MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

ASSIGNMENT 8	**MATH 2050**	WINTER 2018

SOLUTIONS

[3] 1. First observe that

$$\det(2AB^{-1}A^TB^2) = \det(2A)\det(B^{-1})\det(A^T)\det(B^2).$$

First, $\det(2A) = 2^5\det(A) = 96$. Second, $\det(B^{-1}) = \frac{1}{\det(B)} = -\frac{1}{4}$. Third, $\det(A^T) = \det(A) = 3$. Finally, $\det(B^2) = [\det(B)]^2 = 16$. Therefore,

$$\det(2AB^{-1}A^TB^2) = 96 \cdot \left(-\frac{1}{4}\right) \cdot 3 \cdot 16 = -1152.$$

[5] 2. First we reduce A to row-echelon form, keeping track of row interchanges and row multiplications. We have

$$\begin{bmatrix} 1 & -3 & -1 & 2 \\ -2 & 6 & 5 & 3 \\ -1 & 3 & -1 & 2 \\ 4 & -9 & 2 & 2 \end{bmatrix} \xrightarrow[R_4 \to R_4 - 4R_1]{\substack{R_2 \to R_2 - (-2)R_1 \\ R_3 \to R_3 - (-1)R_1}} \begin{bmatrix} 1 & -3 & -1 & 2 \\ 0 & 0 & 3 & 7 \\ 0 & 0 & -2 & 4 \\ 0 & 3 & 6 & -6 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_4} \begin{bmatrix} 1 & -3 & -1 & 2 \\ 0 & 3 & 6 & -6 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 3 & 7 \end{bmatrix} \xrightarrow{R_2 \to \frac{1}{3}R_2} \begin{bmatrix} 1 & -3 & -1 & 2 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 3 & 7 \end{bmatrix}$$

$$\xrightarrow{R_3 \to -\frac{1}{2}R_3} \begin{bmatrix} 1 & -3 & -1 & 2 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 3 & 7 \end{bmatrix} \xrightarrow{R_4 \to R_4 - 3R_3} \begin{bmatrix} 1 & -3 & -1 & 2 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 13 \end{bmatrix}.$$

Let's call this final matrix U, so

$$\det(U) = 1 \cdot 1 \cdot 1 \cdot 13 = 13.$$

We have performed one row interchange, and performed scalar multiplications of rows by $\frac{1}{3}$ and $-\frac{1}{2}$. Thus

$$\det(U) = \det(A) \cdot (-1) \cdot \frac{1}{3} \cdot \left(-\frac{1}{2}\right) = \frac{1}{6}\det(A)$$

and so

$$\det(A) = 6\det(U) = 6 \cdot 13 = 78.$$

[4] 3. (a) We set

$$\det(A-\lambda I) = \begin{vmatrix} 3-\lambda & 1 \\ 2 & 4-\lambda \end{vmatrix} = \lambda^2 - 7\lambda + 10 = (\lambda-2)(\lambda-5) = 0,$$

so $\lambda = 2$ and $\lambda = 5$. For $\lambda = 2$, $A - \lambda I$ is

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

so if $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ then $x_2 = t$ is a free variable and $x_1 = -x_2 = -t$. Thus the eigenspace corresponding to $\lambda = 2$ is the set of all vectors of the form

$$\mathbf{x} = \begin{bmatrix} -t \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

For $\lambda = 5$, $A - \lambda I$ is

$$\begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix}$$

so $x_2 = t$ is a free variable and $x_1 = \frac{1}{2}x_2 = \frac{1}{2}t$. Hence the eigenspace corresponding to $\lambda = 5$ is the set of all vectors of the form

$$\mathbf{x} = \begin{bmatrix} \frac{1}{2}t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

[4] (b) We set

$$\det(A-\lambda I) = \begin{vmatrix} 4-\lambda & -2 \\ 1 & 2-\lambda \end{vmatrix} = \lambda^2 - 6\lambda + 10 = 0,$$

so by the quadratic formula

$$\lambda = \frac{6 \pm \sqrt{36-40}}{2} = 3 \pm i.$$

For $\lambda = 3 + i$, $A - \lambda I$ is

$$\begin{bmatrix} 1-i & -2 \\ 1 & -1-i \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1-i \\ 1-i & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1-i \\ 0 & 0 \end{bmatrix}$$

because $(1-i)(-1-i) = -2$. Therefore if $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ then $x_2 = t$ is a free variable and $x_1 = (1+i)x_2 = (1+i)t$. Thus the eigenspace corresponding to $\lambda = 3 + i$ is the set of all vectors of the form

$$\mathbf{x} = \begin{bmatrix} (1+i)t \\ t \end{bmatrix} = t\begin{bmatrix} 1+i \\ 1 \end{bmatrix}.$$

