac{\partial}{\partial x}\left(\frac{\delta\mathcal{H}}{\delta T}\right)\Big|_z = -\frac{\partial z}{\partial x}\Big|_z = 0$$

all the second terms on the right hand side vanish. Similar considerations also hold for fixing the modified Hamiltonian $\mathcal{G}$.

Chapter 11

Symmetry justification of Lorenz' maximum simplification

Abstract In 1960 Edward Lorenz (1917–2008) published a pioneering work on the 'maximum simplification' of the barotropic vorticity equation. He derived a coupled three-mode system and interpreted it as the minimum core of large-scale fluid mechanics on a 'finite but unbounded' domain. The model was obtained in a heuristic way, without giving a rigorous justification for the chosen selection of modes. In this paper, it is shown that one can legitimate Lorenz' choice by using symmetry transformations of the spectral form of the vorticity equation. The Lorenz three-mode model arises as the final step in a hierarchy of models constructed via the component reduction by means of symmetries. In this sense, the Lorenz model is indeed the 'maximum simplification' of the vorticity equation.

11.1 Introduction

Symmetry is one of the most important concepts in numerous branches of modern natural science. The exploitation of symmetries of dynamical systems may lead to a more efficient treatment of the differential equations describing these systems via a reduction of the information that is necessary in order to account for model dynamics.

There are many ways to utilize symmetries of differential equations, including the systematic construction of exact solutions of PDEs, determination of conservation laws and construction of mappings that relate or linearize differential equations (see e.g. [3, 115, 118]). Also, there is some work on symmetries in the study of dynamical systems and bifurcation theory [48], giving rise to the study of equivariant dynamical systems. In the present work, we will take a related but somewhat different direction to investigate how the Lorenz-1960 model [84] can be derived in a rigorous way.

In this classical work, Lorenz considered the spectral expansion of the barotropic vorticity equation on a torus. In what follows, he sought for the minimum system of coupled ordinary first order differential equations for the Fourier coefficients that is necessary to still account for the nonlinear interaction of modes. In doing so, he first restricted the range of indices in the infinite Fourier series by the values $\{-1, 0, 1\}$. His crucial step to achieve the maximum simplification was the observation that all but three of these remaining coefficients retain their particular values once they are taken. We aim to give a justification of this observation by interpreting it as a condition of symmetry. To be more precise, we show that this simplification is possible due to the corresponding spectral counterparts of point symmetry transformations of the vorticity equation in physical space. These symmetries are preserved under truncations of the infinite Fourier series and are inherited by the spectral set of equations for the Fourier coefficients.

A justification of the selection of modes in finite-mode models by considering inherited symmetries may be potentially useful also in more general situations. It could provide an additional criterion addressing the important question which modes in the reduced model should be retained and which may be neglected. In this sense, using induced symmetries may supply special kinds of truncations that are designed to preserve e.g. the invariants of the parental model also in the truncated dynamics (e.g. [153]). This may lead to more concise and consistent finite-mode representations. As stated above, this approach is to be distinguished from the field of equivariant dynamics. The main difference is that in the latter usually no exhaustive and rigorous calculations of symmetry groups are given.

The organization of this paper is the following: Section 11.2 is devoted to discrete and continuous symmetries of the barotropic vorticity equation in a non-rotating reference frame. We expand the vorticity in a double Fourier series and discuss how the symmetries of the equation in physical space are induced to symmetry transformations in terms of the Fourier coefficients. In section 11.3 the initial model of eight ODEs will be presented amongst a discussion of the induction of the symmetries to this truncated system. Finally, in section 11.4 subgroups of the whole symmetry group of the truncated spectral vorticity equations will be used to derive hierarchies of reduced models describing the evolution of the relevant Fourier coefficients. The main result is that the Lorenz-1960 model is indeed the maximum simplification of the dynamic equations. It is the minimal system that can be obtained by using symmetry subgroups of the spectral vorticity equation for component reduction.

11.2 Symmetries of the barotropic vorticity equation

The inviscid barotropic vorticity equation in an inertial system in stream function form reads

$$\frac{\partial}{\partial t}\nabla^2\psi + \frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\nabla^2\psi - \frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}\nabla^2\psi = 0, \tag{11.1}$$

where ψ is the stream function generating two-dimensional nondivergent flow in the (x, y)-plane. It states the individual or Lagrangian conservation of the vorticity

$$\zeta = \nabla^2\psi.$$

Eqn. (11.1) possesses the eight-element group of discrete symmetries, generated by the elements

$$\begin{aligned} e_1 &: \ (x, y, t, \psi) \to (x, -y, t, -\psi) \\ e_2 &: \ (x, y, t, \psi) \to (-x, y, t, -\psi) \\ e_3 &: \ (x, y, t, \psi) \to (x, y, -t, -\psi). \end{aligned}$$

Note that these transformations are involutive (i.e. $e_i^2 = 1$, $i = 1, 2, 3$) and, moreover, they commute (i.e. $e_ie_j = e_je_i$, $i, j = 1, 2, 3$, $i \neq j$). Each e_i generates a copy of $\mathbb{Z}_2$, the cyclic group of order 2. The group of all discrete symmetry transformations of the vorticity equation may then be written as $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

For sake of completeness, we also present the generators of one-parameter symmetry groups of (11.1), which were computed using the program LIE by A. Head [51]. They read

$$\mathbf{v}_t = \frac{\partial}{\partial t} \qquad \mathbf{v}_u = tx\frac{\partial}{\partial y} - ty\frac{\partial}{\partial x} + \frac{1}{2}\left(x^2+y^2\right)\frac{\partial}{\partial \psi}$$

$$\mathbf{v}_r = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \qquad \mathcal{Z}(h) = h(t)\frac{\partial}{\partial \psi}$$

$$\mathcal{X}_1(f) = f(t)\frac{\partial}{\partial x} - yf'(t)\frac{\partial}{\partial \psi} \qquad \mathcal{X}_2(g) = g(t)\frac{\partial}{\partial y} + xg'(t)\frac{\partial}{\partial \psi}$$

$$\mathcal{D}_1 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2\psi\frac{\partial}{\partial \psi} \qquad \mathcal{D}_2 = t\frac{\partial}{\partial t} - \psi\frac{\partial}{\partial \psi},$$

where f, g and h are arbitrary smooth functions of t. Thus, in addition to the discrete symmetries e_i, $i = 1, 2, 3$, the vorticity equation (11.1) possesses the time translations (generated by $\mathbf{v}_t$), the rotations with constant velocities and on constant angles ($\mathbf{v}_u$ and $\mathbf{v}_r$, respectively), the gauging of stream function with arbitrary summands depending in t ($\mathcal{Z}(h)$), translatory motions with arbitrary (nonconstant) velocities ($\mathcal{X}_1(f)$ and $\mathcal{X}_2(g)$) and (separate) scaling of the space and time variables ($\mathcal{D}_1$ and $\mathcal{D}_2$). In contrast to the other basis operators, the operator $\mathbf{v}_u$ has no counterpart in the three-dimensional case and leads to nonlocal transformations in terms of the fluid velocity and the pressure. This singularity, from the symmetry point of view, of the two-dimensional vorticity equations in terms of the stream function was first observed by Berker [13] and later re-opened (c.f. [7]). In this paper we simultaneously use discrete symmetries and some of transformations from the continuous symmetry group.

We now expand the vorticity in a double Fourier series on the torus,

$$\zeta = \sum_{\mathbf{m}} c_{\mathbf{m}} \exp(i\hat{\mathbf{m}}\cdot\mathbf{x}), \qquad \mathbf{x} = x\mathbf{i} + y\mathbf{j}, \quad \mathbf{m} = m_1\mathbf{i} + m_2\mathbf{j}, \quad \hat{\mathbf{m}} = m_1k\mathbf{i} + m_2l\mathbf{j},$$

where k and l are nonzero constants, $\mathbf{i} = (1,0,0)^{\mathrm{T}}$, $\mathbf{j} = (0,1,0)^{\mathrm{T}}$, m_1 and m_2 run through the set of integers and the coefficient c_{00} vanishes. Inserting this expansion in the vorticity equation (11.1) gives its spectral form [84]:

$$\frac{\mathrm{d}c_{\mathbf{m}}}{\mathrm{d}t} = -\sum_{\mathbf{m}'\neq\mathbf{0}} \frac{c_{\mathbf{m}'}c_{\mathbf{m}-\mathbf{m}'}}{\hat{\mathbf{m}}'^2}\left(\mathbf{k}\cdot[\hat{\mathbf{m}}'\times\hat{\mathbf{m}}]\right), \tag{11.2}$$

where $\mathbf{k} = (0,0,1)^{\mathrm{T}}$. The transformations e_i induce transformations of the Fourier coefficients. Thus, in spectral terms the action $\tilde{\zeta}(x,y,t) = -\zeta(x,-y,t)$ of e_1 on ζ has the form

$$\begin{aligned}\sum_{\mathbf{m}} \tilde{c}_{\mathbf{m}} \exp(i\hat{\mathbf{m}}\cdot\mathbf{x}) &= -\sum_{\mathbf{m}} c_{\mathbf{m}} \exp(i(m_1kx - m_2ly)) \\ &= -\sum_{\mathbf{m}} c_{m_1,-m_2} \exp(i(m_1kx + m_2ly)),\end{aligned}$$

upon changing the summation over m_2. Hence we have $\tilde{c}_{m_1m_2} = -c_{m_1,-m_2}$ as a consequence of the x-reflection e_1. Finally, these and similar computations give the transformations

$$\begin{aligned} e_1\colon &\quad c_{m_1m_2} \to -c_{m_1,-m_2} \\ e_2\colon &\quad c_{m_1m_2} \to -c_{-m_1m_2} \\ e_3\colon &\quad c_{m_1m_2} \to -c_{m_1m_2}, \quad t \to -t. \end{aligned}$$

It is obvious that the induced transformations for the coefficients $c_{\mathbf{m}}$ are symmetries of system (11.2).

Some of continuous symmetries of the vorticity equation also induce well-defined symmetries of system (11.2). Thus, the rotation on the angle π coincides with e_1e_2. (In fact, only one of the reflections e_1 and e_2 is independent up to continuous symmetries.) More nontrivial examples are given by the space translations, which induce the transformations

$$\begin{aligned} p_\varepsilon\colon &\quad c_{m_1m_2} \to e^{im_1k\varepsilon}c_{m_1m_2} \\ q_\varepsilon\colon &\quad c_{m_1m_2} \to e^{im_2l\varepsilon}c_{m_1m_2}. \end{aligned}$$

For the values $\varepsilon = \pi/k$ and $\varepsilon = \pi/l$, respectively we have $p_{\pi/k}\colon c_{m_1m_2} \to (-1)^{m_1}c_{m_1m_2}$ and $q_{\pi/l}\colon c_{m_1m_2} \to (-1)^{m_2}c_{m_1m_2}$. For the sake of brevity, we will set $\hat{p} = p_{\pi/k}$ and $\hat{q} = q_{\pi/l}$.