For $\lambda = 3 - i$, $A - \lambda I$ is

$$\begin{bmatrix} 1+i & -2 \\ 1 & -1+i \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1+i \\ 1+i & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1+i \\ 0 & 0 \end{bmatrix}$$

so $x_2 = t$ is a free variable and $x_1 = (1-i)x_2 = (1-i)t$. Hence the eigenspace corresponding to $\lambda = 3 - i$ is the set of all vectors of the form

$$\mathbf{x} = \begin{bmatrix} (1-i)t \\ t \end{bmatrix} = t\begin{bmatrix} 1-i \\ 1 \end{bmatrix}.$$

[6] (c) We set

$$\det(A - \lambda I) = \begin{vmatrix} -1-\lambda & 2 & 3 \\ 1 & -2-\lambda & -1 \\ -2 & 4 & 4-\lambda \end{vmatrix} = -\lambda^3 + \lambda^2 + 2\lambda = -\lambda(\lambda^2 - \lambda - 2)$$
$$= -\lambda(\lambda - 2)(\lambda + 1) = 0,$$

so $\lambda = 0$, $\lambda = 2$ and $\lambda = -1$. For $\lambda = 0$, $A - \lambda I$ is

$$\begin{bmatrix} -1 & 2 & 3 \\ 1 & -2 & -1 \\ -2 & 4 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & -1 \\ -1 & 2 & 3 \\ -2 & 4 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$
$$\longrightarrow \begin{bmatrix} 1 & -2 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

so if $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ then $x_2 = t$ is a free variable, $x_3 = 0$, and $x_1 = 2x_2 + x_3 = 2t$. Thus the eigenspace corresponding to $\lambda = 0$ is the set of all vectors of the form

$$\mathbf{x} = \begin{bmatrix} 2t \\ t \\ 0 \end{bmatrix} = t\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

For $\lambda = 2$, $A - \lambda I$ is

$$\begin{bmatrix} -3 & 2 & 3 \\ 1 & -4 & -1 \\ -2 & 4 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -4 & -1 \\ -3 & 2 & 3 \\ -2 & 4 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -4 & -1 \\ 0 & -10 & 0 \\ 0 & -4 & 0 \end{bmatrix}$$
$$\longrightarrow \begin{bmatrix} 1 & -4 & -1 \\ 0 & 1 & 0 \\ 0 & -4 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -4 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so $x_3 = t$ is a free variable, $x_2 = 0$, and $x_1 = 4x_2 + x_3 = t$. Hence the eigenspace corresponding to $\lambda = 2$ is the set of all vectors of the form

$$\mathbf{x} = \begin{bmatrix} t \\ 0 \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Finally, for $\lambda = -1$, $A - \lambda I$ is

$$\begin{bmatrix} 0 & 2 & 3 \\ 1 & -1 & -1 \\ -2 & 4 & 5 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 3 \\ -2 & 4 & 5 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 3 \\ 0 & 2 & 3 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & \frac{3}{2} \\ 0 & 2 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

so $x_3 = t$ is a free variable, $x_2 = -\frac{3}{2}x_3 = -\frac{3}{2}t$, and $x_1 = x_2 + x_3 = -\frac{1}{2}t$. Hence the eigenspace corresponding to $\lambda = -1$ is the set of all vectors of the form

$$\mathbf{x} = \begin{bmatrix} -\frac{1}{2}t \\ -\frac{3}{2}t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}.$$

[6] (d) We set

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & -3-\lambda & 5 \\ 3 & -5 & 3-\lambda \end{vmatrix} = -\lambda^3 + 2\lambda^2 - 16\lambda + 32$$

$$= -\lambda^2(\lambda - 2) - 16(\lambda - 2) = -(\lambda - 2)(\lambda^2 + 16) = 0,$$

so $\lambda = 2$, $\lambda = 4i$ and $\lambda = -4i$. For $\lambda = 2$, $A - \lambda I$ is

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -5 & 5 \\ 3 & -5 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 3 & -5 & 1 \\ 0 & -5 & 5 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -\frac{5}{3} & \frac{1}{3} \\ 0 & -5 & 5 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -\frac{5}{3} & \frac{1}{3} \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

so if $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ then $x_3 = t$ is a free variable, $x_2 = x_3 = t$, and $x_1 = \frac{5}{3}x_2 - \frac{1}{3}x_3 = \frac{4}{3}t$. Thus the eigenspace corresponding to $\lambda = 2$ is the set of all vectors of the form

$$\mathbf{x} = \begin{bmatrix} \frac{4}{3}t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 4 \\ 3 \\ 3 \end{bmatrix}.$$