11.3 Discrete symmetries of the truncated system

The restriction of the range of indices in (11.2) by $\{-1, 0, 1\} \times \{-1, 0, 1\}$ leads to the following eight-mode model related to the vorticity equation:

$$\begin{aligned}
\frac{\mathrm{d}c_{11}}{\mathrm{d}t} &= \left(\frac{1}{l^2} - \frac{1}{k^2}\right) klc_{10}c_{01} & \frac{\mathrm{d}c_{10}}{\mathrm{d}t} &= \left(\frac{1}{k^2+l^2} - \frac{1}{l^2}\right) kl\left[c_{11}c_{0,-1} - c_{1,-1}c_{01}\right] \\
\frac{\mathrm{d}c_{1,-1}}{\mathrm{d}t} &= \left(\frac{1}{k^2} - \frac{1}{l^2}\right) klc_{10}c_{0,-1} & \frac{\mathrm{d}c_{01}}{\mathrm{d}t} &= \left(\frac{1}{k^2+l^2} - \frac{1}{k^2}\right) kl\left[c_{10}c_{-11} - c_{11}c_{-10}\right] \\
\frac{\mathrm{d}c_{-1,-1}}{\mathrm{d}t} &= \left(\frac{1}{l^2} - \frac{1}{k^2}\right) klc_{-10}c_{0,-1} & \frac{\mathrm{d}c_{-10}}{\mathrm{d}t} &= \left(\frac{1}{k^2+l^2} - \frac{1}{l^2}\right) kl\left[c_{-1,-1}c_{01} - c_{0,-1}c_{-11}\right] \\
\frac{\mathrm{d}c_{-11}}{\mathrm{d}t} &= \left(\frac{1}{k^2} - \frac{1}{l^2}\right) klc_{-10}c_{01} & \frac{\mathrm{d}c_{0,-1}}{\mathrm{d}t} &= \left(\frac{1}{k^2+l^2} - \frac{1}{k^2}\right) kl\left[c_{-10}c_{1,-1} - c_{10}c_{-1,-1}\right].
\end{aligned} \tag{11.3}$$

System (11.3) consists of first-order ordinary differential equations. Hence the problem on description of its point symmetries is even more difficult than its complete integration (c.f. [115]). At the same time, some symmetries of (11.3) are in fact known since they are induced by symmetries of the vorticity equation. In particular, the symmetric truncation guarantees that the resulting system (11.3) inherits the discrete symmetries of the initial system (11.2), induced by e_1, e_2, e_3 and their compositions. (This is an argument justifying such kind of truncation.) Any truncation also preserves the symmetries p_ε and q_ε. It is easy to see that the transformations $\hat{p}$ and $\hat{q}$ are involutive and commutes with e_1, e_2 and e_3.

The above transformations are of crucial importance for deriving the Lorenz system.

11.4 Component reduction of the truncated system

The transformations e_1, e_2, $\hat{p}$ and $\hat{q}$ and their compositions act only on dependent variables and, therefore, can be used for component reductions of system (11.3).[1] The technique applied is similar to that developed for invariant solutions without transversality (c.f. [4, 118]). However, it is not apparent that the element e_3 can be used for this purpose. We thus restrict ourselves

[1]See the appendix for a depiction of these transformations.

to the group $G \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ generated by e_1, e_2, $\hat{p}$ and $\hat{q}$. We will apply the following subgroups of G:

$$
\begin{aligned}
&S_1 = \{1, e_1\}, \quad S_1' = \{1, e_2\}, \quad S_2 = \{1, \hat{p}\}, \quad S_2' = \{1, \hat{q}\},\\
&S_3 = \{1, \hat{p}e_2\}, \quad S_3' = \{1, \hat{q}e_1\}, \quad S_4 = \{1, \hat{p}\hat{q}\},\\
&S_5 = \{1, \hat{p}e_1\}, \quad S_5' = \{1, \hat{q}e_2\}, \quad S_6 = \{1, e_1e_2\},\\
&S_7 = \{1, \hat{p}e_1e_2\}, \quad S_7' = \{1, \hat{q}e_1e_2\}, \quad S_8 = \{1, \hat{p}\hat{q}e_1\}, \quad S_8' = \{1, \hat{p}\hat{q}e_2\},\\
&S_9 = \{1, \hat{p}\hat{q}e_1e_2\}, \quad S_{10} = \{1, \hat{p}\hat{q}e_1, \hat{p}\hat{q}e_2, e_1e_2\},\\
&S_{11} = \{1, \hat{p}e_1, \hat{q}e_1e_2, \hat{p}\hat{q}e_2\}, \quad S_{11}' = \{1, \hat{q}e_2, \hat{p}e_1e_2, \hat{p}\hat{q}e_1\},\\
&S_{12} = \{1, \hat{p}e_1, \hat{q}e_2, \hat{p}\hat{q}e_1e_2\}.
\end{aligned}
$$

By 1 we denote the identical transformation. There are yet other subgroups of G but as we will see they do not lead to nontrivial reduced systems.

Note that the subgroup S_1' will result in the same reduced system as S_1 (up to the re-notation $(x, k) \leftrightarrow (y, l)$). A similar remark holds also for all the subgroup marked by prime, so we only have to consider reductions with respect to the subgroups without prime. This should be done subsequently.

11.4.1 Trivial reductions

The tuple (c_{ij}) is invariant with respect to the transformation e_1 if and only if the following identifications hold:

$$c_{-1,-1} = -c_{-11}, \quad c_{0,-1} = -c_{01}, \quad c_{1,-1} = -c_{11}, \quad c_{10} = -c_{10}, \quad c_{-10} = -c_{-10}.$$

The last two conditions require that $c_{10} = c_{-10} = 0$. Inserting these identifications in (11.3) leads to a trivial system only. Hence, the subgroup S_1 cannot be used for a component reduction of (11.3). As a consequence, no subgroup of G that contains the transformation e_1 can be used for this purpose.

The same statement is true for the transformations e_2, $\hat{p}$, $\hat{q}$, $\hat{p}e_2$, $\hat{q}e_1$ and $\hat{p}\hat{q}$. Finally, any subgroup of G containing one of the elements e_1, e_2, $\hat{p}$, $\hat{q}$, $\hat{p}e_2$, $\hat{q}e_1$ or $\hat{p}\hat{q}$ gives a trivial reduction.

11.4.2 Reductions in three components

The tuple (c_{ij}) is invariant under the subgroup S_8 generated by the transformation $\hat{p}\hat{q}e_1$ if and only if the following equalities hold:

$$c_{0,-1} = c_{01}, \quad c_{-1,-1} = -c_{-11}, \quad c_{1,-1} = -c_{11}.$$

This transformation allows us to reduce (11.3) to the nontrivial five-component system

$$
\begin{aligned}
\frac{\mathrm{d}c_{10}}{\mathrm{d}t} &= 2\left(\frac{1}{k^2+l^2} - \frac{1}{l^2}\right) klc_{11}c_{01} & \frac{\mathrm{d}c_{11}}{\mathrm{d}t} &= \left(\frac{1}{l^2} - \frac{1}{k^2}\right) klc_{10}c_{01}\\
\frac{\mathrm{d}c_{-10}}{\mathrm{d}t} &= -2\left(\frac{1}{k^2+l^2} - \frac{1}{l^2}\right) klc_{-11}c_{01} & \frac{\mathrm{d}c_{-11}}{\mathrm{d}t} &= \left(\frac{1}{k^2} - \frac{1}{l^2}\right) klc_{-10}c_{01}\\
\frac{\mathrm{d}c_{01}}{\mathrm{d}t} &= \left(\frac{1}{k^2+l^2} - \frac{1}{k^2}\right) kl\left[c_{10}c_{-11} - c_{11}c_{-10}\right].
\end{aligned}
\tag{11.4}
$$

The subgroup S_5 leads to a similar reduction under equating

$$c_{0,-1} = -c_{01}, \quad c_{-1,-1} = c_{-11}, \quad c_{1,-1} = c_{11}.$$

11.4.3 Reductions in four components

Utilizing the subgroup $S_6 = \{1, e_1e_2\}$ by means of similar consideration as in the previous sections leads to the following nontrivial reduced system of (11.3) after equating $c_{-11} = c_{1,-1}$, $c_{-1,-1} = c_{11}$, $c_{0,-1} = c_{01}$ and $c_{-10} = c_{10}$:

$$\begin{aligned} \frac{\mathrm{d}c_{11}}{\mathrm{d}t} &= \left(\frac{1}{l^2} - \frac{1}{k^2}\right) klc_{10}c_{01} & \frac{\mathrm{d}c_{10}}{\mathrm{d}t} &= \left(\frac{1}{k^2+l^2} - \frac{1}{l^2}\right) kl[c_{11} - c_{1,-1}]c_{01} \\ \frac{\mathrm{d}c_{1,-1}}{\mathrm{d}t} &= \left(\frac{1}{k^2} - \frac{1}{l^2}\right) klc_{10}c_{01} & \frac{\mathrm{d}c_{01}}{\mathrm{d}t} &= \left(\frac{1}{k^2+l^2} - \frac{1}{k^2}\right) kl\left[c_{1,-1} - c_{11}\right]c_{10}. \end{aligned} \tag{11.5}$$

Similar four-component reductions are also given by the non-primed subgroups S_7 ($c_{-11} = -c_{1,-1}$, $c_{-1,-1} = -c_{11}$, $c_{0,-1} = c_{01}$ and $c_{-10} = -c_{10}$) and S_9 ($c_{-11} = c_{1,-1}$, $c_{-1,-1} = c_{11}$, $c_{0,-1} = -c_{01}$ and $c_{-10} = -c_{10}$).

11.4.4 Maximal reduction: subgroup S_{10}

Finally, let us derive the reduced system associated with the subgroup S_{10}. This reduced system coincides with the Lorenz-1960 model. Since all transformations mutually commute, there are three equivalent ways for the reduction. The first way is to perform the reduction in a single step from system (11.3) using the subgroup S_{10}. Because the system (11.5) possesses the symmetry transformation induced by $\hat{p}\hat{q}e_1$, we can also start with (11.5) and compute the induced component reduction. Alternatively, it is possible start with (11.4) and apply the transformation induced by e_1e_2. This will give the same reduced system associated with the subgroup S_{10}.

The subgroups S_{11} and S_{12} lead to similar reductions connected with the Lorenz' reduction via transformations generated by $p_{\pi/2k}$ and $q_{\pi/2l}$. Therefore, up to symmetries of (11.3) induced by the symmetries of the vorticity equation, there is a unique reduction in five components.

However, let us note before that we have already employed the transformation

$$(x, y, t, \psi) \to (-x, -y, t, \psi)$$

which accounts for the identification $c_{-\mathbf{m}} = c_{\mathbf{m}}$. Observe that the connection between the complex and real Fourier coefficients is given by

$$a_{\mathbf{m}} = c_{\mathbf{m}} + c_{-\mathbf{m}}, \qquad b_{\mathbf{m}} = i(c_{\mathbf{m}} - c_{-\mathbf{m}}).$$

Hence we can already set

$$c_{\mathbf{m}} = \frac{1}{2}a_{\mathbf{m}}.$$

This explains the first observation by Lorenz [84]: If the imaginary parts of the Fourier coefficients vanish initially, they will vanish for all times. Mathematically, this is justified since we have used the symmetry transformation group S_6.

Now employing the transformation $\hat{p}\hat{q}e_1$ at this stage of simplification, we have the identification

$$c_{1,-1} = -c_{11} \quad \Leftrightarrow \quad a_{1,-1} = -a_{11}.$$

This is the second observation by Lorenz [84], namely that if $a_{1,-1} = -a_{11}$ initially, then the equality will hold for all times. That is, Lorenz heuristically discovered the symmetry transformation $\hat{p}\hat{q}e_1$ to obtain his famous finite-mode model.

It is straightforward to see that the resulting system leads to the Lorenz system upon setting $A = a_{01}, F = a_{10}, G = a_{1,-1}$:

$$\frac{\mathrm{d}A}{\mathrm{d}t} = -\left(\frac{1}{k^2} - \frac{1}{k^2 + l^2}\right) klFG$$

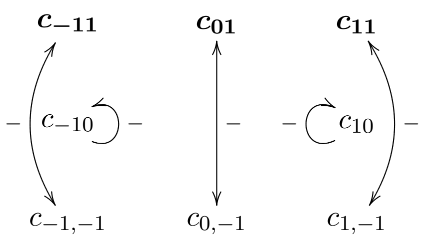
$$\frac{\mathrm{d}F}{\mathrm{d}t} = \left(\frac{1}{l^2} - \frac{1}{k^2 + l^2}\right) klAG$$

$$\frac{\mathrm{d}G}{\mathrm{d}t} = -\frac{1}{2}\left(\frac{1}{l^2} - \frac{1}{k^2}\right) klAF.$$

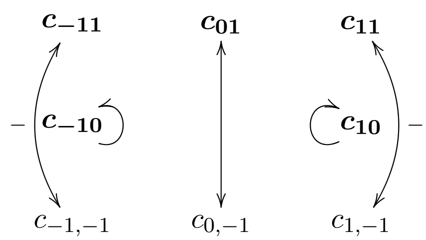
The Lorenz-1960 system is thus the result of invariance of the system (11.3) under the subgroup S_{10}. Hence it is truly the maximum simplification that can be obtained. We have exhausted all the symmetry groups consisting of chosen discrete symmetry transformations and leading to *nontrivial* reduced systems.

Appendix: Depiction of symmetry transformations

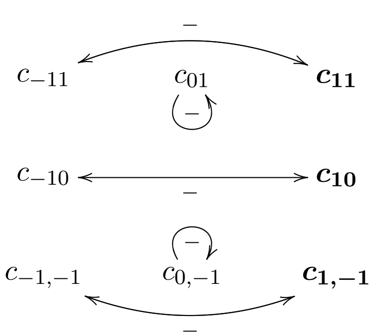
For the readers' convenience, we graphically illustrate some of the symmetry transformations used for the component reduction. An arrow between two components means the identification of the respective coefficients. If there is a minus sign next to an arrow, the corresponding identification is up to minus. For any arrow starting and ending in the same coefficient, the identification up to minus means that this coefficient vanishes. The coefficients remained in the finite-mode models after the identification are displayed in bold.



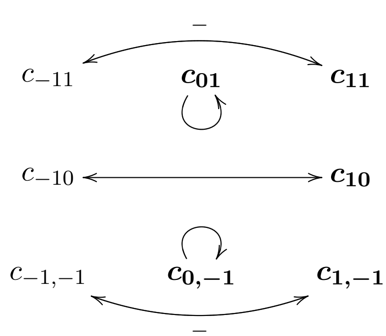
Symmetry transformation e_1



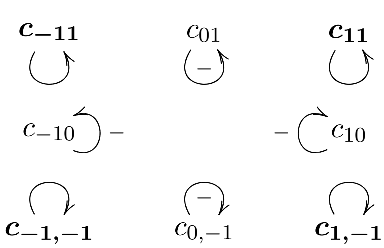
Symmetry transformation $\hat{p}\hat{q}e_1$



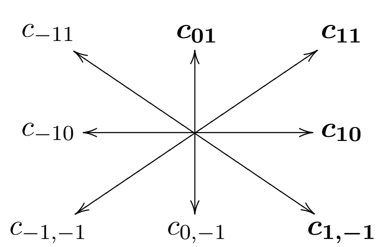
Symmetry transformation e_2



Symmetry transformation $\hat{p}\hat{q}e_2$



Symmetry transformation $\hat{p}\hat{q}$



Symmetry transformation e_1e_2

Chapter 12

Minimal atmospheric finite-mode models preserving symmetry and generalized Hamiltonian structures

Abstract A typical problem with the conventional Galerkin approach for the construction of finite-mode models is to keep structural properties unaffected in the process of discretization. We present two examples of finite-mode approximations that in some respect preserve the geometric attributes inherited from their continuous models: a three-component model of the barotropic vorticity equation known as Lorenz' maximum simplification equations [Tellus, **12**, 243–254 (1960)] and a six-component model of the two-dimensional Rayleigh–Bénard convection problem. It is reviewed that the Lorenz–1960 model respects both the maximal set of admitted point symmetries and an extension of the noncanonical Hamiltonian form (Nambu form). In a similar fashion, it is proved that the famous Lorenz–1963 model violates the structural properties of the Saltzman equations and hence cannot be considered as the maximum simplification of the Rayleigh–Bénard convection problem. Using a six-component truncation, we show that it is again possible retaining both symmetries and the Nambu representation in the course of discretization. The conservative part of this six-component reduction is related to the Lagrange top equations. Dissipation is incorporated using a metric tensor.

12.1 Introduction

Various models of the atmospheric sciences are based on nonlinear partial differential equations. Besides numerical simulations of such models, it has been tried over the past fifty years to capture at least some of their characteristic features by deriving reduced and much simplified systems of equations. A common way for deriving such reduced models is based on the Galerkin approach: One expands the dynamic variables of a model in a truncated Fourier (or some other) series, substitutes this expansion into the governing equations and studies the dynamics of the corresponding system of ordinary differential equations for the expansion coefficients. Although the number of expansion coefficients is usually minimal to allow for an analytic investigation, these reduced models have been used in order to explain some common properties of atmospheric models.

To the best of our knowledge there is up to now no universal criterion for the selection of modes or the choice of truncation of the series expansion. However, at least some cornerstones for the Galerkin approach are already settled. It is desirable for finite-mode models to retain structural properties of the original set of equations, from which they are derived [44, 45]. Such properties are, e.g., quadratic nonlinearities, conservation of energy and one or more vorticity quantities in the nondissipative limit and preservation of the Hamiltonian form.

Recently, an extension of the Hamiltonian structure based on the idea of Nambu [104] to incorporate multiple conserved quantities in a system representation also came into focus. It was shown in [16, 106, 108, 110, 141, 146] that various equations of ideal hydrodynamics and magneto-hydrodynamics allow for a formal Nambu representation. It therefore seems reasonable to derive finite-mode models that also retain this structure. Moreover, almost all models in the atmospheric sciences possess symmetry properties. These symmetries should thus be taken into account in low-dimensional modeling too, which is an issue in the field of equivariant dynamical systems (see, e.g., [48]).

The general motivation for this work is that low-order models are still in widespread use in the atmospheric sciences. It has been mentioned above that their original purpose was to identify characteristic features of the atmospheric flow in the pre-supercomputer era. While the advent of supercomputers partially renders this aim obsolete, finite-mode models are still valuable for testing advanced methods in the atmospheric sciences, related to issues of predictability, ensemble prediction, data assimilation or stochastic parameterization [5, 99, 119, 120]. Such finite-mode models offer the possibility for a conceptual understanding of techniques that are to be used in comprehensive atmospheric numerical models later on. For such testing issues, in turn, it is essential to have finite-mode models that preserve the structure of the underlying set of partial differential equations at least in some minimal way.

In this paper we give two examples of finite-mode models that retain the above mentioned features of their parent model: The first is the three-component Lorenz–1960 model, derived as the maximum simplification of the vorticity equation [84]. The second is a six-component extension of the Lorenz–1963 model [85]. The authors are aware that there exists a great variety of other finite-mode (Lorenz) models, such as e.g. [86, 87, 88], possessing richer geometric structure and allowing to address other important issues in the atmospheric sciences, such as the existence of a slow manifold, atmospheric attractors, balanced dynamics and the initialization problem of numerical weather prediction. Results in these directions can be found, besides in the original papers by Lorenz, e.g., in [25, 27, 159, 162]. The choice to investigate the Lorenz–1960 and Lorenz–1963 models, however, is reasonable since the latter still is one of the most prominent finite-mode models used in dynamic meteorology for testing issues as reviewed above. As we are going to show, the Lorenz–1963 model in various respects does not constitute a sound geometric model, the derivation of a revised version of this system appears to be well justified. The Lorenz–1960 model, on the other hand has been chosen as it is the simplest system for which the techniques to be applied in this paper can be demonstrated.

The Lorenz–1963 model is a dissipative model and as such it necessarily violates conservative properties. On the other hand this is a rather typical situation for more comprehensive atmospheric numerical models too. Usually, the conservative dynamical core of such models is coupled to a number of dissipative processes such as friction, precipitation and radiation. Nonetheless, it is a necessary condition that the numerics for the dynamical core itself do not violate the structural properties of the underlying conservative dynamics [165]. Any valuable toy model of the atmosphere should reflect this, e.g. by consisting of the superposition of a conservative part and a dissipative part. This is one of the guiding principles for our derivation of the generalized Lorenz–1963 model.