For $\lambda = 4i$, $A - \lambda I$ is

$$\begin{bmatrix} 2-4i & 0 & 0 \\ 0 & -3-4i & 5 \\ 3 & -5 & 3-4i \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3-4i & 5 \\ 3 & -5 & 3-4i \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{3}{5}+\frac{4}{5}i \\ 0 & -5 & 3-4i \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{3}{5}+\frac{4}{5}i \\ 0 & 0 & 0 \end{bmatrix}$$

so $x_3 = t$ is a free variable, $x_2 = \left(\frac{3}{5} - \frac{4}{5}i\right) x_3 = \left(\frac{3}{5} - \frac{4}{5}i\right) t$, and $x_1 = 0$. Hence the eigenspace corresponding to $\lambda = 4i$ is the set of all vectors of the form

$$\mathbf{x} = \begin{bmatrix} 0 \\ \left(\frac{3}{5} - \frac{4}{5}i\right) t \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 3 - 4i \\ 5 \end{bmatrix}.$$

Finally, for $\lambda = -4i$, $A - \lambda I$ is

$$\begin{bmatrix} 2+4i & 0 & 0 \\ 0 & -3+4i & 5 \\ 3 & -5 & 3+4i \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3+4i & 5 \\ 3 & -5 & 3+4i \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3+4i & 5 \\ 0 & -5 & 3+4i \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{3}{5} - \frac{4}{5}i \\ 0 & -5 & 3+4i \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{3}{5} - \frac{4}{5}i \\ 0 & 0 & 0 \end{bmatrix}$$

so $x_3 = t$ is a free variable, $x_2 = \left(\frac{3}{5} + \frac{4}{5}i\right) x_3 = \left(\frac{3}{5} + \frac{4}{5}i\right) t$, and $x_1 = 0$. Hence the eigenspace corresponding to $\lambda = -4i$ is the set of all vectors of the form

$$\mathbf{x} = \begin{bmatrix} 0 \\ \left(\frac{3}{5} + \frac{4}{5}i\right) t \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 3 + 4i \\ 5 \end{bmatrix}.$$

[6] (e) We set

$$\det(A - \lambda I) = \begin{vmatrix} -5-\lambda & 8 & -8 \\ -4 & 7-\lambda & -4 \\ 0 & 0 & 3-\lambda \end{vmatrix} = -\lambda^3 + 5\lambda^2 - 3\lambda - 9 = -(\lambda-3)^2(\lambda+1) = 0,$$

where we can use the Rational Roots Theorem and synthetic division (or long division) to carry out the factoring. Thus $\lambda = 3$ and $\lambda = -1$. For $\lambda = 3$, $A - \lambda I$ is

$$\begin{bmatrix} -8 & 8 & -8 \\ -4 & 4 & -4 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & 1 \\ -4 & 4 & -4 \\ 0 & 0 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so if $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ then $x_3 = t$ and $x_2 = s$ are free variables, and $x_1 = x_2 - x_3 = s - t$. Thus the eigenspace corresponding to $\lambda = 3$ is the set of all vectors of the form

$$\mathbf{x} = \begin{bmatrix} s-t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

For $\lambda = -1$, $A - \lambda I$ is

$$\begin{bmatrix} -4 & 8 & -8 \\ -4 & 8 & -4 \\ 0 & 0 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 2 \\ -4 & 8 & -4 \\ 0 & 0 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 2 \\ 0 & 0 & 4 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & -2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

so $x_2 = t$ is a free variable, $x_3 = 0$, and $x_1 = 2x_2 - 2x_3 = 2t$. Hence the eigenspace corresponding to $\lambda = -1$ is the set of all vectors of the form

$$\mathbf{x} = \begin{bmatrix} 2t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

[6] (f) Observe that

$$\det(A - \lambda I) = \begin{vmatrix} -5-\lambda & 8 & -8 \\ -4 & 7-\lambda & -4 \\ 0 & 0 & -1-\lambda \end{vmatrix} = -\lambda^3 + \lambda^2 + 5\lambda + 3 = -(\lambda+1)^2(\lambda - 3) = 0,$$

so $\lambda = -1$ and $\lambda = 3$. For $\lambda = -1$, $A - \lambda I$ is

$$\begin{bmatrix} -4 & 8 & -8 \\ -4 & 8 & -4 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 2 \\ -4 & 8 & -4 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 2 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

so if $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ then $x_2 = t$ is a free variable, $x_3 = 0$, and $x_1 = 2x_2 - 2x_3 = 2t$. Thus the eigenspace corresponding to $\lambda = -1$ is the set of all vectors of the form

$$\mathbf{x} = \begin{bmatrix} 2t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

For $\lambda = 3$, $A - \lambda I$ is

$$\begin{bmatrix} -8 & 8 & -8 \\ -4 & 4 & -4 \\ 0 & 0 & -4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & 1 \\ -4 & 4 & -4 \\ 0 & 0 & -4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & -4 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

so $x_2 = t$ is a free variable, $x_3 = 0$, and $x_1 = x_2 - x_3 = t$. Hence the eigenspace corresponding to $\lambda = 3$ is the set of all vectors of the form

$$\mathbf{x} = \begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$