The organization of the paper is as follows: Properties of discrete and continuous Nambu mechanics are briefly reviewed in section 12.2. Section 12.3 includes a description of the Lorenz–1960 model, establishing its Nambu structure and its compatibility with the admitted point symmetries of the barotropic vorticity equation. In section 12.4, it is shown that the Lorenz–1963 model is neither compatible with the corresponding Nambu (Hamilton) form of the Saltzman convection equations nor with its point symmetries. We hereafter identify the maximum simplification of the Saltzman convection equations [145] that reflects both symmetries and the proper Nambu structure of the continuous model. Finally, in section 12.5 we sum up our results and discuss some open questions.

12.2 Nambu mechanics

Since Nambu mechanics emerged from discrete Hamiltonian mechanics, it is convenient to start with a short description of the latter. The evolution equation of a general n-dimensional Hamiltonian system is given by

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \{F, H\},$$

where $F = F(z_i)$ is an arbitrary function of the phase space variables z_i, $i = 1, \ldots, n$, H is the Hamiltonian function and $\{.,.\}$ is a Poisson bracket, which satisfies bilinearity, skew-symmetry and the Jacobi identity. For discrete Hamiltonian systems, the Poisson bracket is characterized by an antisymmetric rank two tensor that can depend on the coordinates of the underlying phase space. In modern Hamiltonian dynamics, this tensor is allowed to be singular, leading to the notion of a *Casimir* function C, which Poisson-commutes with all arbitrary functions $G(z_i)$

$$\{C, G\} = 0, \quad \forall\, G.$$

Setting $G = H$, it follows that every Casimir is in particular also a conserved quantity.

Guided by Liouville's theorem stating volume-preservation in phase space, Nambu [104] proposed a formalism for discrete mechanical systems allowing *multiple* conserved quantities to determine, at the same level of significance, the evolution of a dynamical system. More precisely, let us consider a point mechanical system with n degrees of freedom and $n-1$ functionally independent conserved quantities $H_j, j = 1, \ldots, n-1$. The evolution equation for an arbitrary function F according to Nambu is

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \frac{\partial(F, H_1, H_2, \ldots, H_{n-1})}{\partial(z_1, z_2, \ldots, z_n)} =: \{F, H_1, H_2, \ldots, H_{n-1}\}.$$

The above bracket operation is called *Nambu bracket*, which due to the properties of the Jacobian is non-singular, multi-linear and totally antisymmetric. It was demonstrated in [151], that a Nambu bracket also fulfills a generalization of the Jacobi identity, which reads

$$\begin{aligned}\{\{F_1, \ldots, F_{n-1}, F_n\}, F_{n+1}, \ldots, F_{2n-1}\} + \{F_n, \{F_1 \ldots, F_{n-1}, F_{n+1}\}, F_{n+2}, \ldots, F_{2n-1}\}\\ + \cdots + \{F_n, \ldots, F_{2n-2}, \{F_1, \ldots, F_{n-1}, F_{2n-1}\}\} = \{F_1, \ldots, F_{n-1}, \{F_n, \ldots, F_{2n-1}\}\}\end{aligned} \tag{12.1}$$

for any set of $2n-1$ functions F_i. Various discrete models that allow for a Nambu formulation were identified, e.g. the free rigid body [104], a system of three point vortices [106], and the conservative Lorenz–1963 model [109], which is discussed in some detail below.

It appears that the application of ideas of discrete Nambu mechanics to field equations was first considered in [14] (and even earlier in a talk [93]), and later independently by Névir and Blender [108]. It was noted that the singularity of many continuous Poisson brackets of fluid mechanics may be formally removed by extending them to tribrackets using explicitly *one* of their Casimir functionals as additional conserved quantity. That is, despite the fact that partial differential equations represent systems with infinitely many degrees of freedom, up to now there only exist models using one additional conserved quantity. This way, the term *continuous Nambu mechanics* (referring to a Nambu representation of field equations) is at once misleading, though it is already used in several papers.

The restriction to tribrackets may be traced back to the underlying Lie algebras on which the Poisson brackets in Eulerian variables are based on [90, 101]. Hence, the fixed relation between the dimension of the phase space and the number of conserved quantities used for a system representation is lost in continuous Nambu mechanics. In the atmospheric sciences, this generalization is called *energy-vorticity theory*, as the employed Casimir functional is frequently related to some vortex integral. Since in the atmospheric sciences the evolution of the rotational wind field is dominant over different scales, the energy-vorticity description may be well suited for a better understanding of e.g. turbulence. Among others, models that can be cast into energy-vorticity form include the inviscid non-divergent 2d and 3d barotropic vorticity equations, the quasi-geostrophic potential vorticity equation and the governing equations of ideal fluid mechanics as well as equations of magnetohydrodynamics [16, 106, 107, 110, 141].

The main problem with continuous Nambu mechanics is that it is up to now not clear whether it is possible to state an appropriate condition analog to the generalized Jacobi identity (12.1) of discrete Nambu mechanics. While this is obviously a serious point assessing the self-reliance of continuous Nambu mechanics compared to usual noncanonical Hamiltonian field theory, for the application of the Nambu formalism this point is not of prior importance. Indeed, the main benefit of a continuous Nambu formulation so far lies in the possibility of the construction of conservative numerical schemes. Namely, numerically conserving the antisymmetric Nambu bracket automatically leads to a numerical conservation of energy and the second constitutive conserved quantity of the bracket. This allows to explain the construction of the celebrated Arakawa discretization [8] of the Jacobian operator and enables to generalize the Arakawa method in a systematic way to other models possessing a Nambu representation [42, 143, 144, 146]. That is, although the Nambu form might appear to be an algebraic curiosity from the theoretical point of view, it is nevertheless of high value in the numerical application. This is also the main reason, why we aim to care about this form in the course of the present paper. The generalized Lorenz–1963 model derived in section 12.4 is of Nambu form and hence it automatically conserves certain representatives of both classes of Casimir functionals of the Rayleigh-Bénard convection equations.

It could be argued, that the numerical preservation of only one additional conserved quantity is only a little success in view of the infinite number of conserved quantities of two-dimensional ideal hydrodynamics. As was demonstrated in [2], not only energy, circulation and enstrophy are statistically relevant for the large scale behavior of ideal fluid mechanics, but also is the third integrated power of the vorticity. Though this objection can hardly be rebutted, the authors are not aware of any truncation conserving an appropriate number of conserved quantities besides the method proposed by Zeitlin [164, 165], which however can neither be adopted for all models in fluid mechanics nor in arbitrary geometries. Although similar objections also hold against the Nambu bracket approach, the Nambu discretization method nevertheless might be considered as an enrichment of existing numerical methods in fluid mechanics (see [37, 147] for a discussion of the Nambu discretization in relation to the statistics). Moreover, the Nambu bracket approach goes beyond various numerical methods for Hamiltonian field equations, in which solely conservation of energy can be assured.

12.3 Structural properties of the vorticity equation

The inviscid barotropic vorticity equation on the f-plane for an incompressible flow can be written in form of a conservation law

$$\frac{\partial \zeta}{\partial t} = -\left[\psi, \zeta\right], \tag{12.2}$$

where $\psi(t, x, y)$ is the stream function generating two-dimensional nondivergent flow, $\zeta = \nabla^2 \psi$ is the vorticity and $[a, b] = \partial a/\partial x \, \partial b/\partial y - \partial a/\partial y \, \partial b/\partial x$ denotes the Jacobian.

12.3.1 The Nambu structure

Eqn. (12.2) possesses an infinite number of conserved quantities, i.e. kinetic energy and all moments of vorticity are preserved (see, e.g., [10]). In [108], the Nambu (or energy-vorticity) bracket

$$\{\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3\} := -\int_\Omega \frac{\delta \mathcal{F}_1}{\delta \zeta} \left[\frac{\delta \mathcal{F}_2}{\delta \zeta}, \frac{\delta \mathcal{F}_3}{\delta \zeta}\right] \mathrm{d}f, \tag{12.3}$$

was introduced for arbitrary functionals $\mathcal{F}_i[\zeta], i = 1, \ldots, 3$. In the above equation, $\delta/\delta\zeta$ denotes the usual variational derivative, $\mathrm{d}f = \mathrm{d}x\mathrm{d}y$ is the area element to be integrated within the 2D-domain Ω. Using appropriate boundary conditions (e.g. cyclic), it can be shown that the above bracket is totally antisymmetric. Geometrically, this Nambu bracket is essentially a reformulation of the singular Lie–Poisson bracket of ideal fluid mechanics, which is based on the infinite-dimensional Lie algebra associated to the group of area preserving diffeomorphisms on Ω.

Using the bracket (12.3) it is possible to reformulate eqn. (12.2) as

$$\frac{\partial \zeta}{\partial t} = \{\zeta, \mathcal{E}, \mathcal{H}\}. \tag{12.4}$$

In the above equation, $\mathcal{H}$ and $\mathcal{E}$ denote the global conserved quantities energy and the second moment of vorticity (enstrophy), respectively, which are given by

$$\mathcal{H}[\zeta] = \frac{1}{2}\int_\Omega (\nabla\psi)^2 \mathrm{d}f, \qquad \mathcal{E}[\zeta] = \frac{1}{2}\int_\Omega \zeta^2 \mathrm{d}f.$$

12.3.2 Maximum simplification

One pioneering work in the field of finite-mode approximations was done by Lorenz [84], who introduced a minimal system of hydrodynamic equations based on (12.2). Using a severe truncation of the Fourier series expansion of ζ, the following set of ordinary differential equations for the remaining three modes A, F, G was derived:

$$\begin{aligned}
\frac{\mathrm{d}A}{\mathrm{d}t} &= \left(\frac{1}{k^2+l^2} - \frac{1}{k^2}\right) klFG, \\
\frac{\mathrm{d}F}{\mathrm{d}t} &= \left(\frac{1}{l^2} - \frac{1}{k^2+l^2}\right) klAG, \\
\frac{\mathrm{d}G}{\mathrm{d}t} &= \frac{1}{2}\left(\frac{1}{k^2} - \frac{1}{l^2}\right) klAF,
\end{aligned} \tag{12.5}$$

where k, l are constant wave numbers. We now show, that this model preserves in some sense the structure of its continuous counterpart, as the above equations can be derived directly from the spectral Nambu bracket of the barotropic vorticity equation. For this purpose, we expand ζ in a double Fourier series on the torus

$$\zeta(c) = \sum_{\mathbf{M}} c_{\mathbf{M}} e^{i\mathbf{M}\cdot\mathbf{x}},$$

where $\mathbf{M} = (m_1 k, m_2 l)^{\mathrm{T}}$ and $\mathbf{x} = (x, y)^{\mathrm{T}}$ are the wavenumber and position vector, respectively. Moreover, $c_{\mathbf{M}} = c^{\dagger}_{\mathbf{-M}} = \frac{1}{2}(A_{m_1,m_2} - iB_{m_1,m_2})$, where $\dagger$ denotes the complex conjugate. The variational derivative may be expanded as

$$\frac{\delta\mathcal{F}[\zeta]}{\delta\zeta(c)} = \frac{1}{\Omega}\sum_{\mathbf{M}} \frac{\partial\mathcal{F}}{\partial c_{\mathbf{M}}} e^{-i\mathbf{M}\cdot\mathbf{x}},$$

where in the right hand side we consider $\mathcal{F}$ as a smooth function of $c_{\mathbf{M}}$. Plugging these expressions into eqn. (12.3) we find the spectral form of the energy–vorticity bracket

$$\{\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3\} = \sum_{\mathbf{K},\mathbf{M}} kl(m_1 k_2 - m_2 k_1) \frac{\partial\mathcal{F}_1}{\partial c_{\mathbf{K}}} \frac{\partial\mathcal{F}_2}{\partial c_{-(\mathbf{M}+\mathbf{K})}} \frac{\partial\mathcal{F}_3}{\partial c_{\mathbf{M}}}. \tag{12.6}$$

To simplify the notation, we have assumed that Ω is the unit square. Enstrophy and energy in their spectral representations are given by $\mathcal{E} = 1/2\sum_{\mathbf{K}} c_{\mathbf{K}} c_{-\mathbf{K}}$ and $\mathcal{H} = 1/2\sum_{\mathbf{K}} \frac{1}{\mathbf{K}^2} c_{\mathbf{K}} c_{-\mathbf{K}}$, respectively. To obtain eqn. (12.5) from bracket (12.6), we have to truncate the conserved quantities $\mathcal{E}$ and $\mathcal{H}$ on the set of indices $k_1, k_2 \in \{-1, 0, 1\}$ under the following restrictions introduced by Lorenz: (*i*) If the coefficients are real at the onset of evolution they remain real for all times. (*ii*) If $c_{1,1} = -c_{1,-1}$ at the onset of evolution this relation holds true for all times. Introducing the new variables $A = \mathrm{Re}(c_{0,1})$, $F = \mathrm{Re}(c_{1,0})$ and $G = \mathrm{Re}(c_{1,-1})$, it is straightforward to recover eqn. (12.5) from eqn. (12.6). The maximum simplification of the vorticity equation in Nambu form then reads

$$\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}t} = kl(\nabla_{\mathbf{z}}\mathcal{E} \times \nabla_{\mathbf{z}}\mathcal{H}) =: \{\mathbf{z}, \mathcal{E}, \mathcal{H}\},$$

with $\mathbf{z} = (A, F, G)^{\mathrm{T}}$, where

$$\mathcal{E} = \frac{1}{2}(A^2 + F^2 + 2G^2), \quad \mathcal{H} = \frac{1}{4}\left(\frac{A^2}{l^2} + \frac{F^2}{k^2} + \frac{2G^2}{k^2+l^2}\right).$$

The above Nambu bracket is based on the Lie–Poisson bracket of $\mathfrak{so}(3)$, turning the Lorenz–1960 model into a particular form of the free rigid body equations. It therefore satisfies all properties of discrete Nambu mechanics. In this respect, the continuous Nambu bracket structure of the vorticity equation passes over to the discrete Nambu bracket structure of the free rigid body. Note, however, that the Lorenz–1960 model represents a restricted class of a free rigid body, as only two moments of inertia are independent. Moreover, it was demonstrated in [19] that the above truncation also respects the maximal set of admitted point symmetries in spectral space. This set of symmetries consists of discrete mirror transformations $(t, x, y, \psi) \mapsto (t, x, -y, -\psi)$ and $(t, x, y, \psi) \mapsto (t, -x, y, -\psi)$, together with combinations of shifts by π in both x and y direction. These shifts are the admitted spectral counterparts of translational symmetries in physical space. In particular, using these transformations the above two observations by Lorenz can be naturally interpreted as conditions of symmetry. That is, for the Lorenz–1960 model preservation of symmetries and preservation of the Nambu structure are mutually compatible. Due to the Nambu representation, eqns. (12.5) also satisfy Liouville's theorem.

As the Lorenz–1960 model inherits the Nambu structure of the vorticity equation and the selection of modes can be justified using the admitted point symmetries of the continuous equation, the notion of a maximum simplification may be regarded as appropriate.

12.4 Structural properties of the Saltzman equations

In this section, we discuss the structural properties of the convection model derived by Saltzman [145]. That is, we discuss the admitted point symmetries and Nambu form and derive the maximum simplification that retains these properties in a minimal form.

The Saltzman equations we base our investigation on read in nondimensional form [55]:

$$\frac{\partial \zeta}{\partial t} + [\psi, \zeta] = R\sigma \frac{\partial T}{\partial x} + \sigma \nabla^4 \psi, \qquad \frac{\partial T}{\partial t} + [\psi, T] = \frac{\partial \psi}{\partial x} + \nabla^2 T. \tag{12.7}$$

As before, ψ is a stream function generating two-dimensional nondivergent flow in the x–z-plane, $\zeta = \nabla^2 \psi$ is the vorticity, T is the temperature departure from a linear conduction profile, σ is the Prandtl number, R is the Rayleigh number and $[a, b] := \partial a/\partial x \, \partial b/\partial z - \partial a/\partial z \, \partial b/\partial x$ denotes the Jacobian.

In what follows, we aim to distinguish between dissipative and nondissipative systems. In the former, we preserve the form of equations as given in (12.7), while in the latter we neglect terms $\nabla^4 \psi$, $\nabla^2 T$. Note, however, that in the second case a different definition of R and σ arises, see [145] for details.

Let us consider the domain $\Omega = [-L, L] \times [0, 1]$. The boundary conditions we adopt are free-slip boundaries at both the top and the bottom of the fluid

$$\psi(t, x, z = 0) = \psi(t, x, z = 1) = 0, \qquad \zeta(t, x, z = 0) = \zeta(t, x, z = 1) = 0,$$

together with

$$T(t, x, z = 0) = T(t, x, z = 1) = 0.$$

Although it could be argued that non-slip boundaries in the vertical would be more natural to this viscous problem, the above choice is motivated to be able to incorporate the Lorenz–1963 model, which is based on free-slip boundaries. In x-direction there are different possibilities, e.g. periodic, free-stress or non-slip boundaries.

12.4.1 Symmetries

We are interested in point symmetries of system (12.7). For this purpose, let us for the moment neglect the impact of the boundary conditions. To compute the maximal Lie invariance algebra we used the Maple package DESOLV [28]. The maximal Lie invariance algebra reads

$$\begin{aligned}&\mathcal{D} = 2t\partial_t + x\partial_x + z\partial_z - (3T - 4Rz)\partial_T, \qquad \partial_t, \qquad \partial_z, \qquad \mathcal{Z}(g) = g(t)\partial_\psi,\\&\mathcal{X}_1(f) = f(t)\partial_z + f(t)R\partial_T + f'(t)x\partial_\psi, \qquad \mathcal{X}_2(h) = h(t)\partial_x - h'(t)z\partial_\psi,\end{aligned} \tag{12.8a}$$

where f, g, h run through the set of real-valued time-dependent functions. Hence, system (12.7) admits scalings, shifts in t and z, respectively, gauging of the stream function and generalized Galilean boosts in z- and x-direction, respectively. Moreover, there are two independent discrete symmetries given by

$$e_1\colon\ (t,x,z,\psi,T)\mapsto(t,x,-z,-\psi,-T), \quad e_2\colon\ (t,x,z,\psi,T)\mapsto(t,-x,z,-\psi,T). \tag{12.8b}$$

The presence of boundary conditions usually restricts the number of admitted symmetries strongly. In the symmetry analysis of differential equations, boundary value problems are rarely considered (see [23] for a discussion of this problem). On the other hand, for Rayleigh–Bénard convection, the consideration of boundaries obviously cannot be omitted. Therefore, we now single out those symmetries, which are admitted by the boundary value problem. We are only interested in symmetries acting on the space geometry of the problem. This is reasonable as transformations acting solely on t, ψ or T in the course of a series expansion do not place restrictions on the Fourier coefficients and thus cannot be used as a criterion for the selection of modes. This at once allows to exclude the transformations generated by ∂_t and $\mathcal{Z}(g)$ from our considerations. Moreover, scaling generated by $\mathcal{D}$ in any case would change the fixed geometry in z-direction, so we can exclude it too. The most general transformation generated by the remaining basis operators in combination with the discrete symmetries is given by

$$(t,x,z,\psi,T)\mapsto(t,\delta_2(x+h\varepsilon_2),\delta_1(z+f\varepsilon_1+\varepsilon_3),\delta_1\delta_2(\psi+f'x\varepsilon_1-h'z\varepsilon_2),\delta_1(T+Rf\varepsilon_1)),$$

where $\varepsilon_i\in\mathbb{R}$ and $\delta_j\in\{-1,1\}$. Acting on the boundaries in z-direction, it is straightforward to determine those transformations preserving their values:

$$\begin{aligned}&(t,x,z,\psi,T)\mapsto(t,\delta_2(x+\varepsilon_2),z,\delta_2\psi,T),\\&(t,x,z,\psi,T)\mapsto(t,\delta_2(x+\varepsilon_2),1-z,-\delta_2\psi,-T).\end{aligned} \tag{12.9}$$

It is now necessary to specify the boundaries in x-direction. A natural choice in the atmospheric sciences are periodic boundary conditions. This way, shifts in x-direction are admitted. On the other hand, this choice singles out the second discrete symmetry, i.e. we have $\delta_2 = 1$. This is the set of point symmetries on which we subsequently base our truncation.

12.4.2 The Nambu structure

To make this paper self-contained, we restate some results given in [16], slightly adapted for the special form of (12.7). The conservative part of system (12.7) can be represented in continuous Nambu form by

$$\begin{aligned}\frac{\partial\zeta}{\partial t} &= -\left[\frac{\delta\mathcal{C}}{\delta T},\frac{\delta\mathcal{H}}{\delta\zeta}\right]-\left[\frac{\delta\mathcal{C}}{\delta\zeta},\frac{\delta\mathcal{H}}{\delta T}\right] = \{\zeta,\mathcal{C},\mathcal{H}\},\\\frac{\partial T}{\partial t} &= -\left[\frac{\delta\mathcal{C}}{\delta\zeta},\frac{\delta\mathcal{H}}{\delta\zeta}\right] = \{T,\mathcal{C},\mathcal{H}\},\end{aligned} \tag{12.10a}$$

with conserved quantities

$$\mathcal{H} = \int_\Omega \left(\frac{1}{2}(\nabla\psi)^2 - R\sigma T z\right) \mathrm{d}f, \quad \mathcal{C} = \int_\Omega \zeta(T-z)\,\mathrm{d}f, \tag{12.10b}$$

representing the total energy and a circulation-type quantity, respectively. In (12.10a), the Nambu bracket $\{\cdot,\cdot,\cdot\}$ is defined for arbitrary functionals $\mathcal{F}_i = \mathcal{F}_i[\zeta, T], i = 1,\dots,3$ by the equation

$$\{\mathcal{F}_1,\mathcal{F}_2,\mathcal{F}_3\} := -\int_\Omega \left(\frac{\delta\mathcal{F}_1}{\delta T}\left[\frac{\delta\mathcal{F}_2}{\delta\zeta},\frac{\delta\mathcal{F}_3}{\delta\zeta}\right] + \frac{\delta\mathcal{F}_1}{\delta\zeta}\left(\left[\frac{\delta\mathcal{F}_2}{\delta T},\frac{\delta\mathcal{F}_3}{\delta\zeta}\right] + \left[\frac{\delta\mathcal{F}_2}{\delta\zeta},\frac{\delta\mathcal{F}_3}{\delta T}\right]\right)\right)\mathrm{d}f. \tag{12.10c}$$

This Nambu bracket is based on the semi-direct product extension of the Lie algebra of area-preserving diffeomorphisms with the vector space of real-valued functions on Ω. The process of extension of a Lie–Poisson bracket is usually done for systems that incorporate more than one field variable. There are different ways how to extend a Lie algebra (see [155] for an excellent overview), but the semi-direct extension is common for systems where one variable is advected by another (as is the temperature departure by the ψ-field). Using the Nambu bracket form, the semi-direct product structure can be cast in completely symmetric form.

We note that there is a second class of Casimir functionals given by

$$\mathcal{S}_g = \int_\Omega g\,\mathrm{d}f, \tag{12.10d}$$

where g is an arbitrary function of $T-z$. This class of Casimirs is not needed for the conservative Nambu representation. However, it plays an important role for a geometric incorporation of dissipation. Namely, using the generalized free energy

$$\mathcal{G} = \mathcal{H} - \mathcal{S} = \int_\Omega \left(\frac{1}{2}(\nabla\psi)^2 - R\sigma T z - \frac{1}{2}R\sigma(T-z)^2\right)\mathrm{d}f, \tag{12.10e}$$

where $\mathcal{S}$ is a realization of the class of Casimirs $\mathcal{S}_g$, it is possible to represent the dissipative system (12.7) via

$$\frac{\partial\zeta}{\partial t} = \{\zeta,\mathcal{C},\mathcal{G}\} + \langle\zeta,\mathcal{G}\rangle, \quad \frac{\partial T}{\partial t} = \{T,\mathcal{C},\mathcal{G}\} + \langle T,\mathcal{G}\rangle, \tag{12.10f}$$

where

$$\langle\mathcal{F}_1,\mathcal{F}_2\rangle := -\int_\Omega \left(\sigma\frac{\delta\mathcal{F}_1}{\delta\zeta}\nabla^4\frac{\delta\mathcal{F}_2}{\delta\zeta} + \frac{1}{R\sigma}\frac{\delta\mathcal{F}_1}{\delta T}\nabla^2\frac{\delta\mathcal{F}_2}{\delta T}\right)\mathrm{d}f$$

is the symmetric bracket of dissipation, which is briefly discussed in the section below.

A delicate problem is to state appropriate boundary conditions making the Nambu bracket (12.10c) twofold antisymmetric. While the antisymmetry $\{\mathcal{F}_1,\mathcal{F}_2,\mathcal{F}_3\} = -\{\mathcal{F}_1,\mathcal{F}_3,\mathcal{F}_2\}$ is always satisfied due to the properties of the Jacobian operator, the antisymmetry $\{\mathcal{F}_1,\mathcal{F}_2,\mathcal{F}_3\} = -\{\mathcal{F}_2,\mathcal{F}_1,\mathcal{F}_3\}$ follows from an integration by parts. Hence, the boundary conditions must be specified in a way such that the resulting boundary terms vanish. For the specified boundary conditions in the vertical and periodical boundaries in x-direction, the Nambu bracket is indeed completely antisymmetric.

12.4.3 Maximum simplification

In this part, we derive the minimal model of (12.7) that retains both point symmetries and the associated Nambu form of the continuous equations. There are several attempts to extend the famous Lorenz–1963 model [85]. The motivation for these extensions is that the Lorenz–1963 model does not represent characteristic features of Rayleigh–Bénard convection properly, as noted e.g. in [32]. Several authors have tried to improve the Lorenz–1963 model by attaching additional modes, but in various cases this lead to models exhibiting nonphysical behavior such as violation of energy or vorticity conservation (e.g. [58]). The problem of energy conservation was solved in [153] where a universal criterion for the truncation to energy-conserving finite-mode models was established. Moreover, truncation to systems in coupled gyrostat form [44, 45] may also lead to models that retain the conservation properties of the original equations. We also note, that a single gyrostat is a Nambu system and hence using such a truncation, conservation of the underlying geometry may be implemented at least in some minimal form.

It was shown in [109] that the conservative part of the Lorenz–1963 model allows for a Nambu representation via

$$\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}t} = \nabla_{\mathbf{z}}\mathcal{H}_1 \times \nabla_{\mathbf{z}}\mathcal{H}_2,$$

where $\mathbf{z} = (x, y, z)^{\mathrm{T}}$. The conserved quantities are

$$\mathcal{H}_1 = \frac{1}{2}x^2 - \sigma z, \quad \mathcal{H}_2 = \frac{1}{2}y^2 + \frac{1}{2}z^2 - rz,$$

where $r = R/R_c$, with R_c being the critical Rayleigh number. However, these two conserved quantities are proportional to spectral forms of energy and $\int_\Omega (T-z)^2 \mathrm{d}f$, respectively, while the spectral expansion of $\mathcal{C}$ under the Lorenz ansatz gives identically zero. Therefore, the Lorenz–1963 truncation only allows for a Nambu form that is not directly related to the continuous Nambu form presented before. Moreover, if we also try to justify the selection of modes of the Lorenz–1963 system using point symmetries, we find that it would be necessary to simultaneously use the symmetries e_2 and shift in x-direction by 1, which in any case would violate the boundary conditions. That is, the selection of modes is not natural from the symmetry point of view in this case.

Additionally, the Lorenz–1963 truncation does not account for the semi-direct product structure of the bracket of the continuous equations. An appropriate discrete realization of this semi-direct product structure is given by the special Euclidean algebra $\mathfrak{se}(3) = \mathfrak{so}(3) \ltimes \mathbb{R}^3$. The associated Lie–Poisson bracket on the dual $\mathfrak{se}(3)^*$ forms the basis of the Hamiltonian (or Nambu) representation of the heavy top equations in the body frame. The Lie–Poisson bracket reads [56, 90]

$$\{F, G\} = -\mathbf{\Pi} \cdot (\nabla_{\mathbf{\Pi}} F \times \nabla_{\mathbf{\Pi}} G) - \mathbf{\Gamma} \cdot (\nabla_{\mathbf{\Pi}} F \times \nabla_{\mathbf{\Gamma}} G + \nabla_{\mathbf{\Gamma}} F \times \nabla_{\mathbf{\Pi}} G),$$

where $\mathbf{\Pi}$ and $\mathbf{\Gamma}$ denote the vectors of angular momentum and the direction of gravity as seen from the body, respectively.

The heavy top model consists of three equations governing the evolution of angular momentum and three equations for the characterization of the direction of gravity as seen from the body. The maximum simplification of the Saltzman model based on the above Lie–Poisson bracket therefore needs a six-component reduction.

We now proceed with the construction of the modified Lorenz–1963 model. The expansion in Fourier series that is compatible with the specified boundary conditions is

$$\begin{aligned}
\psi &= \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} (\phi_{nm} \sin an\pi x + \varphi_{nm} \cos an\pi x) \sin m\pi z, \\
T &= \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} (\vartheta_{nm} \sin an\pi x + \theta_{nm} \cos an\pi x) \sin m\pi z,
\end{aligned} \tag{12.11}$$

where a is the inverse aspect ratio.

For the selection of modes for the six-component model, we employ the concept of symmetry in a similar fashion as in [19, 32, 55]. In these papers it was demonstrated that the admitted point symmetries of the original set of differential equations impose restricting conditions on the Fourier expansion that have to be taken into account in the course of the derivation of finite-mode models. As discussed above, the equations governing Rayleigh–Bénard convection admit an infinite-dimensional symmetry group with finite-dimensional subgroup (12.9) preserving the boundary value problem. However, not all symmetry transformations included in (12.9) may be used for a selection of modes since we cannot use all shifts in x-direction as the spectral space is essentially discrete.

For the selection of modes, we aim to use the transformations

$$\begin{aligned}
t_1\colon\ & (t, x, z, \psi, T) \mapsto (t, x + 1/a, z, \psi, T), \\
t_2\colon\ & (t, x, z, \psi, T) \mapsto (t, x, 1 - z, -\psi, -T).
\end{aligned}$$

The task is now to compute the corresponding implications of these transformations on the Fourier coefficients, which follow from a straightforward application to the expansion (12.11). The transformation t_1 e.g. implies

$$\psi = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} (\phi_{nm} \sin(an\pi x + n\pi) + \varphi_{nm} \cos(an\pi x + n\pi)) \sin m\pi z,$$

and similarly for the transformation of T. Hence t_1 leads to the spectral transformations $(\phi_{nm}, \varphi_{nm}, \vartheta_{nm}, \theta_{nm}) \mapsto ((-1)^n\phi_{nm}, (-1)^n\varphi_{nm}, (-1)^n\vartheta_{nm}, (-1)^n\theta_{nm})$. In a similar fashion, the transformation t_2 is treated. The corresponding transformations in spectral space hence read

$$\begin{aligned}
t_1\colon\ & (\phi_{nm}, \varphi_{nm}, \vartheta_{nm}, \theta_{nm}) \mapsto ((-1)^n\phi_{nm}, (-1)^n\varphi_{nm}, (-1)^n\vartheta_{nm}, (-1)^n\theta_{nm}), \\
t_2\colon\ & (\phi_{nm}, \varphi_{nm}, \vartheta_{nm}, \theta_{nm}) \mapsto ((-1)^m\phi_{nm}, (-1)^m\varphi_{nm}, (-1)^m\vartheta_{nm}, (-1)^m\theta_{nm}).
\end{aligned} \tag{12.12}$$

These two transformations give a restriction on the admitted modes since for all $n = 2k - 1$ and $m = 2k - 1$, respectively, the corresponding Fourier coefficients would violate the symmetry property and hence are not allowed in the truncation.

The discrete symmetry group generated by the transformations (12.12) is $G = \{e, t_1, t_2, t_1t_2\}$, where e denotes the identity transformation. By exhaustively studying the implications of subgroups of G on truncations of (12.11), we can derive different low-dimensional models in a similar fashion as was done in [19] for the barotropic vorticity equation. The list of nontrivial subgroups of G is given by $S_1 = \{e, t_1\}$, $S_2 = \{e, t_2\}$ and $S_3 = \{e, t_2t_3\}$.

Since we already know that the model to be derived must have six coefficients, it remains to select them in accordance with the above subgroups. The first six nonvanishing coefficients under consideration of the subgroup S_1 are φ_{01}, θ_{01}, ϕ_{21}, φ_{21}, ϑ_{21} and θ_{21}. Since these coefficients do not include those of the original Lorenz–1963 model, the resulting model will not be considered here. Selecting the modes using the subgroup S_2 and G itself also does not allow to incorporate the Lorenz–1963 model.

The remaining possibility is given by the subgroup S_3, leading to the choice of coefficients ϕ_{11}, φ_{11}, ϑ_{11}, θ_{11}, φ_{02}, θ_{02}. This choice incorporates both the Lorenz–1963 model and the model in [32] and also gives a sound justification for the selection of modes. In addition to the symmetries, we also aim to preserve the semi-direct product structure of the Nambu representation (12.10c). For this purpose, the selection of the above listed coefficients based on symmetry considerations is still too general. It is necessary to scale these coefficients appropriately. Setting

$$\begin{aligned} &\phi_{11} = bA, && \varphi_{11} = bB, && \varphi_{02} = cC, \\ &\vartheta_{11} = eD, && \theta_{11} = eE, && \theta_{02} = fF, \end{aligned}$$

and plugging the corresponding truncation of (12.11) into the conservative part of system (12.7), it is found that the scaling coefficients have to satisfy

$$c = \frac{1}{2b}, \quad e = \frac{a^3}{\pi^2(1+a^2)}, \quad f = \frac{2a^3}{\pi^2 b^2 (1+a^2)^2},$$

in order to allow for a Nambu representation of heavy top form:

$$\{\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3\} := -\nabla_{\mathbf{\Gamma}}\mathcal{F}_1 \cdot \nabla_{\boldsymbol{\pi}}\mathcal{F}_2 \times \nabla_{\boldsymbol{\pi}}\mathcal{F}_3 - \nabla_{\boldsymbol{\pi}}\mathcal{F}_1 \cdot (\nabla_{\mathbf{\Gamma}}\mathcal{F}_2 \times \nabla_{\boldsymbol{\pi}}\mathcal{F}_3 + \nabla_{\boldsymbol{\pi}}\mathcal{F}_2 \times \nabla_{\mathbf{\Gamma}}\mathcal{F}_3).$$

Here, $\boldsymbol{\pi} = (A, B, C)^{\mathrm{T}}$ and $\mathbf{\Gamma} = (D, E, F)^{\mathrm{T}}$ are the fluid mechanical analogs of the vector of angular momentum and the direction of gravity as seen from the body, respectively.

The resulting six-component model then reads

$$\begin{aligned} \frac{\mathrm{d}A}{\mathrm{d}t} &= \frac{a}{2b\pi(1+a^2)}((a^2-3)\pi^3 BC + 2eR\sigma E), \\ \frac{\mathrm{d}B}{\mathrm{d}t} &= -\frac{a}{2b\pi(1+a^2)}((a^2-3)\pi^3 AC + 2eR\sigma D), \\ \frac{\mathrm{d}C}{\mathrm{d}t} &= 0, \\ \frac{\mathrm{d}D}{\mathrm{d}t} &= \frac{a\pi}{2be}(e\pi CE - 2b^2 f\pi BF - 2b^2 B), \\ \frac{\mathrm{d}E}{\mathrm{d}t} &= -\frac{a\pi}{2be}(e\pi CD - 2b^2 f\pi AF - 2b^2 A), \\ \frac{\mathrm{d}F}{\mathrm{d}t} &= \frac{abe\pi^2}{2f}(BD - AE). \end{aligned} \tag{12.13}$$

The conserved quantities (12.10b) are correspondingly

$$\begin{aligned} \mathcal{H} &= \frac{1}{4ab^2\pi}((1+a^2)b^4\pi^3(A^2+B^2) + 2\pi^3 C^2 + 4Rb^2 f\sigma F), \\ \mathcal{C} &= -\frac{\pi}{2ab}((1+a^2)b^2 e\pi(AD+BE) + 4f\pi CF + 4C). \end{aligned}$$

Using both the heavy top Nambu bracket and the conserved quantities, finite-mode model (12.13) can be cast in Nambu from

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \{\mathbf{x}, \mathcal{C}, \mathcal{H}\},$$

where $\mathbf{x} = (A, B, C, D, E, F)^{\mathrm{T}}$. Note, that this Nambu bracket formulation is now structurally completely analog to the continuous Nambu bracket (12.10c) of the convection equations, as it is based on the Lie–Poisson bracket using the Lie algebra $\mathfrak{se}(3)$. In particular, by fixing one argument in the heavy top Nambu bracket one naturally recovers all the Hamiltonian properties of the Lie–Poisson system, e.g. the Jacobi identity. Bracket (12.10c) therefore also automatically conserves the corresponding truncated forms of the second class of Casimirs (12.10d), such as

$$\mathcal{S} = \frac{R\sigma}{12a\pi}\left(3e^2\pi(D^2+E^2) + 6f^2\pi F^2 + 12fF\right).$$

It is remarkable that the above model has now a mechanical interpretation similar to the Lorenz–1960 model. As the Lorenz–1960 model is a restricted class of the free rigid body equations, the finite-mode model (12.13) in turn may be considered as a restricted class of the heavy top equations that is referred to as *Lagrange top* [56]. This is a heavy top with two moments of inertia being equal (a symmetric top) with the position vector of the center-of-mass pointing in C-direction (which leads to $\mathrm{d}C/\mathrm{d}t = 0$). Correspondingly, a number of results valid for the Lagrange top might already be passed over to the model (12.13), such as issues of stability [56] or numerical algorithms preserving the Lie–Poisson structure of the above bracket [39]. Moreover, due to the additional conserved quantity $C = \mathrm{const}$ the Lagrange top is a prominent example of a Liouville integrable system [11, 43]. In addition, since $\partial \dot{x}_i/\partial x_i = 0$, $\forall i = 1,\ldots,6$, the above set of equations also satisfies the Liouville theorem.

Note that the term linear in C (resp. F) in the expression for $\mathcal{C}$ (resp. $\mathcal{S}$) arises due to the use of variable T instead of $\tilde{T} = T - z$ and correspondingly do the terms linear in A and B in the fourth and fifth equation, respectively.

If $C = 0$ at the onset of evolution, the above model reduces to the five-component model given in [32] upon rescaling of the Fourier modes. Although in any case $C = \mathrm{const}$ during evolution, we find it nevertheless important to retain this component in the above model. Firstly, it enables to cast the reduced model in Lagrange top form. Secondly, the selection of modes based on symmetries does not allow to truncate the Saltzman equations to a five component model, since there is no additional criterion that permits one to predict a priori whether to chose φ_{02} or θ_{02}. Hence, both coefficients must be incorporated in the Fourier series expansion.

The extension to a Nambu-metriplectic finite-mode model is straightforward. Following [100], a discrete metric system can defined via $\mathrm{d}\mathbf{z}/\mathrm{d}t = g\nabla_{\mathbf{z}}P = \langle \mathbf{z}, P\rangle$, where $\mathbf{z}$ is the phase space vector, P denotes a phase space function and g is a tensor. It is further required that $\langle F_1, F_2\rangle = \langle F_2, F_1\rangle, \forall F_1, F_2$, which in turn enforces g to be symmetric. In our case, using the above truncation and correspondingly the generalized free energy (12.10e) as phase space functional, we can incorporate the dissipative terms upon using

$$g = -2a\,\mathrm{diag}\left(\frac{\sigma}{b^2}, \frac{\sigma}{b^2}, 2\sigma b^2, \frac{\pi^2(1+a^2)}{Re^2\sigma}, \frac{\pi^2(1+a^2)}{Re^2\sigma}, \frac{2\pi^2}{Rf^2\sigma}\right)$$

as metric tensor. Then, attaching $g\nabla_{\mathbf{x}}\mathcal{G}$ to system (12.13) gives the maximum simplification of the dissipative Saltzman equations, which reads

$$\begin{aligned}
\frac{\mathrm{d}A}{\mathrm{d}t} &= \frac{a}{2b\pi(1+a^2)}((a^2-3)\pi^3 BC + 2eR\sigma E) - (1+a^2)\pi^2\sigma A,\\
\frac{\mathrm{d}B}{\mathrm{d}t} &= -\frac{a}{2b\pi(1+a^2)}((a^2-3)\pi^3 AC + 2eR\sigma D) - (1+a^2)\pi^2\sigma B,\\
\frac{\mathrm{d}C}{\mathrm{d}t} &= -4\pi^2\sigma C,\\
\frac{\mathrm{d}D}{\mathrm{d}t} &= \frac{a\pi}{2be}(e\pi CE - 2b^2 f\pi BF - 2b^2 B) - (1+a^2)\pi^2 D,\\
\frac{\mathrm{d}E}{\mathrm{d}t} &= -\frac{a\pi}{2be}(e\pi CD - 2b^2 f\pi AF - 2b^2 A) - (1+a^2)\pi^2 E,\\
\frac{\mathrm{d}F}{\mathrm{d}t} &= \frac{abe\pi^2}{2f}(BD - AE) - 4\pi^2 F,
\end{aligned} \tag{12.14}$$

or in the more compact Nambu-metriplectic form

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \{\mathbf{x}, \mathcal{C}, \mathcal{G}\} + \langle \mathbf{x}, \mathcal{G}\rangle.$$

System (12.14) can be considered as a damped Lagrange top and its Nambu-metriplectic form completes the geometric picture of the maximal structure-preserving truncation of the dissipative Saltzman equations (12.10f) discussed in the present paper.

It was noted in [54] that any vorticity field under the Boussinesq approximation has to satisfy the balance equation:

$$\frac{\partial}{\partial t}\int_0^1\int_0^{2/a} \zeta \,\mathrm{d}x\mathrm{d}z = \sigma\left[\frac{\partial}{\partial z}\int_0^{2/a}\zeta\mathrm{d}x\right]\Bigg|_{z=0}^{z=1}.$$

Straightforward computation for the six-component model shows that this balance equation is identically satisfied for any value of the Prandtl number.

It should be emphasized that the results in this section were derived under the assumption of $\delta_2 = 1$ in (12.9). This choice was enforced due to the use of periodic boundary conditions, which are of obvious importance in geophysical fluid dynamics. On the other hand, the alternative choice of $\delta_2 = -1$ would not allow to derive the above discrete convection model, as it would require a different selection of Fourier modes (not including the Lorenz–1963 model). However, this is quite natural as a physical realization admitting this reflection symmetry involves sidewalls at $x = -L, L$ [55]. For such a configuration, the generic Nambu structure (12.10c) has to be supplemented with boundary term contributions, since the second antisymmetry relying on integration by parts is then not automatically fulfilled any more. As usual in continuous Hamiltonian and Nambu mechanics, the presence of nontrivial boundary conditions complicates the appropriate formulation of the models, which in the present case also passes over to the finite-mode simplification.

12.5 Conclusion and outlook

In this paper we have addressed the problem of maximum simplification of atmospheric models by a discussion of the Lorenz–1960 and Lorenz–1963 model. It was reviewed that the Lorenz–1960 model is indeed the maximum simplification of the inviscid barotropic vorticity equation that preserves both point symmetries as well as the inherited Nambu structure of the continuous counterpart. This way, we have also implicitly shown that the selection of modes in accordance with symmetries is compatible with the inherited Nambu form of the discrete model. Inspired from this tutorial example, the Lorenz–1963 model was investigated too. It was found that this model neither preserves the proper Nambu structure nor is it compatible with respect to the underlying symmetries. This may serve as an additional justification of reported unphysical behavior of this model. The proposed extension of the Lorenz–1963 model is a six-component truncation that also includes the model [32] as special case. This model is constructed using a subgroup of the symmetry group of the Saltzman equations preserving the boundary value problem. Moreover, the semi-direct product structure of the Lie–Poisson (or Nambu) bracket of the field equations is retained, hence the model is automatically energy- and Casimir-conserving in the nondissipative limit. This again implicitly shows that for the presented six-component truncation both the Nambu structure and the admitted point symmetries are compatible. Incorporation of dissipation leads to a discrete Nambu-metriplectic model, also conserving the symmetric structure given by the metric part of the continuous bracket. This compatibility of the six-component model with geometric structures may be considered beneficial in view of the testing issues reviewed in section 12.1, for which the newly derived model could be employed. Moreover, due to the preservation of important geometric structures, both models that were presented in this paper may deserve the notion of a maximum simplification.

If one aims to use finite-mode models for physical purposes and not only as toy models, the question of structure-preserving extensions of such minimal systems of equations is of certain interest. It was indicated in section 12.2 that there is merely the method of Zeitlin that allows to construct fully Hamiltonian finite-mode approximations of 2d fluid mechanics. In this method, series of approximated n-dimensional models are derived on the two-dimensional torus T^2, which in the limit $n \to \infty$ converge to the vorticity equation and the equations of a Boussinesq stratified fluid, respectively [164, 165]. This convergent sequence of finite-dimensional models exists due to the property that for a certain representation of $\mathrm{SU}(n)$, in the limit of $n \to \infty$, the group of area-preserving diffeomorphisms (and its semi-direct extension by a vector space, respectively) is recovered. One main benefit of this method is that for each finite-mode model a maximal number of Casimirs is preserved, which makes such models very attractive, e.g. for the investigation of statistical properties of fluid mechanical systems [2]. For more general settings than T^2 or for three-dimensional fluid mechanics, however, the question of a connection between finite- and infinite-dimensional Lie–Poisson systems is not fully answered yet. This points to another still unsolved question, namely whether it is possible to relate discrete and continuous Nambu mechanics in some natural way. Although such a relation was established for two very low-dimensional models in the present paper, the problem of a proper extension of these minimal models without violating the Nambu structure has not been tackled yet. Furthermore, it would be interesting to apply the method used in this paper to other models of fluid mechanics. This way, a list of maximal simplified structure-preserving models could be established. This should be the issue of forthcoming work.

Chapter 13

Summary and conclusions

The main aim of this part of the thesis was to present some examples of how point symmetries can help to derive low-dimensional models of differential equations arising in the atmospheric sciences. Although this is also an issue in the theory of equivariant dynamical systems, we find it necessary to put a stronger focus on a systematic symmetry analysis. Without the precise knowledge of point symmetries of the given differential equation, it might be impossible to exhaustively construct all possible low-dimensional simplifications with a fixed number of coefficients as shown in Chapter 11 for the barotropic vorticity equation. This in turn implies that it is necessary to compute all point symmetries of the given equation first, which leads us back to the field of classical (Lie) symmetry analysis.

In the present thesis we solely focused on spectral models. This restriction was motivated primarily since the overwhelming majority of atmospheric low-dimensional models is based on Fourier or other orthogonal series expansions. This allowed us to investigate both the Lorenz–1960 model and the Lorenz–1963 model using symmetry techniques. The problem of finding an appropriate truncation of the Fourier series is illustrated by various attempts to generalize the Lorenz–1963 model as discussed in Chapter 12. Using symmetry techniques in conjunction with the Hamiltonian and Nambu forms of the underlying field equations we were able to give a geometric motivation for the truncation of the vorticity equation to the Lorenz–1960 model as well as of the Saltzman convection equations to our newly derived extended Lorenz–1963 model.

On the other hand, symmetries can be perfectly used in order to determine appropriate discretization schemes in the usual physical space of the differential equation [81]. In fact, it is a quite recent application of symmetry methods to construct finite-difference approximations of differential equations that have symmetry properties similar to those of the original differential equations. There presently exist two notable techniques for the construction of invariant discretization schemes which explicitly use Lie symmetries. The first technique was developed by Dorodnitsyn and is based on the prolongation of the Lie symmetry generators of a differential equation to the grid points of the selected discretization mesh. Subsequently, the invariants of these prolonged generators are determined, leading to a set of *difference invariance*. Assembling the difference invariance to stable and convergent finite-difference schemes yields invariant approximations of the given differential equation. This method was applied to several $(1+1)$-dimensional evolution equations [12, 34, 35, 157]. The second method rests on the property of *moving frames* to map arbitrary functions to invariant ones [40]. It proceeds as follows. The symmetry group action of a differential equation can be easily extended to a group action on the grid points, on which the differential equation should be approximated. For this extended group action a moving frame can be constructed. The moving frame in turn can be used to transform a prescribed finite-difference approximation of this differential equation to an invariant finite-difference scheme. This way, it is possible to *invariantize* existing numerical schemes rather than setting up new schemes from scratch as done in the method by Dorodnitsyn [33, 71, 72].

The construction of invariant discretization schemes of the equations of the atmospheric sciences using the above specified methods seems promising for several reasons. Firstly, it in some way complements the number of present day's attempts to construct finite-difference approximations of differential equations that numerically retain conservation laws as discussed in Section 12.2. As both symmetries and conservation laws are important properties of differential equations, it seems reasonable to not exclusively focus on the latter. Secondly, it is presumed that hydrodynamical configurations tend to states exhibiting a high degree of symmetry [114]. Even if it might be difficult to prove this conjecture in general, having discretization schemes at hand that are able to account for invariance properties of the set of governing equations is certainly valuable. Moreover, it was described in the second part of the thesis that constructing an approximation of a differential equation always involves both discretization *and* parameterizations. Hence the construction of invariant parameterization schemes is only one step on the route to completely invariant numerical models.

The development of such invariant difference schemes for equations of the atmospheric sciences will be another primary future research perspectives we aim to start working on.

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Acknowledgments

Several people have supported me in my time as a PhD student at the Department of Meteorology and Geophysics and the Faculty of Mathematics. First of all, I should like to acknowledge the enthusiasm of my supervisor Prof. Dr. Michael Hantel and his continuous encouragement over the period of my working. With Prof. Dr. Roman O. Popovych I spend countless hours working on the central parts of this thesis. His support, expertise, patience and believe that also a meteorologist can work in the field of symmetry analysis were the essential ingredients making this thesis possible at all. Furthermore, it is a pleasure to acknowledge the various discussions on advantages and disadvantages of Nambu mechanics with Mag.[a] Katharina Brazda, Dr. Matthias Sommer and Mag. Johannes Staufer. I would also like to express my gratitude to PD Dr. Peter Névir for numerous discussions on fluid mechanics and the energy-vorticity theory, as well as for his comments on the papers about Nambu mechanics in the third part of the thesis. Similarly, I want to extend my thanks to Prof. Dr. Artur Sergyeyev and Prof. Dr. Michael Kunzinger for discussions related to some papers in the first part of the thesis. The help of Prof. Dr. Leopold Haimberger in my period as an assistant at the Department of Meteorology and his support of my work in general are acknowledged likewise. Also the participants of the seminars at the Faculty of Mathematics and at the Department of Meteorology, by their interest, gave me the opportunity to discuss about the ideas presented in the thesis. I am also very grateful to Prof. Georgy I. Burde and Prof. Anatoly G. Nikitin that they agreed to review the thesis.

The main results of this thesis were obtained in the course of three employments: (i) As assistant at the Department of Meteorology and Geophysics (2007/10/01–2008/09/30); (ii) as project member in the FWF project P20632 (2009/10/01–2009/12/31); (iii) and in the course of the DOC-fellowship *Symmetry methods in dynamic meteorology* of the Austrian Academy of Sciences (2009/01/01 – 2010/12/31). I hence acknowledge the financial support by the University of Vienna, by the Austrian Science Foundation and especially by the Austrian Academy of Sciences. The University of Vienna and the Austrian Science Foundation in addition provided a financial basis allowing me to attend several national and international conferences.

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Publications

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